Risk and Risk-Sharing in Two-Country Models

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Overview

Two paths to **variable Pareto weights**
- Capital market frictions
- **Recursive preferences**

Plan of attack
- Two-country model + recursive preferences
- Home bias in consumption, **stochastic volatility**

Intellectual debts
- Colacito and Croce, Kollmann, Tretvoll
- Anderson; Collin-Dufresne, Johannes, and Lochstoer
Recursive preferences
Recursive preferences

**Time aggregator**

\[ U_{jt} = V[c_{jt}, \mu_t(U_{jt+1})] = [(1 - \beta)c_{jt} + \beta\mu_t(U_{jt+1})]^{1/\rho} \]

**Certainty equivalent** function

\[ \mu_t(U_{jt+1}) = [E_t(U_{jt+1})]^{1/\alpha} \]

**Features**

- If \( c_{jt} = c \) is constant \( \Rightarrow U_{jt} = c \)
- \( V, \mu \) both homogeneous of degree one (hd1)
- Intertemporal substitution: \( IES = 1/(1 - \rho) > 0 \)
- Risk aversion: \( RA = 1 - \alpha > 0 \)
- Traditional **additive preferences** if \( \alpha = \rho \)
Recursive preferences (continued)

Intertemporal marginal rate of substitution

\[ m_{jt+1} = \beta \left( \frac{c_{jt+1}}{c_{jt}} \right)^{\rho - 1} \left( \frac{U_{jt+1}}{\mu_t(U_{jt+1})} \right)^{\alpha - \rho} \]

Epstein-Zin term is white noise plus risk adjustment

\[
\begin{align*}
\log U_{t+1} &= E_t(\log U_{t+1}) + \left[ \log U_{t+1} - E_t(\log U_{t+1}) \right] \\
\log \mu_t(U_{t+1}) &= \alpha^{-1} \log E_t(e^{\alpha \log U_{t+1}}) \\
&= E_t(\log U_{t+1}) \\
&\quad + \alpha^{-1} \left[ \log E_t(e^{\alpha \log U_{t+1}}) - E_t(\alpha \log U_{t+1}) \right]
\end{align*}
\]
Two-country model
Two-country model: technology

Production of intermediate goods

\[ y_{jt} = f(k_{jt}, z_{jt}) = [(1 - \eta)k_{jt}^\nu + \eta z_{jt}^\nu]^{1/\nu} \]
\[ y_{1t} = a_{1t} + a_{2t} \]
\[ y_{2t} = b_{1t} + b_{2t} \]

Armington aggregator for final goods

\[ c_{1t} + i_{1t} = h(a_{1t}, b_{1t}) = [(1 - \omega)a_{1t}^\sigma + \omega b_{1t}^\sigma]^{1/\sigma} \]
\[ c_{2t} + i_{2t} = h(b_{2t}, a_{2t}) \]

Capital stocks

\[ k_{jt+1} = (1 - \delta)k_{jt} + i_{jt} \]
Armington aggregator: final goods frontier

![Graph showing Armington aggregator]

- Final goods in country 2
- Final goods in country 1
- Share = 1/2
- Elasticity = 1/2
- Elasticity = 2
- Elasticity = 5
Two-country model: shocks

Productivities

\[
\begin{bmatrix}
\log z_{1t+1} \\
\log z_{2t+1}
\end{bmatrix} = 
\begin{bmatrix}
1 - \gamma & \gamma \\
\gamma & 1 - \gamma
\end{bmatrix}
\begin{bmatrix}
\log z_{1t} \\
\log z_{2t}
\end{bmatrix} + 
\begin{bmatrix}
v_{t}^{1/2} w_{1t+1} \\
v_{t}^{1/2} w_{2t+1}
\end{bmatrix}
\]

Conditional variance ("volatility")

\[
v_{t+1} = (1 - \varphi_v) v + \varphi v_{t} + \tau w_{3t+1}
\]

Innovations \{w_{1t}, w_{2t}, w_{3t}\} independent standard normal
Pareto problem
Pareto problem

Bellman equation \([s_t = (k_{jt}, z_{jt}, v_t)]\)

\[
J(U_t, s_t) = \max_{\{c_{1t}, U_{t+1}\}} V\{c_{1t}, \mu_t[J(U_{t+1}, s_{t+1})]\}
\]

\[
\text{s.t.} \quad V\{c_{2t}, \mu_t(U_{t+1})\} \geq U_t \quad (\lambda_t)
\]

plus resource constraints and shocks

Notation

- \(J\) is agent 1’s utility, \(U\) is agent 2’s utility ("promised utility")
- \(\lambda_t\) is (relative) Pareto weight

Fundamental tradeoff

- Give you more today \((c_{2t})\)
- Give you more in the future \((\mu_t(U_{t+1}))\)
Pareto problem: Pareto weight

First-order conditions

\[
c_{1t}^{\rho-1} / p_{1t} = \lambda_t^* c_{2t}^{\rho-1} / p_{2t}
\]

\[
\beta \left[ J_{t+1}/\mu_t(J_{t+1}) \right]^{\alpha-\rho} \lambda_{t+1}^* = \lambda_t^* \beta \left[ U_{t+1}/\mu_t(U_{t+1}) \right]^{\alpha-\rho}
\]

Additive case \((\alpha = \rho)\)

\[
\lambda_{t+1}^* = \lambda_t^*
\]

Otherwise

\[
\log \lambda_{t+1}^* - \log \lambda_t^* = (\alpha - \rho) \text{ [white noise + risk adjustment]}
\]
Pareto problem: consumption

First-order conditions (repeated)

\[
\frac{c_{1t}^{\rho-1}}{p_{1t}} = \lambda_t^{*} \frac{c_{2t}^{\rho-1}}{p_{2t}} \\
\beta \left[ J_{t+1}/\mu_t(J_{t+1}) \right]^{\alpha-\rho} \lambda_{t+1}^* = \lambda_t^{*} \beta \left[ U_{t+1}/\mu_t(U_{t+1}) \right]^{\alpha-\rho}
\]

Consumption and real exchange rate

\[
e_t = \frac{p_{2t}}{p_{1t}} = \lambda_t^{*} \left( \frac{c_{2t}}{c_{1t}} \right)^{\rho-1}
\]
Numerical examples: exchange economy
Computation

Method adapted from Collin-Dufresne, Johannes, and Lochstoer

- Global projection method
- Implemented in Julia for speed
- State changed from $U_t$ to $s_{at} = a_{1t}/y_{1t}$ or $\lambda^*_t$

P2C2E
Dynamics of the Pareto weight
Consumption and real exchange rate

Log Exchange Rate ($\log(p_2/p_1)$)

Additive
Recursive
Responses to impulse in productivity in country 2 (blue first)

- **Volatility (λ)**: The graph shows the response of volatility over time, with a steady decrease after an initial impulse.
- **Xi (x)**: This graph illustrates the response of xi over time, indicating a gradual decrease.
- **Lambda (λ)**: The lambda graph depicts the percentage change over time, showing a consistent decrease.
- **Alpha (a) and Beta (b)**: These graphs show the response of the parameters a and b, with a gradual decrease over time.
- **C1 and C2**: The graphs for c1 and c2 illustrate the parameters' response, indicating a decrease over time.
- **J and U**: The graphs for J and U show the response of these parameters, with a decrease over time.
Responses to impulse in volatility (blue first, red additive)
Numerical examples: production economy
Investment and volatility

[We’re working on this, harder than we thought]
Stability
Stability of the Pareto weight

What we know

- Colacito and Croce: If $\rho = \sigma = 0$, stable by theorem
- Colacito and Croce, Tretvoll: With some other parameter values, solutions seem stable

Open question

- What configurations of parameter values generate stability?
- Hint at problem: log Pareto weight close to martingale
- Hint at solution: shape of Pareto frontier ($J$ vs $U$) reflects final goods frontier
Last thought

What would you do with this material?