Two Trees the EZ Way

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This version: April 27, 2011

Abstract

We study a general equilibrium model with an endowment economy where aggregate consumption is generated by two independent trees subject to long-run risk and where the investor has Epstein-Zin (EZ) preferences. We find that excess volatility in a two-tree model with EZ preferences is due to the exposure of the assets (trees) to the long-run risk factor, while it is mainly the consequence of the behavior of the risk-free rate in case of the standard specification with constant relative risk aversion (CRRA). The presence of a long-run risk factor generates endogenous return correlation between the assets, which, in our setup, is numerically much more plausible than in the CRRA case, where the extremely high correlation is again due to the extreme behavior of the risk-free rate. Like in the CRRA case very small trees (in terms of their share of aggregate consumption or dividends) have high price-dividend ratios since they are not exposed to systematic consumption risk. With EZ preferences, however, they can nevertheless earn a non-zero risk premium for their exposure to long-run risk, which is not the case with CRRA utility. Finally, time-variation in the drift and volatility of aggregate consumption arising from the setup with two trees become priced risks with EZ preferences and add to the premia for consumption risk of the trees.

Keywords: Epstein-Zin utility, asset pricing, Lucas orchard

JEL: G12

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Earlier versions of this paper were presented at the Annual Symposium of the Society for Nonlinear Dynamics & Econometrics, 2011. The authors would like to thank the conference participants for useful comments and suggestions.
1 Introduction

Two recent innovations in the literature on theoretical asset pricing are the generalization of economies with only one source of aggregate consumption to models with 'multiple trees' and the use of more general preferences than the formerly popular constant relative risk aversion (CRRA) utility. One of the most important representative of this new class of preferences is 'recursive utility', also frequently called 'Epstein-Zin' (EZ) preferences after the two most prominent proponents (see Epstein and Zin (1989)). The key characteristic of this more general type of preferences is that it allows for a separation of the degree of risk aversion and the elasticity of intertemporal substitution (EIS) which are inseparable in a CRRA framework, where one is the inverse of the other. In this paper we study an economy with two tress and EZ preferences to link these two strands of the literature.

The special feature of models with more than one tree is that the relative size of the output of a tree (its 'consumption share') becomes an endogenous state variable. Furthermore, there are interactions and dependencies between the prices of the trees despite the fact that the fundamental consumption processes are independent (see Cochrane, Longstaff, and Santa-Clara (2008) and Martin (2009)). The reason is that the stochastic discount factor depends on aggregate consumption, i.e. the sum of the output quantities of the two trees, so that the price of one tree becomes a function of the output of the other.

When the representative agent has EZ preferences state variables are priced. This means first that there is an additional role for an endogenous state variable like the afore-mentioned consumption share of a tree, and second that further state variables are also priced. It is this fact which makes EZ preferences a popular choice in so-called long-run risk (LRR) models pioneered by Bansal and Yaron (2004), which feature in their basic form a state variable in the form of a small but persistent component in the growth rate of consumption. In addition to generating a premium for the exposure to state variables EZ preferences have the advantage over CRRA that they make the investor react in a plausible fashion to changes in the drift and volatility of consumption, in that asset prices increase with an increasing drift in consumption, while they go down when consumption volatility goes up. This proper reaction of the investor to changes in the investment opportunity set is vital if we want to explain the differential pricing of stocks by differences in their fundamentals, i.e. in their exposures to macroeconomic risk factors.

Our model setup comprises two trees, where each consumption stream is ex-
posed to diffusion risk and a stochastic long-run growth rate. The two trees generate the same type of consumption good so that aggregate consumption is the sum of the two individual consumption streams.\(^1\) As stated above the share of the first tree\(^2\) in aggregate consumption becomes an endogenous state variable, which drives the volatility and, in case the two trees have different growth rates, also the growth rate of aggregate consumption. In line with LRR models, we distinguish between (in our case two) consumption claims and (again two) financial claims, which are written on levered consumption.

We now give a brief summary of our results. It is well-known that models based on EZ preferences do not suffer from the risk-free rate puzzle, i.e. they are able to simultaneously generate levels of the risk-free rate and of the equity risk premium which are compatible with the data. This fact is important, since we find that under CRRA some effects in the two tree economy are basically driven by the risk-free rate puzzle, i.e. by too large a reaction of the risk-free rate to changes in the drift and in the volatility of aggregate consumption. These effects include an overreaction of the aggregate market to shocks in the larger tree (which generates excess volatility in the return on the market portfolio), the fact that the price-consumption ratio increases not only for very small, but also for very large trees relative to the situation when the trees are equally large (which generates excess volatility in the returns on large trees) and finally an implausibly large reaction of asset prices to changes in the long-run growth rate (which induces a very high correlation between the returns on the two financial claims of more than 85%).

In an EZ economy without exogenous state variables the risk-free rate exhibits a moderate reaction to changes in the drift and volatility of aggregate consumption. The downside is that there is also hardly any excess volatility. This seems like bad news for EZ preferences, but the alternative in a CRRA framework would be to 'abuse the risk-free rate puzzle' to generate this effect. Other stylized facts from the two tree setup with CRRA preferences, however, survive in an EZ world. Very small trees still have high price-consumption ratios due to their 'safe haven' property, since they provide a hedge against the consumption risk generated by the very large tree. Furthermore there is a positive cross-exposure of one tree to shocks in the other and thus an endogenously generated positive return correlation even when

\(^1\)Sometimes the two output goods are treated as imperfect substitutes. See Martin (2010) for an example with a two-country model.

\(^2\)Obviously the choice that the state variable is identified via the share of the first tree is without loss of generality.
dividends are uncorrelated. Finally, the expected excess return on a tree increases in its consumption share, since with increasing relative size it represents a larger share of (priced) aggregate risk.

Introducing exogenous state variables like the long-run growth factor will (again) generate excess volatility, which is now due to the exposure of the two trees to the state variables. Note that this represents a completely different channel than the one described above in the CRRA case. The joint dependence of both prices on the state variable ‘long-run growth’ also increases the correlation between the prices of the trees and the correlation between the prices of financial claims written on the trees. In case of CRRA, the negative and in absolute terms very large exposure yields counterfactually high correlations of more than 85%. With EZ preferences, the exposures of the asset prices to the long-run growth rate are positive and moderate, which generates less excess volatility, but also much more plausible levels of correlation.

With EZ-preferences, both systematic risk factors in the economy, aggregate consumption and long-run growth, are priced. This first of all increases expected excess returns on all assets. For a small asset, the exposure to the first factor vanishes, while it is of course still exposed to aggregate growth risk. Despite the latter exposure, small assets are still ‘safe havens’ with very high price-consumption ratios. Furthermore, their non-vanishing exposure to aggregate macroeconomic risk implies that small assets can earn a positive risk premium in case of EZ.\(^3\)

Furthermore, there are additional market prices of risk for consumption compared to the CRRA case. Here the basic argument is that the market prices of consumption risk increase for ‘attractive’ trees which pay off exactly when economic conditions have improved. Small trees are such attractive assets, since a positive shock to a small tree implies that its share increases, which in turn lowers aggregate consumption volatility. Similarly, trees with a higher than average growth rate or a larger exposure to the currently positive growth rate factor (or vice versa) are attractive, since a positive shock to their respective payoffs implies an increase in the growth rate of aggregate consumption.

Finally, differences between the price-consumption ratios, expected returns, or volatilities of assets are driven by differences in their size and also by differences in their exposure to long-run growth risk. With EZ preferences, expected returns

\(^3\)When the price-consumption ratio of a small tree goes to zero, its exposure to aggregate consumption risk also does not vanish and it earns a premium on this exposure, too. See also Martin (2009) for the case of an economy with CRRA.
on equally large assets are larger for those assets which have a larger exposure to priced growth risk, while risk premia are basically independent of the exposure to (non-priced) growth-risk in case of CRRA. Return volatilities are also larger for those assets which have the larger exposure to the growth rate (while basically the opposite holds true with CRRA).

In summary the analysis of the two tree economy with EZ preferences delivers all the interesting results known from the CRRA case concerning expected excess returns and excess volatility, but based on economically much more plausible mechanisms. It also produces some additional effects which are impossible to generate with CRRA utility.

The remainder of the paper is structured as follows. In Section 2 we present the model setup, the equilibrium in our economy is discussed in Section 3, before we analyze the impact of relative risk aversion, elasticity of intertemporal substitution, and some special properties of a two-tree-economy in Section 4. Section 5 concludes.

2 Model Setup

2.1 The Economy

Our economy consists of two trees which produce the single consumption good in quantities $C_1$ and $C_2$, similar to Cochrane, Longstaff, and Santa-Clara (2008), and Martin (2009). The dynamics of the two sources of consumption (‘trees’) are

$$\frac{dC_{1,t}}{C_{1,t}} = (\mu_1 + \phi_1 X_t) \, dt + \sigma_{C1}' dW_t$$  \hspace{1cm} (1)

$$\frac{dC_{2,t}}{C_{2,t}} = (\mu_2 + \phi_2 X_t) \, dt + \sigma_{C2}' dW_t.$$ \hspace{1cm} (2)

The state variable $X$ is the long-run growth rate with dynamics

$$dX_t = -\kappa_X X_t dt + \sigma_X' dW_t.$$  

In these stochastic differential equations $\sigma_{C1}$, $\sigma_{C2}$, and $\sigma_X$ are volatility vectors in $\mathbb{R}^3$, i.e. they represent the sensitivities of $C_1$, $C_2$, and $X$ to the three-dimensional Wiener process $W$. In the following, we set $\sigma_{C1}' = (\sigma_1, 0, 0)$, $\sigma_{C2}' = (0, \sigma_2, 0)$, $\sigma_X' = (0, 0, \sigma_x)$, i.e. the two consumption processes and the state variable $X$ are all locally uncorrelated.
Since the two trees produce identical goods, aggregate consumption is given by \( C_t \equiv C_{1,t} + C_{2,t} \). The consumption share of tree 1, i.e. the proportion of aggregate consumption generated by tree 1, is denoted by \( s_t = \frac{C_{1,t}}{C_{1,t} + C_{2,t}} \) with dynamics

\[
ds_t = \mu_s dt + \sigma_s dW_t,
\]

where

\[
\begin{align*}
\mu_s &= s_t(1 - s_t) [\mu_1 - \mu_2 + (\phi_1 - \phi_2) X_t - s_t \sigma_{C1} \sigma_{C1} + (1 - s_t) \sigma_{C2}^2 \sigma_{C2} + (2s_t - 1) \sigma_{C1} \sigma_{C2}] \\
\sigma_s &= s_t(1 - s_t) (\sigma_{C1} - \sigma_{C2}).
\end{align*}
\]

With our choice for the volatility vectors, one obtains

\[
\begin{align*}
\mu_s &= s_t(1 - s_t) [\mu_1 - \mu_2 + (\phi_1 - \phi_2) X_t - s_t \sigma_1^2 + (1 - s_t) \sigma_2^2] \\
\sigma_s &= \begin{pmatrix} s_t(1 - s_t) \sigma_1 \\ -s_t(1 - s_t) \sigma_2 \\ 0 \end{pmatrix}.
\end{align*}
\]

For \( s = 0 \) and \( s = 1 \), both the drift and the volatility vector of the consumption share are equal to zero, i.e. we are back in a one-tree economy. It is important to note that these boundaries are both absorbing, which means that the equilibrium solutions at the boundaries correspond to those for the one-tree case. With identical trees the consumption share has a positive drift for small \( s \) and a negative one when \( s \) is large, so that the economy tends back to equal consumption shares, but will end up at one of the absorbing boundaries in the long-run.\(^4\) The state variable \( X \) has an impact on the drift of the consumption share if and only if the two trees have different exposures to the long-run growth rate.

Aggregate consumption \( C \) follows the process

\[
\frac{dC_t}{C_t} = \mu_C dt + \sigma'_C dW_t,
\]

where

\[
\begin{align*}
\mu_C &= [s_t \mu_1 + (1 - s_t) \mu_2] + [s_t \phi_1 + (1 - s_t) \phi_2] X_t \\
\sigma_C &= s_t \sigma_{C1} + (1 - s_t) \sigma_{C2}.
\end{align*}
\]

With our choice of the volatilities, we get

\[
\sigma_C = \begin{pmatrix} s_t \sigma_1 \\ (1 - s_t) \sigma_2 \\ 0 \end{pmatrix},
\]

\(^4\)The fact that in the long run one of the two trees will dominate has already been discussed in Cochrane, Longstaff, and Santa-Clara (2008) and Martin (2009).
so that the conditional variance of aggregate consumption is given by
\[
\sigma_C' \sigma_C = s_t^2 \sigma_1^2 + (1 - s_t)^2 \sigma_2^2.
\]
Both the expected growth rate and the volatility of aggregate consumption are stochastic. In case of the growth rate this is due to its dependence on the long-run growth factor \(X\) and, for \(\mu_1 \neq \mu_2\) or \(\phi_1 \neq \phi_2\), also due to stochastic variations in the consumption share \(s\). For the conditional variance of \(C\) only \(s\) matters, while \(X\) does not have an influence. Note that if the two trees have the same volatility (i.e., \(\sigma_1 = \sigma_2\)), the conditional variance of aggregate consumption is lowest when the two trees have the same size.

### 2.2 The Investor

We assume the existence of a representative investor\(^5\) with Epstein-Zin preferences (for details, see e.g. Benzoni, Collin-Dufresne, and Goldstein (2010)) and value function
\[
J_t = E_t \left[ \int_t^\infty f(C_s, J_s) ds \right].
\]
(3)
Here \(f\) denotes the aggregator function
\[
f(C, J) = \frac{\beta C^{1 - \frac{1}{\psi}}}{\left(1 + \frac{1}{\psi}\right) \left((1 - \gamma) \cdot J\right)^{\frac{1}{\beta} - 1}} - \beta \theta J.
\]
where \(\theta = \frac{1 - \gamma}{1 - \psi}\). \(\gamma > 0\) is the coefficient of relative risk aversion, \(\psi > 0\) is the elasticity of intertemporal substitution (EIS), and \(\beta > 0\) is the investor’s time preference rate.

The CRRA case corresponds to \(\gamma = \psi^{-1}\). In the following we assume \(\gamma > 1\) and, whenever \(\gamma \neq \psi^{-1}\), we also assume \(\psi > 1\). This implies that in the EZ case \(\gamma > \frac{1}{\psi}\), so that the investor has a preference for early resolution of uncertainty. Furthermore, it implies \(\theta < 0\).

\(^5\)This is of course conceptually the same as assuming a complete market. We want to focus on the effects generated by the presence of more than one asset and on the implications of different preference specifications, not on the additional issues arising due to incompleteness.
3 Equilibrium

3.1 Wealth-Consumption Ratio

The first fundamental quantity of interest in the model is the wealth-consumption ratio, where the term ‘wealth’ stands for the value of the economy as a whole. From Equation (3), we get

\[ E_t \left[ dJ_t + f(C_t, J_t)dt \right] = 0. \] (4)

We set

\[ J_t = \frac{C_t^{1-\gamma}}{1-\gamma} \beta e^{\theta v_t}. \] (5)

It can be shown\(^6\) that the function \( v \), which is implicitly defined via (5), is the log wealth-consumption ratio. Plugging the expression for \( J \) into Equation (4) and applying Ito to determine \( dJ \) gives a partial differential equation for \( v \equiv v(s, X) \):

\[ 0 = e^{-v} - \beta + \left( 1 - \frac{1}{\psi} \right) \left[ \mu_C - 0.5 \gamma \sigma_C' \sigma_C \right] \\
+ (1 - \gamma) \sigma_C'(v_s \sigma_s + v_x \sigma_X) + v_x \mu_X + v_s \mu_s \\
+ 0.5(v_{xx} + \theta v_x^2)\sigma_X' \sigma_X + 0.5(v_{ss} + \theta v_s^2)\sigma_s' \sigma_s + (v_{sx} + \theta v_s v_x)\sigma_s' \sigma_X. \] (6)

The (natural logs of the) consumption components \( C_1 \) and \( C_2 \) are affine processes. However, their sum, aggregate consumption \( C \), will no longer be affine. This means we cannot rely on the standard affine guess for the log wealth-consumption ratio which is used, e.g., by Eraker and Shaliastovich (2008). Instead, we have to solve for \( v \) numerically.

Concerning the boundary conditions we have already seen in Section 2 that the bounds \( s = 0 \) and \( s = 1 \) of the domain for the consumption share are absorbing, since both the drift and the volatility of \( s \) are equal to zero. Consequently all the terms involving partial derivatives with respect to \( s \) vanish from (6), and we are left with an ordinary differential equation in \( x \), i.e. we are back to the one-tree economy.\(^7\)

For \( x \), there are no such natural boundaries, so that it will not be possible to plug in a known solution at the upper and lower bound of the domain. We will thus have to approximate the behavior of \( v \) in this region via its first and second derivative.\(^8\)

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\(^6\)See Appendix A for details.

\(^7\)Note that we assume a finite wealth-consumption ratio in the limiting one-tree economy. The conditions on the parameters which ensure that this is actually true are stated in Appendix B. Parlour, Stanton, and Walden (2010) provide a discussion of the more general case in an economy with CRRA utility.

\(^8\)See Section 3.3.
The solution of (6) at the boundaries for \( s \) is particularly simple in the special case of the model without the long-run growth factor, since \( x \) disappears from the equation, and we are in the very basic case of a geometric Brownian motion for the one tree generating the consumption good.\(^9\) When \( x \) is present there are basically two strategies to find a solution for the ordinary differential equation resulting from (6), either numerically or by applying log-linearization as in Campbell and Shiller (1988).\(^10\)

### 3.2 Risk-Free Rate and Market Prices of Risk

Following Benzoni, Collin-Dufresne, and Goldstein (2010) the pricing kernel in our economy is

\[
\xi_t = e^{-\beta t - (1 - \theta) \int_0^t I(X_u, s_u)^{-1} du} e^{(\theta - 1) v(s_t, X_t)} C_t^{-\gamma}.
\]

The interest rate is equal to the negative of the drift of the pricing kernel. Applying Ito to the pricing kernel yields

\[
r_t = \beta + \frac{1}{\psi} \mu C - 0.5 \gamma \left( 1 + \frac{1}{\psi} \right) \left( s_t^2 \sigma_1^2 + (1 - s_t)^2 \sigma_2^2 \right) - 0.5(1 - \theta) \left[ v_x^2 s_t^2 (1 - s_t)^2 (\sigma_1^2 + \sigma_2^2) + v_x^2 \sigma_2^2 + 2 v_x s_t (1 - s_t) (s_t \sigma_1^2 - (1 - s_t) \sigma_2^2) \right].
\]

The short rate has the standard form. The terms represent the investor’s impatience, the dependence on the growth rate of economy (multiplied by the inverse of EIS), and the desire for precautionary savings due to the uncertainty about the future evolution of aggregate consumption. With EZ preferences there is an additional precautionary savings term which is caused by uncertainty about the exogenous and endogenous state variables \( X \) and \( s \).\(^11\)

The market prices of risk are collected in the vector \( \lambda_t \) with

\[
\lambda_t = \begin{pmatrix}
\gamma s_t \sigma_1 + (1 - \theta) v_s s_t (1 - s_t) \sigma_1 \\
\gamma (1 - s_t) \sigma_2 - (1 - \theta) v_s s_t (1 - s_t) \sigma_2 \\
(1 - \theta) v_x \sigma_x
\end{pmatrix}.
\]

In the first two components of \( \lambda_t \) the respective first summand represents the standard market price of risk for covariance with aggregate consumption like in the CRRA case. The second summand is special to an economy with EZ preferences.

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\(^9\)The solution is discussed in Appendix B.1.

\(^10\)Details are given in Appendix B.2.

\(^11\)A detailed discussion of the properties of the interest rate will follow in Section 4 below.
State variables, whether endogenous or exogenous, are priced. There is thus a non-zero additional market price of risk for \( s \). For the same reason, the third component of \( \lambda_t \), which represents the market price of risk for the state variable long-run growth rate \( X \), is non-zero with \( EZ \).\(^{12}\)

### 3.3 Price-Dividend Ratios

We are not only interested in pricing the stream of aggregate consumption, i.e. in determining total wealth in the economy, but also in valuing the two consumption trees individually. In addition to them we will also price what we call financial claims, i.e. levered claims on each of the two consumption streams.

As a starting point consider a general asset with dividend stream \( D \) following the dynamics

\[
\frac{dD_t}{D_t} = \mu_D dt + \sigma_D dW_t,
\]

where the drift and the volatility can be a function of \( s \) and \( X \). The partial differential equation for the log price-dividend ratio \( w = w(s, X) \) is\(^{13}\)

\[
-r + \mu_D + \mu_w + 0.5 \sigma'_w \sigma_w - \lambda(\sigma_D + \sigma_w) + \sigma'_D \sigma_w + e^{-w} = 0,
\]

where

\[
\begin{align*}
\mu_w &= w_s \mu_s + w_x \mu_X + 0.5 w_{xx} \sigma'_X \sigma_X + 0.5 w_{ss} \sigma'_s \sigma_s + w_x \sigma'_s \sigma_X \\
\sigma_w &= w_s \sigma_s + w_x \sigma_X.
\end{align*}
\]

Concerning the boundary conditions the situation here is very similar to the one for the claim on aggregate consumption. For \( s = 0 \) and \( s = 1 \) we are back in a one-tree economy so that the asset with dividend stream \( D \) can be priced like any dividend claim in a one-tree economy.\(^{14}\)

The return on this general asset \( D \) is given as

\[
d\ln R_t = dw_t + d\ln D_t + e^{-w} dt.
\]

\(^{12}\)For a detailed discussion see Section 4.

\(^{13}\)The proof is given in Appendix A.

\(^{14}\)See Appendix B.1 for the case without a state variable and Appendix B.2 for the case with the long-run growth rate \( X \).
Plugging in the drift and volatilities of $w$ and of $D$ and using Equation (8) for the risk-free rate gives

$$d \ln R_t = \left[ r + (\sigma_D + \sigma_w)' \lambda - 0.5 (\sigma_D + \sigma_w)' (\sigma_D + \sigma_w) \right] dt$$
$$+ (\sigma_D + w_s \sigma_s + w_x \sigma_x)' dW_t.$$

As usual the expected return is given by the sum of the risk-free rate and the risk premium (and a quadratic Jensen correction term). Return volatility depends on the volatility of the dividend stream $D$ and on the volatility of the price-dividend ratio from Equation (9), which in turn depends on the volatilities of the state variables $s$ and $X$. We will use this formula below to analyze the exposures of returns to innovations in dividends and in $X$ and also to analyze expected excess returns in detail.

The first assets to be priced are of course the claims to the consumption streams generated by the two trees. We will call the price of an individual consumption stream the ‘price of the tree’. It is clear that the price-consumption ratio of a tree which accounts for the whole economy coincides with the wealth-consumption ratio of the market. For the price-consumption ratio of the tree the share of which goes to zero, we can basically have any result concerning its price-consumption ratio, including convergence towards infinity. Note that this is different from the price being infinite, since the problem here arises due to consumption going to zero.\(^{15}\)

In the spirit of LRR models we also want to value levered claims written on the two consumption streams, so-called financial claims. The dynamics of the two dividend streams $D_1$ and $D_2$ are given as

$$\frac{dD_{1,t}}{D_{1,t}} = (\mu_1 + \pi_x \phi_1 X_t) dt + \pi_\sigma \sigma_{C1} dW_t \quad (10)$$
$$\frac{dD_{2,t}}{D_{2,t}} = (\mu_2 + \pi_x \phi_2 X_t) dt + \pi_\sigma \sigma_{C2} dW_t.$$

$\pi_x$ is the leverage parameter for the drift, levering up the exposure of the consumption stream to the long-run growth rate. $\pi_\sigma$ is the analogous factor for the diffusion

\(^{15}\)When the price-consumption ratio goes to infinity in the limit $s \to 0$, we apply the following procedure to determine $w(s, X)$: On the grid for $w(s, X)$ we assume that at the boundary $s = 0$ the first partial derivative of $w(s, X)$ with respect to $s$ is proportional to $s^{-1}$ and the second is proportional to $s^{-2}$. This is certainly a rather ad-hoc approach, but the solutions obtained through it deliver very robust and plausible results. In this case, it also holds true that the exposure of the log price-consumption ratio $w(0, X)$ to shocks in aggregate consumption $C = C_2$ does not go to zero, but is equal to some finite value. For a more detailed discussion of the properties of the small asset in case of CRRA-preferences, see also Martin (2009).
volatilities. To make the two dividend streams actually more risky than consumption we assume \( \pi_x > 1 \) and \( \pi_\sigma > 1 \). Note that we choose equal leverage parameters for the two trees. The specification in (10) assumes perfect correlation between the dividend and the consumption stream. Extending this setup to allow for idiosyncratic risk components in dividend growth is straightforward.\(^{16}\) For the purpose of valuing these claims one then proceeds exactly as described above for the general claim.

4 Quantitative Analysis of the Model

4.1 Parameters

Our parameters are similar to the ones in Bansal, Kiku, and Yaron (2009). The difference between their model and ours is that they work in discrete time and also incorporate stochastic volatility as well as idiosyncratic dividend risk. Our main interest, however, is not in an exact generalization of their model to the case of two trees but rather to match the general behavior of aggregate consumption and the long-run growth rate.\(^{17}\)

The investor’s subjective rate of time preference is set to 0.02, the coefficient of relative risk aversion is \( \gamma = 10 \) and, in the case of EZ preferences, the elasticity of intertemporal substitution is \( \psi = 1.5 \).

We assume \( \mu_1 = \mu_2 = 0.018 \) and \( \sigma_1 = \sigma_2 = 0.0352 \). When both trees are identical and have the same size, the volatility of aggregate consumption is equal to 2.49%, which is the value used in Bansal, Kiku, and Yaron (2009) in a one-tree economy with a long-run growth rate and stochastic volatility.\(^{18}\)

Concerning the dynamics of \( X \) we assume \( \kappa_x = 0.3 \) and \( \sigma_x = 0.0114 \). The long-run mean of \( X \) is equal to zero. The loadings of the drifts of the two consumption sources on \( X \) are equal to one (in the base case) or zero (for the special case when one tree is not exposed to \( X \)).\(^{19}\)

\(^{16}\)We have analyzed the model for this more general setup. The results concerning those quantities we are mainly interested in here remain qualitatively unchanged.

\(^{17}\)Due to the fact that there is no \textit{exogenous} stochastic volatility in our model the overall effects are somewhat smaller than in Bansal, Kiku, and Yaron (2009). Again, we are mainly interested in the general properties of the model rather than the exact numerical output.

\(^{18}\)These parameter values also keep the individual price-consumption ratios in a one-tree economy finite. See Appendix B.1 for the relevant conditions.

\(^{19}\)Like in the case without \( X \) we made sure that the price-consumption ratios are still finite in
The leverage factor in the drift of the two dividend processes is set to $\pi_x = 2.5$, which represents the same exposure to the long-run growth factor as in Bansal, Kiku, and Yaron (2009). Our choice of $\pi_\sigma = 2.5$, the leverage for the diffusion component, is also close to the value chosen by Bansal, Kiku, and Yaron (2009).^{20}

4.2 Identical Trees, No State Variable

The model with identical trees and without the long-run growth factor $X$ naturally serves as a benchmark for the more general cases analyzed below, and the main objective here is to show how EZ preferences (as opposed to CRRA utility) work in this simplest of all setups. Figures 1 and 2 present the results for the CRRA and the EZ case, respectively.

4.2.1 Wealth-Consumption Ratio

The wealth-consumption ratio, representing the value of the claim to future aggregate consumption divided by the current level of this variable, is convex in the consumption share $s$ with CRRA utility and concave with EZ preferences. This behavior of the wealth-consumption ratio can be explained as follows. With identical trees, consumption volatility is lowest for a consumption share of $s = 0.5$. For both CRRA and EZ preferences a low consumption volatility represents a good state, since the indirect utility is decreasing in consumption volatility. Whether a good state implies higher or lower current consumption depends on whether the income effect or the substitution effect dominates. With EZ preferences and an EIS greater than one the substitution effect dominates, resulting in lower current consumption and a higher wealth-consumption ratio. For a CRRA investor with $\gamma > 1$ the EIS is less than one, so that the income effect dominates, implying a lower wealth-consumption ratio. This reaction of the wealth-consumption ratio to a decrease in volatility, however, seems counterintuitive.

The properties of the wealth-consumption ratio are closely linked to the behavior of the risk-free rate. In a one-tree economy, the expected return on the con-

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^{20}Different from Bansal, Kiku, and Yaron (2009), we do not include idiosyncratic dividend risk.
The consumption claim is \(^{21}\)

\[
\text{r} + \gamma \sigma_C' \sigma_C = \beta + \frac{1}{\psi} \mu_C - 0.5 \gamma \left(1 + \frac{1}{\psi}\right) \sigma_C' \sigma_C + \gamma \sigma_C' \sigma_C.
\]

For \(s < 0.5\) the conditional variance of consumption \(\sigma_C' \sigma_C\) is decreasing in \(s\), which implies an increase of the risk-free rate and a decrease in the risk premium. Whether the expected return, i.e., the sum of the risk-free rate and the expected excess return, increases or decreases again depends on the EIS. If it is less than one, like for our CRRA investor, the impact on the risk-free rate dominates, the expected return increases, and the wealth-consumption ratio decreases. The counter-intuitive behavior of the wealth-consumption ratio can thus be traced back to too high a sensitivity of the risk-free rate to consumption volatility and thus to the same mechanism which generates the risk-free rate puzzle. For EZ preferences and \(\psi > 1\) we obtain the opposite result, since the sensitivity of the risk-free rate to consumption volatility is now lower than the sensitivity of the risk premium.

We further find that the wealth-consumption ratio is much higher in the EZ case than for CRRA. Although this relation depends on the specific set of parameters, note that the parameter values we use are rather typical. In the CRRA case the wealth-consumption ratio drops from 7.9 in a one-tree economy to 6.5 when both trees have the same size, while for EZ it increases from 62.2 in a one-tree economy to 66.4 for trees of equal size. To get a first idea of the driving forces for this dramatic difference in levels, we look at the one-tree economy without a state variable again. In this case the wealth-consumption ratio is (see Appendix B.1)

\[
e^v = \frac{1}{\beta - \left(1 - \frac{1}{\psi}\right) \mu_C + 0.5 \gamma \left(1 - \frac{1}{\psi}\right) \sigma_C^2}
\]

It is larger with EZ than with CRRA if and only if \(\mu_C > 0.5 \gamma \sigma_C^2\), which is the case for our parametrization. The intuition for this finding comes from the properties of the risk-free rate analyzed next.

### 4.2.2 Risk-Free Rate

For the risk-free rate we obtain the standard result:

\[
\text{r}_t = \beta + \frac{1}{\psi} \mu_1 - 0.5 \gamma \left(1 + \frac{1}{\psi}\right) \left[s_t^2 + \left(1 - s_t\right)^2\right] \sigma_1^2 - \left(1 - \theta\right) \left[v_s^2 s_t^2 (1 - s_t)^2 + v_s s_t (1 - s_t)(2s_t - 1)\right] \sigma_1^2
\]

\(^{(11)}\)

\(^{21}\)See Appendix B.1 for the risk-free rate and the market prices of risk.
where we use that the trees are identical, i.e. $\mu_1 = \mu_2 = \mu_C$ and $\sigma_1 = \sigma_2$. The first three terms are due to impatience, intertemporal substitution, and precautionary savings due to consumption risk. With EZ preferences there are additional precautionary savings due to the risk generated by the state variables. However, these additional terms are numerically rather small.

The risk-free rate depends on the consumption share $s$ via the precautionary savings terms. It is the volatility of aggregate consumption which has the strongest impact here, so that the risk-free rate is largest when both trees have the same size and the volatility of aggregate consumption is consequently the smallest. Note, however, that the dependence of the risk-free rate on $s$ less pronounced with EZ (where $1/\psi$ is below one) than with CRRA (where $1/\psi = \gamma$ is above one).

The value of the short rate is around 13 to 17% with CRRA and around 2 to 2.5% with EZ. This shows the well-known fact that EZ preferences help to solve the risk-free rate puzzle, which is actually true if $\mu_C > 0.5\gamma\sigma'_C\sigma_C$. In this case, the reduction in the risk-free rate due to the lower factor multiplying the growth rate $\mu_C$ (the inverse of the EIS is $1/\psi$ instead of $\gamma$) dominates the increase due to the lower factor multiplying the variance of consumption.

The analysis of the risk-free rate also provides yet another explanation of the difference in wealth-consumption ratios between CRRA and EZ. In the one-tree economy it holds that

$$e^v = \frac{1}{r + \gamma\sigma_C^2 - \mu_C}.$$  

The risk-free rate is thus the only channel through which the EIS has an impact on the wealth-consumption ratio. The lower risk-free rate with EZ than with CRRA then consequently implies that the wealth-consumption ratio is larger with EZ than with CRRA.

### 4.2.3 Price-Consumption Ratios and Price-Dividend Ratios

The price-consumption ratios of the individual trees are convex functions of the consumption share. Given the concavity of the wealth-consumption ratio for the whole market in the EZ case, this shows that the curvature of wealth-consumption ratio and price-consumption ratio cannot be driven by the same mechanism.

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With realistic parameter values for the consumption process, this condition is met if risk aversion $\gamma$ is moderate. For our parameters, it is met for $\gamma < 28.4$.  

14
The price-consumption ratio of the small tree is very high due to its safe haven property: A small tree is not exposed to systematic consumption risk, because it does not represent a significant share of aggregate consumption (see also Martin (2009)). The price-consumption ratio of the very large tree converges towards the (much lower) wealth-consumption ratio of the market. There are, however, significant differences between CRRA and EZ. For CRRA we observe a significant increase in the price-consumption ratio for \( s \) close to one. Although the slope of the curve is much smaller than at the opposite end of the range for \( s \), the effect is still clearly visible. In contrast to this the price-consumption share goes up a lot less in the EZ case.

A technical explanation for this difference between CRRA and EZ is that, as stated above, the price-consumption ratio of the large tree converges towards the wealth-consumption ratio of the market, but has to be smaller than the latter (since the price-consumption ratio of the small tree is larger). With CRRA the convexity and the U-shape of the wealth-consumption ratio force the price-consumption ratio of large trees to be increasing in their consumption share. In the EZ case the wealth-consumption ratio is concave and therefore does not 'force' the price-consumption ratio of the large tree to increase in the consumption share.

The economic intuition for the behavior of the price-consumption ratio and for the difference between CRRA and EZ can be seen by noting that the price-consumption ratio depends on the expected return, which in turn is equal to the sum of the risk-free rate (depending on \( s \) via the precautionary savings term) and the expected excess return (where \( s \) determines the correlation between the tree and aggregate consumption, i.e. the amount of priced risk). When \( s \) increases the systematic risk in tree 1 increases, and so does its expected excess return, which leads to a decrease in the price-consumption ratio. With increasing \( s \) the riskiness of aggregate consumption first decreases (up to \( s = 0.5 \)) and then increases, so that we first observe an increasing and then a decreasing risk-free rate, which first leads to a decrease and then to an increase in the price-consumption ratio of the tree. When the tree accounts for less than half of the economy in terms of consumption, both effects lead to a decrease in the price-consumption ratio. For \( s = 0.5 \), the risk-free rate is locally independent of \( s \), and the price-consumption ratio decreases in \( s \). Finally, when the tree is large, the impact via the risk-free rate dominates and the price-consumption ratio starts to increase in \( s \).\(^{23}\) The consumption share where

\(^{23}\) The risk premium is basically linear in \( s \), while the risk-free rate is a quadratic function of the consumption share.
both effects offset each other is somewhere between 0.5 and 1.

To explain the differences between CRRA and EZ note that the impact of $s$ on the risk-free rate is larger for CRRA than for EZ, so that the increase in the price-consumption ratio for (very) large trees is stronger and can be observed over a larger range of $s$-values for CRRA than for EZ. The major economic implication here is that the more pronounced U-shape of the price-consumption ratio with CRRA is basically due to the risk-free rate puzzle again.

The prices, especially for small trees, are again larger for EZ than for CRRA preferences. To get the intuition, note that the ratio of price to current cash flow for an arbitrary asset with dividend drift $\mu_D$ and dividend volatility $\sigma_D$ in the one-tree economy is equal to $(r + \gamma \sigma_C \sigma_D - \mu_D)^{-1}$ if that term is positive, and infinity otherwise (see Appendix B.1). For the limiting case of a zero consumption share the cash flows of the small tree are not correlated with aggregate consumption, so its price-consumption ratio tends to $(r - \mu_D)^{-1}$ if that term is positive, and to infinity otherwise. For realistic parameters the risk-free rate is smaller in the EZ economy, which implies that the price-consumption ratio of the small tree is larger with EZ than with CRRA.

Next we turn to the price-dividend ratios of the financial claims. The overall picture is rather similar to the price-consumption ratios of the trees, just with more variation. That the financial claims are less valuable than the underlying tree due to their higher risk is a well-known result. The difference between the two values is largest for large trees.

As the consumption share of the underlying tree goes to zero the price-dividend ratio of the associated financial claim is a function of $r$ and $\mu_D$, while (levered) dividend risk is no longer priced. For finite prices the price-consumption ratio of the underlying tree and the price-dividend ratio of the claim will thus be equal in the limit. When the consumption share goes to one the price-dividend ratio of the financial claim is lower than the price-consumption ratio of the underlying tree due to the additional risk introduced by leverage.

Analogous to the prices of the underlying trees the prices of the financial claims are larger for EZ than for CRRA preferences based on our parameter set. Concerning the convexity and the U-shape the story is the same as for the price-consumption ratios of the trees. However, the increase in the expected excess return with increasing $s$ (leading to a decrease in the price-dividend ratio) is now stronger due to higher risk. The region where the price-dividend ratio decreases in $s$ is thus
wider than for the price-consumption ratio of the tree. For EZ we already observe monotonicity, while for CRRA there is still a narrow range where the price-dividend ratio increases again.

4.2.4 Market Prices of Risk

In the case without a state variable the vector of market prices of risk is

\[
\lambda_t = \gamma \sigma_C + (1 - \theta) v_s \sigma_s
\]

\[
= \left( \begin{array}{c} \gamma s_t \sigma_1 + (1 - \theta) v_s s_i (1 - s_i) \sigma_1 \\ \gamma (1 - s_t) \sigma_2 - (1 - \theta) v_s s_i (1 - s_i) \sigma_2 \end{array} \right). \tag{12}
\]

The first terms in the two elements of \( \lambda_t \) represent the standard premia for covariance with aggregate consumption, which are linear in the consumption share of the respective tree.

The second terms arise due to EZ preferences for which the endogenous state variable \( s \) is also priced. With identical trees the consumption share \( s \) influences the volatility of aggregate consumption only, but not the drift. The additional risk premium is then a compensation for volatility risk. It is positive for a small and negative for a large tree. The technical reason is that for \( s \in (0, 0.5) \) the partial derivative \( v_s \) is positive, while it is negative for \( s > 0.5 \). Economically, a positive shock to the small tree reduces the volatility of aggregate consumption. The economy is in a better state, and a high payoff of an asset (here the small tree) in good states implies an additional (positive) risk premium.

4.2.5 Exposures

The exposures of the assets to the risk factors determine the volatilities of the assets, the amount of (priced) risk and thus the expected excess return, and the correlations between the trees. For the return on the claim to aggregate consumption, the vector of exposures is

\[
\sigma(d \ln R_{c,t}) = \left( \begin{array}{c} s_t \sigma_1 + v_s s_i (1 - s_i) \sigma_1 \\ (1 - s_t) \sigma_2 - v_s s_i (1 - s_i) \sigma_2 \end{array} \right). \tag{13}
\]

The first summand in each of the two terms represents the exposures to aggregate consumption. With a constant wealth-consumption ratio (like in the fundamental one-tree model with CRRA) the return volatility of the asset would be equal to the volatility of the payoff stream.
In the CRRA case the convexity and the U-shape of the wealth-consumption ratio imply an excess exposure to consumption shocks in the large tree, since future payoffs will be higher and the wealth-consumption ratio increases. They also imply a reduced exposure to shocks in the small tree, where the future payoffs are also higher, but the wealth-consumption ratio decreases.

For EZ the effects go exactly the other way around, since the wealth-consumption ratio is concave and has an inverse U-shape as a function of $s$. The exposure to shocks in the large tree is reduced, since now higher future payoffs are accompanied by a decrease in the wealth-consumption ratio due to a higher volatility. At the same time, there is excess exposure to shocks in the small tree, where higher future payoffs come with an increase in the wealth-consumption ratio.

For the return on tree 1, the exposures are

$$\sigma(d \ln R_{c_1,t}) = \left( \sigma_1 + w^{(1)}_s s_t (1 - s_t) \sigma_1 - w^{(1)}_s s_t (1 - s_t) \sigma_2 \right)$$

where $w^{(1)}$ denotes its log price-consumption ratio. As usual, the first summand in the first term represents the exposure of the payoff stream. For small and medium-sized trees the price-consumption ratio is decreasing in $s$ (the consumption share of tree 1), so the exposure to shocks in its own consumption stream is reduced, while there is a positive cross-exposure to shocks in the other tree.

The difference between the two preference specifications shows up when we take a closer look at large trees. For CRRA the price-consumption ratio is increasing in $s$ in this region, which implies an excess exposure to shocks in the own consumption stream and a negative cross-exposure.$^{24}$

With EZ on the other hand the increase in the price-consumption ratio for large trees is negligible, so that we basically always see an underreaction to shocks in the own consumption stream and a positive cross-exposure. Furthermore, the price-consumption ratio of a small stock is a very steep function of $s$, so that the reduction in exposure to the own consumption stream and the cross-exposure to shocks in the other tree is large.$^{25}$ For the returns on the financial claims we obtain the same picture as for the consumption claims to trees, just with higher overall exposure due to the higher volatility of dividends relative to consumption.

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$^{24}$This effect is based on the same mechanism as the risk-free rate puzzle, so it is in a sense 'undesirable'.

$^{25}$If the price-consumption ratio of the small tree is finite for $s \to 0$, the cross-exposure goes to zero as well. If the limiting price-consumption ratio is infinite, however, the limiting cross-exposure is positive.
4.2.6 Excess Volatility

In the CRRA case the two tree setup generates excess volatility at the level of the market for aggregate consumption. It arises from the excess exposure of aggregate consumption to shocks in the large tree, which more than offsets the reduction in the exposure to the small tree. For EZ preferences the model produces an even lower volatility due to the underreaction to shocks in the large tree, but the overall effect is very small.

In terms of the price-consumption ratios of the individual trees we observe excess volatility of large trees in the CRRA case due to an overreaction to shocks in the underlying own consumption stream and a negative cross-exposure. For small trees on the other hand volatility is lower, which is basically a diversification effect arising from the reduced exposure to shocks in the own tree and a positive cross-exposure. With EZ the volatility is lower over nearly the whole range of consumption shares, since the price-consumption ratio hardly exhibits a U-shape so that there is also basically no overreaction to shocks in the own consumption stream.

As already pointed out before the excess volatility in the CRRA case is due to a counter-intuitive reaction of prices. The excess volatility of the aggregate market is rooted in the convex wealth-consumption ratio (caused by the fact that a lower volatility of aggregate consumption lowers the wealth-consumption ratio), while for large trees the reason is the increase of the price-consumption ratio for \( s \to 1 \) (caused by the pronounced sensitivity of the risk-free rate to the volatility of aggregate consumption, which is also the reason for the risk-free rate puzzle).

For EZ, prices exhibit the 'correct' behavior, which eliminates excess volatility. As we will show below in Section 4.3, such excess volatility will be generated by state variables.

4.2.7 Correlations

The return correlation of aggregate wealth with the small tree is driven by the positive cross-exposure of a small tree to innovations in the large one, i.e. the correlation is basically due to the stochastic discount factor, which depends on aggregate consumption. Aggregate consumption in turn changes mostly due to innovations in the large tree, so that these changes then also have an impact on the price of the small tree. The correlation is significantly larger with EZ than with CRRA, since for EZ the price-consumption ratio of the small tree is very steep as a function of \( s \) so
that the positive cross-exposure of small trees is larger for EZ than for CRRA. Not surprisingly aggregate wealth is strongly positively correlated with the large tree.

Despite uncorrelated payoffs returns on the two trees are positively correlated, representing again a stochastic discount factor effect. The correlation is in general larger with EZ than with CRRA. For EZ the cross-exposure to shocks in the other tree is nearly always positive and, due to the high price-consumption ratio of a small tree, also rather pronounced. This generates a significant return correlation between the trees. For CRRA, on the other hand, the cross-exposure is much lower, reflecting the flatter price-consumption ratio as a function of $s$, and can even turn negative for large trees. These two effects together reduce the correlations between the trees to rather low levels.

For financial claims the picture is again very similar. Correlations between them are smaller than for the consumption trees for CRRA and significantly smaller for EZ.

4.2.8 Expected Excess Returns

The expected excess return on the market is given as the scalar product of the vector of the market prices of risk $\lambda_t$ from Equation (12) and the vector of the exposures to the risk factors $\sigma(d\ln R_{c,t})$ from Equation (13). One obtains

$$\lambda'_t\sigma(d\ln R_{c,t}) = \gamma [s_t^2 + (1 - s_t)^2]\sigma_1^2 + \gamma v_s s_t (1 - s_t)(2s_t - 1)\sigma_1^2$$

$$+ (1 - \theta)v_s s_t (1 - s_t)(2s_t - 1)\sigma_1^2 + (1 - \theta)v_s^2 s_t^2 (1 - s_t)^2 2\sigma_1^2.$$

The expected excess return depends on $s$, which drives the level of uncertainty in aggregate consumption and in itself. A more detailed numerical analysis shows that the first term $\gamma [s_t^2 + (1 - s_t)^2]$ dominates. Since the volatility of aggregate consumption is smallest for $s = 0.5$, the expected excess return is a U-shaped function of $s$. It is equal for CRRA and EZ in the limiting case of a one-tree economy (and then given by risk aversion times the variance of aggregate consumption) and when both trees have the same size (since then $v_s = 0$, and the impact of the additional market price of risk for $s$-risk and the additional exposure vanishes). In the other cases it is slightly lower with EZ than with CRRA, i.e. there is a more pronounced U-shape with EZ. However, the differences are numerically rather small.

For the consumption trees CRRA preferences generate expected excess returns which are more or less linear in $s$, while EZ preferences produce an (inverse) S-shape. For small trees there is a reduced exposure to the own underlying consumption.
stream (with a rather low market price of risk) and a positive cross-exposure to the large tree (with a rather high market price of risk) so that expected excess returns go up. This effect is again significant since price-consumption ratios are very steep in $s$. For a large tree the effect goes the other way around so that expected excess returns are reduced.

For the financial claims we obtain the same overall picture. The expected returns are generally higher due to higher volatility.

4.3 Identical Trees with Stochastic Long-Run Growth Rate

4.3.1 General Remarks

In the version of the model discussed in this section we add the long-run growth rate $X$ to the drift terms of the two consumption trees. Our focus is on the changes relative to the base case discussed in the previous section, and we will see below that after the introduction of a state variable EZ preferences will generate a numerically plausible equity risk premium as well as excess volatility. The economic explanations and interpretations will be kept much shorter in this section, since in many cases we can refer the reader back to the more detailed discussion in the previous section. The results are presented graphically in Figures 3 (CRRA) and 4 (EZ).

4.3.2 Wealth-Consumption Ratio and Price-Consumption Ratio

The results here are almost perfectly analogous to the case without state variables. However, overall the additional risk generated by the presence of a state variable leads to higher average prices for CRRA and lower average prices for EZ. The curvature of the wealth-consumption ratio of the market is still concave for EZ and convex for CRRA, and the price-consumption ratios of the individual trees are convex.

A small tree still represents a 'safe haven' which is in high demand and thus has a high price-consumption ratio. As in the case without a state variable the payoffs from small trees are not exposed to systematic consumption risk, but they are now of course exposed to the systematic risk in long-run consumption growth risk. However,

$^{26}$If the wealth-consumption ratio of the small tree is finite (as in our numerical example), the exposure to shocks in the other tree and thus to aggregate consumption goes to zero, as does the expected excess return. If the wealth-consumption ratio is unbounded, this exposure and the expected excess return go to some positive value. See also Martin (2009) for the case of CRRA-preferences.

21
even the almost vanishing exposure to innovations in aggregate consumption (and thus to only one of the two systematic risk factors) is enough to make the small assets highly attractive and to drive up their price-consumption ratios.

Concerning the financial claims note that a small tree and the associated financial claim are still exposed to the systematic risk factor long-run growth rate. The leverage in dividends then implies that the financial claim is less valuable than the tree itself (with no state variable, as in Section 4.2, the price-consumption ratio of the underlying tree and the price-dividend ratio of the financial claim would coincide).

4.3.3 Risk-Free Rate and Market Prices of Risk

In terms of the dependence of the risk-free rate on $s$ we obtain the same picture as before. The risk-free rate now of course increases in $X$. With EZ, the additional precautionary savings due to the riskiness of $X$ reduces it.

The market prices of risk for consumption risk are the same as before. The market price of risk for the diffusion driving $X$ is given by $\lambda_{t,x} = (1 - \theta)\nu_x\sigma_x$. It is of course zero for CRRA preferences ($\theta = 1$), while it is positive for EZ. An increase in $X$ is good news, so that an asset with a positive exposure to $X$ commands a higher risk premium. Since the exposure of the market to $X$ is basically independent of the consumption share and the current value of $X$, the market price of risk for $X$ hardly shows any dependence on $X$.

4.3.4 Exposures

Concerning the exposures to shocks in the two consumption trees the results are close to those obtained for the case without a state variable. The new risk factor to be discussed here is the long-run growth rate $X$.

For the value of the claim to aggregate consumption an increase in $X$ has two effects. An increase in future payoffs due to the higher growth rate will increase its price (cash flow channel), while higher future consumption implies lower future marginal utility, stronger discounting and therefore decreasing prices (stochastic discount factor channel). For CRRA the second channel dominates, so prices will decrease.\(^{27}\) Again, the negative exposure can be traced back to too high a dependence

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\(^{27}\) Note that also for CRRA prices will of course depend on $X$, even if there is no risk premium for this factor.
of the risk-free rate on $X$. For EZ with $\psi > 1$, on the other hand, the cash flow channel dominates, and prices increase in $X$.

For the two consumption trees the results are qualitatively the same as for the aggregate market. For the financial claims the larger exposure to the long-run growth rate enforces the cash flow effect and adds a positive term to the overall exposure. In the CRRA case the previously negative exposure increases (i.e., becomes less negative) and could eventually become positive for very large $\pi_x$, while for EZ the already positive exposure increases.$^{28}$

4.3.5 Expected Excess Returns

Expected excess returns are larger for EZ than for CRRA due to the long-run risk factor $X$, with a difference of around 50 basis points for the consumption claims and 200 basis points for the financial claims. This finding is certainly not new, it is just once more the well-known fact that EZ preferences together with a long-run risk factor can solve the equity premium puzzle.

A new insight that we gain from the setup with a state variable is that now also small trees can earn a positive risk premium even in the case where their limiting price-dividend ratio is finite. Their exposure to aggregate consumption is then basically zero, but they now also have an exposure to the long-run growth rate, and this exposure is independent of the size of the tree. With EZ this long-run risk is priced, which implies positive expected excess returns also on small trees. EZ can thus capture the idea that part of the risk premium is due to a 'general exposure to macro-economic risk' which does not vanish simply because an asset is small. Also note that due to leverage the expected excess returns on the financial claims are higher than those on the consumption trees.

4.3.6 Volatilities

The exposure to the state variable generates additional volatility for CRRA and for EZ. In the CRRA case there is excess volatility for all claims. With EZ we observe only little excess volatility for the aggregate market and for large trees, and a much

$^{28}$The exposure of an asset to $X$ increases in absolute terms when its price-dividend ratio increases, so that the exposure is larger for smaller assets. To get the intuition look once more at the price-dividend ratio in a one tree economy without a state variable (Appendix B.1). It is inversely related to the drift of aggregate consumption, and the smaller the denominator (i.e., the higher the price-dividend ratio), the larger the impact of a change in the drift.
more pronounced excess volatility for the two financial claims independently of their size. Since one of the main objectives of an asset pricing model is to explain exactly this excess volatility for financial claims, this shows that the model produces sensible results.

In general excess volatility is much more pronounced for CRRA than for EZ. While a price volatility exceeding the volatility of the fundamentals might basically be a desirable feature of a model per se, it is generated in a CRRA model by a much too large exposure of prices to \( X \), which is in turn due to a way too strong dependence of the risk-free rate on \( X \). Stated differently one could say that CRRA thus generates a high excess volatility by benefitting from the risk-free rate puzzle. This was also true in the case without state variables, where a small excess volatility was generated by the convexity of the wealth-consumption ratio.

In contrast to this excess volatility in the EZ setup is only caused by the presence of state variables. As argued above the exposure of prices and the risk-free rate to the long-run growth rate is now much more plausible in terms of sign and magnitude (so that the risk-free rate puzzle has been solved). Thus excess volatility is now smaller, but it is caused by an economically plausible mechanism. A larger excess volatility could be achieved by adding further state variables to the model.

### 4.3.7 Correlations

Correlations increase across the board compared to the case with no state variable, since all assets react in the same way to the state variable \( X \). For CRRA, however, the correlations are extremely high (above 85%), and they are caused by the joint negative exposure of prices to \( X \). The correlation between the returns on the financial claims is lower than that between the returns on the two consumption trees, since the prices of the financial claims have a less extreme exposure to the state variable \( X \) than the prices of the consumption trees.

For EZ the joint reaction to the state variable increases the correlation compared to the case without \( X \). The overall level seems okay and is much more realistic than for CRRA. Furthermore, the return correlation between the financial claims is now larger than the one between the consumption trees, since the joint exposure to \( X \) is much larger for the financial claims.
4.4 Only One Tree Exposed to Long-Run Growth Rate

4.4.1 General Remarks

The setup in this section is such that only tree 1 is exposed to $X$, i.e. $\phi_2 = 0$ in Equation (2). In the following, trees 1 and 2 will be called 'risky' and 'safe', respectively. The focus of our analysis is on the following two questions. First, now that the trees are no longer identical, how does a change in the exposure of the whole economy to $X$ over time influence the equilibrium? Second, how do the different exposures to $X$ of the two trees influence their (differential) pricing?

The cases discussed in Sections 4.2 and 4.3 will serve as useful benchmarks here, since an economy with identical trees and no state variables represents the limiting case when the consumption share of the risky tree goes to zero (i.e., for $s \to 0$), while the case with identical trees and a state variable results for the limit $s \to 1$. The results are presented in Figures 5 (for the CRRA economy) and 6 (for the EZ case).

4.4.2 Wealth-Consumption Ratio

The wealth-consumption ratio is convex with CRRA and concave with EZ, so it shows the same qualitative behavior as before. In the CRRA case the wealth-consumption ratio is larger when the risky tree dominates ($s = 1$), since its higher risk implies a higher price in this preference framework. For EZ it is exactly the other way around, i.e. the wealth-consumption ratio is smaller when the risky tree dominates, since with EZ a higher risk implies a lower price. The overall exposure of the economy to the long-run growth factor $X$ thus matters.

4.4.3 Price-Consumption Ratio and Price-Dividend Ratio

Comparing large and small trees in terms of their price-consumption ratios we first see that obviously the price-consumption ratio of the large tree converges to the wealth-consumption ratio of the market. For the small tree the price-consumption ratio is much higher than for the large tree, which as usual reflects its safe haven property for both CRRA and EZ preferences.

For a small safe tree not only the exposure to aggregate consumption but also the one to long-run risk vanishes, which additionally increases its price-consumption.

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29Tree 2, the 'safe' tree, is of course still exposed to consumption risk.
ratio. When the small tree is risky it still has an exposure to the long-run growth rate. The main difference to the analysis in Section 4.3 is that now the long-run growth rate represents \textit{idiosyncratic} instead of systematic risk. So the tree exhibits a higher overall risk than the market, but all the risk it is exposed to is unsystematic and thus not priced. Since the small tree is thus no longer exposed to systematic risk, it represents a safe haven with a high price-consumption ratio.

The relative pricing of large risky versus large safe trees depends on the overall risk of these trees, since this risk basically defines systematic risk. For CRRA we obtain the (by now familiar) result that additional risk increases prices, since a large risky tree is more valuable than an equally large safe tree, while the opposite is true for EZ.

The price-dividend ratios of the financial claims behave very much like the price-consumption ratios of the two trees, which is the same result as for the other setups.

### 4.4.4 Market Prices of Risk

The general expression for the vector of market prices of risk is given in Equation (7). The first two components there represent the compensations per unit of consumption risk coming from the two trees, while the third is the market price for $X$-risk.

The market prices of consumption risk are mainly driven by the exposure of aggregate consumption to shocks in the two trees, and for CRRA, they are linear in consumption share. With EZ preferences the endogenous state variable $s$ is also priced, which creates an additional term in the market prices of risk for consumption risk. The consumption share $s$ drives the volatility of aggregate consumption. As discussed above via this channel the market price of risk for small trees is increased, and the one for large trees is reduced. In the setup here $s$ also has an impact on the drift of aggregate consumption, since the trees do not have identical growth rates ($\phi_2 = 0$).

To analyze the impact which the dependence of the drift of aggregate consumption on $s$ has on the pricing of shocks to the two trees, first assume that $\mu_1 \neq \mu_2$. Then there is an additional positive risk premium for consumption risk of the attractive tree (with high growth rate) and negative risk premium for consumption risk of the unattractive tree (with low growth rate). The technical explanation for this effect is that a higher growth rate implies a higher price-consumption ratio for EZ (but not for CRRA). So when tree 1 is the attractive one, this leads to an increase
in $v_s$, which in turn leads to an additional risk premium on the consumption risk of tree 1.

From an economic point of view a positive shock to consumption from the attractive tree increases the growth rate of aggregate consumption, representing a good state. The attractive tree thus has a positive return in good states, which leads to an additional positive risk premium for its consumption risk. By the same token there is a reduction in the market price of risk for the consumption risk of the unattractive tree.

Second, we look at the implications of the different exposures to the long-run growth factor $X$, since $\phi_1 = 1$ and $\phi_2 = 0$. Then $s$ also has an impact on the exposure of aggregate consumption to $X$. A positive shock to tree 1 increases the exposure of aggregate consumption to $X$. If this happens when $X > 0$, the growth rate of aggregate consumption increases, while with $X < 0$ the opposite will be true. Finally, independent of the value of $X$, the volatility of the growth rate of aggregate consumption increases in $s$.

Putting the results together shows that for $X < 0$, both the lower drift and the higher volatility of aggregate consumption are 'bad news', implying a negative contribution to the market price of risk for consumption from the tree which is exposed to $X$ (and therefore unattractive for negative $X$). For $X$ around zero there is still a slightly negative contribution to the market price of risk for consumption from the exposed tree due to the variance effect described above. For large and positive $X$, the positive drift effect will outweigh the negative variance contribution, resulting in a positive contribution to the market price of risk for consumption from the tree which is exposed to $X$ (and which is therefore attractive).

The market price of risk for the exposure to the long-run growth rate increases in the consumption share of the risky tree, which measures the exposure of aggregate consumption to the long-run growth rate. While the exposure of aggregate consumption to the long-run growth rate is linear in $s$, the exposure of the wealth-consumption ratio and thus the market price of risk is not. When the consumption share of tree 1 is small, its contribution to aggregate wealth is larger than its contribution to aggregate consumption. The pricing of $X$-risk depends on its impact on

---

30This change in $v_s$ will be positive as long as the tree is not too large. See the discussion in Section 4.2 for the potentially counteracting effect of volatility.

31The impact is admittedly rather small. Nevertheless the economic argument highlights the role of EZ preferences together with the endogenous state variable $s$ in the pricing of risky assets in the two tree economy.
the wealth-consumption ratio. A higher contribution of tree 1 to aggregate wealth than its consumption share $s$ implies that also the market price of risk for $X$ is larger than suggested by $s$. When tree 1 is large its contribution to aggregate wealth is smaller than its contribution to aggregate consumption, so that the market price of risk is smaller than suggested by $s$.

4.4.5 Exposures

In terms of the exposure to consumption risk the story is the same as before. The part to discuss is again the exposure to the long-run growth rate.

This exposure is zero for the market when $s = 0$, since there is no $X$-risk in this case, and it increases in $s$ in absolute terms. As in the case with identical trees it is negative for CRRA and positive for EZ. Finally, for EZ preferences the exposure is greater than $s\sigma_x$ when the risky tree is small and vice versa. For CRRA the behavior is basically the same but with opposite sign.

For the safe tree and the financial claim written on it the cash flows are not exposed to $X$. There is an indirect exposure to $X$, however, through the stochastic discount factor, or, stated differently, through the dependence of the risk-free rate on $X$. This exposure is negative for both EZ and CRRA and decreases in the consumption share of the risky tree, i.e. in the exposure of aggregate consumption to $X$. Quantitatively the negative exposure is much larger for CRRA than for EZ, since the risk-free rate has a much larger exposure to $X$ with CRRA than with EZ. In particular the exposure of the large safe tree is even more negative than the exposure of the market as a whole for CRRA. For EZ the exposure of the large safe tree is negative, while that of the market is positive.

Turning to the risky tree and the associated financial claim it is clear that when the risky tree is small there is only a cash flow effect, resulting in a positive exposure to $X$ for both CRRA and EZ (and these exposures are even identical when the share of the risky tree is zero). When the consumption share of the risky tree increases, the effect via the stochastic discount factor kicks in, leading to a negative and extreme exposure of medium and large trees for CRRA, while the exposure stays positive for EZ.
4.4.6 Risk-Free Rate and Expected Excess Returns

For EZ the risk-free rate is smaller when the risky tree dominates. An increasing share of the risky tree induces additional precautionary savings due to the risk factor $X$, although this effect is numerically rather small.

While it hardly matters for CRRA in terms of expected excess returns which tree currently dominates the economy, this is highly relevant for EZ, since here the share of the risky tree determines that the market price of risk for $X$.

The expected excess return on aggregate wealth is still U-shaped as before, i.e. it is lowest if both trees have roughly the same size and the volatility of aggregate consumption is lowest. In the CRRA case it is equal for $s = 0$ and $s = 1$. For EZ it is larger when the risky asset dominates, since the risk premium on the (then systematic) long-run growth rate adds to the expected excess return.

For the consumption trees and the financial claims expected excess returns are increasing in the consumption share of the respective tree (as in the previous sections). With CRRA utility there are again hardly any differences between the safe and the risky tree, once we control for the consumption share. With EZ the expected excess return on a small tree increases fast in $s$ due to large cross-exposure to the priced shocks in the large tree, with the effect again being due to very steep price-consumption ratio as a function of $s$.

Differences between the tree are caused by long-run growth risk and EZ preferences. The risky tree has a positive exposure to $X$ which starts to become systematic (and thus priced) with increasing $s$. The safe tree, on the other hand, has a negative exposure to $X$, and the market price of risk for $X$ decreases in its consumption share, so that the (negative) risk premium on $X$ increases in its consumption share. Put together, the premium earned on a (risky or safe) tree for $X$-risk increases in its consumption share, and the expected excess return on a risky tree is larger than the expected excess return on an equally large safe tree. The difference is particularly pronounced for the levered financial claims.

4.4.7 Return Volatility

In the CRRA case there is again excess volatility for all assets, and the degree to which volatility is excessive increases significantly in the consumption share of the risky tree. In contrast to this we observe excess volatility in the EZ economy only for those assets which are (sufficiently) exposed to long-run macro-economic risk.
This shows up in the fact that there is excess volatility for the risky tree and the associated financial claim, due to their exposure to $X$. On the other hand, there is no excess volatility for the safe tree and the financial claim written on it. They are exposed to $X$ through the stochastic discount factor, but this exposure is not large enough to generate excess volatility.

4.4.8 Correlations

Also in the setup of this section the return correlation between the consumption trees is larger than that between the financial claims, where the difference is more pronounced for CRRA than for EZ. When the risky tree is small, correlations can become negative for CRRA, but stay positive with EZ.

If the risky tree is small CRRA preferences lead to opposite reactions of the two trees to shocks in the risky tree 1, a positive reaction to shocks in the safe tree 2, and again opposite reactions to shocks in $X$. The ultimate sign of the return correlation thus depends on which of these forces is stronger. For small $s$ the exposure to $X$ dominates, implying a negative correlation. With EZ preferences prices react positively to shocks in both trees, but exhibit opposite reactions to shocks in $X$. The exposures to $X$ are moderate, and the correlation is then determined by the very large cross-exposure of the risky tree due to the very steep slope of the price-consumption ratio as a function of $s$, yielding a positive correlation. So it is again the implausibly large exposure to $X$ which generates the negative correlation in the CRRA case, while the correlation between the trees is still positive with EZ.

When the risky tree dominates the economy with CRRA-preferences again suffers from the problem of too high an exposure to $X$, causing correlations which seem implausibly large. For EZ the correlations are positive, but moderate. There is still room for the risky and the safe asset to move independently of each other.

5 Summary and Conclusion

In this paper we have analyzed an economy with two trees which produce a single consumption good and a representative investor with EZ preferences. The dynamics of the two consumption trees are exposed to a stochastic component in the drift which is the long-run growth factor known from LRR-models introduced by Bansal and Yaron (2004). Our paper thus links the literature on multiple tree models to the one on long-run risk.
A model with multiple trees and CRRA preferences will generate effects like excess volatility, non-zero return correlation despite uncorrelated cash flow innovations or exposures of returns to risk factors the underlying cash flows are not exposed to. However, we show in our paper that the economic mechanisms behind these effects are not always plausible. The fundamental reason is that with CRRA preferences it is not possible to choose the investor’s risk aversion and EIS separately. This can be done with EZ preferences, and we show that the basic characteristics of a two tree model (like spillovers between the trees) are still present, while at the same time the model can explain additional effects (like positive risk premia on small trees with still finite price-dividend ratios) and delivers economically more plausible results (like positive but moderate correlations between the prices of single trees). From a very fundamental perspective our paper thus highlights the role of risk aversion and the elasticity of intertemporal substitution in asset pricing.

The lack of separability of these two quantities in a CRRA framework has far-reaching consequences, affecting basically all dimension of asset pricing. In a CRRA world prices tend to fall when consumption growth increases or when volatility decreases, while the reaction is exactly reverse in an EZ world and thus much more in line with economic intuition. The very high sensitivity of the interest rate to expected consumption growth makes the returns on assets like consumption trees much more correlated in the CRRA case than for EZ, with the correlation reaching a value of more than 0.85, which seems rather implausible given that the cash flow streams are uncorrelated. For EZ preferences asset returns are also positively correlated, but to a much lower degree. Excess volatility in the CRRA model is generated by pretty much the same mechanism as the risk-free rate puzzle, while in an EZ framework it is due to the presence of state variables only.

Another interesting effect concerns the risk premium for a small asset. With CRRA state variables are not priced in equilibrium, so that the only source for a risk premium on an asset is its covariation with aggregate wealth. For a very small tree (with a finite price-dividend ratio) which represents only a very small fraction of aggregate consumption this implies a zero risk premium. With EZ there are risk premia for state variables, so that even such a small tree can have a non-zero expected excess return. EZ preferences thus capture the intuition that small assets can earn a large risk premium if they have a high exposure to general macroeconomic risk.

Our analysis is only the first step into a detailed analysis of the properties of multiple tree models under more general preferences. In future work one can look at additional state variables, in particular sources of stochastic uncertainty.
like stochastic volatility or, in a model with jumps, stochastic intensity, or at the implications of learning.
References


A Equilibrium in the Two-Tree Economy

To solve for the equilibrium with EZ-preferences, we follow Benzoni, Collin-Dufresne, and Goldstein (2010). From Equation (3) for the indirect utility $J$, we get

$$E_t [dJ_t + f(C_t, J_t) dt] = 0. \quad (A.1)$$

We set

$$J_t = \frac{C_t^{1-\gamma}}{1-\gamma} \beta^\theta e^{\theta v_t},$$

i.e. we express $J_t$ via the (still unknown) function $v_t$. Plugging the expression for $J_t$ into Equation (A.1) and applying Ito to determine $dJ$ gives a partial differential equation for $v \equiv v(s, X_t)$:

$$0 = e^{-v} - \beta + (1 - \gamma) \sigma_C'(v_s \sigma_s + v_x \sigma_X) + \left(1 - \frac{1}{\psi}ight) [\mu_C - 0.5 \gamma \sigma_C' \sigma_C] \quad (A.2)$$

$$+ v_x \mu_X + v_s \mu_s + 0.5(v_{xx} + \theta v_x^2) \sigma_X' \sigma_X + 0.5(v_{ss} + \theta v_s^2) \sigma_s' \sigma_s + (v_{sx} + \theta v_s v_x) \sigma_s' \sigma_X.$$

The pricing kernel then follows, similar to the case of CRRA-preferences, from an indifference argument. It is given by (see, e.g., Benzoni, Collin-Dufresne, and Goldstein (2010))

$$\xi_t = e^{-\beta \theta t - (1-\theta) \int_0^t \{(X_{t_s} - s) - X_t \} dt} C_t^{-\gamma}.$$

Its dynamics follow by applying Ito, and we obtain

$$\frac{d\xi_t}{\xi_t} = \mu_\xi dt + \sigma_\xi dW_t,$$

where

$$\mu_\xi = -\beta \theta - (1 - \theta) e^{-v} + \gamma (\mu_C + 0.5 (1 + \gamma) \sigma_C' \sigma_C)$$

$$+ (\theta - 1) [v_x \mu_X + v_s \mu_s + 0.5 v_{xx} \sigma_X' \sigma_X + 0.5 v_{ss} \sigma_s' \sigma_s + v_{sx} \sigma_s' \sigma_X$$

$$+ 0.5 (\theta - 1) v_x^2 \sigma_X' \sigma_X + 0.5 (\theta - 1) v_s^2 \sigma_s' \sigma_s + (\theta - 1) v_x v_s \sigma_s' \sigma_X]$$

$$- \gamma (\theta - 1) \sigma_C'(v_s \sigma_s + v_x \sigma_X)$$

$$\sigma_\xi = -\gamma \sigma_C - (1 - \theta) (v_s \sigma_s + v_x \sigma_X).$$

The interest rate is the negative of the drift of the pricing kernel, i.e., $r_t = -\mu_\xi$, which gives

$$r_t = \beta \theta + (1 - \theta) e^{-v} + \gamma (\mu_C - 0.5 (1 + \gamma) \sigma_C' \sigma_C)$$

$$+ (1 - \theta) [v_x \mu_X + v_s \mu_s + 0.5 v_{xx} \sigma_X' \sigma_X + 0.5 v_{ss} \sigma_s' \sigma_s + v_{sx} \sigma_s' \sigma_X$$

$$+ 0.5 (\theta - 1) v_x^2 \sigma_X' \sigma_X + 0.5 (\theta - 1) v_s^2 \sigma_s' \sigma_s + (\theta - 1) v_x v_s \sigma_s' \sigma_X]$$

$$+ \gamma (\theta - 1) \sigma_C'(v_s \sigma_s + v_x \sigma_X).$$
This equation can be simplified by using Equation (A.2), and we get

\[
r_t = \beta + \frac{1}{\psi} \mu_C - 0.5 \gamma \left( 1 + \frac{1}{\psi} \right) \sigma_C' \sigma_C - 0.5(1 - \theta) \left[ v_s^2 \sigma_s' \sigma_s + v_x^2 \sigma_x' \sigma_x + 2v_s v_x \sigma_s' \sigma_x + 2 \sigma_C' (v_s \sigma_s + v_x \sigma_x) \right] (A.3)
\]

The market prices of risk follow from the volatilities of the pricing kernel, i.e., from the negative of the exposures \( \sigma_\xi \) of the pricing kernel to the Wiener process \( W \). They are thus given by

\[
\lambda_t = -\sigma_\xi
= \gamma \sigma_C + (1 - \theta) (v_x \sigma_X + v_s \sigma_s)
= \gamma \left[ s_t \sigma_{D1} + (1 - s_t) \sigma_{D2} \right] + (1 - \theta) \left[ v_x \sigma_X + v_s \sigma_s (1 - s_t) (\sigma_{D1} - \sigma_{D2}) \right]
= \begin{pmatrix}
\gamma s_t \sigma_1 + v_s (1 - \theta) s_t (1 - s_t) \\
\gamma (1 - s_t) \sigma_2 - v_s (1 - \theta) s_t (1 - s_t) \sigma_2
\end{pmatrix}
\]

where we have plugged in the specific volatilities of consumption and of the long-run growth rate in the last step.

Given the pricing kernel, we can determine the prices of all assets. Consider a general asset with dividend stream \( D \), where

\[
\frac{dD_t}{D_t} = \mu_D dt + \sigma_D' dW_t.
\]

The price of the claim to all future dividends is equal to

\[
D_t e^{w_t} = E_t^\mathbb{P} \left[ \int_t^\infty \xi_s D_s \, ds \right]
\]

where \( w \) denotes the log price-dividend ratio of the asset. Applying the theorem of Feynman-Kac to this equation gives a partial differential for \( w = w(s, X) \):

\[
-r_t + \mu_D + \mu_w + 0.5 \sigma_w' \sigma_w - (\sigma_D + \sigma_w)' \lambda + \sigma_D' \sigma_w + e^{-w} = 0, \quad (A.4)
\]

where the drift and the volatilities of \( w \) follow via Ito.

\[
\begin{align*}
\mu_w &= w_s \mu_s + w_x \mu_X + 0.5 w_s \sigma_X' \sigma_X + 0.5 w_{ss} \sigma_s' \sigma_s + w_{sx} \sigma_s' \sigma_X \\
\sigma_w &= w_s \sigma_s + w_x \sigma_X.
\end{align*}
\]

The first asset to price is the claim to aggregate consumption. Plugging in \( v \) and using the partial differential equation (A.2) for \( v \) and Equation (A.3) for the risk-free rate shows that \( v \) indeed solves the partial differential equation (A.4) for the claim to aggregate consumption. This proves that the function \( v \) defined above is indeed the log wealth-consumption ratio.


B Equilibrium in the One-Tree Economy

B.1 No State Variables

In the special case of a one-tree economy with no state variables, the dynamics of consumption are

\[
\frac{dC_t}{C_t} = \mu_c dt + \sigma_c dW_{c,t}.
\]

In a one-tree economy, the state variable \(s\) vanishes from the partial differential equation (A.2) for \(v\), and with no state variable \(X\), we are left with an equation which can immediately be solved for the log wealth-consumption ratio:

\[
e^v = \frac{1}{\beta - \left(1 - \frac{1}{\psi}\right)\mu_c + 0.5\gamma \left(1 - \frac{1}{\psi}\right)\sigma_c^2},
\]

(B.1)

In case of CRRA-preferences, the wealth-consumption ratio is finite if it holds that\(^{32}\)

\[
\beta + (\gamma - 1)\mu_c > 0.5\gamma (\gamma - 1)\sigma_c^2.
\]

In case of EZ-preferences, the wealth-consumption ratio is finite if it holds that

\[
0.5\gamma \left(1 - \frac{1}{\psi}\right)\sigma_c^2 > \left(1 - \frac{1}{\psi}\right)\mu_c - \beta.
\]

Finally, the wealth-consumption ratio is larger with EZ-preferences (with \(\gamma > 1\), \(\psi > 1\)) than with CRRA-preferences if

\[
\mu_c > 0.5\gamma\sigma_c^2.
\]

The risk-free rate follows from Equation (A.3) which gives

\[
r = \beta + \frac{1}{\psi}\mu_c - 0.5\gamma \left(1 + \frac{1}{\psi}\right)\sigma_c^2
\]

The market price of risk for consumption risk is given by \(\lambda_c = \gamma\sigma_c\).

We can also price general claims with dividends given by

\[
\frac{dD_t}{D_t} = \mu_d dt + \sigma_{dc} dW_{c,t} + \sigma_{dd} dW_{d,t}
\]

where we assume that the Wiener processes \(W_c\) and \(W_d\) are uncorrelated. The market price of risk for \(W_d\), which does not have any impact on aggregate consumption, is of course equal to zero both for CRRA- and for EZ-preferences. The partial differential equation (A.4) again simplifies to a normal equation which can be solved for \(w\), since both \(s\) and \(X\) vanish. The price-dividend ratio is given by

\[
e^w = \begin{cases} 
1 & r + \gamma\sigma_c\sigma_{dc} - \mu_D > 0 \\
\infty & r + \gamma\sigma_c\sigma_{dc} - \mu_D \leq 0
\end{cases}
\]

\(^{32}\)This restriction is violated if \(\gamma\) is too large. Parlour, Stanton, and Walden (2010) discuss an economy with one risky and one risk-free tree and show that the wealth-consumption ratio is always finite in this economy, which allows them to analyze the pricing implications of high levels of risk aversion.
B.2 State Variables: Long-Run Growth Rate

In the special case of a one-tree economy with state variable ‘long-run growth rate’, the dynamics of consumption and of the state variable are

\[ d\ln C_t = (\mu_c + X_t - 0.5\sigma_c^2) \, dt + \sigma_c dW_{c,t} \]
\[ dX_t = -\kappa X_t dt + \sigma_x dW_{x,t}. \]

The Wiener processes \( W_c \) and \( W_x \) are uncorrelated.

The model is affine, and we follow Eraker and Shaliastovich (2008) who use a Campbell-Shiller approximation to linearize the returns on the claim to aggregate consumption and on the dividend-claims. The log wealth-consumption ratio is \( v = A_0 + A_{1x} X_t \), where the coefficients \( A_0 \) and \( A_{1x} \) and the linearizing constant \( k_1 \) solve the non-linear system of equations

\[ A_{1x} = \frac{1 - \frac{1}{\psi}}{1 - k_1 + k_1 \kappa_x} \]
\[ A_0 = \ln \frac{k_1}{1 - k_1} \]
\[ 0 = \beta - \ln k_1 + \left( 1 - \frac{1}{\psi} \right) \left( \mu_c - 0.5\sigma_c^2 \right) + 0.5 \theta \left[ \left( 1 - \frac{1}{\psi} \right)^2 \sigma_c^2 + k_1^2 A_{1x}^2 \sigma_x^2 \right] \]

The linearizing constant \( k_1 \) is equal to \( e^\bar{v} (1 + e^\bar{v})^{-1} \), where \( \bar{v} \) is the average wealth-consumption ratio. Thus, \( k_1 \) is between zero and one and usually rather close to one.

We can also price general claims with dividends given by

\[ d\ln D_t = (\mu_d + \phi X_t - 0.5\sigma_d^2) \, dt + \sigma_d dW_{c,t} + \sigma_x dW_{x,t} + \sigma_d dW_{d,t}. \]

The Wiener processes \( W_c, W_x, \) and \( W_d \) are uncorrelated, so that the standard deviation of the log dividend growth is

\[ \sigma_d = (\sigma_{dc}^2 + \sigma_{dx}^2 + \sigma_{dd}^2)^{0.5}. \]

Eraker and Shaliastovich (2008) again rely on a Campbell-Shiller approximation for the return on the dividend claim. They show that the log price-dividend ratio is approximately equal to \( w = A_{d0} + A_{d1x} X_t \). The coefficients \( A_{d0} \) and \( A_{d1x} \) and the linearizing constant \( k_{d1} \) solve the non-linear system of equations

\[ A_{d1x} = \frac{\phi - \frac{1}{\psi}}{1 - k_{d1} + k_{d1} \kappa_x} \]
\[ A_{d0} = \ln \frac{k_{d1}}{1 - k_{d1}} \]
\[ 0 = \theta \beta + (1 - \theta) \ln k_{d1} - \ln k_{d1} - \gamma \left( \mu_c - 0.5\sigma_c^2 \right) + (\mu_d - 0.5\sigma_d^2) \]
\[ + 0.5 \left[ \gamma^2 \sigma_c^2 + (k_{d1} A_{d1x} - (1 - \theta) k_1 A_{1x})^2 \sigma_c^2 + \sigma_d^2 \right] \]
\[ - 2\gamma \sigma_c \sigma_{dc} + 2 (k_{d1} A_{d1x} - (1 - \theta) k_1 A_{1x})^2 \sigma_x \sigma_{dx} \].

Analogously to \( k_1 \), \( k_{d1} \) is between zero and one and usually rather close to one.
Identical trees, state variables

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Identical trees, no state variables

$\phi_1 = \phi_2 = 0.0$

Different exposure to $X$

$\phi_2 = 0.0$

One risk-free tree

$\phi_2 = \sigma_2 = 0.0$

Table 1: Parameter

The table gives the parameters for the dynamics of consumption and dividends.
Figure 1: No state variables, identical trees: CRRA-investor

The figure shows the characteristics of the economy, the prices of the consumption and the dividend claims, and their dynamics. The preference parameters are $\gamma = 10$, $\delta = 0.1$, $\beta = 0.02$. The parameters of the economy are $\mu_1 = \mu_2 = 0.018$, $\sigma_1 = \sigma_2 = 0.035$. 
Figure 2: No state variables, identical trees: EZ-investor

The figure shows the characteristics of the economy, the prices of the consumption and the dividend claims and their dynamics. The preference parameters are $\gamma = 10$, $\delta = 1.5$, $\beta = 0.02$. The parameters of the economy are $\mu_1 = \mu_2 = 0.018$, $\sigma_1 = \sigma_2 = 0.035$. 
The figure shows the characteristics of the economy, the prices of the consumption and the dividend claims and their dynamics. The preference parameters are $\gamma = 10$, $IES = 0.1$, $\beta = 0.02$. The parameters of the economy are $\mu_1 = 0.018$, $\sigma_1 = 0.035, \phi_1 = 1.0$; $\mu_2 = 0.018$, $\sigma_2 = 0.035$, $\phi_2 = 1.0$; $\kappa_x = 0.30$, $\sigma_x = 0.0114$; $\pi_1 = 2.5$, $\pi_2 = 2.5$. 

Figure 3: Long-run growth rate, identical trees: CRRA-investor
Figure 4: Long-run growth rate, identical trees: EZ-investor

The figure shows the characteristics of the economy, the prices of the consumption and the dividend claims and their dynamics. The preference parameters are $\gamma = 10$, $IES = 1.5$, $\beta = 0.02$. The parameters of the economy are $\mu_1 = 0.018$, $\sigma_1 = 0.035$, $\phi_1 = 1.0$; $\mu_2 = 0.018$, $\sigma_2 = 0.035$, $\phi_2 = 1.0$; $\kappa_x = 0.30$, $\sigma_x = 0.0114$; $\pi_1 = 2.5$, $\pi_2 = 2.5$. 

42
The figure shows the characteristics of the economy, the prices of the consumption and the dividend claims and their dynamics. The preference parameters are $\gamma = 10$, $IES = 0.1$, $\beta = 0.02$. The parameters of the economy are $\mu_1 = 0.018$, $\sigma_1 = 0.035$, $\phi_1 = 1.0$; $\mu_2 = 0.018$, $\sigma_2 = 0.035$, $\phi_2 = 0.0$; $\kappa_x = 0.30$, $\sigma_x = 0.0114$; $\pi_1 = 2.5$, $\pi_2 = 2.5$. 

Figure 5: Long-run growth rate $X$, only one tree is exposed to $X$: CRRA-investor
Figure 6: Long-run growth rate $X$, only one tree is exposed to $X$: EZ-investor

The figure shows the characteristics of the economy, the prices of the consumption and the dividend claims and their dynamics. The preference parameters are $\gamma = 10$, $IES = 1.5$, $\beta = 0.02$. The parameters of the economy are $\mu_1 = 0.018$, $\sigma_1 = 0.035$, $\phi_1 = 1.0$; $\mu_2 = 0.018$, $\sigma_2 = 0.035$, $\phi_2 = 0.0$; $\kappa_x = 0.30$, $\sigma_x = 0.0114$; $\pi_1 = 2.5$, $\pi_2 = 2.5$. 