Bounds on the autocorrelation of admissible stochastic discount factors

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Article Info

Article history:
Received 19 May 2010
Accepted 3 March 2012
Available online 13 March 2012

JEL classification:
G12
C52

Keywords:
Stochastic discount factor
Autocorrelation bounds
Asset pricing models
Time variation

1. Introduction

Under the assumption that the law of one price holds, the price of any asset is given by the expected discounted value of its future payoffs. The discounting is done using a random variable known as a pricing kernel, a stochastic discount factor (SDF) or an intertemporal marginal rate of substitution (IMRS), depending on the context. This discount factor representation, also called the canonical asset pricing equation, is quite general since there is a corresponding SDF for all asset pricing models. Furthermore, as argued by Hansen and Richard (1987), all asset pricing implications of a model are contained within the canonical asset pricing equation. Thus, studying the properties of SDFs admissible under the pricing equation is crucial to explain the behavior of asset prices.

The difficulty is that admissible SDFs are not observable. In a seminal paper, Hansen and Jagannathan (1991) get around this problem by using asset market data to restrict the admissible region for the unconditional variance of the SDF. Building on this contribution, there is now a considerable literature on SDF bounds. For example, Snow (1991) derives lower bounds on selected higher unconditional moments of the SDF. Bansal and Lehmann (1997) present a bound on the expectation of the log SDF. Techniques to tighten the Hansen–Jagannathan variance bound have also been proposed. (See Gallant et al. (1990), Balduzzi and Kallal (1997), Ferson and Siegel (2003), Bekaert and Liu (2004) and Kan and Zhou (2006) among others.)

A common aspect of this literature is that it studies cross-sectional properties of admissible SDFs. For instance, the Hansen–Jagannathan bound looks at the variability of the SDF across states of the economy. While such an investigation is of great importance, the cross-sectional properties only exploit asset pricing restrictions from one angle. The SDF is a random variable following a process through time restricted by the canonical asset pricing equation. Specifically, the discount factor representation indicates that the process for an admissible SDF and an asset return should combine so that their product is one plus an unpredictable error. The time-series properties of admissible SDFs have received little attention in the SDF bounds literature.

In this paper, we derive a bound which focuses on an important time-series characteristic of the SDF. Specifically, we show how to use observable asset market data to restrict the admissible region for the first-order autocorrelation of the SDF. We interpret this statistic as a measure of a model’s economic time variation across two periods. Estimating bounds for nominal and real SDFs at monthly and quarterly frequencies, we find that the admissible autocorrelations are significantly negative, but greater than \(-0.02\), implying that the bounds impose a strong restriction on candidate SDFs. We illustrate the relevance of these findings by showing that some widely used consumption-based models are misspecified with respect to the autocorrelation bound. Finally, we examine the implications of our results for the admissibility of linear factor models and the appropriateness of empirical pricing factors.

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In particular, the sign of the autocorrelation gives information on whether or not recessionary states are expected to be followed by expansionary states. Their magnitude further provides a measure of the importance of the predictable variation in the business cycle due to the previous SDF. We also show the role of the SDF autocorrelation in linking the conditional versus unconditional versions of the SDF risk premium, the assets’ excess returns per unit of beta risk and the mean–variance frontier of the investment opportunity set. Finally, the SDF autocorrelation is inversely related to the term premium of two-period versus one-period risk-free assets and it considers the importance of this term premium relative to the SDF risk premium. Put together, these associations lead us to interpret the autocorrelation bound as a measure of admissible economic time variation.

Using the generalized method of moment of Hansen (1982), we estimate autocorrelation bounds for real and nominal SDFs at monthly and quarterly frequencies. The asset market data include one- and two-period risk free discount bond prices from the Fama files in CRSP, the returns on the CRSP value-weighted index, and the returns on a low and a high book-to-market portfolios. Our data sets cover the period June 1959–December 2000. Our empirical results show that the autocorrelation bounds are relatively tight. Our estimates are statistically negative, but greater than −0.02, thus providing a strong restriction on the SDF. This restriction is not only important for asset pricing theory, but it also represents a new yardstick for empirical asset pricing models. Given that autocorrelation estimates can be inaccurate, vary over subperiods or present unreliable standard errors, we confirm the robustness of our bound results with respect to an alternative data set, a subperiod analysis and finite sample concerns.

To further demonstrate the usefulness of our bound, we examine its implications for two general classes of candidate asset pricing models. The first class is consumption-based models, where the SDF is the IMRS of consumers. We show that our bound represents a measure of admissible short-term time variation in the risk premium, and that it provides a joint examination of the equity premium puzzle of Mehra and Prescott (1985) and the term premium puzzle of Backus et al. (1989). We illustrate these points by studying analytically and empirically an external habit model and an internal habit model. These models generate time-varying risk premium because the underlying agents have time-varying risk aversion. We show that both models are misspecified with respect to the autocorrelation bound, even though they are within the Hansen and Jagannathan (1991) bound for reasonable parameters. The models prescribe a risk premium that is sufficiently large, but too predictable.

The second class is linear factor models, where the SDF is a linear function of pricing factors, including hedging factors and information variables. The autocorrelation bound is useful to shed light on the debates on whether the predictability motivating these models is spurious and can be expected rationally. We illustrate this by studying 13 linear factor models that Hodrick and Zhang (2001) find “interesting” because they obtain some success in explaining the cross-section of returns by capturing the time variation in the business cycle. We show that 11 of the 13 “interesting models” are significantly misspecified with respect to the autocorrelation bound, casting doubts on their ability to correctly capture the time-varying economic conditions. Our autocorrelation restriction is particularly relevant for so-called scaled-factor models, in which the SDF factor loadings are linear function of some lagged information variables. We show that the loading assumption results in a SDF that will more likely generate non-admissible time variation. We finally demonstrate how the autocorrelation bound can be used to examine the appropriateness of the empirical factors by studying their autocorrelation structure and time variation contributions. Our results point out a number of problematic factors commonly used in empirical models, although we cannot determine to what degree this finding comes from aggregation biases in published macroeconomic variables or truly inadmissible time variation in the unobservable fundamental economic factors.

This paper is organized as followed. First, we provide the derivation and economic interpretation of the SDF autocorrelation bound. Next, we present our estimation technique, our data sets and our empirical results regarding the estimation of the bound. Then, we examine the implications of our findings for candidate asset pricing models. Finally, we provide our conclusion.

2. Derivation and interpretation of the bound

Using a general framework, this section derives a bound on the first-order autocorrelation of admissible SDFs and then examines its economic interpretation.

2.1. Derivation

Under the law of one price, there exists a random variable $m_{t+1}$, called the SDF, such that

$$E[x_{t+1}m_{t+1}|l_t] = q_t,$$  

(1) where $x_{t+1}$ is an $N$-vector of asset payoffs with a corresponding $N$-vector of prices $q_t$, and $l_t$ is the information available at time $t$. Eq. (1) is often termed the basic or canonical pricing equation as it is valid under most asset pricing theories. $m_{t+1}$ is either a real or a nominal SDF depending on whether the payoffs are in units of the consumption basket or in units of currency. As conditional moments are difficult to estimate, since it is not possible to consider all available information, it is often more convenient to work with unconditional moments. Applying the law of iterated expectation to Eq. (1), the unconditional basic pricing equation is given by

$$E[x_{t+1}m_{t+1}] = E[q_t].$$  

(2)

As suggested by Hansen and Richard (1987), it is possible to incorporate conditioning information as part of the restrictions imposed by Eq. (2) by viewing $x_{t+1}$ and $q_t$ as payoffs and prices to managed strategies involving information variables available in $l_t$.

An important result from Hansen and Jagannathan (1991) is that the payoff on a particular portfolio can also serve as a SDF,

$$m_{t+1} = x_{t+1}E[x_{t+1}x_{t+1}^T]^{-1}E[q_t].$$  

(3)

If there is an asset with a constant unit payoffs, Eq. (2) implies that $E[m_{t+1}]$ is equal to the mean price of this asset, which puts a restriction on the mean of all SDFs, including $m_{t+1}$.

Besides $m_{t+1}$, there are an infinite number of SDFs that satisfy Eq. (2), or in the terminology of Hansen and Jagannathan (1991), that are admissible. The celebrated Hansen–Jagannathan bound places a lower bound on the variance of all admissible SDFs by showing that $m_{t+1}$ has the smallest variance among all such SDFs,

$$\text{Var}[m_{t+1}] = \text{Var}[m_{t+1}] = E[q_tE[x_{t+1}x_{t+1}^T]^{-1}E[x_{t+1}E[x_{t+1}x_{t+1}^T]^{-1}E[q_t].$$  

(4)

where $\text{Var}[x_{t+1}]$ is the covariance matrix of the asset payoffs.

The key theoretical insight of this paper is to combine the Hansen–Jagannathan bound with the restrictions on admissible SDFs imposed by the basic pricing equation under the assumption that payoffs and prices on conditionally risk-free assets of different maturities are available in the economy. Specifically, we assume that the asset market data represented by $x_{t+1}$ and $q_t$ include an...

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1 Hansen and Jagannathan (1991) discuss technical assumptions needed to obtain Eq. (3). In particular, the necessary moments, including second moments of $m_{t+1}$ and $x_{t+1}$, are defined and the assets in $x_{t+1}$ are nonredundant.
asset with a unit payoff and a price denoted by $B_{1,t}$, and an asset with payoff $B_{2,t}$, and price $B_{2,t}$,

$$x_{t+1} = \left( \begin{array}{c} 1 \\ x_{t+1}^{X_2} \end{array} \right), \quad q_t = \left( \begin{array}{c} B_{1,t} \\ B_{2,t} \end{array} \right),$$

(5)

where $x_{t+1}^{X_2}$ and $q_t^{X_2}$ are respectively $(N - 2)$-vectors of payoffs and prices for the remaining available assets. In terms of observable financial assets, the first two assets in $x_{t+1}$ represent respectively a one-period and a two-period risk-free discount (or zero-coupon) bonds. Of course, a two-period zero-coupon bond has a next-period payoff equal to the price of a one-period bond one period later.

**Proposition 1.** Suppose an economy where Eq. (1) holds for the asset payoffs and prices given in (5). Let $\rho_{m_0/m_1}$ denote the correlation coefficient between $m_{t+1}$ and $m_{t+2}$. Then

$$0 \leq \rho_{m_1/m_2} \leq \rho_{m_1/m_2}^{*} \leq \rho_{m_1/m_2} \leq \rho_{m_1/m_2}^{*} \leq \rho_{m_1/m_2} \leq \rho_{m_1/m_2}^{*} \leq \rho_{m_1/m_2}$$

and

$$\rho_{m_1/m_2}^{*} \leq \rho_{m_1/m_2} \leq 0 \quad \text{if} \quad E[|B_{2,t}|] - (E[B_{1,t}])^2 \leq 0,$$

where

$$\rho_{m_1/m_2}^{*} = \frac{E[|B_{2,t}|] - (E[B_{1,t}])^2}{\sqrt{\text{Var}[m_{t+1}]/\text{Var}[m_{t+2}]}}.$$

**Proof.** The proof is straightforward. By definition, under stationarity, the unconditional correlation coefficient between $m_{t+1}$ and $m_{t+2}$ is given by

$$\rho_{m_1/m_2} \equiv \frac{\text{Cov}[m_{t+1}, m_{t+2}]}{\sqrt{\text{Var}[m_{t+1}] \text{Var}[m_{t+2}]}} = \frac{E[m_{t+1}m_{t+2}] - (E[m_{t+1}])^2}{\text{Var}[m_{t+1}]}.$$

The numerator of the autocorrelation is the autocovariance of the SDF. Using Eq. (1), the law of iterated expectation and the availability of bond payoffs and prices, it can be rewritten as

$$E[m_{t+1}m_{t+2}] - (E[m_{t+1}])^2 = E[m_{t+1}(E[m_{t+2} | I_{t+1}]) - (E[m_{t+1}])^2]$$

$$= E[m_{t+1}B_{t+1} - (E[B_{t+1}])^2]$$

$$= E[B_{2,t}] - (E[B_{1,t}])^2.$$

The denominator of the autocorrelation is the variance of the SDF. It is unobservable, but Eq. (4) provides a lower bound on its value. Dividing the autocovariance of the SDF by the lower bound on its variance gives $\rho_{m_1/m_2}^{*}$, the bound on the autocorrelation of the SDF. Of course, as explicitly stated in the proposition, the sign of $\rho_{m_1/m_2}^{*}$, and whether it represents a lower or upper bound, depend on the sign of the autocovariance of the SDF.

**Proposition 1** looks relatively simple. Using the insight of Hansen and Jagannathan (1991), it places a restriction on the first-order autocorrelation of the SDF by making minimal assumptions on the economy. In particular, it does not make assumptions on the SDF functional form or the distribution of payoffs. Because the bound is formed with observable assets, it has the same appealing features as the Hansen–Jagannathan bound in that it can potentially shed light on why a particular class of models fail to explain returns and indicate steps to improve them. In contrast to the Hansen–Jagannathan bound, which is concerned with the variance of the SDF across states, the autocorrelation bound focuses on an important time-series property. Hence, the autocorrelation bound differs substantially from the Hansen–Jagannathan bound, as well as other bounds motivated by cross-sectional properties of the SDF, like the bounds of Snow (1991) on selected higher moments or the bound of Bansal and Lehmann (1997) on the expectation of the log SDF.

While the autocorrelation bound is formed with observable assets, its derivation does not allow the identification of the admissible SDF at the bound. In particular, the autocorrelation of $m_{t+1}$, the minimum variance SDF, is unlikely to equal $\rho_{m_1/m_2}$ and could be outside the bound. The autocovariance of the SDF is derived by using the fact that $E[m_{t+1} | I_{t+1}] = B_{t+1}$ under Eq. (1). While $m_{t+1}$ is such that $E[m_{t+1}^2 | I_{t+1}] = E[B_{t+1}^2]$ and $E[m_{t+1}B_{t+1}] = E[B_{t+1}]$, it is unlikely that $E[m_{t+1}^2 | I_{t+1}] = B_{t+1}$. Hence, $m_{t+1}$ is an admissible SDF for the unconditional Eq. (2), but is not likely admissible for the conditional version in Eq. (1). Hence, it does not necessarily provide a relevant autocorrelation. Given that $\rho_{m_1/m_2}$ employs $\text{Var}[m_{t+1}]$ as denominator even though $m_{t+1}$ might not have an admissible autocorrelation, the (unobservable) autocorrelations of admissible SDFs are probably closer to zero than the autocorrelation bound indicates.

Even though the autocorrelation bound requires information on the SDF variance, the driving moment of the bound is the first-order autocovariance of the SDF. An autocorrelation can be thought of as a standardized autocovariance. Hence, studying either moment leads to similar conclusions. Our discussion mainly focuses on the autocorrelation as this statistic has well-known advantages over the autocovariance for interpretation. Specifically, it is the slope coefficient in a linear regression of a variable on a constant and its first lag, is always between plus and minus one, and is independent of the scale of the variable. These characteristics explain why the autocorrelations of stock returns, and not their autocovariances, have been studied widely. As shown next, these characteristics also make the autocorrelation of the SDF easier to interpret economically.

### 2.2. Economic interpretation

The autocorrelation is a measure of linear predictability of the SDF from its immediate past. To illustrate explicitly, we can use a stationary first-order autoregressive or AR(1) model,

$$m_{t+1} = E[m_{t+1}] + \rho_{m_1/m_2} (m_t - E[m_{t+1}]) + \epsilon_{t+1},$$

with $E[\epsilon_{t+1}] = 0$ and $E[\epsilon_{t+1}\epsilon_{j}] = 0$ for $j \neq 0$. As a coefficient of predictability, the autocorrelation of the SDF has an economic role in the predictable time variation of the economy. We now highlight this role by focusing on the implications of the sign and magnitude of the autocorrelation.

The sign of the autocorrelation coefficient can be useful in understanding the expected direction of the business cycle changes over time. For example, in consumption-based asset pricing models, the SDF is the IMRS of consumers and is a decreasing function of aggregate consumption (representing the state of the economy). Hence, it is inversely related to the desirability of the state of the economy. Simply put, payoffs received in recession are worth more than payoffs received in expansion. If the SDF is

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1. Cochrane (1992, Fig. 4) provides some evidence on this possibility. For price-dividend ratios payoffs, he reports that the autocorrelations of the SDFs at the variance bound are generally greater than 0.5. The next sections will make clear economically and empirically that these values are not admissible.

2. Notice that it is also possible to obtain a tighter autocorrelation bound by using strategies that tighten the variance bound. Such strategies include incorporating a positivity constraint as in Hansen and Jagannathan (1991), the implications of economic factors as in Balduzzi and Kallal (1997) and the role of state variables in Kan and Zhou (2006), and using conditioning information efficiently as in Gallant et al. (1990), Ferson and Siegel (2003), Bekaert and Liu (2004) and Abhyankar et al. (2007).
positively (negatively) autocorrelated, then a good or bad economic state will tend to be followed by a similar (opposite) economic state. So the autocovariance and autocorrelation of the SDF provide an indication of whether or not recessionary states are expected to be followed by expansionary states.

In addition, the magnitude of the autocorrelation coefficient is an indication of the importance of the predictable variation due to the previous SDF. Specifically, the coefficient of determination $R^2 = \rho_{m_1,m_2}^2$ in the AR(1) model. Thus, $\rho_{m_1,m_2}$ gives the proportion of the variability of the SDF explained by the previous SDF, and

$$Var[m_{t+1}] = \frac{E[Var[\epsilon_{t+1}]]}{1-\rho_{m_1,m_2}} = \frac{E[Var[m_{t+1}]]}{1-\rho_{m_1,m_2}} = \frac{E[Var[m_{t+1}]]}{1-\rho_{m_1,m_2}},$$

where $Var[m_{t+1}] = Var[\epsilon_{t+1}/\mu_0, \mu_1]$. In the AR(1) model, $Var[\epsilon_{t+1}]$ represents the unpredictable variability of the SDF and is equal to the mean conditional variability of the SDF $E[Var[m_{t+1}]]$. As the autocorrelation gets away from zero, the predictable variation of the SDF increases, and thus the business cycle changes become more predictable in IMRS models. When $\rho_{m_1,m_2} = 0$, 25% of $Var[m_{t+1}]$ is explained by the previous SDF, while when $\rho_{m_1,m_2} \rightarrow 1$, almost 100% of the variation in the SDF is predictable. The autocorrelation bound can thus provide an important diagnostic tool to examine if the predictable time variation implicit in a model is admissible.

Given that the SDF is an economy-wide variable central in pricing all assets, its autocorrelation can be related to time variation in returns. In particular, as only its unpredictable representation varies in equilibrium, we can relate the autocorrelation of the SDF to risk premium and other measures of investment opportunities. The rest of this section presents these alternatives.

### 2.2.1. Linear beta representation and equity premium

What is the effect of the autocorrelation of the SDF, and its associated economic time variation, on expected equity returns? One simple way to understand this effect is to relate expressions for the unconditional linear beta representation, on expected equity returns? One rest of this section presents these alternatives.

By looking at the mean-standard deviation representation, we can obtain a result for Sharpe ratios and the mean–variance frontier of the investment opportunity set (IOS hereafter) that parallels the result of the previous subsection. Specifically, we can rewrite Eq. (2) as

$$Sh = \rho_{m_1,m_2} R_i Std[m_{t+1}],$$

where

$$Sh = \frac{E[R_{t+1}^i] - R_i}{Std[R_{t+1}^i]} \cdot$$

is the Sharpe or reward-to-volatility ratio of asset $i$. In this representation, a greater variability of the SDF means a higher excess return per unit of standard deviation for risky assets, with the importance of systematic versus diversifiable risk is measured by the correlation of the asset’s return with the negative SDF. The portfolios of assets for which this correlation is one represent the most fully diversified portfolios and provide the highest Sharpe ratio, denoted by $Sh^{MV}$. In the expected return–standard deviation space, they have the highest slope of a line from the risk-free asset and thus sit on the mean–variance frontier of the IOS. For available assets, as demonstrated by Hansen and Jagannathan (1991), the Hansen–Jagannathan standard deviation bound $Std[m_{t+1}^i]$ is proportional to the highest achievable Sharpe ratio, $Sh = R_i Std[m_{t+1}^i]$. Conditional versions for the Sharpe ratios and the mean–variance frontier are straightforward to obtain.

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4. Notice that $R_i$ is not equal to the expected one-period risk-free rate since, due to Jensen’s inequality, $R_i = 1/E[B_{t+1}] < E[R_{t+1}] = E[R_{t+1}^i]$.

5. Notice that the expression for $\beta'$ can also be written as the slope coefficient of a regression of $R_{t+1}^i$ on $-m_{t+1}/R_t^i$. $\beta'$ represents the mean conditional variability of the SDF $Var[m_{t+1}^i]$.

Eq. (7) suggests two ways to increase the SDF risk premium leading to a higher unconditional equity premium. The first way is by increasing the conditional SDF variability $Var[m_{t+1}^i]$, which also leads to a higher conditional equity premium. The second way is by making the SDF more predictable, through a larger-magnitude SDF autocorrelation. As predictable variation in the SDF does not represent risk, the second way can be thought as “spuriously” generating a higher unconditional equity premium. The autocorrelation bound provides a restriction on the magnitude of the predictable variation in the SDF risk premium, and hence limits the second way. More specifically, using the result from Eq. (7) with the linear beta representations, we obtain

$$E[R_{t+1}^i] = \left(1 - \rho_{m_1,m_2}^2\right) R_f$$

The autocorrelation of the SDF relates the mean conditional risk premium $E[R_{t+1}^m]$ with the unconditional risk premium $R_f$. Equivalently, for any two assets $i$ and $j$, it relates the asset $i$’s mean excess conditional return per unit of beta risk with the asset $j$’s unconditional excess return per unit of beta risk. A square SDF autocorrelation $\rho_{m_1,m_2}$ too high generates a difference too large between the conditional and unconditional SDF risk premiums or assets’ excess returns (per unit of beta risk) implied by the model.

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6. It is important to note that, although the conditional linear beta representation holds more generally (using the full information set $i_t$ in the conditional operators), we focus on the restricted set $i_t = m_t$ to highlight the role of the SDF autocorrelation. Given that the set of information at time $t$ is potentially much larger, our analysis does not consider other sources of time variation in returns and should be interpreted accordingly.
Using Eq. (7) and focusing on unit correlation portfolios (i.e., the mean–variance frontier), we can relate the slopes of the conditional and unconditional mean–variance frontiers as

\[ E\left(\frac{Sh_{m_2}^{MV}}{R_{m_2}^{1}}\right)^2 = \left(1 - \rho_{m_1,m_2}^{2}\right) E\left(\frac{Sh_{m_1}^{MV}}{R_{m_1}^{1}}\right)^2. \]

From this expression, the further \( \rho_{m_1,m_2} \) is from zero, the lower is the mean square slope of the conditional IOS frontier (for a given risk-free rate) due to the predictability of the SDF. In the limit, as \( \rho_{m_1,m_2} \to 1 \), the investment opportunities become totally predictable, resulting in zero conditional Sharpe ratios in equilibrium.\(^7\)

Hence, a model with a square SDF autocorrelation \( \rho_{m_1,m_2}^2 \) above the bound generates an unacceptable difference between its implied conditional and unconditional mean–variance frontiers.

### 2.2.3. Term premium on the two-period risk-free asset

Given that the autocovariance of the SDF is determined by the expected discount bond prices of differing maturity, we can also establish a relation between the autocovariance of the SDF and the one-period returns on risk-free assets of different maturities. Specifically, we examine the expected term premium implicit in the one-period return of the two-period risk-free bond, defined as

\[ E[TP^n] = \frac{R^n_{m_2}}{R^n_{m_1}} - \rho_{m_1,m_2}^{2}. \]

This expression shows that if the autocorrelation (or autocovariance) of the SDF is equal to zero, there is no term premium on the two-period risk-free asset. Otherwise, they are of opposite signs, and given that \( R^n_{m_2} \) should be close to one, the autocovariance of the SDF is of similar magnitude to the term premium. Furthermore, since the volatility of the SDF represents the premium for the maturity of the assets, as given by the term premium on the two-period risk-free bond \( E[TP^n] \), then the autocorrelation bound will be close to zero, providing a strong restriction on admissible SDFs. Thus, the autocorrelation bound implicitly examines the importance of the term premium relative to the risk premium prescribed by admissible SDFs.

### 2.2.5. Related literature

In summary, the autocorrelation of SDF represents a useful measure of term variation implicit in a model, with implications for risk premium and other representations of investment opportunities. By restricting its admissible region, the autocorrelation bound thus provides an important diagnostic tool to examine if the time variation implicit in a candidate model is admissible.

While we are not aware of another study that directly considers the time variation in risk implied by the SDF autocorrelation, Ferson (1989) presents an investigation closely related to ours. He studies the time variation in expected returns and risk due to changes in the level of interest rates. In regressions of security returns on one-month treasury bill rates, he shows that if the intercepts and slopes are not equal, then Eq. (1) implies that the risk of the assets (measured by their covariances with the SDF) must be time-varying. Empirically, Ferson (1989) documents that the risk of numerous assets is time-varying. Since the SDF autocorrelation is equal to zero if the risk-free rate is constant, we also examine the time variation in expected returns and risk due to changes in the risk-free rate. Our bound gives a direct measure of the magnitude of admissible time variation that can be associated with changes in the risk-free rate.

Finally, there exists a large literature that attempts to explain stock and bond returns simultaneously.\(^9\) Given that it links stock and bond returns over one and two periods, the SDF autocorrelation provides a convenient statistic to summarize the difficulty of most models in this literature, and it clarifies their misspecification of the equilibrium time-varying risk-return relation.

### 3. Estimation of the bound

The usefulness of the autocorrelation bound provided in Eq. (6), and restated in (8), is an empirical question. After all, it is even possible that the estimate of \( \rho_{m_1,m_2} \) is not between plus and minus one, which would make the bound useless given the Cauchy–Schwarz inequality restrictions on \( \rho \). Even if the bound meets this criteria, a tighter bound is always more desirable in terms of restriction on the SDF. This section presents our estimation technique, our data sets, and our resulting empirical estimates of the SDF autocovariance and autocorrelation bound.

#### 3.1. Estimation technique

A simple strategy to estimate the autocorrelation bound is to map the problem into moment conditions and use the generalized method of moments (GMMs) of Hansen (1982). For a sample of size \( T \), we implement this strategy with the following moments:

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\(^7\) We can present these implications in a simper way by assuming that the conditional variance of the AR(1) residual is constant over time. Then the expression simplifies to \( \rho_{m_1,m_2}^{2} - \frac{1}{T} \rho_{m_1,m_2}^{2} \).

\(^8\) For the rest of the paper, the term premium on the two-period risk-free asset \( E[TP^n] \) will be called the term premium as a shortcut. We emphasize that term premium should not be confused with the term premium commonly used empirically, which refers to the premium that a long-term risk-free asset command over a short-term one.

\(^9\) This literature is far too large so a partial list is included here. Consumption-based models have been proposed by Campbell (1986), Breeden et al. (1989), Ferson and Constantinides (1991), Ferson and Harvey (1992) and Abel (1999). Linear factor models have been proposed by Campbell (1996) and Fama and French (1993). Evidence on predictability has been presented by Keim and Stambaugh (1986), Campbell (1987, 2000) and Fama and French (1989). Singleton (1990) and Ferson (2003) are related reviews.
Table 1
Summary statistics. This table shows the average (Avg), the standard deviation (Std), the minimum (Min), the maximum (Max) and the first-order autocorrelation ($\rho_1$) for our monthly and quarterly data sets. The sample size $T$ of each data set is also reported. The price data for the one-period and two-period bonds ($B_1$ and $B_2$) are taken from the Fama Treasury Bill Term Structure File in CRSP. The other asset payoffs are the (gross) holding period returns on the value-weighted index taken from CRSP ($R_{vw}$), and some services (SIC codes 5000–5999 and 7000–7999), finance (SIC codes 6000–6999) and other sectors (agriculture, mines, oil, construction, transportation, telecommunication, health and legal services). The (gross) inflation rates implied by the Consumer Price Index ($\pi_t$) is obtained from CRSP. The data cover the period June 1959–December 2000.

<table>
<thead>
<tr>
<th></th>
<th>Monthly ($T = 499$)</th>
<th>Quarterly ($T = 167$)</th>
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<tr>
<td></td>
<td>Avg</td>
<td>Std</td>
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<tr>
<td>$B_1$</td>
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<td>$B_2$</td>
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<td>1.0099</td>
<td>0.0422</td>
</tr>
<tr>
<td>$R_{bm}$</td>
<td>1.0096</td>
<td>0.0476</td>
</tr>
<tr>
<td>$R_{qbm}$</td>
<td>1.0130</td>
<td>0.0459</td>
</tr>
<tr>
<td>$R_{gmv}$</td>
<td>1.0103</td>
<td>0.0448</td>
</tr>
<tr>
<td>$R_{gmv}$</td>
<td>1.0090</td>
<td>0.0396</td>
</tr>
<tr>
<td>$R_{gmv}$</td>
<td>1.0115</td>
<td>0.0562</td>
</tr>
<tr>
<td>$R_{gmv}$</td>
<td>1.0112</td>
<td>0.0509</td>
</tr>
<tr>
<td>$R_{gmv}$</td>
<td>1.0090</td>
<td>0.0454</td>
</tr>
<tr>
<td>$R_{gmv}$</td>
<td>1.0036</td>
<td>0.0033</td>
</tr>
</tbody>
</table>

The data are constructed from bonds with a maximum mismatch of four days and are aligned such that each month lasts 30.4 days. To obtain results for the real SDF, we follow Campbell et al. (1997, Section 11.2.1) by making the assumption that the inflation is independent of the real SDF. Under this assumption, $B_{1,t}^r = B_{1,t} - P_t \pi_t [1/P_{t-1}] [1/h]$ and $B_{2,t}^r = B_{2,t} - P_t \pi_t [1/P_{t-1}] [2/h]$, where $B_{1,t}^r$ is the nominal discount bond price of maturity $i$, $B_{2,t}^r$ is the corresponding real discount bond price, and $P_t$ is the price index at time $t$. Then, under rational expectations, we compute the real risk-free discount bond prices by multiplying the nominal bond prices by the gross inflation realized up to the maturity of the bonds. Hence, our real discount bonds provide a return to maturity equal to the ex post real risk-free return. The realized inflation is defined as the growth in the Consumer Price Index taken from CRSP.

We examine two sets of asset payoffs to represent $\chi_{t-1}^{2}$ in Eq. (5). To obtain a reasonable but parsimonious sample of the cross-sectional differences in the equity market, the asset payoffs in the first set, denoted $\chi_{t-1}^{2}$, are the gross returns on the value-weighted index taken from CRSP, a low and a high book-to-market portfolios: $\chi_{t-1}^{2} = (R_{vw}, R_{bm}, R_{qbm})$. The book-to-market portfolios are value-weighted portfolios of the firms with the 30% lowest and highest book-to-market, respectively. Since book-to-market portfolios might be problematic as an ex ante reflection of the equity investment opportunity set (see Lo and MacKinlay (1990), MacKinlay (1995) and Ferson et al. (1999)), our second set of asset payoffs $\chi_{t-1}^{2}$ consists of the gross returns on five value-weighted industry portfolios: $\chi_{t-1}^{2} = (R_{gmv}, R_{gmv}, R_{gmv}, R_{gmv}, R_{gmv})$. The portfolios regroup firms respectively in manufacturing (SIC codes 2000–3999), utilities (SIC codes 4900–4999), wholesale, retail and some services (SIC codes 5000–5999 and 7000–7999), finance (SIC codes 6000–6999) and other sectors (agriculture, mines, oil, construction, transportation, telecommunication, health and legal services). For the real SDF, the data are adjusted to take into account the inflation implied by the Consumer Price Index taken from CRSP.

10 More precisely, our quarterly data are from the quarter ending in June 1959 to the one ending in December 2000.

11 We agree with them that this common and convenient assumption may be unrealistic. We argue in Section 3.3.3 that relaxing it would likely not change our estimates of the autocovariance and autocorrelation of the real SDF.

12 We would like to thank Ken French for making the portfolio returns available on his web site.
3.3. Empirical results

3.3.1. Parameter estimates

Table 2 gives the parameter estimates of Eqs. (9)–(13) using the data sets just described. The table shows the estimated values of the mean, autocovariance, Hansen–Jagannathan variance bound and autocorrelation bound for real and nominal SDFs at monthly and quarterly frequencies. It also provides Newey and West (1987) standard errors. Panel A presents the results for the total sample. Panels B and C present the results for two approximately

### Table 2

Moments of admissible SDFs. This table shows the mean, the first-order autocovariance, the Hansen–Jagannathan variance bound and the autocorrelation bound for admissible SDFs. Panel A gives the results for the total sample. Panels B and C give the results for two subsamples with October 1979 as the cutoff date. Moments of the real and nominal SDFs at the monthly and quarterly frequencies from two sets of asset payoffs are presented. The values are estimated using GMM from the moment conditions $\frac{1}{T} \sum_{t=1}^{T} \left( R_{t} - \mu_{t} \right) = 0$, $\frac{1}{T} \sum_{t=1}^{T} \left( R_{t} - \mu_{t} \right) \left( R_{t} - \mu_{t} \right) = 0$, $\frac{1}{T} \sum_{t=1}^{T} \left( R_{t} - \mu_{t} \right) \left( R_{t} - \mu_{t} \right) \left( R_{t} - \mu_{t} \right) = 0$, with the last three moment conditions used alternatively. The data, described in Table 1, cover the period June 1959–December 2000. For the real SDFs, the data are adjusted to account for the inflation.

<table>
<thead>
<tr>
<th>Period in months (T)</th>
<th>Real SDF</th>
<th>Nominal SDF</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1 (T = 499)</td>
<td>3 (T = 167)</td>
</tr>
<tr>
<td><strong>Panel A: Total sample</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\mu_{n}$</td>
<td>0.9912***</td>
<td>0.9960***</td>
</tr>
<tr>
<td>(s.e.)</td>
<td>(0.00017)</td>
<td>(0.00077)</td>
</tr>
<tr>
<td>$\sigma_{n}$</td>
<td>-0.0065***</td>
<td>-0.00139***</td>
</tr>
<tr>
<td>(s.e.)</td>
<td>(0.00010)</td>
<td>(0.00335)</td>
</tr>
<tr>
<td>$\sigma_{n}^{2}$</td>
<td>0.10107***</td>
<td>0.15182***</td>
</tr>
<tr>
<td>(s.e.)</td>
<td>(0.02214)</td>
<td>(0.04702)</td>
</tr>
<tr>
<td>$\rho_{n,m,n}$</td>
<td>-0.00642***</td>
<td>-0.00917***</td>
</tr>
<tr>
<td>(s.e.)</td>
<td>(0.00162)</td>
<td>(0.00376)</td>
</tr>
<tr>
<td><strong>Panel B: Before October 1979</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\mu_{n}$</td>
<td>0.99923</td>
<td>0.99615***</td>
</tr>
<tr>
<td>(s.e.)</td>
<td>(0.00021)</td>
<td>(0.00084)</td>
</tr>
<tr>
<td>$\sigma_{n}$</td>
<td>-0.00355***</td>
<td>-0.00133***</td>
</tr>
<tr>
<td>(s.e.)</td>
<td>(0.0013)</td>
<td>(0.0044)</td>
</tr>
<tr>
<td>$\sigma_{n}^{2}$</td>
<td>0.06482***</td>
<td>0.15576***</td>
</tr>
<tr>
<td>(s.e.)</td>
<td>(0.02727)</td>
<td>(0.05573)</td>
</tr>
<tr>
<td>$\rho_{n,m,n}$</td>
<td>-0.00544*</td>
<td>-0.00851*</td>
</tr>
<tr>
<td>(s.e.)</td>
<td>(0.00323)</td>
<td>(0.00464)</td>
</tr>
<tr>
<td><strong>Panel C: From October 1979</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\mu_{n}$</td>
<td>0.99827***</td>
<td>0.99287***</td>
</tr>
<tr>
<td>(s.e.)</td>
<td>(0.00022)</td>
<td>(0.00095)</td>
</tr>
<tr>
<td>$\sigma_{n}$</td>
<td>-0.00993***</td>
<td>-0.00148***</td>
</tr>
<tr>
<td>(s.e.)</td>
<td>(0.00015)</td>
<td>(0.00055)</td>
</tr>
<tr>
<td>$\sigma_{n}^{2}$</td>
<td>0.18792***</td>
<td>0.21802**</td>
</tr>
<tr>
<td>(s.e.)</td>
<td>(0.04265)</td>
<td>(0.11074)</td>
</tr>
<tr>
<td>$\rho_{n,m,n}$</td>
<td>-0.00497***</td>
<td>-0.00677***</td>
</tr>
<tr>
<td>(s.e.)</td>
<td>(0.00128)</td>
<td>(0.00419)</td>
</tr>
<tr>
<td>$\sigma_{n}^{2}$</td>
<td>0.22043***</td>
<td>0.21565*</td>
</tr>
<tr>
<td>(s.e.)</td>
<td>(0.05180)</td>
<td>(0.11647)</td>
</tr>
<tr>
<td>$\rho_{n,m,n}$</td>
<td>-0.00424***</td>
<td>-0.00684***</td>
</tr>
<tr>
<td>(s.e.)</td>
<td>(0.00105)</td>
<td>(0.00432)</td>
</tr>
</tbody>
</table>

* The significance for this coefficient is with respect to one.
** The coefficient is significant at 10% level.
*** The coefficient is significant at 5% level.
***** The coefficient is significant at 1% level.
Bias-adjusted moments of admissible SDFs. This table shows the bias-adjusted Hansen–Jagannathan variance bound and autocorrelation bound for admissible SDFs, using the adjustment proposed by Ferson and Siegel (2003). Moments of the real and nominal SDFs at the monthly and quarterly frequencies from two sets of asset payoffs are presented. The values are estimated using GMM from the moment conditions

\[
\sum_{t=1}^{T} \left[ R_{i,t} - \mu_{i} - \rho_{i-1,t-1} \left( R_{i,t-1} - \mu_{i-1} - \rho_{i-2,t-2} \left( R_{i,t-2} - \mu_{i-2} \right) \right) \right] = 0,
\]

for each parameter in panel A of Table 2 represent our estimates of the moments of admissible SDFs.

### Table 3

<table>
<thead>
<tr>
<th>Period in months (T)</th>
<th>Real SDF</th>
<th>Nominal SDF</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 (T = 499)</td>
<td>0.08965**</td>
<td>0.00898**</td>
</tr>
<tr>
<td>3 (T = 167)</td>
<td>0.00917**</td>
<td>0.00917**</td>
</tr>
<tr>
<td></td>
<td>0.00898**</td>
<td>0.00898**</td>
</tr>
<tr>
<td></td>
<td>0.00917**</td>
<td>0.00917**</td>
</tr>
</tbody>
</table>

*The coefficient is significant at 10% level.
**The coefficient is significant at 5% level.
***The coefficient is significant at 1% level.

equal sub-samples, using October 1979 as the cutoff date, which is the time of the Federal Reserve monetary policy shift toward inflation (see Ferson and Merrick (1987) and Ghysels and Hall (1990) among others).

The SDF autocovariance represents a measure of admissible time variation in an economic model. Panel A of Table 2 indicates that the real and nominal SDF autocovariances are equal to $-0.00065$ at a monthly frequency and $-0.00139$ at a quarterly frequency. All values remain significantly negative in subperiods, although the monthly SDF autocovariances are slightly smaller in the first subperiod compared to the second one. The estimates of the Hansen–Jagannathan variance bounds, which identify the minimum admissible premium for risk, are significantly positive in all panels. The values are similar to other estimates found in the literature.

The estimates of $\rho_{m_{1},m_{2}}$, in panel A of Table 2 represent our main empirical results. For the first set of asset payoffs, the estimates of the monthly autocorrelation bounds are $-0.00644$ and $-0.00444$ for real and nominal SDF, respectively. The corresponding quarterly values are $-0.00897$ and $-0.00897$. The estimates are similar for our second set of asset payoffs. Using the industry portfolios, the real and nominal estimates of the autocorrelation bounds are $-0.00732$ and $-0.00532$ at a monthly frequency, and $-0.01098$ and $-0.01479$ at a quarterly frequency. All estimates are significantly negative at the 5% level. The subperiod results reveal that the estimates of $\rho_{m_{1},m_{2}}$ are relatively stable through time. As expected from the smaller sample size, their standard errors are generally larger, but they remain significantly negative at least at the 12% level.

#### 3.3.2. Finite sample issues

This section addresses two issues that arise because the parameters are estimated from a finite sample. The first issue is the upward bias in the estimates of the Hansen–Jagannathan variance bound documented by Burnside (1994), Cecchetti et al. (1994), Ferson and Siegel (2003) and Kan and Robotti (2007). Ferson and Siegel (2003) show that the bias increases with the number of observations and increases with the number of asset payoffs. They also provide a correction which is exact for normally distributed returns and simulations to demonstrate that it works well for non-normal returns. Since our autocorrelation bound is inversely related to the variance bound, this literature implies that our bound will be biased toward zero. To see the importance of the bias, we examine bias-adjusted estimates of the autocorrelation bound. The results are reported in Table 3, with the bias-adjusted GMM moments provided in the table description.

Table 3 shows that the autocorrelation bound remains of similar magnitude after the bias correction and thus supports our earlier findings. The adjustment has two effects on our estimates. First, as expected, they become more negative. Second, the autocorrelation bound standard errors increase, thus reducing the significance of our estimates. However, these effects are generally small. For example, the size of the bias is less than 0.003 and the increase in the standard error is less than 0.002 for all but two estimates. Consistent with the findings of Ferson and Siegel (2003), the effects are more pronounced in our quarterly data sets (where the sample size is the smallest), and when using the industry portfolios (where the number of assets is the largest).

The second issue is the finite sample properties of the GMM estimators of the moments of admissible SDFs. We examine this issue with Monte Carlo simulations. Two observations motivate this investigation. First, asymptotic standard errors are generally underestimates in finite sample (see Ferson and Foerster (1994) among others). Second, the bond prices used in the GMM system are highly persistent (see their autocorrelations in Table 1). This near nonstationarity could be problematic as the asymptotic theory of Hansen (1982) assumes strict stationarity of the data.

The simulations are conducted as follow. We estimate a first-order vector autoregression (VAR(1)) for the bond prices and asset returns needed to form $e_{t}$ and $q_{t}$. We then use the estimated coefficient matrix as parameters of the model and generate 10,000 replications with a sample size equal to the number of observations in our data sets. The artificial data generated this way are found to match closely the means, the variances and covariances, and the first-order autocorrelations and serial cross-correlations in our data sets. In each simulation trial, we estimate the parameter values using GMM from the moment conditions in Eqs. (9)–(13). Table 4 gives for each parameter the average, the standard deviation, the 5th and 95th percentiles. The results are reported in Table 4, with the bias-adjusted GMM moments provided in the table description.

---

13 Bekaert et al. (1997) document a finite sample bias in VAR(1) coefficients estimated from highly persistent variables. Following Bekaert and Hodrick (2001), we perform a large scale simulation of the estimated VAR(1) model to estimate the size of the bias. We find that the bias is negligible and thus proceed with the original coefficient estimates. We also reproduce our simulations with a VAR(2) model and find the same results.
Table 4
Finite sample properties of the GMM estimators of the moments of admissible SDFs. This table examines the finite sample properties of the GMM estimators of the mean, the first-order autocovariance, the Hansen–Jagannathan variance bound and the autocorrelation bound for admissible SDFs. For each parameter, the table reports the average (avg), the standard deviation (std), the 5th percentile (P5) and the 95th percentile (P95) taken across 10,000 replications. For the variance bound and the autocorrelation bound, the table also provides the large-scale average using 10,000 + \( T \) observations. Artificial data are generated from a VAR(1) model estimated from the asset payoffs formed from the data described in Table 1. In each simulation trial, the parameter values are estimated using GMM from the moment conditions provided in Table 2. Moments of the real and nominal SDFs at the monthly and quarterly frequencies from two sets of asset payoffs are presented.

<table>
<thead>
<tr>
<th>Period in months (T)</th>
<th>Real SDF</th>
<th>Nominal SDF</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1 (T = 499)</td>
<td>3 (T = 167)</td>
</tr>
<tr>
<td>( \mu_t )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>avg</td>
<td>0.99912</td>
<td>0.99601</td>
</tr>
<tr>
<td>(std)</td>
<td>(0.00024)</td>
<td>(0.00118)</td>
</tr>
<tr>
<td>P5</td>
<td>0.99872</td>
<td>0.99408</td>
</tr>
<tr>
<td>P95</td>
<td>0.99952</td>
<td>0.99794</td>
</tr>
<tr>
<td>( \sigma_{\mu_{t-1}} )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>avg</td>
<td>-0.00664</td>
<td>-0.00140</td>
</tr>
<tr>
<td>(std)</td>
<td>(0.00004)</td>
<td>(0.00023)</td>
</tr>
<tr>
<td>P5</td>
<td>-0.00701</td>
<td>-0.00179</td>
</tr>
<tr>
<td>P95</td>
<td>-0.00057</td>
<td>-0.00102</td>
</tr>
<tr>
<td>( \sigma^2_{\mu_{t-2}} )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>avg</td>
<td>0.10757</td>
<td>0.17722</td>
</tr>
<tr>
<td>(std)</td>
<td>(0.02536)</td>
<td>(0.05906)</td>
</tr>
<tr>
<td>P5</td>
<td>0.07075</td>
<td>0.09406</td>
</tr>
<tr>
<td>P95</td>
<td>0.13565</td>
<td>0.28468</td>
</tr>
<tr>
<td>LS avg</td>
<td>0.09876</td>
<td>0.14785</td>
</tr>
<tr>
<td>( \rho_{\mu_{t-1}, \mu_{t-1}} )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>avg</td>
<td>-0.00626</td>
<td>-0.00872</td>
</tr>
<tr>
<td>(std)</td>
<td>(0.00140)</td>
<td>(0.00310)</td>
</tr>
<tr>
<td>P5</td>
<td>-0.00875</td>
<td>-0.01454</td>
</tr>
<tr>
<td>P95</td>
<td>-0.00424</td>
<td>-0.00469</td>
</tr>
<tr>
<td>LS avg</td>
<td>-0.00648</td>
<td>-0.00946</td>
</tr>
<tr>
<td>( \sigma^2_{\mu_{t-2}} )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>avg</td>
<td>0.09991</td>
<td>0.16028</td>
</tr>
<tr>
<td>(std)</td>
<td>(0.02537)</td>
<td>(0.07308)</td>
</tr>
<tr>
<td>P5</td>
<td>0.06405</td>
<td>0.07491</td>
</tr>
<tr>
<td>P95</td>
<td>0.14631</td>
<td>0.30753</td>
</tr>
<tr>
<td>LS avg</td>
<td>0.08580</td>
<td>0.12013</td>
</tr>
<tr>
<td>( \rho_{\mu_{t-1}, \mu_{t-1}} )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>avg</td>
<td>-0.00677</td>
<td>-0.00961</td>
</tr>
<tr>
<td>(std)</td>
<td>(0.00153)</td>
<td>(0.00396)</td>
</tr>
<tr>
<td>P5</td>
<td>-0.00955</td>
<td>-0.01715</td>
</tr>
<tr>
<td>P95</td>
<td>-0.00454</td>
<td>-0.00479</td>
</tr>
<tr>
<td>LS avg</td>
<td>-0.00746</td>
<td>-0.01160</td>
</tr>
</tbody>
</table>

percentile and the 95th percentile taken across 10,000 replications. For the variance bound and the autocorrelation bound, the table also reports the large-scale average using 10,000 + \( T \) observations. As shown by Person and Siegel (2003), the large-scale average provides an unbiased estimate of the bounds.

The simulation results in Table 4 again support our earlier findings and confirm that they are robust to the issue of the finite sample properties of the GMM estimators. First, consistent with the literature on an upward bias in the variance bound, the finite sample average estimates of the variance bound are greater than their large-scale averages, and hence the autocorrelation bound estimates are closer to zero than their large-scale averages. However, the large-scale (unbiased) averages are very similar to the finite sample average estimates (as well as our estimates in Table 2), and their values are within the variability of the data across simulations trials. Second, while the empirical standard errors are generally slightly larger than the corresponding asymptotic standard errors in Table 2, the inferences on the significance of the SDF moment coefficients are not materially affected by the use of the finite sample empirical distributions.

3.3.3. Summary and further issues
Our empirical results document that the autocovariance and autocorrelation bound of admissible SDFs are very small and significantly negative. For example, our full sample autocorrelation bound estimates are between –0.004 and –0.015. We also show that these results are robust to two alternative sets of assets, to a subperiod analysis and to finite sample concerns. From these findings, two points are noteworthy. First, given the negative SDF autocovariances, the autocorrelation bound estimates represent lower bounds. Thus, the values of the first-order SDF autocorrelation for admissible models should be between our estimates and zero. Second, given the small magnitude of the estimates, the autocorrelation bounds are very tight. Hence, there is little gain in further tightening the autocorrelation bounds by following the strategies mentioned in footnote 3.

In economic terms, our results have numerous interpretations, as discussed in Section 2.2. Taken as a whole, they indicate little admissible time variation in a SDF based on its own past. Such findings are consistent with our knowledge on risk premium, Sharpe ratios and term premium in a nominal economy. Furthermore, while we assume that the inflation is independent from the real SDF to obtain real estimates, they are consistent with our economic expectation in a real economy. Brandt and Wang (2003) and Ang et al. (2008) argue that inflation risk is an important determinant of the term premium, suggesting that the absolute autocorrelation of the real SDF could be even smaller than our estimates. Hence, we believe that our results would not change materially if we could truly observe real discount bond prices.\(^{14}\)

Overall, our empirical results demonstrate that the autocorrelation bound provides a strong and economically sensible restriction on the SDF.

4. Implications for candidate asset pricing models
The autocorrelation bound allows us to study the admissibility of a candidate SDF, denoted \( y_{t+1} \), from a new perspective. We now examine the implications of our autocorrelation bounds for two general classes of asset pricing models: consumption-based models and linear factor models.

4.1. Consumption-based models

In consumption-based asset pricing models, the SDF is the IMRS of consumers,

\[ y_{t+1} = \beta MU_{t+1} / MU_t, \]

where \( \beta \) is the subjective time-discount factor and \( MU_t \) is the marginal utility of consumption at time \( t \). We next examine the implications of our bounds for these models and then provide an investigation for some widely-used habit-formation models.

\(^{14}\) Another potential concern over the use of short-term risk-free bonds (especially at the one-month frequency), as discussed by Duffee (1996) and Longstaff (2000), is that the associated term premium could be partly caused by liquidity or immediacy issues. The term premium could thus potentially be lower after controlling for these issues (which is beyond the scope of this paper). As a lower term premium corresponds to a negative SDF autocovariance of smaller magnitude, this concern should not be problematic for our findings of an almost zero SDF autocorrelation.
4.1.1. Implications

The marginal utility of consumption reflects the desirability of the last unit of consumption. Risk-averse agents like to smooth consumption across the states of the economy. Their marginal utility is low in good times and high in bad times. The SDF autocorrelation bound finds that the marginal utility growth should have an autocorrelation near zero. Hence, there is almost no linear association between the growth in desirability of the last unit of consumption across two adjacent periods.

SDF bounds can provide a distributional assumption-free examination of the puzzling features of the time-separable power utility (or constant relative risk aversion) model. In particular, Hansen and Jagannathan (1991) use their variance bound to provide an alternative characterization of the equity premium puzzle of Mehra and Prescott (1985). They show that the IMRS of a representative consumer with power utility, denoted $y^{PW}$, presents too little variation to be admissible, so that

$$\text{Var}[y^{PW}_{t+1}] \leq \text{Var}[m_{t+1}].$$

Hence, the power utility model is unable to explain the premium for risk observed in equity.

One feature of the equity premium is that it has varied considerably over time. (See Mehra and Prescott (2003).) As the SDF autocorrelation represents a useful measure of admissible time variation, it allows an investigation of whether there is an equity premium time variation puzzle. Our empirical estimates indicate that the mean of the risk premium conditional on the previous SDF is almost equal to the unconditional risk premium. Our economic interpretation highlights that there is a complementarity between the variance and the autocorrelation of the SDF. A model can misspecify the conditional risk premium because it does not prescribe a sufficiently large unconditional risk premium and/or because it implies an inadequate time variation. The autocorrelation bound thus provides a complementary restriction to the SDF variance bound. It represents an additional hurdle for evaluating candidate IMRSs proposed as explanation for the equity premium puzzle.

As the SDF autocorrelation is related to the term premium, our bound implicitly considers another puzzling feature of the power utility model. Backus et al. (1989) show that the model is unable to match the observed term premium for quarterly data. As our bound looks at the importance of the term premium relative to the risk premium prescribed by admissible SDFs, it provides a joint examination of the term and risk premium puzzles. Given our empirical estimates, a candidate IMRS should prescribe a small term premium compared to its risk premium.

4.1.2. Empirical analysis

To illustrate these implications, we examine two widely used habit-formation models. Consumers with habit-formation preferences base their utility on their consumption compared to a reference point, called the habit. Habit-formation models have been proposed as a partial resolution of the equity premium puzzle. For example, Cochrane and Hansen (1992) show that a simple model where the habit is a linear function of past consumption is within the Hansen–Jagannathan bound for reasonable parameters. The short-term time variation in the risk premium of habit-formation models and their potential for solving the term premium puzzle has been largely unexplored.

We study two models with utility specification given by

$$U_t = E \left[ \sum_{s=0}^{\infty} \beta^s (C_{t+s} - H_t)^{1-\gamma} - 1 \right]_{t+1}, \quad i = \{EH, IH\}.$$
aggregate per capita consumption growth is assumed to follow an AR(1) process:

\[ \text{IMRSs are provided in Appendix A.} \]

obtain higher consumption relative to their habit (good times) in

from such bad times since they have a lower habit in the next per-

low consumption relative to their habit, they also benefit indirectly

can potentially solve the term premium puzzle. Intuitively, while

models have smaller IMRS autocovariances than the

PW

\[ \beta = 1 - \rho_t + \rho_t^2 \sigma_t^2 \], \( K = \frac{\text{Var}(y_t)}{\text{Var}(y_t)^2} \). The approximations come from the use of first and/or second-order Taylor’s expansions. The derivations of the analytical

IMRSs are provided in Appendix A.

Table 5

<table>
<thead>
<tr>
<th>Moments</th>
<th>PW model</th>
<th>EH model</th>
<th>IH model</th>
</tr>
</thead>
<tbody>
<tr>
<td>( E[y_{t+1}] )</td>
<td>( \beta \left( \delta + \frac{1}{1 - \rho_t^2} \right) )</td>
<td>( \beta \left( \delta + \frac{1}{1 - \rho_t^2} (1 + \frac{2(1-\eta)}{1-\rho_t^2}) \right) )</td>
<td>( \beta \left( \delta + \frac{1}{1 - \rho_t^2} (1 + \frac{2(1-\eta)}{1-\rho_t^2}) + 2K(1-\phi) \left( K - \frac{1}{K} \right) \right) )</td>
</tr>
<tr>
<td>( \text{Var}[y_{t+1}] )</td>
<td>( \frac{\sigma_t^2}{1 - \rho_t^2} )</td>
<td>( \frac{\sigma_t^2}{1 - \rho_t^2} (1 + \frac{2(1-\eta)}{1-\rho_t^2}) )</td>
<td>( \frac{\sigma_t^2}{1 - \rho_t^2} (1 + \frac{2(1-\eta)}{1-\rho_t^2}) + 2K(1-\phi) \left( K - \frac{1}{K} \right) )</td>
</tr>
<tr>
<td>( \text{Cov}[y_{t+1}, y_{t+2}] )</td>
<td>( \rho_{y_{t+1}, y_{t+2}} )</td>
<td>( \rho_{y_{t+1}, y_{t+2}} )</td>
<td>( \rho_{y_{t+1}, y_{t+2}} )</td>
</tr>
<tr>
<td>( \rho_{y_{t+1}, y_{t+2}} )</td>
<td>( \phi )</td>
<td>( \phi )</td>
<td>( \phi \</td>
</tr>
</tbody>
</table>

Fig. 1. Monthly IMRS variances and autocorrelations. Variances (top panels) and autocorrelations (bottom panels) of admissible and candidate IMRSs at the monthly frequency. Each panel provides the empirical moment (continuous line with symbols), its 95% confidence interval (dotted lines with symbols), the analytical moment (dashed line with symbols), the bounds (continuous lines without symbol) and their 95% confidence interval (dotted lines without symbol). Gamma is the curvature parameter \( \gamma \).

\( i = \{EH, IH\} \), where \( K_i^\eta \) is positive and increasing in \( \eta \) and \( \gamma \). The habit models have smaller IMRS autocovariances than the PW model, implying that they prescribe a greater term premium. Hence, they can potentially solve the term premium puzzle. Intuitively, while agents with high habit level fear bad times, which provide them low consumption relative to their habit, they also benefit indirectly from such bad times since they have a lower habit in the next period. Bad times in the current period increase the likelihood that they obtain higher consumption relative to their habit (good times) in the next period, resulting in a negatively autocorrelated IMRS.

Figs. 1 and 2 illustrate our findings from the second step at monthly and quarterly frequencies, respectively. The top panels give the Hansen–Jagannathan variance bound along with the variance of the PW, EH and IH IMRSs, while the bottom panels give our autocorrelation bound along with the autocorrelation of the IMRSs. Continuous lines with symbols represent empirical moments while dashed lines with symbols provide analytical moments. Dotted lines with symbols give 95% asymptotic confidence intervals for the empirical moments.\(^{15}\) The bounds and their 95% asymptotic confidence intervals are given by continuous and dotted lines without symbol, respectively. We show the moments for reasonable values of \( \gamma \); assuming \( \beta = 1, \eta = 0.8 \) for the EH model and \( \eta = 0.6 \) for the

\(^{15}\) To compute the intervals, we use the simplified results derived in Fuller (1996, Theorem 6.3.6 and Corollary 6.3.6.1) that, under normality (see Corollary 6.2.1.1), \( \sqrt{T} (\hat{\sigma}^2 - \sigma^2) \overset{\Delta}{\sim} N(0, 2\sigma^4) \) and that, assuming an AR(1) process (see Eq. (6.2.9)), \( \sqrt{T} (\hat{\rho} - \rho) \overset{\Delta}{\sim} N(0, 1 - \rho^2) \).
Finally, we take the values for the real SDF using the first set of assets, in panel A of Table 2, for the bounds and use their standard errors to compute the confidence intervals.

In both figures, the top panels show that the habit-formation models represent a partial resolution to the equity premium puzzle. While the PW model is always below the Hansen–Jagannathan bound, the EH and IH models are reasonably close to the bound with γ = 8 at the monthly frequency and with γ = 10 at the quarterly frequency. These results are consistent with the literature. The agents’ concerns over maintaining their habit lead them to demand a greater premium for risk.

In contrast, the bottom panels show that the habit models are puzzling with respect to their risk premium time variation and their term premium. Their IMRS autocorrelations are significantly below the bound, with values not far from −0.5. Bad times are too likely followed by good times for habit agents, with near 25% predictability. The models generate a high equity premium in part by making the IMRS too predictable. They also generate strongly economically misspecified term premiums. For example, at γ = 8, the approximate term premiums of the EH and IH models are respectively 2.7% and 3.7% at the monthly frequency, and 2.6% and 1.6% at the quarterly frequency. The observed values are 0.065% in monthly data and 0.139% in quarterly data.

Importantly, the figures show that the IMRS autocorrelations of the habit models are not greatly affected by variation in γ. This relation holds well at the quarterly frequency, but is also seen at the monthly frequency for γ < 10. All analytical autocorrelations in Table 5 are independent of γ, implying that the weak positive relation between the IMRS autocorrelations and γ is a second or third order effect. Thus, although γ can be picked to match either the observed IMRS variance or autocovariance, our results indicate that the EH and IP models cannot jointly match both.17

To further demonstrate this finding, we use our analytical results in Table 5 to find the level of habit needed to match the autocorrelation bound, given the consumption growth AR(1) parameters. Figs. 1 and 2 reveal that the analytical expressions are adequate approximations of their corresponding empirical moments, as (except for the monthly habit models with γ < 10) they usually fall within the confidence intervals. Solving for η such that ρ_1 = −ρ_2 = ρ_3, we obtain η = −0.24 at the monthly frequency and η = 0.21 at the quarterly frequency. To match the autocorrelation bound, the habit level has to be very small or even negative (implying a model with durability of consumption similar to one studied by Ferson and Constantinides (1991)), making the habit models similar to the PW model in terms of their (inadequate) risk premium.

Table 6 considers numerous variations of the EH and IH models to examine the robustness with respect to the consumption measure, the habit level value and the linear habit specification. It provides the IMRS autocovariance and autocorrelation with γ varying

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16 We choose η such that the IMRSs remain positive for all values of γ. As can be seen from their expressions in Appendix A, the empirical IMRSs for the PW and EH models are straightforward to compute. The IH IMRS is more difficult to estimate because of the presence of conditional expectations. However, the variable D_i^γ is a stationary process. As in Daniel and Marshall (1997), we fit an ARMA model for D_i^γ and use the one-period-ahead forecast as estimate of E[D_i^γ|γ]. For model selection, we estimate all possible ARMA (p, q) models with 0 ≤ (p,q) ≤ 12, and choose the one with the smallest Schwarz’s Bayesian information criterion.

17 Focusing on the relation between the IMRS variance and the term premium, Dunn and Singleton (1986) show that a similar tradeoff exists in a model where agents have preferences for the durability of goods.
from 2 to 10. Panels A and B look at the monthly and quarterly models, respectively. In each panel, the first $EH$ and $IH$ models provide the results for the base models studied in Figs. 1 and 2.

First, we consider alternative ways of measuring consumption. While our base models use non-durable and services expenditures, the second and third $EH$ and $IH$ models consider them separately. Also, while our base quarterly models use the third monthly observation in each quarter to measure growth, the fourth $EH$ and $IH$ quarterly models take the sum of the three monthly observations in each quarter. For all these models, the IMRS autocovariances and autocorrelations are too negative. Hence, our findings are robust to alternative ways of measuring consumption.

Second, we consider greater values of the habit level $\eta$ motivated by the high values estimated by Ferson and Constantinides (1991). For the $EH$ models, we report the results for $\eta = 0.9$ and $\eta = 0.95$. For the $IH$ models, since the empirical IMRS does not remain positive for larger values, we report the results for $\eta = 0.8$. The IMRS autocovariances and autocorrelations remain too negative. We also investigate a quarterly $IH$ model with $\eta = 0.9$ and $γ = 1$, which corresponds well to the models in Ferson and Constantinides (1991). Its IMRS autocovariance and autocorrelation are $-0.187$ and $-0.368$, respectively. Our findings are thus robust to higher values for the habit level.

Third, we consider two models with alternative specifications of the linear habit function. The first model, denoted by $5Y–EH$, is the five-year linear external habit model studied by Li (2001). Its habit approaches an equal-weighted average of the consumption over the past five years. With $\eta = 0.98$, Li (2001) argues that the $5Y–EH$ model performs well in fitting the cyclical behavior of equity returns. We find its IMRS autocorrelation too negative at monthly frequency and too positive at quarterly frequency. As the five-year average results in a habit not autocorrelated, the $5Y–EH$ IMRS inherits the time-series properties of the underlying consumption growth.

The second model, denoted by $SEH$ and $SIH$, is the seasonal habit model proposed by Ferson and Harvey (1992). Its habit is proportional to the consumption a year ago and quarterly non-seasonally adjusted consumption data are used. Ferson and Harvey (1992) document that a $SIH$ model performs better than an $IH$ model relative to the Hansen–Jagannathan bound. Table 6 presents the results for the $SEH$ models with $\eta = 0.8$ and $\eta = 0.9$, and for the $SIH$ model with $\eta = 0.8$. The IMRS autocorrelations for the seasonal habit models are too negative. We also examine a $SIH$ model with $\eta = 0.9$ and $γ = 1$, values similar to the ones in Ferson and Harvey (1992, table 7). We obtain an IMRS autocovariance and autocorrelation of $-0.113$ and $-0.430$, respectively. Our findings are thus robust to two alternative linear habit specifications.

It is well known that sampling error and time aggregation in consumption data can lead to a spurious autocorrelation in consumption growth. (See Breeden et al. (1989), Wilcox (1992) and
Heaton (1993). It is reassuring that our findings are similar with monthly and quarterly consumption growth data, which have an autocorrelation $\phi$ of opposite sign and magnitude. The analytical expressions allow to further examine the effect of this coefficient. First, we compute the IMRS autocorrelations when $\phi = 0$. In all cases, we obtain $\rho_{yt,yt+1} = -0.49$. Hence, they are not outside the bounds because of a spurious autocorrelation in consumption growth. Second, when we solve for $\phi$ such that $\rho_{yt,yt+1} = \rho_{m_1,m_2}$, we obtain $\phi = 0.79$ for the EH model with $\eta = 0.8$ and $\phi = 0.59$ for the IH model with $\eta = 0.6$. To meet the bound, $\phi$ has to be implausibly large. Finally, the IMRS autocorrelation expressions are independent of the variance of consumption growth. Hence, while time aggregation can generate a lower variability of consumption growth, it does not affect our findings on the IMRS autocorrelation.

4.2. Linear factor models

In general linear factor models, the SDF is a time-varying linear function of the pricing factors,

$$y_{t+1} = b_0 + b_1 f_{t+1},$$

where $b_0$ is the time-varying intercept, $b_1$ is the time-varying $K$-vector of factor loadings, and $f_{t+1}$ is a $K$-vector of factors. This model is known as the conditional linear factor model. For practicality, most studies focus on two special cases. First, the unconditional factor model,

$$y_{t+1} = b_0 + b_1 f_{t+1},$$

assumes constant parameters. Second, the scaled-factor model is obtained by assuming $b_{0x}$ and $b_1$ are linear function of some information variable $z_t$ in $l_t$,

$$y_{t+1} = (b_{0x} + b_2 z_t) + (b_{1x} + b_{1z} z_t) f_{t+1},$$

$$= b_{0x} + b_2 z_t + b_{1x} f_{t+1} + b_{1z} f_{t+1} z_t.$$

The last equality shows that this model reduces to an unconditional factor model where the factors are scaled by the information variable.\(^{19}\) We next examine the implications of the autocorrelation bounds for these models and then provide an empirical analysis using a wide variety of models.

4.2.1. Implications

The linear factor models are usually motivated by the Intertemporal Capital Asset Pricing Model (ICAPM) of Merton (1973). The ICAPM accounts for the predictable time variation in risk to explain the cross-section of returns. It recognizes the dynamic nature of the economy in two ways. First, as emphasized by Campbell (1996), factors that are good forecasters of future returns on wealth (so-called hedging factors) should have larger prices of risk. Hence, hedging factors should predict changes in the investment opportunity set (IOS). Second, the quantity of risk of an asset and the risk premium in the economy can vary over the business cycle. The conditional ICAPM captures this effect by explaining conditional returns using conditional moments. Information variables that help predict changes in the IOS become natural choices to account for their variation. The scaled-factor model illustrates that a conditional model generally leads to an unconditional model with more factors. As argued by Jagannathan and Wang (1996), these additional factors are important to explain the cross-section of returns, even if no hedging factors are considered. Supported by the empirical evidence, starting with Keim and Stambaugh (1986), Conrad and Kaul (1988), Fama and French (1989), Chen (1991) and Ferson and Harvey (1991), models with hedging factors and time-varying risk premiums are now commonplace and have served in numerous applications.

However, there are at least two important unresolved issues with respect to modelling the time variation in expected returns. First, the existence and rationality of the predictability are still under debate. There is a large literature on “data mining” and spurious regressions that casts doubts on the existence of predictability. Ferson et al. (2003) provide an overview and show that both effects reinforce each other, concluding that many results may be spurious. MacKinlay (1995) argues that the Sharpe ratios of some predictability-based trading strategies are too large to be explained by a rational multifactor model. There are also studies on market efficiency and behavioral finance (see Schwert (2003) and Barberis and Thaler (2003) for reviews) that argue for irrational (or mispricing) explanations of the predictability. Second, the ICAPM offers little theoretical guidance on the choice of factors and information variables. Fama (1991) calls this theory a “fishing license” as it can justify the inclusion of about any desirable factor. In discussing this issue, Cochrane (2001, chapter 7) advocates a disciplined approach in the choice of factors involving compelling economic motivation as well as statistical restraints.

By providing a new restriction on the time-series of the SDF, the autocorrelation bound can shed light on these two unresolved issues. First, the autocorrelation bound is a useful way to examine the admissibility of the time variation implicit in the linear factor models. While hedging factors and information variables should individually be good forecasters of the IOS, the autocorrelation bound indicates that jointly, they should not be able to forecast the next period SDF. Models that pass this strong restriction are not only less likely to include variables that spuriously predict changes in the IOS, but they also meet one important criterion for the rationality of their time variation. Second, the autocorrelation bound gives some indications on the appropriateness of the empirical factors. In particular, the time-series properties of the empirical factors become an important statistical consideration in the selection of hedging factors and information variables.

We can demonstrate these points by studying the analytical expression for the SDF autocovariance of a candidate unconditional factor model with $K$ factors, possibly including scaled factors,

$$\text{Cov}(y_{t+1}, y_{t+2}) = (b_1 \circ \text{Std}(f_{t+1})) \rho_{b_{1x},b_{1z}}(b_1 \circ \text{Std}(f_{t+1})).$$

(14)

where $\text{Std}(f_{t+1}) = \left(\text{Std}(f_{1x,t+1}) \cdots \text{Std}(f_{Kx,t+1})\right)$, $b_1 \circ \text{Std}(f_{t+1})$ is the element-by-element multiplication of $b_1$ and $\text{Std}(f_{t+1})$, and $\rho_{b_{1x},b_{1z}}$ is the autocorrelation matrix of the factors, defined as

$$\rho_{b_{1x},b_{1z}} = \left(\begin{array}{cccc}
\rho_{b_{1x},b_{1x}}^{1,1} & \cdots & \rho_{b_{1x},b_{1x}}^{1,K} \\
\vdots & \ddots & \vdots \\
\rho_{b_{1x},b_{1z}}^{K,1} & \cdots & \rho_{b_{1x},b_{1z}}^{K,K}
\end{array}\right).$$

From Table 2, the autocovariance of admissible SDFs is close to zero. For a candidate SDF to generate a reasonable time variation, Eq. (14) reveals that a linear combination of the autocorrelation matrix elements should be close to zero. The most straightforward way is to choose empirical factors with elements close to zero. In fact, in a one-factor model, the factor should have approximately zero autocorrelation. In a multifactor model, a highly-autocorrelated factor should be “balanced” by factors with opposite autocorrelation or serial cross-correlation effects. Hence, a model with just one strongly autocorrelated factor is likely to fail the autocorrelation restriction.
A particularly interesting case occurs for the scaled-factor model, where at least one factor is an information variable \( z_e \). The variable is usually lagged by one period, one column of the autocorrelation matrix becomes the simple correlations between the factors and the information variable \( \rho_{i,t+1, z_e} \). Since these contemporaneous correlations are likely significant, they might play an important role in the misspecification of the risk premium time variation of scaled-factor model. A similar analysis holds for the scaled factors, as columns of the autocorrelation matrix involve the correlations \( \rho_{i,t+1, z_e} \). The autocorrelation bound thus raises concerns on the validity of the common assumption about the use of lagged information variables.

To highlight the role of each autocorrelation matrix element, we can rewrite Eq. (14) as

\[
\text{Corr}[y_{t+1},y_{t+2}] = \sum_{i=1}^{K} \sum_{j=1}^{K} C_{ij}^i,
\]

where \( C_{ij}^i \equiv b_i^{'i} \beta_i^{'i} \beta_j^{'i} \beta_j^{'i} \) is the contribution of the serial cross-correlation \( \rho_{i,t+1, j,t+2} \) to the SDF autocovariance. Hence, \( C_i \) represents the contribution of \( \rho_{i,t+1, j,t+2} \) to the time variation associated with the candidate SDF autocovariance. Two types of contribution are helpful in understanding the role of a factor in generating this time variation. The first type, denoted by \( C_{ij}^i \), is the partial contribution of a factor. \( C_i \) gives the portion of the SDF autocovariance related to the serial correlation of factor \( i \). It provides the SDF autocovariance that a model would generate under the assumption that the other factors are useless (i.e., \( b_i^{'i} = 0 \)). The second type is the total contribution of a factor, defined as

\[
C_i \equiv \sum_{j=1}^{K} (C_{ij}^i + C_{ji}^i) - C_{ii}^i.
\]

\( C_i \) is the sum of all contributions related to factor \( i \). It represents the decrease in the SDF autocovariance, and its associated time variation, if the serial correlation and the serial cross-correlations of factor \( i \) would be zero, or if factor \( i \) is assumed useless (i.e., \( b_i^{'i} = 0 \)). Large values for \( C_i \) and \( C_i \) indicate that factor \( i \) generates unrealistic time variation in the risk premium.

By studying the autocorrelation matrix of the factors, along with their partial and total contributions, we can understand why a candidate SDF fails to meet the autocorrelation bound, in the sense that the factor(s) causing the rejection can be identified, and thus provide empirical guidance on the choice of factors and information variables.

4.2.2. Empirical analysis

To examine these issues empirically, we start from the results of Hodrick and Zhang (2001), a comprehensive study of linear factor models. They examine the specification errors of 21 monthly and 32 quarterly scaled and nonscaled factor models. They use the methodology of Hans and Jagannathan (1997) and a common set of test assets, namely the 25 Fama–French size and book-to-market portfolios and the Treasury bill. The nonscaled models are a constant, the CAPM, a linearized CCAPM, the conditional CAPM of Jagannathan and Wang (1996) (the JW model), a linear version of the intertemporal model of Campbell (1996), a linearized version of the production-based model of Cochrane (1996), and the Fama and French (1993) three-factor model (the FF3 model) and five-factor model (the FF5 model). Scaled-factor models are formed by using the cyclical part of the industrial production index (the IP variable), the cyclical component of real GNP (the GNP variable), the consumption-wealth ratio of Lettau and Ludvigson (2001) (the CAY variable) and a January dummy (the JAN variable) to capture movements over the business cycle.

Given their large set of models, Hodrick and Zhang (2001) display the parameter estimates only for “interesting models”. According to their definition, these models pass the test of zero Hansen and Jagannathan (1997) distance at the 1% level and have scaling parameters (for scaled-factor models) jointly significant at the 5% level. They obtain 7 monthly models and 5 quarterly models that are “interesting”. They also report the monthly FF3 nonscaled model for comparison. These 13 models either contain factors whose role is to act as a hedge against changes in the IOS, or present parameters that move with the changing economic conditions.

We now re-estimate these 13 models to test their admissibility with respect to the autocorrelation bound.21 The results are in Table 7, with monthly models in panel A and quarterly models in panel B. The second column of Table 7 gives the SDF \( y \), with parameter estimates obtained following the methodology of Hodrick and Zhang (2001). Appendix B provides the definitions and construction details of the required variables. Our estimates are similar in sign and magnitude to the ones in Hodrick and Zhang (2001, table 4), with discrepancies attributed to different data periods.22 The third column of Table 7 presents the SDF autocovariances \( \sigma_{y,t+1, y,t+2} \). The last column gives the SDF autocovariances \( \rho_{i,t+1, y,t+2} \), as well as their asymptotic statistical significance.

Table 7 shows that the autocovariances of most “interesting” factor models are far from the autocorrelation bounds, which are \(-0.00444\) (monthly) and \(-0.00934\) (quarterly) for the nominal SDF with the first set of assets (panel A of Table 2). Only two models, the monthly FF3 with factors scaled by JAN and Cochrane’s quarterly model with factors scaled by JAN, do not have autocovariances significantly different from the bound. On the other hand, 11 autocovariances are significant at the 1% level, with nine values significantly positive, implying a time variation where “bad times” are followed by more “bad times” too often. The autocovariances confirm that these models are strongly economically misspecified, and even the two models with insignificant autocovariances obtain values that are economically far from the estimated values in Table 2.24

Tables 8 and 9 explore which factors contribute significantly to the misspecification of the “interesting models”. Table 8 gives the autocovariance matrices \( \rho_{i,t+1, y,t+2} \) of the factors used in the monthly models (panel A) and the quarterly models (panel B). To keep their size reasonable, we report the results for the individual factors and information variables, and thus eliminate the scaled factors. Furthermore, to facilitate the reading, we put the diagonal elements (the autocovariances of the individual factors) in bold. Table 8 is helpful in understanding why the autocovariances of many

21 Note that by picking models that were shown to be “interesting” in part because they capture well the movements in the economic cycle, our results are biased against finding any significant misspecification of the SDF autocorrelation. However, if the time variation contributes to an improved risk adjustment at the cost of misspecifying the time series properties of the SDF, then the autocorrelation bound is well suited to detect this situation.

22 We use data from June 1959 to December 2000, while Hodrick and Zhang (2001) use data from January 1952 to December 1997 for monthly models and from January 1953 to December 1997 for quarterly models.

23 The significance comes from the results of Fuller (1996, Corollary 6.3.6.1) given earlier, under zero autocorrelation. Hamilton (1994, Section 8.3) shows that the asymptotic distribution of the autocorrelation is the same for any consistent estimate of \( y_t \). Thus, the significance is unaffected by the separate estimation of the SDF parameters.

24 As a robustness check, we re-compute \( \sigma_{y,t+1, y,t+2} \) and \( \rho_{i,t+1, y,t+2} \), using the parameter estimates reported in Hodrick and Zhang (2001, Table 4). We obtain similar findings and the same conclusions. The three worthy differences are the results for the monthly CCAPM scaled by IP, now significantly positive, the monthly CCAPM scaled by JAN, now insignificant, and Cochrane’s quarterly model scaled by JAN, now significantly negative at the 10% level.
Table 7
SDF autocovariances and autocorrelations for factor models. This table presents the SDF autocovariances and autocorrelations for the “interesting models” identified by Hodrick and Zhang (2001). Panel A shows the results for the monthly models. Panel B shows the results for the quarterly models. The candidate SDFs are estimated following Hodrick and Zhang (2001) with data covering the period June 1959-December 2000. Details on the definitions and construction of the factors are provided in Appendix B. The coefficient is significantly different from zero at the 1% level. The coefficient is significantly different from zero at the 5% level. The coefficient is significantly different from zero at the 10% level.

<table>
<thead>
<tr>
<th>Model</th>
<th>y</th>
<th>$\sigma_{yt, t_2}$</th>
<th>$\rho_{yt, t_2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Panel A: Monthly models</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>CAPM scaled by IP</td>
<td>1.051 - 0.169RW - 0.181IP + 0.0434RVW + 0.1881IP + 0.280Ac + IP</td>
<td>0.104</td>
<td>0.371***</td>
</tr>
<tr>
<td>CCAPM scaled by JAN</td>
<td>1.014 + 0.240Ac - 0.874fAN + 0.783Ac - IP + JAN</td>
<td>-0.151</td>
<td>-0.322***</td>
</tr>
<tr>
<td>JF's model scaled by AC</td>
<td>1.036 - 0.004AC - 0.534fANX + 0.117fAN + 0.598IP</td>
<td>-0.23</td>
<td>-0.184***</td>
</tr>
<tr>
<td>JWY's model scaled by JAN</td>
<td>0.130 + 0.040fAN - 0.810fANX + 2.125fAN - 0.108fAN - 1.008fAN + JAN</td>
<td>0.547</td>
<td>0.673***</td>
</tr>
<tr>
<td>Campbell's model nonscaled</td>
<td>-0.37 - 0.008RW - 0.059LR + 0.588DV - 2.974RTB - 0.859fAN</td>
<td>1.287</td>
<td>0.962***</td>
</tr>
<tr>
<td>FF3 nonscaled</td>
<td>1.052 - 0.045RVW - 0.016SMB - 0.080HML</td>
<td>0.011</td>
<td>0.193***</td>
</tr>
<tr>
<td>FF3 scaled by JAN</td>
<td>-0.189fAN + 0.465fANX - 0.213HML + JAN</td>
<td>-0.004</td>
<td>-0.011</td>
</tr>
<tr>
<td>Panel B: Quarterly models</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>JWY's model nonscaled</td>
<td>-1.127 + 0.005RVW + 0.204fANX + 1.208fAN</td>
<td>0.502</td>
<td>0.466***</td>
</tr>
<tr>
<td>Campbell's model nonscaled</td>
<td>0.068 - 0.009fAN + 0.314LRB + 0.348DV - 0.342RTB - 0.707fAN</td>
<td>0.655</td>
<td>0.665***</td>
</tr>
<tr>
<td>Cochrane's model scaled by GNP</td>
<td>-0.137fANX + 0.138fANX - 0.333GNP</td>
<td>0.331</td>
<td>0.405***</td>
</tr>
<tr>
<td>Cochrane's model scaled by JAN</td>
<td>0.725fANX + 0.509fANX + JAN</td>
<td>-0.148</td>
<td>-0.109</td>
</tr>
<tr>
<td>FF3 nonscaled</td>
<td>1.629 + 0.055fAN + 0.013SMB - 0.062HML - 0.158fANX + 0.605fANX</td>
<td>0.070</td>
<td>0.349***</td>
</tr>
</tbody>
</table>

Table 8
Autocorrelation matrices of the factors. This table presents the autocorrelation matrices for the factors and information variables used in the “interesting models” identified by Hodrick and Zhang (2001). Panel A shows the results for the monthly models. Panel B shows the results for the quarterly models. The data cover the period June 1959-December 2000. Details on the definitions and construction of the factors are provided in Appendix B. The diagonal elements are in bold. Correlations with absolute values greater than 0.088 at the monthly frequency and 0.152 at the quarterly frequency are significantly different from zero at the 5% level.

<table>
<thead>
<tr>
<th>$f_{t+1} \times f_{t+2}$</th>
<th>$R_{RVW}$</th>
<th>$R_{fAN}$</th>
<th>$R_{SMB}$</th>
<th>$R_{HML}$</th>
<th>$R_{IP}$</th>
<th>$R_{JAN}$</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>$R_{RVW}$</strong></td>
<td>0.05</td>
<td>0.05</td>
<td>0.07</td>
<td>-0.02</td>
<td>0.00</td>
<td>0.03</td>
</tr>
<tr>
<td>$R_{fAN}$</td>
<td>0.12</td>
<td>-0.35</td>
<td>-0.10</td>
<td>-0.01</td>
<td>-0.21</td>
<td>-0.08</td>
</tr>
<tr>
<td><strong>$R_{fAN}$</strong></td>
<td>0.09</td>
<td>-0.07</td>
<td>0.97</td>
<td>0.09</td>
<td>-0.12</td>
<td>0.61</td>
</tr>
<tr>
<td><strong>$R_{SMB}$</strong></td>
<td>0.15</td>
<td>-0.04</td>
<td>0.09</td>
<td>0.54</td>
<td>-0.08</td>
<td>0.23</td>
</tr>
<tr>
<td>$R_{HML}$</td>
<td>0.02</td>
<td>-0.08</td>
<td>0.60</td>
<td>0.24</td>
<td>0.06</td>
<td>0.98</td>
</tr>
<tr>
<td><strong>$R_{IP}$</strong></td>
<td>0.10</td>
<td>-0.07</td>
<td>0.26</td>
<td>0.21</td>
<td>-0.06</td>
<td>-0.03</td>
</tr>
<tr>
<td><strong>$R_{JAN}$</strong></td>
<td>0.11</td>
<td>0.06</td>
<td>0.00</td>
<td>-0.24</td>
<td>0.11</td>
<td>-0.12</td>
</tr>
<tr>
<td><strong>$R_{IP}$</strong></td>
<td>0.09</td>
<td>0.01</td>
<td>0.02</td>
<td>0.06</td>
<td>0.04</td>
<td>0.10</td>
</tr>
<tr>
<td><strong>$R_{JAN}$</strong></td>
<td>-0.04</td>
<td>0.00</td>
<td>0.00</td>
<td>-0.01</td>
<td>-0.06</td>
<td>0.00</td>
</tr>
<tr>
<td><strong>$R_{JAN}$</strong></td>
<td>-0.11</td>
<td>0.05</td>
<td>-0.21</td>
<td>0.14</td>
<td>0.13</td>
<td>-0.06</td>
</tr>
</tbody>
</table>

| **$R_{fAN}$** | 0.00 | 0.04 | 0.00 | 0.00 | 0.00 | 0.00 | 0.10 | 0.10 | 0.10 | 0.01 | 0.00 | **0.09** |

candidate factor models are significantly positive. At the monthly frequency, the autocorrelations of the factors $R_{fAN}$, $R_{SMB}$, $Df$, $fAN$, $TRM$, and $TRM_{TERM}$ are greater than 0.4. At the quarterly frequency, the same finding holds for $R_{fANX}$, $R_{SMB}$, $Df$, $fAN$, $TRM$, $fANX$, and $TRM_{TERM}$. Their serial cross-correlations are also often significant. Problematic autocorrelations are seen as well for the information variables $fAN$ and $fAN_{fAN}$. Furthermore, their corresponding matrix columns, presenting the contemporaneous correlations $\rho_{yt, t+1}$,
illustrate a problem inherent in scaled-factor models: the use of lagged information variables creates significant cross-correlations, and thus plays a role in the misspecification of their SDF autocorrelations.

Table 9 further identifies factors with significant roles in generating unrealistic time variation. It examines the large contributions of the autocorrelation matrix elements by listing the factors with partial contribution $C_i^j$ and total contribution $C_i$ greater than 0.05 (denoted by “”) or 0.1 (denoted by “++”), or less than −0.05 (denoted by “−”) or −0.1 (denoted by “−−”). Monthly and quarterly models are covered in panel A and panel B, respectively. For example, the table singles out $IP^t$ for the monthly CAPM with factors scaled by IP. This model generates a SDF autocovariance equal to 0.104. Given that the partial contribution $Ci^j/IP^t > 0.1$, the serial correlation of $IP^t$ (equal to 0.94) generates an autocovariance greater than 0.1 by itself. Moreover, the total contribution $Ci/IP^t > 0.05$ implies that the autocovariance would fall by more than 0.05 if $IP^t$ is eliminated. Hence, $IP^t$ is largely responsible for the misspecified time variation. Overall, Table 9 shows numerous factors with both large partial and total contributions. Only three models have no factor with a contribution greater than 0.05 in absolute value. In addition, Table 9 shows that the information variables $IP$ and $GNP$ and their associated scaled factors frequently generate abnormally large time variation contributions. The table reinforces the findings of Table 8 in that the factors previously identified as highly autocorrelated generate inadequate time variation. It thus confirms that these empirical variables are not appropriate as risk factors because of their time-series properties.

5. Conclusion

This paper focuses on the first-order autocorrelation of the SDF, a statistic related to the economic time variation across two adjacent periods. Using observable asset market data, we derive a bound on the SDF autocorrelation. We empirically estimate real and nominal versions of the bound for monthly and quarterly SDFs. We find that the bounds are relatively tight as the autocorrelations of admissible SDFs are significantly negative, but greater than −0.02. We therefore document that the SDF, which can be closely related to the growth in the marginal utility of investors, is an uncorrelated or very slightly mean-reverting process at the monthly and quarterly intervals. This conclusion holds even after a series of robustness checks to alleviate issues related to the accuracy, stability and precision of our estimates in finite samples.

We examine the implications of the bound for two large classes of asset pricing models. On consumption-based models, we show that the bound provides a joint examination of the equity premium puzzle of Mehra and Prescott (1985) and the term premium puzzle of Backus et al. (1989). Furthermore, it allows us to investigate if there is an equity premium time variation puzzle. Using two habit-formation models, we illustrate that the bound results in a restriction that is different, but complementary to one presented by Hansen and Jagannathan (1991). We also provide evidence that the time variation implicit in the linear habit-formation models is not admissible. On linear factor models, we show that 11 of the 13 models deemed “interesting” by Hodrick and Zhang (2001) are significantly misspecified with respect to the autocorrelation bound. We interpret this finding as evidence that the time variation implicit in these models is not rational. Hence, their success in explaining the cross-section of returns might be wrongly attributed to their ability to capture the variation of risk over the business cycle. We furthermore demonstrate how the bound can be used to study the appropriateness of the empirical factors.

Our examination of consumption-based and linear factor models establishes that the autocorrelation bound is a useful restriction on the SDF. We can formulate two general recommendations from our analysis. First, in asset pricing theory, more efforts should be made to model correctly the serial correlations and cross-correlations of the state variables as they determine the time-series properties of the SDF, and hence the time variation in the risk premium. Second, in asset pricing tests, the choice of empirical factors should be influenced by the time-series properties of the variables, especially if the model and testing framework include time-varying specifications. Depending on the testing objectives, this point

---

Table 9

<table>
<thead>
<tr>
<th>Model</th>
<th>$f$ with $C_i^j &lt; -0.05$ or $C_i &gt; 0.05$</th>
<th>$f$ with $C_i &lt; -0.05$ or $C_i &gt; 0.05$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Panel A: Monthly models</td>
<td></td>
<td></td>
</tr>
<tr>
<td>CAPM scaled by IP</td>
<td>$IP^{t}$</td>
<td>$IP^{t}$</td>
</tr>
<tr>
<td>CCAPM scaled by IP</td>
<td>$\Delta C_i / \Delta C_i \times IP^{t}$</td>
<td>$\Delta C_i / \Delta C_i \times IP^{t}$</td>
</tr>
<tr>
<td>JW’s model scaled by IP</td>
<td>$IP^{t} / C_i^{IP,RSL}$</td>
<td>$IP^{t} / C_i^{IP,RSL}$</td>
</tr>
<tr>
<td>JW’s model scaled by JAN</td>
<td>$R_{IP,JAN} \times JAN^{t}$</td>
<td>$R_{IP,JAN} \times JAN^{t}$</td>
</tr>
<tr>
<td>Campbell’s model nonscaled</td>
<td>$\text{DIV}^{t} \times \text{RKT}^{t}$</td>
<td>$\text{DIV}^{t} \times \text{RKT}^{t}$</td>
</tr>
<tr>
<td>F1F nonscaled</td>
<td></td>
<td></td>
</tr>
<tr>
<td>F1F scaled by JAN</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Panel B: Quarterly models</td>
<td></td>
<td></td>
</tr>
<tr>
<td>JW’s model nonscaled</td>
<td>$R_{CN}^{t}$</td>
<td>$R_{CN}^{t}$</td>
</tr>
<tr>
<td>Campbell’s model nonscaled</td>
<td>$\text{DIV}^{t} \times \text{TRM}^{t}$</td>
<td>$\text{DIV}^{t} \times \text{TRM}^{t}$</td>
</tr>
<tr>
<td>Cochrane’s model scaled by GNP</td>
<td>$\text{RINV}^{t} \times \text{GPN}^{t}$</td>
<td>$\text{RINV}^{t} \times \text{GPN}^{t}$</td>
</tr>
<tr>
<td>Cochrane’s model scaled by JAN</td>
<td>$\text{NRINV}^{t} \times \text{JAN}^{t}$</td>
<td>$\text{NRINV}^{t} \times \text{JAN}^{t}$</td>
</tr>
<tr>
<td>FF5 nonscaled</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

A factor with $0.1 < C_i > 0.05$.  
++ A factor with $C_i > 0.1$.  
−− A factor with $-0.1 < C_i < -0.05$.  
−−− A factor with $C_i < -0.1$.  

---

25 Our analysis is complementary to the ones of Lewellen and Nagel (2006), who argue that the time variation in betas and equity premium of conditional CAPM models would have to be implausibly large to explain the cross-section of stock returns, and of Kan and Robotti (2009), who find that commonly used macroeconomic factors are too noisy for conditional models to outperform unconditional models using the Hansen-Jagannathan distance.
might involve formal considerations of measurement issues, like aggregation biases in published macroeconomic variables, as they generate spurious autocorrelation and hence spurious economic time variation.

There are numerous avenues for future research related to the autocorrelation bound. First, it is straightforward to examine the implications of the bound for other interesting asset pricing models. The bound is general enough to be applicable in most asset pricing contexts. Second, it is possible to estimate model parameters using the autocorrelation bound restriction. Such estimation would likely reduce the loadings on factors that are highly autocorrelated. Finally, it is feasible to extend our restriction to consider the term structure of risk-free bonds. This extension of the autocorrelation bound can then be used to examine the admissibility of term structure models.

Acknowledgments

I thank an anonymous referee, Dong-Hyun Ahn, Mike Chernov, Mike Cliff, Frank Coggs, Jennifer Conrad, Nadia Massoud, Ike Mathur (the editor), Marcel Rindisbacher, Ken Singleton, Oumar Sy, Kevin Wang and seminar participants at the Australasian Finance and Banking Conference, European Financial Management Meetings, FMA European Meetings, NFA Meetings and WFA Meetings, and at Laval University for helpful comments and discussions. I am also grateful to Kalok Chan (Department Head) and the Department of Finance at the Hong Kong University of Science and Technology, where part of this research was conducted while I was a Visiting Associate Professor of Finance. I acknowledge financial support from the Institut de Finance Mathématique de Montréal, and the Investors Group Chair in Financial Planning and Faculty of Business Administration at Laval University.

Appendix A. Analytical IMRSs for consumption-based models

This appendix presents the derivations of the approximate analytical IMRSs for the power utility, the external habit and the internal habit models. The derivations are based on the existence of a representative agent with aggregate per capita consumption as his consumption choice. Throughout, with $c_{t}^{1} = \ln c_{t}^{A}$ and $\bar{c}_{t} = c_{t}^{A} - c_{t-1}^{A}$, we assume that

$$\bar{c}_{t} = \bar{c}(1 - \phi) + \phi c_{t-1} + \xi_{t},$$

with $E[\xi_{j}] = 0, Var[\xi_{j}] = \sigma^{2}$ and $E[\xi_{j}\xi_{j-n}] = 0$ for $n \neq 0$. To arrive at reasonable approximate analytical expressions, we use the Taylor’s series expansions $e^{x} = e^{x_{0}} + e^{x_{0}}(x - x_{0}) + \frac{1}{2}e^{x_{0}}(x - x_{0})^{2} + \cdots$ and $\ln x = \ln x_{0} + \frac{1}{x_{0}}(x - x_{0}) - \frac{1}{2x_{0}^{2}}(x - x_{0})^{2} + \cdots$, with first and/or second order expansions being sufficient for our purpose. Using the approximate analytical IMRS, it is straightforward to obtain analytical expressions for the mean, variance, autocovariance and autocorrelation of the IMRS.

A.1. Power utility model

Investors maximize the utility specification given by

$$U_{t} = E \left[ \sum_{s=0}^{\infty} \beta^{s+1} \frac{C_{t+s}^{1} - 1}{1 - \gamma} \right],$$

and their corresponding IMRS is

$$y_{t+1} = \beta \left( \frac{C_{t+1}^{A}}{C_{t}^{A}} \right)^{-\gamma} = \beta e^{-\gamma(c_{t+1}^{A} - c_{t}^{A})} \approx \beta \left( 1 - \gamma \bar{c}_{t+1} + \frac{1}{2} \gamma^{2} \bar{c}_{t+1}^{2} \right).$$

For the representative investor, the approximate analytical IMRS is:

$$y_{t+1} = \beta \left( \frac{C_{t+1} - \eta C_{t}}{C_{t} - \eta C_{t-1}} \right)^{-\gamma}.$$

A.2. External habit model

Investors maximize the utility specification given by

$$U_{t} = E \left[ \sum_{s=0}^{\infty} \beta^{s} \frac{\left[ C_{t+s} - \eta C_{t+s-1} \right]^{1-\gamma} - 1}{1 - \gamma} \right],$$

and their corresponding IMRS is

$$y_{t+1} = \beta \left( \frac{C_{t+1} - \eta C_{t}}{C_{t} - \eta C_{t-1}} \right)^{-\gamma}.$$

For the representative investor, the approximate analytical IMRS is:

$$y_{t+1} = \beta \left( \frac{C_{t+1}^{A} - \eta C_{t}^{A}}{C_{t}^{A} - \eta C_{t-1}^{A}} \right)^{-\gamma} = \beta e^{-\gamma \left[ \ln \left( \frac{C_{t+1}^{A} - \eta C_{t}^{A}}{C_{t}^{A} - \eta C_{t-1}^{A}} \right) \right]} \approx \beta \left[ 1 - \gamma \bar{c}_{t+1} + \frac{1}{2} \gamma^{2} \left( \bar{c}_{t+1}^{2} \right) \right].$$

A.3. Internal habit model

Investors maximize the utility specification given by

$$U_{t} = E \left[ \sum_{s=0}^{\infty} \beta^{s} \frac{\left[ C_{t+s} - \eta C_{t+s-1} \right]^{1-\gamma} - 1}{1 - \gamma} \right],$$

and the corresponding IMRS can be written as\footnote{See Daniel and Marshall (1997).}

$$y_{t+1} = \beta \left( \frac{E_{t+1}[D_{t+2}]}{E_{t}[D_{t+1}]} \right) \left( \frac{C_{t+1} - \eta C_{t}}{C_{t} - \eta C_{t-1}} \right)^{-\gamma}$$

with

$$D_{t} = 1 - \eta \beta \left( \frac{C_{t} - \eta C_{t-1}}{C_{t-1} - \eta C_{t-2}} \right).$$

For the representative investor, the approximate analytical IMRS is:

$$y_{t+1} = \beta \left( \frac{E_{t+1}[D_{t+2}^{A}]}{E_{t}[D_{t+1}^{A}]} \right) \left( \frac{C_{t+1}^{A} - \eta C_{t}^{A}}{C_{t}^{A} - \eta C_{t-1}^{A}} \right)^{-\gamma}$$

$$\approx 1 - \eta \beta \left[ 1 - \gamma \bar{c}_{t+1} + \frac{1}{2} \gamma^{2} \left( \bar{c}_{t+1}^{2} \right) \right].$$

$$E_{t-1}[D_{t}^{A}] \approx 1 - \eta \beta \left[ 1 - \gamma \left( g_{t}(1 - \phi) + (\phi - \eta)g_{t-1} \right) + \frac{1}{2} \gamma^{2} \left( g_{t}(1 - \phi) + (\phi - \eta)g_{t-1} \right) \right] \left[ 1 - \gamma \left( g_{t}(1 - \phi) + (\phi - \eta)g_{t-1} \right) \right]^{2}.$$
Appendix B. Variables used in the factor models

This appendix defines the variables used in the factor models and provides details on their construction. Our goal is to replicate the construction of Hodrick and Zhang (2001).

B.1. Factors

The factors needed for the seven asset pricing models are as follows:

- **$R_{MW}$**: the return on the value-weighted CRSP index in excess of the one-month risk-free rate. For the quarterly model, the compounded version is used;
- **$AC$: the growth rate in real nondurables consumption, computed using the personal consumption expenditures for non-durable goods and the consumer price index from the Bureau of Economic Analysis (BEA) of the US Department of Commerce;
- **$R_{REBM}$**: the difference between the yields on $baa$ and $aaa$ corporate bonds from the Fed Board of Governors. For the quarterly model, the third observation in each quarter is used;
- **$R_{LG}$**: the labor income growth, measured as $R_{LG} = (L_{t-1} + L_{t-2}) / (L_{t-2} + L_{t-3})$, where $L$ is labor income per capita calculated as the difference between personal income and dividend income per capita. The data are obtained from BEA. For the quarterly model, the quarterly growth rate in labor income is used;
- **$LBR$: the labor income factor, constructed as the monthly growth rate in real labor income, with data obtained from BEA. For the quarterly model, the quarterly growth rate in real labor income is used;
- **$DIV$: the dividend yield on $R_{MW}$. For the quarterly model, the third observation in each quarter is used;
- **$KTB$: the relative bill rate, calculated as the difference between the one-month T-bill rate and its one-year backward moving average. For the quarterly model, the third observation in each quarter is used;
- **$TRM$: the yield spread between long and short-term government bonds, calculated as the difference in yields on the 30-year government bond and on the one-year government bond. The data is from the Fed Board of Governors. For the quarterly model, the third observation in each quarter is used;
- **$SMB$: the small minus big factor, calculated as the difference in returns on a small size portfolio and a large size portfolio, obtained from Ken French's web site. For the quarterly model, the compounded version is used;
- **$HML$: the high minus low factor, calculated as the difference in returns on a high book-to-market portfolio and a low book-to-market portfolio, obtained from Ken French's web site. For the quarterly model, the compounded version is used;
- **$NRINV$: the growth rate on real nonresidential investment, from BEA;
- **$RINV$: the growth rate on real residential investment, from BEA;
- **$TERM$: the term structure factor, calculated as the difference between the yield on a 30-year government bond and the yield on the one-month bill. For the quarterly model, the third observation in each quarter is used.

B.2. Information variables

The following information variables are used in the scaled factor models:

- **$IP$: the lagged cyclical part of the natural logarithm of the industrial production index, obtained from the FRED database at the Federal Reserve Bank of St. Louis. The Hodrick and Prescott (1997) filter on the first five years is used to initialize the cyclical series. The smoothing parameter is set to 6400. The procedure is then used recursively on all available data to find the subsequent elements for the cyclical series. The method guarantees that each element is in the time $t$ information set;
- **$JAN$: the January dummy, which takes the value of one for each January and zero otherwise. For the quarterly model, $JAN$ takes the value of one for the first quarter and is zero otherwise;
- **$GNP$: the lagged cyclical component of real GNP from FRED. The Hodrick and Prescott (1997) filter with smoothing parameter equal to 1600 is used. Because GNP is not announced until the following quarter, GNP is lagged once so that it is in the time $t$ information set.

References


