Multifrequency jump-diffusions: An equilibrium approach

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Received 21 December 2006; received in revised form 5 June 2007; accepted 5 June 2007
Available online 15 September 2007

Abstract

This paper proposes that equilibrium valuation is a powerful method to generate endogenous jumps in asset prices. We specify an economy with continuous consumption and dividend paths, in which endogenous price jumps originate from the market impact of regime-switches in the drifts and volatilities of fundamentals. We parsimoniously incorporate regimes of heterogeneous durations and verify that the persistence of a shock endogenously increases the magnitude of the induced price jump. As the number of frequencies driving fundamentals goes to infinity, the price process converges to a novel stochastic process, which we call a multifractal jump-diffusion.

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JEL classification: G0; C5

Keywords: Endogenous jumps; General equilibrium; Markov regime-switching; Multifrequency; Fat tails; Stochastic volatility; Time deformation; Volatility component

1. Introduction

In continuous-time settings, jumps in financial prices seem necessary to account for thick tails in asset returns, and the corresponding implied volatility smiles in near-maturity options.\textsuperscript{1} In a seminal contribution, Merton (1976) assumes that the stock price follows an exogenous jump-diffusion with constant volatility. Subsequent research considers econometric refinements such as: stochastic volatility, priced jumps, jumps in volatility, correlation between jumps in returns and volatility, and infinite activity.\textsuperscript{2,3} For example, Bakshi et al. (1997) and Bates (2000) consider price processes

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\textsuperscript{1} Numerous studies provide evidence for jumps in the valuation of stocks and other financial securities, based on the series of either the assets themselves or their derivative claims. See, for example, Andersen et al. (2002), Ball and Torous (1985), Bates (1996), Carr et al. (2002), Carr and Wu (2003), Eraker et al. (2003), Jarrow and Rosenfeld (1984), Jorion (1988), Maheu and McCurdy (2004), and Press (1967).

\textsuperscript{2} Jump processes are classified as having finite or infinite activity depending on whether the number of jumps in a bounded time interval is finite or infinite.

with exogenous jumps and stochastic volatility, concluding that additional discontinuities in volatility are necessary to match option valuations. Duffie et al. (2000) consequently analyze an extension with discrete volatility changes while exogenously specifying the relation between volatility and returns. A related line of research (e.g., Madan et al., 1998) advocates in favor of pure jump processes, which permit infinite activity with many small events and fewer large discontinuities. In all of this literature, jumps in valuations, and their relation to volatility, are exogenously specified.4

In this paper, we propose that equilibrium valuation is a powerful method to generate endogenous discontinuities in asset prices, as well as a number of return characteristics that prior literature specifies exogenously. Our approach builds on a standard consumption-based asset-pricing economy with homogeneous investors, where dividends and consumption may be identical as in Lucas (1978), or imperfectly correlated as in Campbell and Cochrane (1999). Consumption and dividends follow continuous diffusions, but their drift rates and volatilities can undergo discrete Markov switches. This generates endogenous jumps in stock prices, and equilibrium feedback between volatility fluctuations and price discontinuities. Our paper therefore bridges a gap between exogenously specified jump-diffusions and the discrete-time volatility feedback literature, which relates exogenous movements in dividend news volatility to endogenous returns (e.g., Abel, 1988; Barsky, 1989; Calvet and Fisher, 2007; Campbell and Hentschel, 1992).

We then incorporate shocks of heterogeneous durations into our economy by adopting the Markov-switching multifractal (MSM) of Calvet and Fisher (2001, 2004). Under this assumption, dividend volatility is the product of a vector of state components that follow independent regime-switching processes. The components have identical marginal distributions and heterogeneous durations. Thus, some may switch on average only once every several years or decades, while others can have durations measured in days or less. This multifactor volatility specification is highly parsimonious and requires only a few parameters regardless of the size of the state vector. Previous research shows that MSM is consistent with the slowly declining autocovariograms, fat tails, and power variations of financial series. It further provides a closed form likelihood, and substantially outperforms benchmark volatility models in- and out-of-sample.5

The present paper embeds an MSM specification for dividends and consumption within a continuous-time equilibrium model and explores the consequences for endogenous prices. The stock price displays jumps of heterogeneous frequencies, and the largest jump sizes are endogenously triggered by the most persistent volatility shocks. The model thus produces many small jumps and fewer large jumps, as in Madan et al. (1998), Carr et al. (2002), and others. Our equilibrium contributes to this literature by endogenizing the heterogeneity of jump sizes and the association between jump size and frequency.

We investigate the limiting behavior of the economy when the number of volatility components goes to infinity. Under mild conditions, the dividend process weakly converges to a multifractal diffusion,6 building on results from Calvet and Fisher (2001). Even more striking, the equilibrium price:dividend ratio converges to an infinite intensity pure jump process with heterogeneous frequencies. Prices are then the sum of a continuous multifractal diffusion and an infinite intensity pure jump process, yielding a new stochastic process that we accordingly call a multifractal jump-diffusion. A jump in stock prices occurs in the neighborhood of any instant, but the process is continuous almost everywhere.

For simplicity, the majority of the paper focuses on time-separable preferences. The stochastic discount factor is then continuous, and endogenous jumps in stock valuations are “unpriced” in the sense that they do not affect expected excess stock returns (e.g., Merton, 1976). In the final section of the paper, we obtain priced jumps by considering non-separable preferences. Specifically, previous work in discrete time (Calvet and Fisher, 2007) uses Epstein–Zin utility and consumption switches to match the equity premium with reasonable levels of risk aversion. We now show that in continuous time, stochastic differential utility implies endogenous discontinuities in the stochastic discount factor and priced jumps, providing additional flexibility in structural modeling of jump-diffusions.

The main goal of paper is to show that equilibrium conditions can help to generate a parsimonious but rich model of asset prices, including jumps in valuations, multifrequency volatility shocks, a negative correlation between jumps
and volatility, infinite activity, and priced jumps. For expositional clarity, the model is not presented at the highest level of generality. We anticipate that extensions to heterogeneous investors, incomplete markets, and more general preferences will broaden the applicability of our approach.

The remainder of the paper is structured as follows. Section 2 sets out the general consumption-based model with regime-switching dividends and endogenous price jumps. Section 3 introduces the multifrequency dividend specification, and shows that the size of price jumps is inversely related to the persistence of exogenous shocks. Section 4 considers the limiting equilibrium as the number of frequencies goes to infinity. Section 5 extends the approach to stochastic differential utility and endogenous jump premia. Unless stated otherwise, all proofs are given in Appendix A.

2. An equilibrium model with endogenous price jumps

This section develops a continuous-time equilibrium model with regime-shifts in the mean and volatility of consumption and dividend growth.7

2.1. Preferences, information and income

We consider an exchange economy with a single consumption good defined on the set of instants \( t \in [0, \infty) \). The information structure is represented by a filtration \( \{\mathcal{F}_t\} \) on the probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \).

The economy is specified by two independent stochastic processes: a bivariate Brownian motion \( Z_t = (Y(t), Z_D(t)) \in \mathbb{R}^2 \) and a random state vector \( M_t \in \mathbb{R}^k \), where \( k \) is a finite integer. The processes \( Z \) and \( M \) are mutually independent and adapted to the filtration \( \{\mathcal{F}_t\} \). The bivariate Brownian \( Z \) has zero mean and covariance matrix:

\[
\begin{pmatrix}
1 & \rho_{Y,D} \\
\rho_{Y,D} & 1
\end{pmatrix},
\]

where the correlation coefficient \( \rho_{Y,D} = \text{Cov}(dY, dD)/dt \) is strictly positive. The vector \( M_t \) is a stationary Markov process with right-continuous sample paths.

The economy is populated by a finite set of identical investors \( h \in \{1, \ldots, H\} \), who have homogeneous information, preferences and endowments. Investors observe the realization of the processes \( Z \) and \( M \), and have information set \( I_t = \{(Z_s, M_s); s \leq t\} \). The common utility is given by

\[
U_t = \mathbb{E} \left[ \int_0^{+\infty} e^{-\delta s} u(c_{t+s}) \, ds \bigg| I_t \right],
\]

where the discount rate is a positive constant: \( \delta \in (0, \infty) \). The Bernoulli utility \( u(\cdot) \) is twice continuously differentiable, and satisfies the usual monotonicity and concavity conditions: \( u' > 0 \) and \( u'' < 0 \). Furthermore, the Inada conditions hold: \( \lim_{c \to 0} u'(c) = +\infty \) and \( \lim_{c \to +\infty} u'(c) = 0 \).

Every agent continuously receives the exogenous endowment stream \( Y_t \in (0, \infty) \). The process \( Y_t \) is identical across the population, and follows a geometric Brownian motion with stochastic drift and volatility. Specifically, let \( g_Y(\cdot) \) and \( \sigma_Y(\cdot) \) denote deterministic measurable functions defined on \( \mathbb{R}^+_+ \) and taking values on the real line.

Assumption 1 (Income). The moments \( \mathbb{E}[\int_0^t |g_Y(M_s)| \, ds] \) and \( \mathbb{E}[\int_0^t \sigma_Y^2(M_s) \, ds] \) are finite, and the exogenous income stream is given by

\[
\ln(Y_t) \equiv \ln(Y_0) + \int_0^t \left[ g_Y(M_s) - \frac{\sigma_Y^2(M_s)}{2} \right] \, ds + \int_0^t \sigma_Y(M_s) \, dY(s)
\]
at every instant \( t \in [0, \infty) \).

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7 Following the seminal contribution of Hamilton (1989), several authors have considered discrete-time settings with regime-shifts in the drifts and volatilities of consumption and/or dividends (e.g., Calvet and Fisher, 2007; Cechetti et al., 1990; Garcia et al., 2003; Lettau et al., in press). In continuous time, Veronesi (1999, 2000) and David and Veronesi (2002) investigate the impact of investor learning about Markov switches in the drift rate of a Lucas economy and an IID consumption economy. In their settings, information diffuses slowly, and hence beliefs and prices have continuous sample paths.
The moment conditions guarantee that the stochastic integrals are well-defined. By Itô’s lemma, the income flow satisfies the stochastic differential equation
\[
\frac{dY_t}{Y_t} = g_Y(M_t) \, dt + \sigma_Y(M_t) \, dZ_Y(t).
\]

### 2.2. Financial markets and equilibrium

Agents can trade two financial assets: a bond and a stock. The bond has an instantaneous rate of return \( r_f(t) \), which is endogenously determined in equilibrium. Its net supply is equal to zero.

The stock is a claim on the stochastic dividend stream \( \{D_t\}_{t \geq 0} \).

**Assumption 2 (Dividend process).** The dividend stream is given by
\[
\ln(D_t) = \ln(D_0) + \int_0^t \left[ g_D(M_s) - \frac{\sigma_D^2(M_s)}{2} \right] \, ds + \int_0^t \sigma_D(M_s) \, dZ_D(s),
\]
where \( g_D(\cdot) \) and \( \sigma_D(\cdot) \) are measurable functions defined on \( \mathbb{R}_k^\infty \) and valued in \( \mathbb{R} \) such that \( E[\int_0^t |g_D(M_s)| \, ds] < \infty \) and \( E[\int_0^t \sigma_D^2(M_s) \, ds] < \infty \) for all \( t \).

We infer from Itô’s lemma:
\[
\frac{dD_t}{D_t} = g_D(M_t) \, dt + \sigma_D(M_t) \, dZ_D(t).
\]

The dividend process \( D_t \) has continuous sample paths, but its drift and volatility can exhibit discontinuities. Every agent is initially endowed with \( N_s \in \mathbb{R}_+ \) units of stock, where one unit represents one claim on the flow \( D_t \). We treat the difference \( Y_t - N_s D_t \) as the non-tradable component of the endowment flow, which is identical across agents and implicitly defined by Assumptions 1 and 2.

Each agent selects a consumption-portfolio strategy \((c^h, N^h, B^h)\) defined on \( \Omega \times [0, \infty) \) and taking values on \( \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \), where \( c^h(\omega, t), N^h(\omega, t) \) and \( B^h(\omega, t) \) respectively denote consumption, stockholdings and bondholdings in every date-event \( (\omega, t) \). A strategy is called admissible if it is adapted, self-financing, and implies nonnegative wealth at all times.

**Definition.** A general equilibrium consists of a stock price process \( P \), an interest rate process \( r_f \), and a collection of individual admissible consumption-portfolio plans \((c^h, N^h, B^h)\) \( 1 \leq h \leq H \), such that

(i) For every \( h \), \((c^h, N^h, B^h)\) maximizes utility over all admissible plans.

(ii) Goods markets and securities markets clear:
\[
\frac{1}{H} \sum_{h=1}^H c^h(t, \omega) = Y(t, \omega), \quad \frac{1}{H} \sum_{h=1}^H N^h(t, \omega) = N_s, \quad \text{and} \quad \frac{1}{H} \sum_{h=1}^H B^h(t, \omega) = 0
\]

for almost all \((t, \omega)\).

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8 Distinguishing between aggregate endowment flow and dividends on public equity is common in asset pricing settings. For example, Brennan and Xia (2001) invoke non-traded labor income as a wedge between the aggregate endowment flow and dividends. In our economy, the difference \( Y_t - N_s D_t \) can be either positive or negative since \( Y_t \) and \( D_t \) are imperfectly correlated diffusions, and we hence prefer the more general interpretation of an unmodeled endowment flow shock. Recent literature (e.g., Cochrane et al., in press; Santos and Veronesi, 2005) proposes methods to ensure positivity of the non-traded portion of the endowment while maintaining tractable asset prices.

9 The wealth nonnegativity constraint prevents agents from using doubling strategies to create arbitrage profits (Dybvig and Huang, 1988).
Continuous-time securities markets have been the object of a wide literature. An influential line of research characterizes the properties of asset prices in dynamically complete equilibria (e.g., Bick, 1990; Cox et al., 1985; Duffie and Skiadas, 1994; He and Leland, 1993). Raimondo (2005) proves that a securities market equilibrium does indeed exist when exogenous fundamentals (individual utility and endowment, asset payoffs) satisfy standard regularity conditions, without making a priori assumptions on the dynamic completeness or incompleteness of endogenous prices. Anderson and Raimondo (2006) extend the results to heterogeneous investors.

In our setting, agents are fully symmetric, and autarky is therefore the unique equilibrium. Individual consumption coincides with individual income:

\[ c_t(t, \omega) = y_t(t, \omega) \]

for every \( h, t, \omega \), and it is convenient to denote \( C_t \equiv Y_t, Z_C \equiv Y, g_C(\cdot) \equiv g_Y(\cdot), \sigma_C(\cdot) \equiv \sigma_Y(\cdot), \) and \( \rho_{C,D} \equiv \rho_{Y,D} \). The special case where \( g_C \) and \( \sigma_C \) are constant implies IID consumption growth - a standard assumption in asset pricing that is broadly consistent with postwar US data (e.g. Campbell, 2003). Recently, Bansal and Yaron (2004), Hansen et al. (2005), Lettau et al. (in press) and others have argued that consumption may contain small, highly persistent components with large price impacts. The more general dynamics (2.1) accommodate this possibility.

The stochastic discount factor (SDF) is equal to instantaneous marginal utility:

\[ \Lambda_t = e^{-\delta u'(C_t)}. \] (2.3)

It satisfies the stochastic differential equation:

\[ \frac{d\Lambda_t}{\Lambda_t} = -r(M_t) dt - \alpha(C_t) \sigma_C(M_t) dZ_C(t), \]

where \( \alpha(c) \equiv -cu''(c)/u'(c) \) denotes the coefficient of relative risk aversion and \( \pi(c) \equiv -cu'''(c)/u''(c) \) is the coefficient of relative risk prudence. The instantaneous interest rate:

\[ r_t(M_t) = \delta + \alpha(C_t) g_C(M_t) - \frac{\alpha(C_t) \pi(C_t) \sigma_C^2(M_t)}{2} \] (2.4)

increases with investor impatience and the growth rate of the economy, and is reduced by the precautionary motive.

In equilibrium, the stock price \( P_t \) is given by

\[ \frac{P_t}{D_t} = \mathbb{E} \left[ \int_{0}^{+\infty} e^{-\delta s} u'(C_{t+s}) \frac{D_{t+s}}{u'(C_t)} ds \right] \]

The joint distribution of \((C_{t+s}, D_{t+s}/D_t)\) depends on the state \( M_t \) and the consumption level \( C_t \), but not on the initial dividend \( D_t \). The \( P/D \) ratio is therefore a deterministic function of \( M_t \) and \( C_t \), which will henceforth be denoted by \( Q(M_t, C_t) \).

Shifts in the state \( M_t \) induce discontinuous changes in the \( P/D \) ratio and the stock price. We use lower cases for the logarithms of all variables.

**Proposition 1** (Equilibrium stock price). *The stock price follows a jump-diffusion, which can be written in logs as the sum of the continuous dividend process and the price:dividend ratio:*

\[ p_t = d_t + q(M_t, C_t). \]

A price jump occurs when there is a discontinuous change in the Markov state \( M_t \), driving the continuous dividend and consumption processes.

The endogenous price jumps contrast with the continuity of the fundamentals and the SDF.

### 2.3. Equilibrium dynamics under isoelastic utility

As noted by Campbell (2003), consumption and wealth have increased manyfold over the past two centuries, but real interest rates, risk premia and valuation ratios have not consistently trended up or down. To capture these

\[ \text{See Duffie (1996) and the references therein for the analysis of continuous-time Arrow–Debreu economies.} \]
empirical regularities, power utility is often useful in a representative agent setting because of its scale invariance. We correspondingly assume that every investor has the same constant relative risk aversion \( \alpha \in (0, \infty) \), i.e.:

\[
u(c) \equiv \begin{cases} c^{1-\alpha} / (1 - \alpha) & \text{if } \alpha \neq 1, \\ \ln(c) & \text{if } \alpha = 1. \end{cases}
\]

The \( P/D \) ratio then simplifies as follows.

**Proposition 2 (Equilibrium with isoelastic utility).** The \( P/D \) ratio is a deterministic function of the Markov state:

\[
q(M_t) = \ln \mathbb{E}_t \left( \int_0^{+\infty} e^{-\int_0^s \left(r(M_{t+h}) - gD(M_{t+h}) + \alpha \sigma C(M_{t+h}) \sigma D(M_{t+h}) \rho C,D \right) dh} ds \right),
\]

where \( \mathbb{E}_t \) denotes the conditional expectation given \( M_t \).

Over an infinitesimal time interval, the stock price changes by

\[
d(p_t) = d(d_t) + \Delta(q_t).
\]

where \( \Delta(q_t) \equiv q_t - q_{t-} \) denotes the finite variation of the price:dividend ratio in case of a discontinuous regime change. Consider the effect of a Markov switch that increases the volatility of current and future dividends (without impacting consumption). The \( P/D \) ratio falls and induces a negative realization of \( \Delta(q_t) \). Market pricing can thus generate an endogenous negative correlation between volatility changes and price jumps. This contrasts with earlier jump models where the relation between discontinuities and volatility is exogenously postulated (e.g. Duffie et al., 2000; Carr and Wu, 2004).

Under isoelastic utility, our results can be made robust to some degree of investor heterogeneity. Assume that in addition to the stock and bond, a complete set of traded financial assets exists, and investors can hedge the risks implicit in the state vector \( M_t \). If agents have heterogeneous coefficients of relative risk aversion \( \alpha_h \) and homogeneous discount rates \( \delta > 0 \), Huang (1987) and Duffie and Zame (1989) show that equilibrium asset prices are supported by an isoelastic representative investor. Thus, when markets are complete, the SDF (2.3) is consistent with heterogeneity in risk aversion. Extensions to investor heterogeneity under incomplete markets are likely to lead to more novel implications (e.g., Constantinides and Duffie, 1996; Calvet, 2001), and are well-deserving of further research.

3. A multifrequency jump-diffusion for equilibrium stock prices

We now parsimoniously incorporate multifrequency shocks into the economy. We specify in Section 3.1 a dividend process \( D_t \) with a finite number of volatility frequencies, and discuss in Section 3.2 how to select the consumption process \( C_t \). The resulting multifrequency jump-diffusion for prices is investigated in Section 3.3.

3.1. Dividends with multifrequency volatility

We introduce shocks of multiple frequencies by assuming that dividends follow an MSM process (Calvet and Fisher, 2001, 2004), as is now explained. The construction is based on the Markov state vector:

\[
M_t = (M_{1,t}, M_{2,t}, \ldots, M_{\bar{k},t}) \in \mathbb{R}_{\bar{k}}^+,
\]

which has components of heterogeneous durations. Persistence is highest for the first component, and progressively diminishes with the component index \( k \).

We assume that each component \( M_{k,t} \) is itself a Markov process. For parsimony, the components are mutually independent: \( M_{k,t} \) and \( M_{k',t} \) are statistically independent if \( k \neq k' \). Given the Markov state \( M_t \) at date \( t \), the dynamics over an infinitesimal interval are defined as follows. For each \( k \in \{1, \ldots, \bar{k}\} \), a change in \( M_{k,t} \) may be triggered by a

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11 There is of course abundant evidence that individual investors do not have isoelastic utility. For instance, richer agents tend to invest a higher share of their financial wealth in risky assets as compared to poorer agents (e.g. Carroll, 2002). The implications of such wealth effects lie outside the scope of the present paper.
Poisson arrival with intensity $\gamma_k$. The component $M_{k,t+dt}$ is drawn from a fixed distribution $M$ if there is an arrival, and otherwise remains at its current value: $M_{k,t+dt} = M_{k,t}$. The construction can be summarized as

\[ M_{k,t+dt} \text{ drawn from distribution } M \quad \text{with probability } \gamma_k \, dt \]
\[ M_{k,t+dt} = M_{k,t} \quad \text{with probability } 1 - \gamma_k \, dt. \]

The Poisson arrivals and new draws from $M$ are independent across $k$ and $t$.

We observe that the components $M_{k,t}$ differ in their arrival intensities but not in their marginal distribution $M$. Each component therefore follows a Markov process that is identical except for time scale. These features greatly contribute to the parsimony of the model. As with any process driven by Poisson arrivals, the sample paths of a component $M_{k,t}$ are ‘cadlag’, i.e. are right-continuous and have a limit point to the left of any instant.\(^{12}\)

The construction can accommodate any distribution $M$ with positive support. For parsimony, we henceforth consider that components are drawn from a family of distributions specified by a single parameter $m_0 \in \mathbb{R}$. We also tightly parameterize the arrival intensities by requiring:

\[ \gamma_k = \gamma_1 b^{k-1}, \quad k \in \{1, \ldots, \tilde{k}\}. \quad (3.1) \]

The parameter $\gamma_1$ determines the persistence of the lowest frequency component, and $b$ the spacing between component frequencies.

Stochastic volatility is the renormalized product

\[ \sigma_D(M_t) \equiv \tilde{\sigma}_D \left( \prod_{k=1}^{\tilde{k}} M_{k,t} \right)^{1/2}, \quad (3.2) \]

where $\tilde{\sigma}_D$ is a positive constant.\(^{13}\) The components of the state vector interact multiplicatively, and for this reason are called *multipliers*. We normalize the distribution $M$ by imposing that $E(M) = 1$. The parameter $\tilde{\sigma}_D$ is then the unconditional standard deviation of the dividend growth process: $\text{Var}(dD_t/D_t) = \tilde{\sigma}_D^2 \, dt$. For simplicity, we now assume that dividends have the constant growth rate:

\[ g_D(M_t) \equiv \tilde{g}_D, \]

which focuses attention on the rich volatility dynamics embedded in (3.2).

These conditions conclude the specification of Markov-switching multifractal (MSM) volatility. The dividend structure can be summarized as follows.

**Assumption 3 (Multifrequency dynamics).** The dividend process has a constant drift and an MSM volatility with a finite number $\tilde{k}$ of frequencies.

\(^{12}\) Cadlag is a French acronym for *continue à droite, limites à gauche*. We refer the reader to Billingsley (1999) for further details.

\(^{13}\) The conditions $E(\int_0^t |g_D(M_s)| \, ds) = |g_D| \cdot t < \infty$ and $E(\int_0^t \tilde{\sigma}_D(M_s) \, ds) = \tilde{\sigma}_D^2 \cdot t < \infty$ are then trivially satisfied.

\(^{14}\) The normalization $E(M) = 1$ implies that $m_1 = 2 - m_0$. Earlier research shows the empirical usefulness of the binomial distribution in MSM (e.g. Calvet and Fisher, 2004).
Fig. 1. Construction of multifractal volatility. This figure illustrates the construction of multifractal volatility with three volatility components and $T = 10,000$ periods. The first panel shows the randomly drawn values of the lowest frequency component $M_{1,t}$ over time. The second and third panels respectively show the middle frequency component $M_{2,t}$ and the high frequency component $M_{3,t}$. The last panel gives the variance $\sigma_t^2 = \sigma_D^2 M_{1,t} M_{2,t} M_{3,t}$, where we set $\sigma_D = 1$ so that the variance equals the product of the components displayed in the top three panels. The simulation uses the binomial MSM construction with $m_0 = 1.4$, $b = 2$, and $y_1 = 0.0002$.

This panel now shows much greater detail with more pronounced peaks and intermittent bursts of volatility. The larger number of volatility components accommodates a broad range of long-run, medium-run, and short-run dynamics. The second panel illustrates the impact of these various frequencies on dividend growth. Finally, the last panel reports the dividend $D_t$. The last two panels confirm that MSM generates both short and long swings in volatility and thick tails in the dividend growth series, while by design there are no jumps in dividends themselves.

We finally emphasize two theoretical properties of our approach. First, MSM permits the state space to be very large. For instance with a binomial distribution $M$, the number of states is equal to $2^k$. The example considered in Fig. 2 is based on $2^8$ or 256 states. Second, MSM is parsimonious. In a general Markov chain, the size of the transition matrix is equal to the square of the number of states. For instance a general Markov chain with $2^8$ states generally needs to be
Fig. 2. The multifractal dividend process. This figure illustrates the construction of a multifractal dividend path over \( T = 10,000 \) periods. The first panel shows a simulation of multifractal volatility with \( \bar{k} = 8 \) volatility components. The volatility parameters \( m_0 = 1.4, b = 2 \) and \( \gamma_1 = 0.0002 \) are identical to Fig. 1, and \( \bar{\sigma}_D = 0.01 \). The random draws used for the first three components \( M_{1,t}, M_{2,t}, \) and \( M_{3,t} \) are also identical to Fig. 1. Hence, the displayed volatility in the first panel is the outcome of following the construction in Fig. 1 to a higher level of \( \bar{k} \) and rescaling for a different \( \bar{\sigma}_D \). The second and third panels then show how the volatility process maps into dividend growth and dividends. The second panel displays dividend growth, \( \Delta d_t = (\bar{g}_D - \sigma^2 t / 2) \Delta + \sigma_t \epsilon_t \), where \( \epsilon_t \) are standard iid normals, \( \Delta = 1 \), and \( \bar{g}_D = 0.0001 \). The third panel shows the logarithm of dividends, \( d_t = d_0 + \sum_{s=1}^t \Delta d_s \).

parametrized by \( 2^{16} = 65,536 \) elements. In comparison, the MSM dividend dynamics are fully characterized by

\[
(\bar{g}_D, \bar{\sigma}_D, m_0, \gamma_1, b) \in \mathbb{R}^5,
\]

where \( \bar{g}_D \) and \( \bar{\sigma}_D \) quantify the mean and standard deviations of dividend growth, \( m_0 \) parameterizes the distribution \( M \), \( \gamma_1 \) is the intensity of the most persistent component, and \( b \) quantifies the growth rates of intensities. The five-parameter specification of the dividend process accommodates an arbitrary number of frequencies.
3.2. Multifrequency economies

We now turn to the specification of aggregate consumption, which will close the description of the exchange economy.

**Specification 1: Lucas tree economy.** The stock is a claim on aggregate consumption: $D_t = C_t$. The seven parameters $(\tilde{g}_D, \tilde{\sigma}_D, m_0, \gamma_1, b, \alpha, \delta)$ then fully specify the jump-diffusion price process. By Proposition 2, the $P/D$ ratio is given by

$$
\mathbb{E}_t \left( \int_0^{+\infty} e^{-[\delta-(1-\alpha)\tilde{g}_D]s-(\alpha(1-\alpha)/2)\int_0^s \sigma_D^2(M_{t+h}) dh} ds \right).
$$

(3.3)

An increase in volatility reduces the price:dividend ratio only if $\alpha < 1$, which is consistent with earlier research in discrete time (e.g., Barsky, 1989; Abel, 1988).\(^\text{15}\)

**Specification 2: IID consumption.** We can alternatively assume that consumption has a constant drift and volatility.\(^\text{16}\) The interest rate (2.4) is then constant, and the price:dividend ratio is equal to

$$
\mathbb{E}_t \left( \int_0^{+\infty} e^{-(r_l-\tilde{g}_D)s-\alpha \sigma_C \delta \int_0^s \sigma_D(M_{t+h}) dh} ds \right).
$$

High volatility feeds into low asset prices for any choices of relative risk aversion $\alpha$. This approach fits well with the discrete-time volatility feedback literature, which suggests that aggregate stock prices decrease with the volatility of dividend news (e.g. Bekaert and Wu, 2000; Calvet and Fisher, 2007; Campbell and Hentschel, 1992; French et al., 1987; Pindyck, 1984).

**Specification 3: Multivariate MSM.** We develop in Appendix B a multivariate extension of MSM that permits more flexible specifications of consumption. This approach helps to construct SDF models with a stochastic volatility only partially correlated to the stochastic volatility of dividends. While this construction is appealing for empirical applications, we focus for expositional simplicity on Specifications 1 and 2 in the remainder of the paper.

3.3. The equilibrium stock price

Jumps in equilibrium prices are triggered by regime changes in the volatility state vector. Since the components have heterogeneous persistence levels, the model avoids the difficult choice of a unique frequency and size for “rare events,” which is a common issue in specifying traditional jump-diffusions.\(^\text{17}\)

The relation between the frequency and size of a jump is easily quantified by loglinearizing the price:dividend ratio. Consider the parametric family of state processes $M_t(\epsilon) = 1 + \epsilon (\nu_t - 1), t \in \mathbb{R}_+, \epsilon \in [0, 1)$, where $\nu$ is itself a fixed MSM state vector.

**Proposition 3** (First-order expansion of $P/D$). The log of the $P/D$ ratio is approximated around $\epsilon = 0$ by the first-order Taylor expansion:

$$
q[M_t(\epsilon)] = \bar{q} - q_1 \sum_{k=1}^{\bar{k}} \frac{M_{k,t}(\epsilon) - 1}{\delta^k + \gamma_k} + o(\epsilon).
$$

(3.4)

---

\(^{15}\) When future consumption becomes riskier, two opposite economic effects impact the $P/D$ ratio, as can be seen in (2.5). First, investors perceive an increase in the covariance $\sigma_C(M_{t+h})\sigma_D(M_{t+h})|D_t$ between future consumption and dividends (systematic risk), which reduces the price:dividend ratio. Second, the precautionary motive increases the expected marginal utility of future consumption, which lowers future interest rates $r_l(M_{t+h})$ and increases $P/D$. The negative impact of systematic risk dominates when $\alpha < 1$.

\(^{16}\) The equilibrium model is then specified by the dividend parameters $(\tilde{g}_D, \tilde{\sigma}_D, m_0, \gamma_1, b)$, the utility coefficients $(\alpha, \delta)$, the consumption parameters $(\tilde{g}_C, \tilde{\sigma}_C)$, and the correlation $\rho_{C,D}$.

\(^{17}\) In the simplest exogenously specified jump-diffusions, it is often possible that discontinuities of heterogeneous but fixed sizes and different frequencies can be aggregated into a single collective jump process with an intensity equal to the sum of all the individual jumps, and a random distribution of sizes. A comparable analogy can be made for the state vector $M_t$ in our model, but due to the equilibrium linkages between jump size and the duration of volatility shocks, and the state dependence of price jumps, no such reduction to a single aggregated frequency is possible for the equilibrium stock price.
The Lucas tree economy (Specification 1) implies \( \delta' = \delta - (1 - \alpha)\bar{g}_D + \alpha(1 - \alpha)\bar{\sigma}_D^2/2, \bar{q} = -\ln(\delta') \) and \( q_1 = \alpha(1 - \alpha)\bar{\sigma}_C^2\bar{\sigma}_D/2. \) When consumption is IID (Specification 2), the parameters instead are \( \delta' = r_f - \bar{g}_D + \alpha \rho_{C,D} \bar{\sigma}_C \bar{\sigma}_D, \bar{q} = -\ln(\delta') \) and \( q_1 = \alpha \rho_{C,D} \bar{\sigma}_C \bar{\sigma}_D/2. \)

When the distribution \( M \) is close to unity, the \( P/D \) ratio is approximated by a persistence-weighted sum of the volatility components. Low-frequency multipliers deliver persistent and discrete switches, which have a large impact on the \( P/D \) ratio. By contrast, higher frequency components have no noticeable effect on prices, but give additional outliers in returns through their direct effect on the tails of the dividend process. The price process is thus characterized by a large number of small jumps (high frequency \( M_{k,t} \)), a moderate number of moderate jumps (intermediate frequency \( M_{k,t} \)), and a small number of very large jumps. Earlier empirical research suggests that this is a good characterization of the dynamics of stock returns.

We illustrate in Fig. 3 the endogenous multifrequency pricing dynamics of the model, in the case where consumption is IID. The top two panels present a simulated dividend process, in growth rates and in logarithms of the level respectively. The middle two panels then display the corresponding stock returns and log prices. The price series exhibits much larger movements than dividends, due to the presence of endogenous jumps in the \( P/D \) ratio. To see this clearly, the bottom two panels show consecutively: (1) the “feedback” effects, defined as the difference between log stock returns and log dividend growth and (2) the price:dividend ratio. Consistent with Proposition 3, we observe a few infrequent but large jumps in prices, with smaller but more numerous small discontinuities. The simulation demonstrates that the difference between stock returns and dividend growth can be large even when the \( P/D \) ratio varies in a plausible and relatively modest range (between 26 and 33 in the figure). The pricing model thus captures multifrequency stochastic volatility, endogenous multifrequency jumps in prices, and endogenous correlation between volatility and price innovations.

![Fig. 3. Equilibrium price and return dynamics. This figure illustrates the relation between exogenous dividends and equilibrium prices when consumption is iid. The top two panels display simulated dividend growth rates and dividend levels, constructed in the same manner as Fig. 2. The parameters used in the specification are \( m_0 = 1.35, \bar{\sigma}_D = 0.7, b = 2.2, \) and \( \bar{g}_D = 0.0001. \) The middle two panels demonstrate the result of equilibrium pricing. In these panels we use the preference and consumption parameters \( \alpha = 25, \delta = 0.00005, \bar{g}_C = 0.00005, \rho_{C,D}\bar{\sigma}_D = 0.0012. \) The left-hand side displays returns, and the right side shows the log price realization. Both show more variability, and in particular jumps, relative to the dividend processes. To isolate the endogenous pricing effects in returns and prices, the bottom left panel shows the volatility “feedback” effect, defined as the difference between log returns and log dividend growth, i.e., \( \Delta p_t - \Delta d_t, \) or the difference between the middle left and top left panels. To show the same endogenous pricing effects in levels, the bottom right hand panel shows the price:dividend ratio.](image-url)
4. Price dynamics with an infinity of frequencies

We now investigate how the price diffusion evolves as $\bar{k} \to \infty$, i.e. as components of increasingly high frequency are added into the state vector. This can help guide our judgement about the number of components that are useful in empirical applications. Two apparently contradictory observations can be made. On the one hand, Figs. 1 and 2 suggest that the volatility process $\sigma_{D}(M_{t})$ exhibits increasingly extreme behavior as $\bar{k}$ goes up. On the other hand, the equilibrium jump-diffusion for prices seems to be quite insensitive to higher frequency components. We show in this section how these two observations can be reconciled by deriving the limit behavior of the price dynamics.

4.1. Time deformation

We begin by reviewing the limit behavior of MSM when the number of high-frequency components $\bar{k}$ goes to infinity. The parameters $(\bar{g}_{D}, \bar{\sigma}_{D}, m_{0}, \gamma_{1}, b)$ are fixed. Let $M_{t} = (M_{k,t})_{k=1}^{\infty} \in \mathbb{R}_{+}^{\infty}$ denote an MSM Markov state process with countably many frequencies. The process $M_{t}$ is defined for $t \in [0, \infty)$, has mutually independent components, and each component $M_{k,t}$ is characterized by the arrival intensity $\gamma_{k} = \gamma_{1}b^{k-1}$. For a finite $\bar{k}$, stochastic volatility is defined as the product of the first $\bar{k}$ components of the state vector: $\sigma_{D,\bar{k}}(M_{t}) \equiv \bar{\sigma}_{D} (M_{1,t}, M_{2,t}, \ldots, M_{\bar{k},t})^{1/2}$.

Since instantaneous volatility $\sigma_{D,\bar{k}}(M_{t})$ depends on an increasing number of components, the differential representation (2.2) becomes unwieldy as $\bar{k} \to \infty$. We instead characterize dividend dynamics by the time deformation

$$\theta_{\bar{k}}(t) = \int_{0}^{t} \sigma_{D,\bar{k}}^{2}(M_{s}) \, ds.$$  \hspace{1cm} (4.1)

Given a fixed instant $t$, the sequence $\{\theta_{\bar{k}}(t)\}_{\bar{k}=1}^{\infty}$ is a positive martingale with bounded expectation. By the martingale convergence theorem, the random variable $\theta_{\bar{k}}(t)$ converges to a limit distribution when $\bar{k} \to \infty$. A similar argument applies to any vector sequence $\{\theta(t_{1}), \ldots, \theta(t_{l})\}$, guaranteeing that the stochastic process $\theta_{\bar{k}}$ has at most one limit point. We verify that a limit process does indeed exist by checking that the sequence $\{\theta_{\bar{k}}\}$ is tight.\footnote{We refer the reader to Billingsley (1999) for a detailed exposition of weak convergence in function spaces.} Intuitively, tightness prevents the process from oscillating too wildly as $\bar{k} \to \infty$. As shown in Calvet and Fisher (2001), the tightness property holds on a bounded time interval $[0, T]$ under the following sufficient condition.

Assumption 4. $\mathbb{E} (M^{2}) < b$.

This inequality restricts the fluctuations of the time deformation process by requiring that volatility shocks be sufficiently small or have durations decreasing sufficiently fast. When $T$ is finite, the sequence $\theta_{\bar{k}}$ then weakly converges to a limit process $\theta_{\infty}$, which generates continuous sample paths (Calvet and Fisher, 2001).\footnote{Because volatility exhibits increasingly extreme behavior as $\bar{k}$ goes up, the time deformation $\theta_{\infty}$ cannot be computed by taking the pointwise limit of the integrand $\sigma_{D,\bar{k}}^{2}(M_{t})$ in Eq. (4.1). Specifically, $\sigma_{D,\bar{k}}^{2}(M_{t})$ converges almost surely to zero as $\bar{k} \to \infty$ (by the Law of large numbers), suggesting that $\theta_{\infty} \equiv 0$. This conclusion would of course be misleading. For every fixed $t$, Assumption 4 implies that $\mathbb{E} \{\theta_{\bar{k}}^{2}(t)\} < \infty$ (Calvet and Fisher, 2001), and the sequence $\{\theta(t)\}_{t}$ is therefore uniformly integrable. Hence $\mathbb{E} \theta_{\infty}(t) = \mathbb{E} \theta_{\infty}(t) = \mathbb{E} \sigma_{D,t}^{2} > 0$. We refer the reader to Calvet and Fisher (2001, 2002) for a more detailed exposition of the rich properties of multifractal diffusions.}

We now check that the same results hold when the time domain is unbounded. Consider the space $D(0, \infty)$ of cadlag functions defined on $[0, \infty)$, and let $d_{\infty}$ denote the Skohorod distance.

Proposition 4 (Time deformation with countably many frequencies). Under Assumption 4, the sequence $\{\theta_{\bar{k}}\}_{\bar{k}=1}^{\infty}$ weakly converges as $\bar{k} \to \infty$ to a measure $\theta_{\infty}$ defined on the metric space $(D(0, \infty), d_{\infty})$. Furthermore, the sample paths of $\theta_{\infty}$ are continuous almost surely.

The limiting process has a Markov structure analogous to MSM with a finite $\bar{k}$. We interpret $M_{t} = (M_{k,t})_{k=1}^{\infty}$ as the state vector of the limiting time deformation $\theta_{\infty}$.

Using time-deformation, the dividend process for a finite $\bar{k}$ is represented by

$$d_{k}(t) \equiv d_{0} + \bar{g}_{D}t - \frac{\theta_{\bar{k}}(t)}{2} + B[\theta_{\bar{k}}(t)],$$
where \( B \) is a standard Brownian. By Proposition 4, \( d_k(t) \) converges to
\[
d_\infty(t) \equiv d_0 + \bar{g}Dt - \frac{\theta_\infty(t)}{2} + B[\theta_\infty(t)]
\]
as \( \bar{k} \to \infty \).

The dividend process converges even though volatility \( \sigma_{D,k}(M_t) \) exhibits a degenerate behavior as \( \bar{k} \to \infty \). This apparent contradiction is best understood by examining the local properties of the limit dividend process. The local variability of a sample path is characterized by the local Hölder exponent:
\[
\beta(t) = \max\{\beta \in \mathbb{R}_+ \text{ s.t. } |d(t + \Delta t) - d(t)| = O(|\Delta t|^\beta)\}.
\]
The local Hölder exponent quantifies the order of variation around instant \( t \). In jump-diffusions, the coefficient \( \beta(t) \) equals 0 at points of discontinuity, and 1/2 otherwise. In contrast, the continuous dividend process with countably many frequencies implies that \( \beta(t) \) takes a continuum of values in any time interval, which is a defining property of a multifractal diffusion.\(^{20}\)

4.2. Limiting equilibrium price process

We now examine the equilibrium impact of permitting increasingly many frequencies in the volatility of dividends. A particularly striking example is provided by the Lucas tree economies discussed in Section 3.2. We consider

Assumption 5. \( \alpha \leq 1 \) and \( \rho = \delta - (1 - \alpha)\bar{g}D > 0 \).

For finite \( \bar{k} \), the equilibrium price:dividend ratio is given by (3.3), or equivalently
\[
q_k(t) = \ln \mathbb{E} \left[ \int_0^{+\infty} e^{-\rho s - \frac{\alpha(1-\alpha)}{2} [\theta_k(t+s)-\theta_k(t)]} \mathrm{d}s \right] (M_{k,t})_{k=1}^{\bar{k}}.
\]
The price process has therefore the same distribution as
\[
p_k(t) \equiv d_k(t) + q_k(t).
\]
When the number of frequencies goes to infinity, the dividend process has a well-defined limit. We check in Appendix A that the P/D ratio (4.2) is a positive submartingale, which also converges to a limit as \( \bar{k} \to \infty \).

**Proposition 5** (Jump-diffusion with countably many frequencies). Consider the maintained Assumptions 1–5. When the number of frequencies goes to infinity, the log-price process weakly converges to
\[
p_\infty(t) \equiv d_\infty(t) + q_\infty(t),
\]
where
\[
q_\infty(t) = \ln \mathbb{E} \left[ \int_0^{+\infty} e^{-\rho s - \frac{\alpha(1-\alpha)}{2} [\theta_\infty(t+s)-\theta_\infty(t)]} \mathrm{d}s \right] (M_{k,t})_{k=1}^{\infty}
\]
is a pure jump process. The limiting price is thus a jump-diffusion with countably many frequencies.

In an economy with countably many frequencies, the log-price process is the sum of (1) the continuous multifractal diffusion \( d_\infty(t) \) (2) the pure jump process \( q_\infty(t) \). We correspondingly call \( p_\infty(t) \) a multifractal jump-diffusion.

When \( \bar{k} = \infty \), the state space is a continuum while the Lucas tree economy is still specified by the seven parameters \((\bar{g}D, \bar{\sigma}_D, m_0, \gamma_1, b, \alpha, \delta)\). The equilibrium P/D ratio \( q_\infty(t) \) exhibits rich dynamic properties. Within any bounded time interval, there exists almost surely (a.s.) at least one multiplier \( M_{k,t} \) that switches and triggers a jump in the stock price. This property implies that a jump in price occurs a.s. in the neighborhood of any instant. The number of switches is

\(^{20}\) Multifractal diffusions were introduced in Calvet et al. (1997) and Calvet and Fisher (2002). We refer the reader to this earlier work for a more detailed discussion of local properties.
also countable a.s. within any bounded time interval, implying that the process has infinite activity and is continuous almost everywhere. Equilibrium valuation therefore generates a limit $P/D$ ratio that follows an infinite intensity pure jump process.

We illustrate in Fig. 4 the convergence of the equilibrium price processes as $\bar{k}$ becomes large. The first panel shows a simulation with $\bar{k} = 2$ volatility components, and the following panels consecutively add higher frequency components to obtain paths with $\bar{k} = 4$, $\bar{k} = 6$, and $\bar{k} = 8$ components. Consistent with the theoretical construction, the figure is obtained by randomly drawing a trajectory of the Brownian motion $B$ in stage $\bar{k} = 0$, which is thereafter taken as fixed. Similarly, each multiplier $M_{k,t}$ is drawn only once, so that $(M_{k,t})_{k=1}^{\bar{k}}$ does not vary when we move from stage $\bar{k}$ to stage $\bar{k} + 1$. The figure suggests that the price process becomes progressively insensitive to the addition of new high-frequency components, and the sample path of the price process stabilizes. This illustrates the main result of Proposition 5. For low $\bar{k}$, adding components has a significant impact, and as $\bar{k}$ increases the process converges.

The results of this section provide useful guidance on the choice of the number of frequencies in theoretical and empirical applications. On the one hand, the convergence of the price process implies that the marginal contribution of additional components is likely to be small in applications concerned with fitting the price or return series. It is then convenient to consider a number of frequencies $\bar{k}$ that is sufficiently large to capture the heteroskedasticity of financial series, but sufficiently small to remain tractable. On the other hand, countably many frequencies might prove useful in more theoretical contexts, in which the local behavior of the price process needs to be carefully understood. Examples could include the construction of learning models or the design of dynamic hedging strategies.

Fig. 4. Convergence to multifractal jump-diffusion. This figure illustrates convergence of the equilibrium price process as the number of high-frequency volatility components becomes large. The panels show consecutively simulations of the log price process $p_{\bar{k}}(t) = d_0 + \bar{\gamma}_D t - \theta L(t)/2 + B(\theta L(t) + q L(t))$ for $\bar{k} = 2, 4, 6, 8$. All panels hold constant the Brownian $B(t)$. The multipliers $M_{k,t}$ are also drawn only once, and then held constant as higher level multipliers are added. The construction is thus recursive in $\bar{k}$, with each increment requiring the previously drawn non-deformed dividends and multipliers from the preceding level, plus new random draws for the next set of (higher frequency) multipliers being incorporated. We observe large differences between the panels corresponding to $\bar{k} = 2$ and $\bar{k} = 4$, more moderate changes between $\bar{k} = 4$ and $\bar{k} = 6$, and only modest differences between $\bar{k} = 6$ and $\bar{k} = 8$. In this set of simulations, we use the Lucas economy specification with $T = 2500$, $m_0 = 1.4$, $d = 3.25$, $\gamma_l = 0.25 b^2 \approx 0.0001$, $\bar{\sigma}_C = 0.0125$, $\bar{\gamma}_D = 0.00008$, $\delta = 0.00003$, and $\alpha = 0.5$. The results are consistent with Proposition 5, which ensures that as the number of frequencies $\bar{k}$ grows, the log price $p_{\bar{k}}(t)$ weakly converges to a multifractal jump-diffusion.
5. Recursive utility and priced jumps

In discrete-time, Calvet and Fisher (2007) show that when investors have Epstein–Zin utility and consumption undergoes regime-switches, the equity premium can be matched with reasonable levels of risk aversion. The recursive preference equilibrium easily generalizes to continuous time, permitting discontinuities in the SDF and priced jumps, as we now show.

Agents have a stochastic differential utility \( V_t \) (Duffie and Epstein, 1992), which is specified by a normalized aggregator \( f(c, v) \) and satisfies the fixed point equation:

\[
V_t = E_t \left[ \int_t^T f(C_s, V_s) \, ds + V_T \right]
\]

for any instants \( T \geq t \geq 0 \). The aggregator is given by

\[
f(c, v) = \frac{\delta}{1 - \psi^{-1}} \frac{e^{1 - \psi^{-1}} - [(1 - \alpha)v]^{\theta}}{[1 - \psi^{-1}]^{\theta - 1}},
\]

where \( \alpha \) is the coefficient of relative risk aversion, \( \psi \) is the elasticity of intertemporal substitution, and \( \theta = (1 - \psi^{-1})/(1 - \alpha) \). The case where \( \theta = 1 \) corresponds to isoelastic utility as considered previously.

Under the consumption process in Assumption 1, the recursive utility has functional form \( V(c, M_t) = \varphi(M_t)^{-1-\alpha}/(1 - \alpha) \).21 The stochastic discount factor is then \( \Lambda_t = \delta^{-1} \exp(\int_0^t f(c_s, V_s) \, ds) f(c, V_t) \) (Duffie and Epstein, 1992; Duffie and Skiadis, 1994), or equivalently

\[
\Lambda_t = [\varphi(M_t)]^{1-\theta} C_t^{-\alpha} e^{-\delta t + \theta(1 - \alpha)(1 - \alpha)(1 - \theta) - \int_0^t [\varphi(M_s)]^{-\theta} \, ds}.
\]

Consumption and the exponential expression are continuous. On the other hand, the first factor in the equation is a function of \( \varphi(M_t) \), which depends on the Markov state and can therefore vary discontinuously through time. In the simplifying case where \( \theta = 1 \) (power utility), the factor \( [\varphi(M_t)]^{1-\theta} \) drops out and the SDF has continuous sample paths, reducing to \( \Lambda_t = e^{-\delta t} C_t^{-\alpha} \). On the other hand, if \( \theta \neq 1 \), the term \( [\varphi(M_t)]^{1-\theta} \) is discontinuous and the marginal utility of consumption depends on the current state. Discrete changes in the state vector \( M_t \) thus cause jumps in the SDF.

Since switches in \( M_t \) trigger simultaneous jumps in the stochastic discount factor and the \( P/D \) ratio, they impact expected returns, and hence are “priced” in equilibrium. To see this, we denote by \( \gamma = \sum_{k=1}^N \gamma_k t_k \) the intensity that at least one arrival occurs. Furthermore, let \( E_{t,A} \) denote the expectation operator conditional on: (1) the investor information set \( I_t \); (2) the occurrence of at least one arrival between \( t \) and \( t + dt \). The conditional equity premium is then:

\[
-\frac{1}{dt} E_{t,A} \left[ \frac{d\Lambda_t}{\Lambda_t} \frac{dP_t}{P_t} \right] = \alpha \sigma_C(M_t) \sigma_D(M_t) \rho_{C,D} + \gamma E_{t,A} \left[ -\frac{\Delta \Lambda_t}{\Lambda_t} \frac{\Delta P_t}{P_t} \right].
\]

When \( \theta \neq 1 \) the final term is generally non-zero, confirming that the possibility of a discontinuity modifies the expected return required by investors on the stock. Jumps thus represent a priced risk in equilibrium.

The ability of our framework to accommodate priced jumps is potentially useful for empirical applications. For example, in discrete-time Calvet and Fisher (2007) use non-separable preferences to obtain priced switches in a calibration that simultaneously fits, with reasonable levels of risk-aversion, the equity premium, equity volatility, and the drifts and volatilities of consumption and dividends. Further, in a recent contribution, Bhamra et al. (2006) extend our framework by considering levered claims on the priced asset. They find that the ability to capture priced jumps is empirically important in simultaneously reconciling the equity premium, default spreads, and empirically

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21 The fixed point equation (5.1) can be written as \( f(C_t, V_t) \, dt + E_t(dV_t) = 0 \). Let \( \varphi_1, \ldots, \varphi_d \) denote the value of \( \varphi \) in all possible states \( m^1, \ldots, m^d \). The fixed point equation is then

\[
\delta \frac{\partial}{\partial \varphi} [\varphi^{-\theta} - \varphi] + \varphi \left[ (1 - \alpha) g_C(m') + \frac{\alpha(\alpha - 1)}{2} \sigma^2_C(m') \right] + \sum_{j \neq i} a_{i,j}(\varphi_j - \varphi_i) = 0,
\]

where \( a_{i,j} = P(M_{i+1,d} = m^j | M_t = m^i) / dt \). Existence and uniqueness can then be analyzed using standard methods.
observed default rates. We anticipate that future work will use our structural approach to modeling priced jumps in other applications, including, for example, pricing options and other derivatives.

6. Conclusion

We have developed in this paper a continuous-time asset-pricing economy with endogenous multifrequency jumps in stock prices. Equilibrium valuation gives a number of appealing features that are often assumed exogenously in previous literature, including: (1) heterogeneous jump sizes with many and frequent small jumps and few large jumps; (2) endogenous correlation between jumps in prices and volatility. Further, jumps are priced in equilibrium under non-separable preferences.

We consider the weak limit of our economic equilibrium as the number of components driving fundamentals becomes large. Under appropriate conditions, the stock price converges to a new mathematical object called a multifractal jump-diffusion. The equity value can be decomposed into a multifractal diffusion related to the exogenous dividend process, and an infinite-intensity pure jump process corresponding to endogenous variations in the price:dividend ratio. Stock price jumps occur in the neighborhood of any instant, but sample paths are continuous almost everywhere.

Our results focus on two special cases of consumption-based asset pricing economies: IID consumption growth and a Lucas tree. In Appendix B, we show how the economy may be generalized to accommodate intermediate cases where consumption and dividend growth state variables are correlated but not identical. Future work may further develop these specifications.

Acknowledgements

We received helpful comments from Bernard Cornet (the editor), Rose-Anne Dana, Darrell Duffie, René Garcia, John Geanakoplos, Jim Hamilton, Jean Jacod, Guido Kuersteiner, Guy Laroque, Tomasz Michalski, Chris Shannon, two anonymous referees, and seminar participants at CREST and the 2005 NSF/CEME Mathematical Economics Conference in Honor of Gérard Debreu held at UC Berkeley. We are very appreciative of financial support provided for this project by the Agence Nationale de la Recherche, the HEC Foundation, and the Social Sciences and Humanities Research Council of Canada.

Appendix A. Proofs

Proof of Proposition 2. The price dividend ratio satisfies:

\[ Q(M_t) = \mathbb{E} \left( \int_0^{+\infty} \frac{\Lambda_{t+s}}{\Lambda_t} \frac{D_{t+s}}{D_t} \, ds \right) M_t \).

Since

\[ d \ln \Lambda_t = \left[ -r_t(M_t) - \frac{\alpha^2 \sigma_C^2(M_t)}{2} \right] dt - \alpha \sigma_C(M_t) dZ_C(t), \]

\[ d \ln D_t = \left[ g_D(M_t) - \frac{\sigma_D^2(M_t)}{2} \right] dt + \sigma_D(M_t) dZ_D(t), \]

we infer that

\[ \ln \frac{\Lambda_{t+s}}{\Lambda_t} + \ln \frac{D_{t+s}}{D_t} = \int_0^s \left[ g_D(M_{t+h}) - r_t(M_{t+h}) - \frac{\sigma_D^2(M_{t+h})}{2} \right] dh \]

\[ + \int_0^s [\sigma_D(M_{t+h}) dZ_D(t + h) - \alpha \sigma_C(M_{t+h}) dZ_C(t + h)] \]
is conditionally Gaussian with mean \( \int_0^t [g_D(M_{t+h}) - r_t(M_{t+h}) - \sigma_D^2(M_{t+h})] \, dh \) and variance \( \int_0^t [\alpha^2 \sigma_C^2(M_{t+h}) + \sigma_D^2(M_{t+h}) - 2\alpha \sigma_C \sigma_D(M_{t+h})] \, dh \). We then easily check that
\[
\mathbb{E} \left( \frac{A_{t+s}}{A_t} \frac{D_{t+s}}{D_t} \bigg| M_t \right) = \mathbb{E} \left| \int_0^t [g_D(M_{t+h}) - r_t(M_{t+h}) - \alpha \sigma_C \sigma_D(M_{t+h})] \, dh \right|
\]
and conclude that Eq. (2.5) holds.

**Proof of Proposition 3.** Given an initial state \( v_t \), the P/D ratio of the Lucas tree economy (Specification 1) with random state \( (M_s)_{s \geq 0} \) can be written as
\[
Q(\varepsilon) = \mathbb{E} \left( \int_0^{+\infty} e^{-\delta s - \frac{\alpha(1-\alpha)}{2} \int_0^s (\sigma_C^2(M_{t+h}) - \sigma_D^2) \, dh} \, ds \bigg| v_t \right).
\]
We note that \( Q(0) = 1/\delta' \). By the dominated convergence theorem, the function \( Q \) is differentiable and
\[
Q'(0) = -q_1 \mathbb{E} \left\{ \int_0^{+\infty} e^{-\delta s} \left[ \int_0^s \sum_{k=1}^k (v_{k,t+h} - 1) \, dh \right] \, ds \bigg| v_t \right\}.
\]
Since \( \mathbb{E}_t(v_{k,t+h} - 1) = e^{-\gamma h} (v_{k,t} - 1) \), we infer that
\[
Q'(0) = -q_1 \sum_{k=1}^k (v_{k,t} - 1) \left( \int_0^{+\infty} e^{-\delta s} \int_0^s e^{-\gamma h} \, dh \, ds \right) = -q_1 \sum_{k=1}^k \frac{v_{k,t} - 1}{\delta'(\delta' + \gamma_k)}.
\]
Hence
\[
Q(\varepsilon) = Q(0) \left( 1 - q_1 \sum_{k=1}^k \frac{v_{k,t} - 1}{\delta'(\delta' + \gamma_k)} \right) + o(\varepsilon).
\]
We take the log and conclude that (3.4) holds. A similar argument holds in the IID consumption case (Specification 2).

**Proof of Proposition 4.** We showed in Calvet and Fisher (2001) that the restriction of \( \theta_k \) on any bounded subinterval \([0, T]\) is uniformly equicontinuous and has a continuous limiting process. Theorem 16.8 in Billingsley (1999) implies that the sequence \( \theta_k \) is also tight on \( D[0, \infty) \). We conclude that the sequence \( \theta_k \) converges in \( D[0, \infty) \) to a limit process \( \theta_\infty \) with continuous sample paths.

**Proof of Proposition 5.** Consider
\[
Q_k(t) \equiv \mathbb{E} \left[ \int_0^{+\infty} e^{-\rho s} e^{-\lambda(t_{k+1} - t_k)} \, ds \bigg| M_t \right],
\]
where \( \lambda = \alpha(1 - \alpha)/2 > 0 \). We easily check that \( Q_k(t) \) is a positive and bounded submartingale:
\[
Q_k(t) \leq \mathbb{E}_k \left[ Q_{k+1}(t) \right] \leq \frac{1}{\rho}.
\]
The P/D ratio \( Q_k(t) \) therefore converges to a limit distribution, which we now easily characterize.

Consider the function \( \Phi : D[0, \infty) \to D[0, \infty) \) defined for every cadlag function \( f \) by the integral transform:
\[
(\Phi f)(t) = \int_0^{+\infty} \exp(-\rho s - \lambda [f(t+s) - f(t)]) \, ds.
\]
The function \( \Phi \) is bounded with respect to the Skohorod distance since \( (\Phi f)(t) \in [0, 1/\rho] \) for all \( t \). We also check that it is continuous. Since \( \theta_k \to \theta_\infty \), we infer that \( \Phi \theta_k \) weakly converges to \( \Phi \theta_\infty \). Hence \( Q_k(t) \to Q_\infty(t) \), and the proposition holds.
Appendix B. Multivariate extensions

The asset pricing models in Section 3 are based on univariate MSM, and assume either IID consumption or Lucas tree economies. We now introduce an extension of MSM that permit intermediate co-movements of consumption and dividends.

B.1. Bivariate MSM in continuous time

We begin by generalizing to a continuous-time setting the multivariate discrete-time specification of MSM (Calvet et al., 2006). Consider two economic processes \( \alpha \) and \( \beta \), which could for instance correspond to consumption and dividends. For every frequency \( k \), the processes have volatility components:

\[
M_{k,t} = \begin{bmatrix} M_{\alpha,k,t}^g \\ M_{\beta,k,t}^g \end{bmatrix} \in \mathbb{R}_+^2.
\]

The period-\( t \) volatility column vectors \( M_{k,t} \) are stacked into the \( 2 \times \bar{k} \) matrix

\[
M_t = (M_{1,t}, M_{2,t}, \ldots, M_{\bar{k},t}).
\]

As in univariate MSM, we assume that \( M_{1,t}, M_{2,t}, \ldots, M_{\bar{k},t} \) at a given time \( t \) are statistically independent. The main task is to choose appropriate dynamics for each vector \( M_{k,t} \).

Economic intuition suggests that volatility arrivals are correlated but not necessarily simultaneous across economic series. For this reason, we allow arrivals across series to be characterized by a correlation coefficient \( \rho^* \in [0, 1] \). Assume that the volatility vector \( M_{k,s} \) has been constructed up to date \( t \). Over the following interval of infinitesimal length \( d\tau \), each series \( c \in \{\alpha, \beta\} \) is hit by an arrival with probability \( \gamma_k d\tau \). If \( \rho^* = 0 \), the arrivals are assumed to be independent. On the other hand if \( \rho^* \in (0, 1] \), the probability of an arrival on \( \beta \) conditional on an arrival on \( \alpha \) is \( \rho^* \), as is the probability of no \( \beta \) arrival conditional on no \( \alpha \) arrival.

The construction of the volatility components \( M_{k,t} \) is then based on a bivariate distribution \( M = (M^\alpha, M^\beta) \in \mathbb{R}_+^2 \). \( ^{22} \) If arrivals hit both series, the state vector \( M_{k,t+d\tau} \) is drawn from \( M \). If only series \( c \in \{\alpha, \beta\} \) receives an arrival, the new component \( M_{c,k,t+d\tau} \) is sampled from the marginal \( M^c \) of the bivariate distribution \( M \). Finally, \( M_{k,t+d\tau} = M_{k,t} \) if there is no arrival.

As in the univariate case, the transition probabilities \( (\gamma_1, \gamma_2, \ldots, \gamma_k) \) are defined as

\[
\gamma_k = \gamma_1 b^{k-1},
\]

where \( \gamma_1 > 0 \) and \( b \in (1, \infty) \). This completes the specification of bivariate MSM in continuous time.

B.2. Asset pricing model

We assume that the consumption and dividend processes have constant drifts but stochastic volatilities

\[
\sigma_C(M_t) = \tilde{\sigma}_C(M_{1,t}^\alpha, M_{2,t}^\alpha, \ldots, M_{\tilde{k},t}^\alpha)^{1/2}, \quad \sigma_D(M_t) = \tilde{\sigma}_D(M_{1,t}^\beta, M_{2,t}^\beta, \ldots, M_{\tilde{k},t}^\beta)^{1/2},
\]

where \( \tilde{k} \leq \bar{k} \). The specification permits correlation in volatility across series through the bivariate distribution \( M \), and correlation in returns through the Brownian motions \( Z_C \) and \( Z_D \). This flexible setup permits to construct a more general class of jump-diffusions for stock prices. We note that the model reduces to IID consumption (Specification 2) if \( \tilde{k} = 0 \), and to a Lucas tree economy (Specification 1) if \( \tilde{k} = \bar{k} \) and \( \rho^* = 1 \).

\( ^{22} \) We can for instance choose a bivariate binomial. See Calvet et al. (2006) for further details.
The generalized model might also be useful for option pricing. In our environment, the price of a European option \( f(P_T) \) is therefore given by\(^\text{23}\)

\[
f_0 = \mathbb{E}_0 \left[ \frac{\Lambda_T}{\Lambda_0} f(P_T) \right]
\]

As in Hull and White (1987), let \( f((M_t)_{t \in [0,T]} = \mathbb{E}_0[\Lambda_T f(P_T)/\Lambda_0 | (M_t)_{t \in [0,T]}] \) denote the option price conditional on the state history. The law of iterated expectations implies:

\[
f_0 = \mathbb{E}_0 f \left( (M_t)_{t \in [0,T]} \right).
\]

The empirical implications of this setup will be the object of further research.

**References**


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\(^{23}\) See Anderson and Raimondo (2005), David and Veronesi (2002), Garcia et al. (2003) and Garleanu et al. (2006) for recent work on consumption-based option pricing.


