Nonparametric identification of positive eigenfunctions∗

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Abstract

Important objects for analysis in economics, and especially in asset pricing theory, can be expressed as positive eigenfunctions of positive linear operators on function spaces. This note provides nonparametric identification and existence conditions for positive eigenfunctions. These conditions are readily verifiable and have general application to nonparametric models in economics. Two sets of conditions are provided: one set for operators on \(L^p(\mu)\) spaces and a second set for operators on Banach lattices. The use of these results is illustrated by application to the decomposition of stochastic discount factors in dynamic asset pricing models into their permanent and transitory components, in both discrete- and continuous-time environments.

Keywords: Nonparametric identification, Nonparametric models, Perron-Frobenius theory, Asset pricing, Markov processes.

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1 Introduction

Recent work in economics, and especially in asset pricing theory, has shown that important objects for analysis can be expressed as positive eigenfunctions of positive linear operators on function spaces. The eigenvalues corresponding to these positive eigenfunctions also have important economic interpretations. For example, positive eigenfunctions of pricing operators in economies characterized by Markov state processes can be used to decompose the stochastic discount factor (SDF) into its permanent and transitory components, and to approximate the prices of long-term state-contingent payoffs, with the eigenvalue embodying the yield on these payoffs (Hansen and Scheinkman, 2009; Hansen, 2012). Moreover, positive eigenfunctions are used in the study of the term structure of risk in economies with recursive preferences and stochastic growth (Hansen, Heaton, and Li, 2008; Hansen and Scheinkman, 2012a,b). Habit formation components in a semiparametric consumption capital asset pricing model (CAPM) can be expressed as a positive eigenfunction, with the eigenvalue representing the reciprocal of the time preference parameter (Chen, Chernozhukov, Lee, and Newey, 2012). Moreover, the market-implied marginal utilities of a representative agent investor consistent with panels of derivative security prices can be expressed as a positive eigenfunction (Ross, 2013). The analysis of these models, and any subsequent econometric implementation, relies critically on the identification and existence of the positive eigenfunction.

The purpose of this note is to provide nonparametric identification and existence conditions for the positive eigenfunction. Work on the identification of positive eigenfunctions in economics has been relatively limited to date. The genesis of this analysis is the classical Perron-Frobenius Theorem, which says that a positive matrix has a unique positive eigenvector. Primitive identification conditions for the special case of positive integral operators on $L^p(\mu)$ spaces have been provided by Chen et al. (2012) and Christensen (2013) in the context of a semiparametric consumption CAPM with habit formation, and the Hansen and Scheinkman (2009) SDF decomposition in discrete-time environments, respectively. These two papers assume the operators can be written as integral operators and place primitive conditions on the integrability and positivity of the integral kernel, from which identification follows by application of an integral-operator analogue of the Perron-Frobenius Theorem. Taking a somewhat different tack, Hansen and Scheinkman (2009) use Markov process theory to

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1 More generally the quantities can be expressed as a positive eigenvector of a linear operator on a vector space. The term eigenfunction is used when the quantity of interest is assumed to lie in a function space. As most spaces of interest in economics are function spaces, the term eigenfunction is used henceforth.

2 These conditions identify the positive eigenfunction up to scale (any positive multiple of a positive eigenfunction is a positive eigenfunction). If the positive eigenfunction is normalized to have unit norm, then these conditions are sufficient for the point identification of the normalized positive eigenfunction.
establish identification of the positive eigenfunction of pricing operators in continuous-time Markov environments.

On the other hand, identification of the positive eigenfunction of positive operators on a Banach lattice has been well studied in abstract functional analysis in the mathematics literature (see, e.g., Kreăin and Rutman (1950); Schaefer (1960)). However, these function-analytic conditions are typically very high level, relating to irreducibility of the operator and properties of the resolvent of the operator.

This note aims to help to fill the gap between the very high-level conditions in functional analysis, and the primitive conditions for integral operators, by providing a reasonably tractable set of sufficient conditions for existence and identification of the positive eigenfunction of positive operators on a Banach lattice. Some of these conditions turn out to have a clear interpretation in economic contexts. When applied to the special case of integral operators on $L^p(\mu)$ spaces, the identification conditions herein are weaker than those previously provided by Chen et al. (2012) and Christensen (2013) and do not rely on the existence of stationary and transition densities.

Additionally, the conditions provided in this note show several important facts beyond identification which pave the way for subsequent econometric implementation. In particular, the conditions show that the eigenvalue with which the positive eigenfunction is associated is the largest eigenvalue of the operator, is separated from the rest of the spectrum, and has multiplicity one. This implies the largest eigenvalue is continuous with respect to perturbations of the underlying operator. If one can construct an estimator of the operator that is “close” in an appropriate sense to the true operator, then the maximum eigenvalue of the estimator, and its eigenfunction, should be close to the true eigenvalue/eigenfunction.

These identification conditions are illustrated by application to the Hansen and Scheinkman (2009) SDF decomposition. The general conditions derived herein are applied to provide identification conditions for SDF decomposition in discrete- and continuous-time environments. These conditions are shown to have a connection with no-arbitrage conditions. These conditions might also be applied to extend the Recovery Theorem of Ross (2013) from a discrete-state environment to a setting with continuous statespace.

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This note was first written in October 2012 presented in November 2012. In June 2013, the author of this paper and the authors of Escanciano and Hoderlein (2012) became aware of each other’s independent works, which make use of similar tools from functional analysis. Escanciano and Hoderlein (2012) consider nonparametric identification of marginal utilities in consumption-based asset pricing models whereas the present paper considers application to stochastic discount factor decomposition. Moreover, Escanciano and Hoderlein (2012) rely heavily on the representation of their operator as an integral operator with positive kernel.
Section 2 provides identification conditions for positive operators on \( L^p(\mu) \) spaces. Section 3 then applies these conditions to study identification of the positive eigenfunction of pricing operators in discrete- and continuous-time environments. Section 4 contains the general set of conditions for a positive operator on a Banach lattice from which the earlier results follow. An appendix contains relevant definitions and all proofs.

## 2 Positive operators on \( L^p(\mu) \) spaces

This section provides sufficient conditions for the existence and nonparametric identification of positive eigenfunctions of positive linear operators on \( L^p(\mu) \) spaces. The special case \( L^2(\mu) \) is particularly relevant to semi/nonparametric estimation due to the attractive properties of Hilbert spaces.

Let \((X, \mathcal{X}, \mu)\) be a \(\sigma\)-finite measure space and let \(L^p(\mu)\) denote the space \(L^p(X, \mathcal{X}, \mu)\). Let \(T : L^p(\mu) \rightarrow L^p(\mu)\) be a bounded linear operator. The operator \(T\) is said to be positive if \(Tf \geq 0\) a.e.-[\(\mu\)] whenever \(f \geq 0\) a.e.-[\(\mu\)]. Let \(\sigma(T)\) denote the spectrum of \(T\) and \(r(T) := \sup\{|\lambda| : \lambda \in \sigma(T)\}\) denote the spectral radius of \(T\).

Identification when \(1 \leq p < \infty\) and when \(p = \infty\) are treated separately due to the different topological properties of these spaces.

### 2.1 Case 1: \(1 \leq p < \infty\)

Sufficient conditions for the existence and nonparametric identification of the positive eigenfunction of a bounded linear operator \(T : L^p(\mu) \rightarrow L^p(\mu)\) when \(1 \leq p < \infty\) are first provided.

**Assumption 2.1.** (i) \(T\) is positive, (ii) \(T^n\) is compact for some \(n \geq 1\), (iii) \(r(T) > 0\), and (iv) for each \(f \in L^p(\mu)\) such that \(f \geq 0\) a.e.-[\(\mu\)] and \(f \neq 0\) there exists \(n \geq 1\) such that \(T^n f > 0\) a.e.-[\(\mu\)].

These conditions will be referred to as *positivity, power compactness, non-degeneracy*, and *eventual strong positivity*. The first positivity condition (Assumption 2.1(i)) is typically an easy condition to motivate, either from the economic context of the problem or the structure

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4The space \(L^p(\mu) := L^p(X, \mathcal{X}, \mu)\) with \(1 \leq p \leq \infty\) consists of all (equivalence classes of measurable) functions \(f : X \rightarrow \mathbb{R}\) such that \(\int |f|^p \, d\mu < \infty\). The space \(L^\infty(\mu) := L^\infty(X, \mathcal{X}, \mu)\) consists of all (equivalence classes of measurable) functions \(f : X \rightarrow \mathbb{R}\) such that \(\text{ess sup} |f| < \infty\).
of the operator. The power compactness condition is trivially satisfied if \( T \) is compact, though this more general condition suffices. A sufficient condition for the non-degeneracy condition \( r(T) > 0 \) is that there exists a nonzero \( f \in L^p(\mu) \) with \( f \geq 0 \) a.e.-\([\mu]\) for which \( T^n f \geq \delta f \) a.e.-\([\mu]\) for some \( n \geq 1 \) and \( \delta > 0 \) \cite{Schaefer1960}. This condition is also satisfied if \( T \) has an eigenfunction corresponding to a nonzero eigenvalue. Both this condition and the eventual strong positivity condition (Assumption 2.1(iv)) have a particular intuitive interpretation in the asset pricing context discussed subsequently.

For the special case of operators that can be written as integral operators, \cite{Chen2012} and \cite{Christensen2013} assume \( T \) is compact and has a strictly positive integral kernel. This latter condition implies \( T \) satisfies the remaining conditions of Assumption 2.1.

The following Theorem shows that Assumption 2.1 is sufficient for the identification of the positive eigenfunction of both \( T \) and its adjoint \( T^* \). Let \( q = \infty \) if \( p = 1 \), otherwise let \( q \) be such that \( q^{-1} + p^{-1} = 1 \). The space \( L^q(\mu) \) can be identified as the dual space of \( L^p(\mu) \). The adjoint \( T^* : L^q(\mu) \rightarrow L^q(\mu) \) of \( T \) is defined as the bounded linear operator such that

\[
\int g(Tf) \, d\mu = \int (T^*g)f \, d\mu
\]

for all \( f \in L^p(\mu) \) and \( g \in L^q(\mu) \).

**Theorem 2.1.** Under Assumption 2.1, there exist eigenfunctions \( \tilde{f} \in L^p(\mu) \) and \( \tilde{f}^* \in L^q(\mu) \) of \( T \) and \( T^* \) with eigenvalue \( r(T) \), such that \( \tilde{f}, \tilde{f}^* > 0 \) a.e.-\([\mu]\) and \( \tilde{f}, \tilde{f}^* \) are the unique eigenfunctions of \( T \) and \( T^* \) that are non-negative a.e.-\([\mu]\). Moreover, \( r(T) \) is an isolated point of \( \sigma(T) \) and has multiplicity one.

Note that Theorem 2.1 shows not just that there is a unique positive eigenfunction of \( T \), but that it is also the unique non-negative eigenfunction of \( T \).

### 2.2 Case 2: \( p = \infty \)

Sufficient conditions for the existence and identification of the positive eigenfunction of a bounded linear operator \( T : L^\infty(\mu) \rightarrow L^\infty(\mu) \) are now provided. There are two important differences from the \( 1 \leq p < \infty \) case. First, the eventual strong positivity condition needs to be strengthened. Second, identification of the adjoint eigenfunction is not considered, as the dual space of \( L^\infty(\mu) \) is typically identified with a space of signed measures (rather than a function space).
Assumption 2.2. (i) $T$ is positive, (ii) $T^n$ is compact for some $n \geq 1$, (iii) $r(T) > 0$, and (iv) for each $f \in L^\infty(\mu)$ such that $f \geq 0$ a.e.-$[\mu]$ and $f \neq 0$ there exists $n \geq 1$ such that $\text{ess inf } T^n f > 0$.

As with the preceding case, positivity may be motivated by the economic context of the problem or the structure of the operator. Assumption 2.2(iv) appears stronger than Assumption 2.1(iv). This modification is required because the topological properties of the space $L^\infty(\mu)$ are quite different from those of the $L^p(\mu)$ spaces with $1 \leq p < \infty$.

Theorem 2.2. Under Assumption 2.2, there exists an eigenfunction $\bar{f} \in L^\infty(\mu)$ of $T$ with eigenvalue $r(T)$, such that $\text{ess inf } \bar{f} > 0$ and $\bar{f}$ is the unique eigenfunction of $T$ that is non-negative a.e.-$[\mu]$. Moreover, $r(T)$ is an isolated point of $\sigma(T)$ and has multiplicity one.

3 Application: SDF decomposition

Equilibrium conditions for dynamic asset pricing models impose restrictions on asset prices. In environments characterized by Markov state processes these restrictions allow a family of operators to be constructed that price risky payoffs over alternative investment horizons. Hansen and Scheinkman (2009) and Hansen (2012) show that a positive eigenfunction of this family of pricing operators can be used to extract information about the relation between prices of state-contingent payoffs and the investment horizon. Central to their analysis, and indeed any subsequent econometric implementation of their analysis, is the existence and identification of the positive eigenfunction of this family of operators.

A set of sufficient conditions for identification of the positive eigenfunction in continuous-time environments is provided by Hansen and Scheinkman (2009), who draw heavily on the theory of continuous-time Markov processes. Specifically, they require that their (continuous-time) SDF process is strictly positive, and that the state process when discretely-sampled is irreducible, Harris recurrent under a change of measure induced by the permanent component of the SDF, and that there exists a stationary distribution for the conditional expectations induced by the change of measure. They show that the positive eigenfunction compatible with these stochastic stability conditions must be unique (up to scale), though they do not rule out the existence of other positive eigenfunctions.

Alternative sufficient conditions for identification are now proposed for both discrete- and continuous-time environments by applying the preceding results to in this context. These
conditions relate to the inherent positivity of prices assigned to claims to non-negative payoffs. Although stronger assumptions are imposed than in \cite{Hansen and Scheinkman (2009)} (for example, here a power compactness condition is imposed on the pricing operator and the environment is assumed to be stationary), these conditions guarantee uniqueness of the positive eigenfunction. Existence of the positive eigenfunction in discrete-time environments is also established here under a natural condition on the yield on zero-coupon bonds.

### 3.1 Identification in discrete-time environments

The discrete-time environment is first described, following \cite{Hansen and Scheinkman (2013)}. Consider an environment characterized by a strictly stationary, discrete-time (first-order) Markov process \( \{(X_t, Y_t)\}_{t=0}^{\infty} \) with support \( X \times Y \) where \( X \subseteq \mathbb{R}^{d_x} \) and \( Y \subseteq \mathbb{R}^{d_y} \) are Borel sets, such that the joint distribution of \((X_{t+1}, Y_{t+1})\) conditioned on \((X_t, Y_t)\) depends only on \( X_t \). The import of this latter condition is that only payoffs that are contingent on future values of \( X_t \) are relevant for the long-run analysis.

Following \cite{Hansen and Scheinkman (2009)} and \cite{Hansen (2012)}, suppose that at each time \( t \), the price of a claim to \( \psi(X_{t+1}) \) units of the numeraire at time \( t+1 \) satisfies

\[
T\psi(X_t) = E[m(X_t, X_{t+1}, Y_{t+1})\psi(X_{t+\tau})|X_t]
\]  

(2)

for some non-negative measurable \( m \). The term \( m(X_t, X_{t+1}, Y_{t+1}) \) is the SDF for pricing single-period claims at date \( t \). Suppose further that for each \( n \geq 1 \) the date-\( t \) price of a claim to \( \psi(X_{t+n}) \) units of the numeraire at time \( t+n \) satisfies

\[
T_n\psi(X_t) = E\left[\prod_{s=0}^{n-1} m(X_{t+s}, X_{t+s+1}, Y_{t+s+1})\psi(X_{t+n})|X_t\right].
\]  

(3)

For example, if \( \psi = 1 \) then \( T_n\psi(X_t) \) is simply the price of a \( n \)-period zero-coupon bond at time \( t \) given the state \( X_t \). Relations (2) and (3) reflect the standard Euler equations or no-arbitrage equations in discrete-time dynamic asset pricing models, with some extra restrictions on the form of the SDF. The operator \( T_n \) factorizes as \( T_n = T^n \) as a consequence of the assumptions on the state process. A canonical example of this framework is the representative-agent consumption-based model with CRRA preferences in which

\[
m(X_t, X_{t+1}, Y_{t+1}) = \beta \left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}
\]  

(4)
where $C_t$ is aggregate consumption at time $t$ and $C_{t+1}/C_t = g(X_t, X_{t+1}, Y_{t+1})$ for some measurable function $g$, as in the substantial literature following Bansal and Yaron (2004).

Let $Q$ denote the stationary distribution of $X$ on $\mathcal{X}$. Stationarity of $X$ suggests that a reasonable function space in which to work is $L^p(Q) := L^p(\mathcal{X}, \mathcal{X}, Q)$ for some $1 \leq p < \infty$ where $\mathcal{X}$ is the Borel $\sigma$-algebra on $\mathcal{X}$. Extension of the following analysis to the space $L^\infty(Q)$ is straightforward.

**Definition 3.1.** $\phi \in L^p(Q)$ is a principal eigenfunction of the family of pricing operators $\{T_n : n \geq 1\}$ if $\phi > 0$ a.e.-$[Q]$ and $T_n \phi = \rho^n \phi$ for each $n \geq 1$ and some $\rho \in \mathbb{R}$.

Hansen and Scheinkman (2009) show that a principal eigenfunction $\phi$ and its eigenvalue $\rho$ can be used to extract much information about the term-structure of risk in such an economy. First, $(\rho, \phi)$ can be used to decompose the SDF into the product of its permanent and transitory components, viz.

$$
\left( \prod_{s=0}^{n-1} m(X_{t+s}, X_{t+s+1}, Y_{t+s+1}) \right) = \rho^{-n} \left( \prod_{s=0}^{n-1} m(X_{t+s}, X_{t+s+1}, Y_{t+s+1}) \right) \frac{\phi(X_{t+n})}{\phi(X_t)} \times \rho^n \frac{\phi(X_t)}{\phi(X_{t+n})}
$$

where the first term on the right-hand side of (5) is the permanent component and the second term is the transitory component. The permanent component is a multiplicative martingale with respect to the history generated by the history of the state process $X$, that is,

$$
E \left[ \rho^{-n} \left( \prod_{s=0}^{n-1} m(X_{t+s}, X_{t+s+1}, Y_{t+s+1}) \right) \frac{\phi(X_{t+n})}{\phi(X_t)} \bigg| X_t \right] = 1
$$

almost surely, for each $n \geq 1$. The permanent component can therefore be used to define a change of measure, from which the prices of long-horizon state-contingent payoffs can be approximated. Under this approximation, $\rho$ encodes the yield on long-term state-contingent payoffs, and $\phi$ captures state-dependence of prices of such payoffs. The eigenvalue $\rho$ is also related to the implied premium on risky assets in excess of long-term bonds (see also Alvarez and Jermann (2003); Backus, Chernov, and Zin (2012); Christensen (2013)). Identification of $\phi$ (up to scale) is therefore sufficient for the identification of the the permanent and transitory components of the SDF.

**Remark 3.1.** $\phi$ is a principal eigenfunction of $\{T_n : n \geq 1\}$ if and only if $\phi$ is a positive eigenfunction of $T$.

As a consequence, Theorem 2.1 may be applied to provide identification conditions for the
principal eigenfunction in this discrete-time setting. A boundedness/compactness condition is required.

**Assumption 3.1.** (i) $T : L^p(Q) \to L^p(Q)$ is a bounded linear operator, and (ii) $T_n$ is compact for some $n \geq 1$.

It remains to verify the remaining conditions of Theorem 2.1 in this context. It is clear that $T : L^p(Q) \to L^p(Q)$ as defined in (2) is positive: the conditional expectation operator is a positive operator and $m$ takes values in $[0, \infty)$. Eventual strong positivity is less trivial to verify, and is essentially a joint condition on the non-negativity of $m$ and the dependence properties of the state process. There exist models in which strong positivity fails to hold. For example, if there is a $N_0 \subseteq \mathcal{X}$ with $Q(N_0) > 0$ such that

$$m(X_t, X_{t+1}, Y_{t+1}) = 0 \text{ whenever } X_{t+1} \in N_0$$

(7) then the condition is violated (take $\psi$ to be the indicator function of the set $N_0$). Therefore, eventual strong positivity may fail to hold in models in which the SDF has the property (7), such as in models with endogenous default (see, e.g., Arellano (2008)).

The principle of no-arbitrage asserts that any claim to a non-negative payoff which is positive with positive conditional probability must command a positive price (see, e.g., Hansen and Renault (2010) and references therein). It is helpful to think of the collection

$$\Psi = \{\psi \in L^p(Q) : \psi \geq 0 \text{ a.e.-}[Q], \psi \neq 0\}$$

(8) as a menu of claims to state-contingent payoffs. That is, at date $t$ each pair $(\psi, n) \in \Psi \times \mathbb{N}$ represents a claim to $\psi(X_{t+n})$ units of the numeraire at time $t+n$. The eventual strong positivity condition can be motivated in terms of the prices the model assigns to each claim represented by $(\psi, n) \in \Psi \times \mathbb{N}$.

As eventual strong positivity depends on both the non-negativity of $m$ and the dependence properties of the state process, the following assumptions are made.

**Assumption 3.2.** For any $S \subset \mathcal{X}$ with $Q(S) > 0$ there exists $n \geq 1$ and a set $N_S \subseteq \mathcal{X}$ with $Q(N_S) = 1$ such that $\Pr(X_{t+n} \in S | X_t = x) > 0$ for all $x \in N_S$.

**Assumption 3.3.** The pricing system implied by (2) and (3) satisfies no-arbitrage, in the sense that for each $(\psi, n) \in \Psi \times \mathbb{N}$ there exists a set $N_{\psi,n} \subseteq \mathcal{X}$ with $Q(N_{\psi,n}) = 1$ such that
for each $x \in N_{\psi,n}$:

$$E[\psi(X_{t+n})|X_t = x] > 0 \implies T_n\psi(x) > 0. \quad (9)$$

Assumption 3.2 is similar to the irreducibility condition (in the sense of Markov processes) assumed for the discretely-sampled state process in Hansen and Scheinkman (2009). Chen et al. (2012) and Christensen (2013) assume the transition density of $X_{t+1}$ given $X_t$ exists and is strictly positive, which implies Assumption 3.2. Assumption 3.3 is similar to the assumption of strict positivity of the continuous-time SDF process as assumed in Hansen and Scheinkman (2009).

**Lemma 3.1.** $T$ satisfies eventual strong positivity under Assumptions 3.2 and 3.3.

An identification result is presented first. An existence and identification result then follows under an additional mild condition.

**Theorem 3.1.** Under Assumptions 3.1, 3.2, and 3.3, if $\phi \in L^p(Q)$ is a principal eigenfunction of $\{T_n : n \geq 1\}$ then $\phi$ is the unique eigenfunction of $\{T_n : n \geq 1\}$ in $L^p(Q)$ that is non-negative a.e.-$[Q]$. Moreover, its eigenvalue $\rho$ is positive, has multiplicity one, and is the largest element in $\sigma(T)$.

Theorem 3.1 shows that if one can be guaranteed of the existence of a principal eigenfunction of the family of operators $\{T_n : n \geq 1\}$ then this principal eigenfunction must be unique.

The following condition is sufficient for existence, in addition to uniqueness, of the principal eigenfunction.

**Assumption 3.4.** There exists $n \geq 1$ and a set $N_B \subseteq X$ with $Q(N_B) = 1$ such that the yield $y_n(x)$ on a zero-coupon bond with maturity $n$ is bounded uniformly for $x \in N_B$.

**Lemma 3.2.** $r(T) > 0$ under Assumption 3.4.

The following existence and identification result obtains by application of Theorem 2.1.

**Theorem 3.2.** Under Assumptions 3.1, 3.2, 3.3, and 3.4, there exists a $\phi \in L^p(Q)$ such that $\phi$ is a principal eigenfunction of $\{T_n : n \geq 1\}$, and $\phi$ is the unique eigenfunction of $\{T_n : n \geq 1\}$ in $L^p(Q)$ that is non-negative a.e.-$[Q]$. Moreover, its eigenvalue $\rho$ is positive, has multiplicity one, and is the largest element in $\sigma(T)$. 
Remark 3.2. Under the conditions of Theorem 3.1 or 3.2, there exists an eigenfunction \( \phi^* \in L^q(Q) \) of \( T^*: L^q(Q) \to L^q(Q) \) defined in (4) corresponding to the eigenvalue \( \rho \) such that \( \phi^* > 0 \) a.e.-[Q] and \( \phi^* \) is the unique eigenfunction of \( T^* \) that is non-negative a.e.-[Q].

Remark 3.3. Assumption 3.4 can be weakened to require only that there exists \( \psi \in \Psi \) and \( n \in \mathbb{N} \) and a set \( N_B \subseteq \mathcal{X} \) with \( Q(N_B) = 1 \) such that the price of a claim to \( \psi(X_{t+n}) \) units of the numeraire at time \( t \) is is bounded below by a constant multiple \( \delta > 0 \) of \( \psi(X_t) \) when \( X_t = x \), for each \( x \in N_B \).

3.2 Identification in continuous-time environments

The sufficient conditions for identification established in discrete-time environments have a natural extension to continuous-time environments, thereby forming an alternative set of identification conditions to those in Hansen and Scheinkman (2009).

Consider an environment characterized by a strictly stationary, continuous-time Markov process \( \{Z_t\}_{t \in [0,\infty)} \) with support \( \mathcal{Z} \subset \mathbb{R}^d \) where \( \mathcal{Z} \) is a Borel set, and whose sample paths are right continuous with left limits. Following Hansen and Scheinkman (2009), consider a class of model in which for each \( \tau \geq 0 \) the price \( T_\tau \psi(Z_t) \) of a claim to \( \psi(Z_{t+\tau}) \) units of the numeraire at time \( t + \tau \) is, at time \( t \), given by

\[ T_\tau \psi(Z_t) = E[M_\tau(Z_t)\psi(Z_{t+\tau})|Z_t] \quad (10) \]

for each \( \tau \in [0,\infty) \). The SDF is now a stochastic process \( M_\tau(Z_t) \) which is assumed to be a multiplicative functional of the sample path \( \{X_s : t \leq s \leq t + \tau\} \). This restriction implies that the family of pricing operators \( \{T_\tau : \tau \in [0,\infty)\} \) factorizes as \( T_{\tau+s} = T_\tau T_s \) for \( \tau, s \geq 0 \) and that \( T_0 = I \), the identity operator.

Let \( Q \) denote the stationary distribution of \( Z \) on \( \mathcal{Z} \). The following conditions are for identification in \( L^p(Q) := L^p(\mathcal{Z}, \mathcal{Z}', Q) \) for some \( 1 \leq p < \infty \) where \( \mathcal{Z}' \) is the Borel \( \sigma \)-algebra on \( \mathcal{Z} \). Extension of the following analysis to the space \( L^\infty(Q) \) is straightforward.

Definition 3.2. \( \phi \in L^p(Q) \) is a principal eigenfunction of the family of pricing operators \( \{T_\tau : \tau \in [0,\infty)\} \) if \( \phi > 0 \) a.e.-[Q] and \( T_\tau \phi = \rho^\tau \phi \) for each \( \tau \geq 0 \) and some \( \rho \in \mathbb{R} \).

\( ^5 \)Hansen and Scheinkman (2009) allow for the state process to be nonstationary. However, the assumption of stationarity is not necessarily restrictive in an asset pricing context, and may be achieved by a change of variables. For example, consumption-based asset pricing models are typically written in terms of consumption growth rather than consumption.
The principal eigenfunction $\phi$ and its eigenvalue $\rho$ may be used to decompose the SDF in a manner analogous to expression (5), and this pair contains information about the yield and state-dependence of price of long-term state-contingent payoffs in a manner analogous to the discrete-time case.

**Remark 3.4.** $\phi$ is a principal eigenfunction of $\{T_\tau : \tau \in [0, \infty)\}$ only if $\phi$ is a positive eigenfunction of $T_{\tau^*}$ for each $\tau^* > 0$.

Alternative identification conditions can therefore be provided by applying the discrete-time results to the “skeleton” of operators $\{T_{n\tau^*} : n \geq 1\}$ for some fixed positive $\tau^*$. Analysis of continuous-time Markov processes by means of their discretely-sampled skeleton is common practice. Indeed, [Hansen and Scheinkman (2009)] cast their continuous-time identification conditions in terms of the skeleton of $\{Z_t\}_{t \in [0, \infty)}$ under a change of measure induced by the permanent component. The following assumptions are made.

**Assumption 3.5.** (i) $T_{\tau^*} : L^p(Q) \to L^p(Q)$ is a bounded operator, and (ii) there exists $n \geq 1$ such that $T_{n\tau^*}$ is compact.

**Assumption 3.6.** Assumptions 3.2 and 3.3 are satisfied with $\{Z_{n\tau^*}\}_{n=0}^\infty$ in place of $\{X_t\}_{t=0}^\infty$ and $T_{\tau^*}$ in place of $T$.

Assumption 3.5 and stationarity of the state process is more than [Hansen and Scheinkman (2009)] assume, but it helps to ensure uniqueness of the principal eigenfunction. Assumption 3.6 is similar to the assumptions of strong positivity of the SDF process and irreducibility of the state process in [Hansen and Scheinkman (2009)].

**Theorem 3.3.** Under Assumptions 3.2 and 3.6, if $\phi \in L^p(Q)$ is a principal eigenfunction of $\{T_\tau : \tau \in [0, \infty)\}$ then $\phi$ is the unique eigenfunction of $\{T_\tau : \tau \in [0, \infty)\}$ in $L^p(Q)$ that is non-negative a.e.-$[Q]$. Moreover, the eigenvalue $\rho$ is positive, has multiplicity one, and $\rho^{\tau^*}$ is the largest element in $\sigma(T_{\tau^*})$.

Unlike the discrete-time case, proving existence of a positive eigenfunction of $T_{\tau^*}$ is is not enough to show existence of an eigenfunction of the whole family $\{T_\tau : \tau \in [0, \infty)\}$ because this family is not characterized fully by $T_{\tau^*}$.

4 Positive operators on Banach lattices

The theoretical underpinning of the preceding identification theorems is an extension by Schaefer (1960) of the famous theorem of [KreĐn and Rutman (1950)] to positive linear oper-
ators on Banach lattices. Identification of positive eigenfunctions of positive linear operators on Banach lattices is well studied in mathematics, as evidenced by the large body of work following Kreĭn and Rutman (1950). However, identification conditions are typically cast in terms of very high-level conditions on the irreducibility of the operator and properties of the resolvent of the operator. This section provides a particularization of these results to a set of sufficient conditions that are arguably more applicable in economics.

Let $E$ be a Banach lattice and $T : E \to E$ be a bounded linear operator. Examples of Banach lattices of functions include the $L^p(\mu)$ spaces as dealt with in Section 2, the Banach spaces $C_b(\mathcal{X})$ of bounded continuous functions $f : \mathcal{X} \to \mathbb{R}$ on a (completely regular) Hausdorff space $\mathcal{X}$, and, if $\mathcal{X}$ is also compact, the space $C(\mathcal{X})$ of continuous functions $f : \mathcal{X} \to \mathbb{R}$.

Let $E_+$ and $E_{++}$ denote the positive cone in $E$ and its quasi-interior. Let $T^*$ denote the adjoint of $T$, $E^*$ denote the dual space of $E$, and $E^*_+$ and $E^*_{++}$ denote the positive cone in $E^*$ and its quasi-interior. Again let $\sigma(T)$ and $r(T)$ denote the spectrum and spectral radius of $T$. $T$ is said to be positive if $TE_+ \subseteq E_+$ and irreducible if $\sum_{n=1}^{\infty} \lambda^{-n}T^n f \in E_{++}$ for some $\lambda > r(T)$ and each $f \in (E_+ \setminus \{0\})$. The resolvent of $T$ is defined as $R(T, z) = (T - zI)^{-1}$ for all $z \in (\mathbb{C} \setminus \sigma(T))$.

The following high-level assumption and identification result are key.

**Assumption 4.1.** (i) $T$ is positive, (ii) $r(T)$ is a pole of the resolvent of $T$, and (iii) $T$ is irreducible.

**Theorem 4.1** (Schaefer (1960), Theorem 2; see also Schaefer (1999), p. 318). Under Assumption 4.1, $r(T) > 0$, $r(T)$ is a pole of the resolvent of $T$ of order one, there exist eigenfunctions $\bar{f} \in E_+$ and $\bar{f}^* \in E^*_+$ of $T$ and $T^*$ corresponding to eigenvalue $r(T)$, and the multiplicity of $r(T)$ as an eigenvalue of $T$ is one.

**Corollary 4.1.** Under Assumption 4.1, the eigenfunctions $\bar{f}$ and $\bar{f}^*$ are the unique eigenfunctions of $T$ and $T^*$ belonging to $E_+$ and $E^*_+$.

The difficulty in application of Theorem 4.1 and Corollary 4.1 to establish existence and nonparametric identification of positive eigenfunctions in economics is that the Assumptions 4.1(ii) and 4.1(iii) appear difficult to motivate in an economic context. However it is possible to provide more tractable sufficient conditions for these assumptions. These sufficient

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6The terminology of a function space is used because the related identification problems in economics are typically in the context of operators on function spaces. More generally, there exist other Banach lattices that are not function spaces. The subsequent results apply equally to such spaces, despite maintaining the terminology of a function space.
conditions also yield a stronger result than Theorem 4.1 and Corollary 4.1 with regards to the separation of \( r(T) \) in the spectrum of \( T \), which is useful in the econometric analysis of positive eigenfunctions of \( T \).

**Assumption 4.2.** (i) \( T \) is positive, (ii) \( T^n \) is compact for some \( n \geq 1 \), (iii) \( r(T) > 0 \), and (iv) for each \( f \in (E_+ \setminus \{0\}) \) there exists \( n \geq 1 \) such that \( T^n f \in E_{++} \).

Assumption 4.2 is sufficient for existence and identification of the positive eigenfunctions of \( T \) and \( T^* \), and shows that the eigenvalue with which these are associated is the largest element of \( \sigma(T) \).

**Theorem 4.2.** Under Assumption 4.2 there exist eigenfunctions \( \tilde{f} \in E_{++} \) and \( \tilde{f}^* \in E^*_{++} \) of \( T \) and \( T^* \) with eigenvalue \( r(T) \), and \( \tilde{f} \) and \( \tilde{f}^* \) are the unique eigenfunctions of \( T \) and \( T^* \) belonging to \( E_+ \) and \( E^*_+ \).

Moreover, \( r(T) \) is an isolated point of \( \sigma(T) \), has multiplicity one, and is the unique point in \( \sigma(T) \) belonging to the circle \( \{ z \in \mathbb{C} : |z| = r(T) \} \).

## 5 Conclusion

This note provided sufficient conditions for the nonparametric identification and existence of the positive eigenfunction of a positive linear operator. Application to SDF decomposition into its permanent and transitory components is discussed for discrete- and continuous-time environments.

## A Relevant definitions

### A.1 Banach lattices and positive cones

All definitions are as in Schaefer (1999). A vector space \( E \) over the real field \( \mathbb{R} \) endowed with an order relation \( \leq \) is an ordered vector space if \( f \leq g \) implies \( f + h \leq g + h \) for all \( f, g, h \in E \) and \( f \leq g \) implies \( \lambda f \leq \lambda g \) for all \( f, g \in E \) and \( \lambda \geq 0 \). If, in addition, \( \sup\{f, g\} \in E \) and \( \inf\{f, g\} \in E \) for all \( f, g \in E \) then \( E \) is a vector lattice. If there exists a norm \( \| \cdot \| \) on an ordered vector lattice \( E \) under which \( E \) is complete and \( \| \cdot \| \) satisfies the lattice norm property, namely \( f, g \in E \) and \( |f| \leq |g| \) implies \( \|f\| \leq \|g\| \), then \( E \) is a Banach lattice.
Let $E_+$ denote the positive cone of $E$ defined with respect to the order relation on $E$. If $E = L^p(\mu)$ with $1 \leq p \leq \infty$ then $E_+ = \{ f \in E : f \geq 0 \text{ a.e.-}[\mu] \}$. If $E = C_b(\mathcal{X})$ or $C(\mathcal{X})$ then $E_+ = \{ f \in E : f(x) \geq 0, \text{ for all } x \in \mathcal{X} \}$. An element $f \in E_+$ belongs to the quasi-interior $E_{++}$ of $E_+$ if $\{ g \in E : 0 \leq g \leq f \}$ is a total subset of $E$. For example, if $E = L^p(\mu)$ with $1 \leq p < \infty$ and $f \in E$ is such that $f > 0$ a.e.-$[\mu]$ then $f \in E_{++}$. If $E = L^\infty(\mu)$ and $f \in E$ is such that $\text{ess inf} \ f > 0$ then $f \in E_{++}$. If $E = C_b(\mathcal{X})$ or $C(\mathcal{X})$ and $f \in E$ is such that $\inf_{x \in \mathcal{X}} f(x) > 0$ then $f \in E_{++}$.

Let $E^*$ denote the dual space of $E$ (the set of all bounded linear functionals on $E$). For $f \in E$, $f^* \in E^*$, define the evaluation $\langle f, f^* \rangle := f^*(f)$. The dual cone $E_+^* := \{ f^* \in E^* : \langle f, f^* \rangle \geq 0 \}$ whenever $f \in E_+$ is the set of positive linear functionals on $E$. An element $f^* \in E_+^*$ is strictly positive if $f \in E_+$ and $f \neq 0$ implies $\langle f, f^* \rangle > 0$. The set of all strictly positive elements of $E_+^*$ is denoted $E_{++}^*$.

### A.2 The spectrum of operators on Banach spaces

Let $E$ be a Banach space and $T : E \to E$ be a bounded linear operator. The definitions are presented in the case that $E$ is a Banach space over $\mathbb{C}$. When $E$ is a Banach space over $\mathbb{R}$, the complex extension $T(x + iy) = T(x) + iT(y)$ for $x, y \in E$ of $T$ is defined on the complexification $E + iE$ of $E$. The spectrum and associated quantities of $T$ when $E$ is a Banach space over $\mathbb{R}$ are obtained by applying the complex Banach space definitions to the complex extension of $T$.

The resolvent set $\rho(T) \subseteq \mathbb{C}$ of $T$ is the set of all $z \in \mathbb{C}$ for which the resolvent operator $R(T,z) := (T - zI)^{-1}$ exists as a bounded linear operator on $E$ (where $I : E \to E$ is the identity operator given by $Ix = x$ for all $x \in E$). The spectrum $\sigma(T)$ is defined as the complement in $\mathbb{C}$ of $\rho(T)$, i.e. $\sigma(T) := (\mathbb{C} \setminus \rho(T))$. The point spectrum $\pi(T) \subseteq \sigma(T)$ of $T$ is the set of all $z \in \mathbb{C}$ for which $\ker(T - zI)$ is nontrivial. Clearly $(T - zI)^{-1}$ fails to exist for $z \in \pi(T)$. When $\pi(T)$ is nonempty, each $\lambda \in \pi(T)$ is an eigenvalue of $T$. The dimension of $\ker(T - \lambda I)$ is the (geometric) multiplicity of the eigenvalue $\lambda$. Any $\psi \in \ker(T - \lambda I)$, $\psi \neq 0$ is an eigenvector of $T$ corresponding to $\lambda$. The spectral radius $r(T)$ of $T$ is $r(T) = \sup \{ |\lambda| : \lambda \in \sigma(T) \}$. 

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B Proofs

Proof of Theorem 2.1. Immediate by application of Theorem 4.2.

Proof of Theorem 2.2. Immediate by application of Theorem 4.2.

Proof of Lemma 3.1. Take any \( \psi \in \Psi \). Let \( N_\psi = \{ x \in X : \psi(x) > 0 \} \) denote the support of \( \psi \). Assumption 3.2 shows that \( E[\psi(X_{t+n}|X_t)] > 0 \) a.e.-\( [Q] \) for some \( n \geq 1 \). Assumption 3.3 then implies that \( T_n\psi(X_t) > 0 \) a.e.-\( [Q] \).

Proof of Theorem 3.1. The proof is by application of Theorem 2.1. Assumption 3.1, 3.2 and 3.3 satisfy Assumptions 2.1(i), 2.1(ii) and 2.1(iv). For Assumption 2.1(iii), \( r(T) \geq \rho \) by definition and \( \rho > 0 \) since \( T_n\phi = \rho^n\phi > 0 \) a.e.-\( [Q] \) for some \( n \in \mathbb{N} \) by Lemma 3.1.

Proof of Lemma 3.2. Assumption 3.4 implies
\[
T^n\mathbf{1}(x) = T_n\mathbf{1}(x) = (1 + y_n(x))^{-1} \geq \epsilon \times \mathbf{1}  \tag{A.1}
\]
for each \( x \in N_B \) and some \( \epsilon > 0 \). The result follows by application of Proposition 3 of Schaefer (1960).

Proof of Theorem 3.2. Immediate by application of Theorem 2.1.

Proof of Theorem 3.3. Apply Theorem 3.1 with \( T_{\tau^*} \) in place of \( T \).

Proof of Corollary 4.1. The eigenvalue \( r(T) \) has multiplicity one by Theorem 4.1, so any other eigenfunction (i.e. one that is not a constant multiple of \( \tilde{f} \)) must correspond to an eigenvalue different from \( r(T) \). Suppose \( \bar{g} \in (E_+ \backslash \{0\}) \) be an eigenfunction of \( T \) corresponding to \( \lambda \neq r(T) \). Then, with \( \langle f, f^* \rangle := f^*(f) \) for \( f^* \in E^* \), \( f \in E \),
\[
\lambda \langle \bar{g}, \tilde{f}^* \rangle = \langle T\bar{g}, \tilde{f}^* \rangle = (\tilde{f}^* \circ T)(\bar{g}) = T^* (\tilde{f}^*)(\bar{g}) = r(T)\tilde{f}^*(\bar{g}) = r(T)\langle \bar{g}, \tilde{f}^* \rangle \tag{A.2}
\]
which is a contradiction, since \( \tilde{f}^* \in E_{++}^* \) and \( g \in (E_+ \backslash \{0\}) \) implies \( \langle \bar{g}, \tilde{f}^* \rangle > 0 \).

As the multiplicities of eigenvalues of finite multiplicity of \( T \) are preserved under adjoints (Dunford and Schwartz, 1958, Exercise VII.5.35), \( r(T) \) is an eigenvalue of \( T^* \) with multi-
plicity one. That \( \tilde{f}^* \in E^*_+ \) is the unique eigenfunction of \( T^* \) in \( E^*_+ \) follows by a similar argument.

**Proof of Theorem 4.2.** Assumption 4.1(ii) is satisfied under Assumptions 4.2(i)–(iii) because \( r(T) \in \sigma(T) \) since \( T \) is positive (Schaefer, 1999, p. 312) and each nonzero element of \( \sigma(T) \) is a pole of the resolvent of \( T \) since \( T^n \) is compact (Dunford and Schwartz, 1958, p. 579). Assumption 4.2(iv) implies that for each \( f \in (E_+ \setminus \{0\}) \) and \( f^* \in E^*_+ \) there exists \( n \geq 1 \) such that \( \langle T^n f, f^* \rangle > 0 \), so \( T \) is irreducible (Schaefer, 1974, Proposition III.8.3). The first part then follows immediately from Theorem 4.1 and Corollary 4.1.

For the second part, compactness of \( T^n \) for some \( n \geq 1 \) and the fact that \( \sigma(T)^n = \sigma(T^n) \) implies that the only possible limit point of elements of \( \sigma(T) \) is zero (Dunford and Schwartz, 1958, p. 579). Therefore \( r(T) \) is an isolated point of \( \sigma(T) \). That \( r(T) \) has multiplicity one is immediate from Theorem 4.1. Finally, that \( r(T) \) is the unique point in \( \sigma(T) \) belonging to the circle \( \{ z \in \mathbb{C} : |z| = r(T) \} \) follows under Assumption 4.2(iv) by application of Proposition V.5.6 of Schaefer (1974).

**References**


