Optimal control of a stochastic system with an exponential-of-integral performance criterion

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Abstract: An approach based on the theory of positive semigroups is introduced for the analysis of the infinite-horizon, exponential-of-integral optimal control problem for a stochastic system of the Ito form. Existence conditions for state-feedback admissible controllers are formulated and optimality conditions are derived. Connections between the exponential-of-integral optimal control problem and stochastic differential games are discussed.

Keywords: Optimal stochastic control; exponential-of-integral cost; risk-sensitive control; differential games.

1. Introduction

In this paper we consider the Ito type system
\[
dx_t = b(x_t, u_t)dt + \sigma(x_t)dw_t,
\]
where \(x_t \in \mathbb{R}^n\) is the state, \(u_t \in \mathbb{R}^m\) is the control and \(w_t\) is an \(r\)-dimensional standard Brownian motion. The cost functional (to be minimized) is the infinite-horizon or long-run average exponential-of-integral (also called risk-sensitive) performance criterion
\[
J(u) = \lim_{T \to \infty} \frac{1}{T} \ln E \left\{ \exp \left[ \theta \int_0^T c(x_t, u_t)dt \right] \right\},
\]
where \(c(x_t, u_t)\) is a nonnegative continuous function and \(\theta\) is a scalar. For \(\theta < 0\), the cost functional is said to be risk-seeking and risk-averse for \(\theta > 0\). The risk-neutral case \(\theta = 0\) is equivalent to the standard long-run average cost functional. In this paper we consider only the risk-averse case \(\theta > 0\). Furthermore, in order to simplify the notation we take \(\theta = 1\). The techniques presented in the paper can be extended in a straightforward way to the risk-seeking case. However, in the interest of keeping the paper within the required size limitations we consider the risk-averse case only.

Problems of form (1.1) and (1.2) arise in numerous applications, e.g. economics [8] and missile guidance [13]. In the linear-quadratic case, \(b(x, u) = Ax + Bu, \sigma(x) = E, c(x, u) = x^T Qx + u^T Ru\), an important connection with differential games and \(H_\infty\) control has been established. The relationship between the exponential-of-integral optimal control problem and differential games was first suggested in [9] for the finite-horizon, linear-quadratic case. The infinite-horizon, linear-quadratic case was discussed in [6, 7, 11, 14]. The general finite-horizon exponential-of-integral optimal control problem is discussed in [4, 10, 15].

In [12], the relationship between the infinite-horizon exponential-of-integral control problem and a certain infinite-horizon stochastic differential game was rigorously established. The approach was based on the theory of large deviations. In this paper, we derive simple conditions for the existence of admissible (stabilizing) controls and optimality for problem (1.1) and (1.2). By inspection, we establish the above-mentioned relationship between the infinite-horizon exponential-of-integral optimal control problem and the infinite-horizon stochastic differential game. We restrict our attention to the class of stationary Markov controls, i.e. \(u_t = u(x_t)\).

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emphasis are different and the class of systems is more restricted.

The paper is organized as follows: In Section 2, we outline the basics of our approach, in Section 3, we present conditions for the existence of admissible controls and optimality conditions, and in Section 4, we discuss the connection with differential games. Conclusions are formulated in Section 5.

2. Preliminaries

Consider the Ito system
dx_t = b(x_t)dt + \sigma(x_t)dw_t, \quad x_0 = x, \tag{2.1}

where $x \in \mathbb{R}^n$, $w_t$ is a standard $r$-dimensional Brownian motion defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and $b(x)$ and $\sigma(x)$ satisfy a global Lipschitz condition, i.e. there exists a constant $k > 0$ such that

$$\|b(y) - b(x)\| + \|\sigma(x) - \sigma(y)\| \leq k|x - y|.$$ \tag{2.2}

Then the solution of (2.1) is well defined for all $t \geq 0$ and is a Markov diffusion process on $\mathbb{R}^n$. We assume that there exists a unique invariant measure $\nu$ for (2.1), i.e. if $P(t, x, \cdot)$ is the transition function corresponding to (2.1) and $U_t\mu(\cdot) = \int_{\mathbb{R}^n} P(t, x, \cdot) \mu(dx)$, then $\nu$ is the unique probability measure on $\mathbb{R}^n$ such that $U_t\nu(\cdot) = \nu(\cdot)$.

Let $c: \mathbb{R}^n \to \mathbb{R}$ be a nonnegative continuous function and define

$$I_T(x) = \mathbb{E}_x \left[ \exp \left\{ \int_0^T c(x_s)ds \right\} \right]. \tag{2.3}

We are interested in calculating the limit

$$J = \lim_{T \to \infty} \frac{1}{T} \ln I_T(x). \tag{2.4}

Consider the functional

$$(K_t h)(x) = \mathbb{E}_x \left[ \exp \left\{ \int_0^t c(x_s)ds \right\} h(x_t) \right]. \tag{2.5}

where $h: \mathbb{R}^n \to \mathbb{R}$ is a continuous function. Note that $I_T(x) = (K_T 1)(x)$ where $1(x)$ is the function identically equal to one. It is well known that $K_t$ is a semigroup, i.e. $K_{t+s}h = K_tK_sh$. The generator $A$ of $K_t$ is defined by

$$Av = \lim_{t \to 0^+} \frac{K_tv - v}{t} \tag{2.6}

and the domain of $A$, $D(A)$, is the set of functions for which the above limit exists. For the twice continuously differentiable functions, the operator $A$ coincides with differential operator

$$L + c = \sum_{i=1}^n b_i \frac{\partial}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^n (\sigma \sigma')_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + c. \tag{2.7}

Note that (2.5) is the Feynman–Kac formula for the solution of the ‘abstract’ evolution equation

$$\frac{\partial s}{\partial t} = As, \quad s(0, x) = h(x), \tag{2.8}

i.e. $s(t, x) = (K_t h)(x)$. The following lemma is needed in the subsequent analysis.

**Lemma 2.1.** Let $v \in D(A)$. Then for any $\gamma \in \mathbb{C}$,

$$e^{-\gamma t} K_t v = v + \int_0^t e^{-\gamma s} K_s (A - \gamma)v \, ds. \tag{2.9}

**Proof.** See [2, p. 195]. \hfill \square

3. The optimal control problem

Consider now the controlled system
dx_t = b(x_t, u_t)dt + \sigma(x_t)dw_t, \quad x_0 = x, \tag{3.1}

where, as before, $x \in \mathbb{R}^n$, $u_t \in \mathbb{R}^m$ is the control and $w_t$ is a standard $r$-dimensional Brownian motion. We assume that $b(x, u)$ is globally Lipschitz in $x$ and $u$, and $\sigma(x)$ is globally Lipschitz in $x$, i.e. there exists a constant $k > 0$ such that

$$\|b(x, u) - b(y, v)\| + \|\sigma(x) - \sigma(y)\| \leq k(|x - y| + |u - v|), \quad \forall x, y \in \mathbb{R}^n, \quad u, v \in \mathbb{R}^m. \tag{3.2}

Define the class $\tilde{U}$ as functions $u: \mathbb{R}^n \to \mathbb{R}^m$ which satisfy the following conditions:

(A) There exists a finite constant $k_u > 0$ such that

$$|u(x) - u(y)| \leq k_u|x - y|, \quad \forall x, y \in \mathbb{R}^n. \tag{3.3}

(B) The closed-loop system with control $u(x)$ has a unique invariant measure $\nu_u$. 

Let \( c: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \) be a nonnegative function which is continuous in both of its arguments. Define the cost functional

\[
J(u) = \lim_{T \to \infty} \frac{1}{T} \ln E^x_T \left[ \exp \left( \int_0^T c(x_s, u(x_s)) ds \right) \right].
\]

(3.4)

Here, \( E^x_T[\cdot] \) denotes the expectation associated with the closed-loop system with control \( u \in U \). A control \( u \in U \) for which \( J(u) < \infty \) is said to be admissible, denoted \( u \in \bar{U} \).

The basic control problem we are interested in solving is to find a control \( u \in \bar{U} \) that minimizes \( J(u) \), i.e.

\[
J^* = \inf_{u \in \bar{U}} J(u).
\]

(3.5)

We begin by giving sufficient conditions for the existence of an admissible control. Let \( L^u \) be the differential generator of (3.1) with control \( u \in U \) and define

\[
\tilde{L}^u = L^u + \frac{1}{2} \sum_{i,j=1}^n (\sigma(x)\sigma'(x))_{ij} \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j}.
\]

(3.6)

For \( \rho > 0 \) define \( \zeta(\rho) = \inf_{|x| > \rho} \inf_{u} c(x,u) \).

**Theorem 3.1.** Let \( u \) satisfy (3.3). Assume there exists a twice continuously differentiable function \( \phi(x) \) which is bounded below and positive constants \( \lambda, \rho_0 \) such that

(i) \( \zeta(\rho) > \lambda \) for all \( \rho > \rho_0 \),

(ii) for all \( t > 0 \),

\[
E_x \left[ \int_0^t \left| \sigma'(x_s) \frac{\partial \phi}{\partial x} (x_s) \right|^2 ds \right] < \infty,
\]

(3.7)

(iii) for all \( x \in \mathbb{R}^n \),

\[
\tilde{L}^u \phi(x) + c(x, u(x)) \leq \lambda.
\]

(3.8)

Then \( u \in \bar{U} \) and \( J(u) \leq \lambda \).

**Proof.** First we show that there exists a unique invariant measure. Note that

\[
\sum_{i,j=1}^n (\sigma(x)\sigma'(x))_{ij} \frac{\partial \phi}{\partial x_i} \frac{\partial \phi}{\partial x_j} = \left( \frac{\partial \phi}{\partial x} \right)' \sigma(x) \frac{\partial \phi}{\partial x} \geq 0.
\]

(3.9)

Therefore, from (3.8) we get

\[
\tilde{L}^u \phi(x) \leq \tilde{L}^u \phi(x) \leq \lambda - c(x, u(x)).
\]

(3.10)

Next note (i) implies that for \( |x| > \rho_0 \) we have

\[
L^u \phi(x) \leq \tilde{L}^u \phi(x) \leq \lambda - \zeta < 0.
\]

(3.11)

This, combined with (3.7) implies that there exists a unique invariant measure \( \nu^* \) for (3.1) [17].

We now show that \( J(u) < \infty \). Define \( \nu(x) = \exp\{\phi(x)\} \). Then a simple calculation shows that it follows from (3.8) that

\[
(L^u + c(x, u(x))) \nu(x) \leq \lambda \nu(x).
\]

(3.12)

Note that \( A^u = L^u + c(x, u(x)) \) is the differential generator of the semigroup \( (K^u_t)_{t \geq 0} \).

\[
e^{-\zeta t} K^u_t \nu = \nu + \int_0^t e^{-\zeta s} K^u_t (A^u - \gamma) \nu ds,
\]

(3.13)

for all \( \gamma \in \mathbb{R} \). Using (3.12) in (3.13) gives

\[
e^{-\zeta t} K^u_t \nu \leq \nu + (\lambda - \gamma) \int_0^t e^{-\zeta s} K^u_t \nu ds.
\]

(3.14)

Since \( e^{-\zeta t}(K^u_t \nu)(x) \geq 0 \) for all \( x \in \mathbb{R}^n \), it follows from the Bellman–Gronwall inequality that

\[
e^{-\zeta t} K^u_t \nu \leq e^{(\lambda - \gamma)t} \nu
\]

(3.15)

or

\[
K^u_t \nu \leq e^{\lambda t} \nu.
\]

(3.16)

Define \( \eta = \inf_{x \in \mathbb{R}^n} \nu(x) > 0 \). Then a simple calculation shows that

\[
K^u_t \nu \geq \eta K^u_t \nu.
\]

(3.17)

Combining (3.16) and (3.17) gives

\[
K^u_t \nu \leq \frac{\nu}{\eta} e^{\lambda t}
\]

(3.18)

and, therefore,

\[
J(u) = \lim_{T \to \infty} \frac{1}{T} \ln K_T \nu \leq \lambda.
\]

(3.19)

The proof is complete. \( \square \)

**Example 3.1.** Consider the linear system

\[
b(x, u) = Ax + Bu, \quad \sigma(x) = E,
\]

(3.20)

with the quadratic cost functional

\[
c(x, u) = \frac{1}{2}(x'Qx + u'Ru), \quad Q \geq 0, \quad R > 0.
\]

(3.21)
Note first that condition (i) becomes \( \frac{1}{2} x' Q x > \lambda \) for \( |x| > \rho \). Thus, we need \( Q > 0 \) (as we see below, this is actually more than is necessary). Let \( u = K x \). Then equation (3.8) becomes

\[
x'(A + BK)' \frac{\partial \phi}{\partial x} + \frac{1}{2} \sum_{i,j=1}^{n} (EE')_{ij} \frac{\partial^2 \phi}{\partial x_i \partial x_j} + \frac{1}{2} x'(Q + K' RK)x \leq \lambda.
\]

Let \( \phi(x) = \frac{1}{2} x'Mx \). Then (3.22) becomes

\[
x'[(A + BK)'M + M(A + BK) + MEE'M + Q + K' RK]x + \text{Tr } EE'M \leq 2\lambda.
\]

In order for \( \phi(x) \) to be bounded below, we need \( M \geq 0 \). Next note that the left-hand side of (3.7) is now

\[
E_x \left[ \int_0^t |E'Mx_s|^2 ds \right] = E_x \left[ \int_0^t x_s'MEE'Mx_s ds \right]
\]

\[
= \int_0^t \text{Tr } MEE'Mx_s(x_s')ds = \text{Tr } MEE'M \int_0^t X_s ds,
\]

where \( X_t \) is the covariance of \( x_t \). It is well known that \( X_t \) satisfies the Lyapunov equation

\[
\dot{X}_t = (A + BK)X_t + X_t(A + BK)' + EE',
\]

\[
X_0 = 0
\]

and it is easy to see that (3.7) is satisfied for all \( t > 0 \). Therefore, if the matrix inequality

\[
(A + BK)'M + M(A + BK) + MEE'M + Q + K' RK \leq 0
\]

has a positive-semi-definite solution, all the conditions of Theorem 3.1 are satisfied and \( u \in \mathcal{U} \).

Conditions for an inequality of the form (3.26) to have a positive-definite solution have been studied in [16]. It is shown, in particular, that if \( A + BK \) is Hurwitz, \( (A + BK, E) \) is controllable and \( (Q, A + BK) \) is observable, then a positive-definite solution exists if and only if

\[
C(joI - (A + BK))^{-1} EE' \times (-joeI - (A + BK))^{-1} C' \leq I,
\]

where \( C \) is chosen so that \( C'C = Q + K' RK \).

Note, in particular, that (3.27) implies that \( \| H \|_\infty \leq 1 \), where \( H(s) = (sl - A - BK)^{-1} E \) is the transfer function from the noise input to the output.

\[
z = \frac{\sqrt{Q_x}}{\sqrt{Kx}} = Cx.
\]

The relationship between the linear-quadratic exponential-of-integral optimal control problem and \( H_x \) control is discussed in [6, 7].

Having established sufficient conditions for the existence of an admissible control, we now turn to optimality conditions for problem (3.5).

Theorem 3.2. Assume there exists a twice continuously differentiable function \( \phi(x) \) and a constant \( \lambda > 0 \) such that

\[
\hat{\lambda} = \inf_{u \in \mathcal{U}} [\tilde{L}_u \phi + c(x, u)].
\]

Then \( \hat{\lambda} \leq \hat{\lambda}^* \). Furthermore, if \( \hat{\lambda} \) is bounded below and there exists a \( u^* \in \mathcal{U} \) such that

\[
\hat{\lambda} = \tilde{L}^{u^*} \phi + c(x, u^*),
\]

then \( u^* \) is optimal and \( \hat{\lambda} = \hat{\lambda}^* \).

Proof. It follows from (3.29) that

\[
\tilde{L}_u \phi + c(x, u) \geq \hat{\lambda}, \quad \forall u \in \mathcal{U}.
\]

Let \( \hat{\lambda} = \exp[\tilde{L}_u \phi(x)] \). Then it follows, using a similar argument as in the proof of Theorem 3.1, that

\[
(K_u \hat{\lambda})(x) \geq e^{\hat{\lambda}t} \hat{\lambda}(x).
\]

Define

\[
\hat{\lambda}_R(x) = \begin{cases} \hat{\lambda}(x) & \text{if } |x| \leq R, \\ 0 & \text{otherwise}. \end{cases}
\]

Then it follows from (3.32) that

\[
(K_u \hat{\lambda}_R)(x) \geq e^{\hat{\lambda}_R} \lambda_R(x).
\]

It is easy to show that

\[
(K_u \hat{\lambda}_R)(x) \leq \sup_{x \in \mathbb{R}^n} \hat{\lambda}_R(x)(K_u 1)(x)
\]

\[
= \| \hat{\lambda}_R \|_\infty (K_u 1)(x).
\]

Combining (3.34) and (3.35) gives

\[
(K_u 1)(x) \geq e^{\hat{\lambda}_R} \| \hat{\lambda}_R \|_\infty.
\]
and from (3.36), we get
\[
J(u) = \lim_{T \to \infty} \frac{1}{T} \ln(K_T 1)(x) \geq \hat{\lambda}, \quad \forall u \in \mathcal{U}.
\] (3.37)

Finally, if (3.30) holds, we have
\[
(K_T^* \delta)(x) = e^{t \delta(x)}. \quad (3.38)
\]

Note that
\[
(K_T^* \delta)(x) \geq \inf_{\varepsilon \in \mathbb{R}^n} \delta(z)(K_T^* 1)(x)
\]
\[
= \hat{\eta}(K_T^* 1)(x) > 0.
\] (3.39)

Therefore,
\[
(K_T^* 1)(x) \leq \frac{(K_T^* \delta)(x)}{\hat{\eta}} = e^{t \delta(x)}
\] (3.40)

and
\[
\lim_{t \to -\infty} \frac{1}{t} \ln(K_T^* 1)(x) \leq \hat{\lambda}.
\] (3.41)

Combining (3.37) and (3.41) gives
\[
J(u^*) = \lim_{T \to \infty} \frac{1}{T} \ln(K_T^* 1)(x) = \hat{\lambda}.
\] (3.42)

The proof is complete. □

**Example 3.2.** We consider again the linear system considered in Example 3.1. Performing the minimization in (3.29) gives
\[
u = -R^{-1}B' \frac{\partial \hat{\phi}}{\partial x}.
\] (3.43)

Substituting (3.43) into (3.29) gives
\[
(Ax) \frac{\partial \phi}{\partial x} + \frac{1}{2} \sum_{i,j=1}^n (EE')_{ij} \frac{\partial^2 \phi}{\partial x_i \partial x_j} + \frac{1}{2} \left( \frac{\partial \phi}{\partial x} \right) (EE' - BR^{-1} B') \frac{\partial \phi}{\partial x} + \frac{1}{2} x' Q x = \hat{\lambda}.
\] (3.44)

This equation has solution
\[
\hat{\phi}(x) = \frac{1}{2} x' M x, \quad \hat{\lambda} = \frac{1}{2} \text{Tr} EE' M,
\] (3.45)

where \( M \) satisfies
\[
A'M + MA + M(EE' - BR^{-1} B') M + Q = 0.
\] (3.46)

Equations of the form (3.46) have been extensively studied in the literature in recent years. In particular, it is shown in [3] that if (3.46) has a solution \( M > 0 \) such that matrix \( A - (BR^{-1} B' - EE')M \) is Hurwitz, then \( A - BR^{-1} B'M \) is also Hurwitz and \( M < M \), where \( M \) is any other solution of (3.46). If, in addition, the pair \( (A - BR^{-1} B'M, E) \) is controllable then \( u \in \mathcal{U} \) and, since \( M > 0 \), by Theorem 3.2 this control is optimal.

4. Connection with stochastic differential games

In this section, we illustrate that the infinite-horizon exponential-of-integral control problem is equivalent to a certain stochastic differential game problem. The connection between the exponential-of-integral cost control problem and differential games was originally observed in [9] for the finite-horizon linear-quadratic case.

Consider the system
\[
x_t = (b(x_t, u_t) + a(x_t)v_t)dt + a(x_t)dw_t,
\] (4.1)

where \( b(x, u) \) and \( a(x) \) are as before and \( v_t \in \mathbb{R}^r \) is a second control input. Let \( u \in \mathcal{U} \) and \( v \in \mathcal{V}_u \), where \( \mathcal{V}_u \) is the class of all \( v: \mathbb{R}^n \to \mathbb{R}^r \) such that (4.1) has a unique invariant measure, and define
\[
K(u, v) = \lim_{T \to \infty} \frac{1}{T} \int_0^T E \left[ c(x_t, u_t) - \frac{1}{2} |v_t|^2 \right] dt,
\] (4.2)

where \( c(x, u) \) is as in Section 3. Consider the stochastic differential game with payoff \( K(u, v) \), minimizing player \( u \) and maximizing player \( v \). Then the lower value of the game is defined as
\[
\lambda = \inf_{u \in \mathcal{U}} \sup_{v \in \mathcal{V}_u} K(u, v).
\] (4.3)

The infinite-horizon Isaacs equation for this problem is
\[
\lambda = \inf_{u \in \mathcal{U}} \sup_{v \in \mathcal{V}_u} \left[ L^{u,v} \phi(x) + c(x, u) - \frac{1}{2} |v|^2 \right],
\] (4.4)

where \( L^{u,v} \) is the differential operator corresponding to (4.1), i.e.
\[
L^{u,v} = L^u + v' \sigma^i \frac{\partial}{\partial x^i}.
\] (4.5)

**Remark 4.1.** Note that both the drift vector \( b(x, u) + \sigma(x) v \) and the integrand of the cost functional \( c(x, u) - \frac{1}{2} |v|^2 \) are separable in \( u, v \). Thus, the Isaacs condition holds for the payoff \( K(u, v) \) and, therefore, \( \lambda \) is the value of the stochastic game, i.e.
\[
\lambda = \inf_u \sup_v K(u, v) = \sup_v \inf_u K(u, v) \quad [1, \text{p. 348}].
\]
Solving the maximization problem in (4.4) gives

$$v^* = \sigma \frac{\partial \phi}{\partial x}.$$  \hfill (6.4)

Substituting into (4.4) gives

$$\hat{\lambda} = \inf_{u \in U} \left[ L^\nu \phi(x) + \frac{1}{2} \left( \frac{\partial \phi(x)}{\partial x} \right)' \sigma(x) \sigma'(x) \frac{\partial \phi(x)}{\partial x} + c(x, u) \right]$$

$$= \inf_{u \in U} \left[ \tilde{L}^\nu \phi(x) + c(x, u) \right].$$ \hfill (6.7)

Comparing (6.7) with Theorem 3.2 we see that \(\hat{\lambda} = \lambda\). In summary we have the following results.

**Proposition 4.1.** Under the assumptions of Theorem 3.2, the infinite-horizon exponential-of-integral optimal control problem (3.1), (3.4) and (3.5) and infinite-horizon stochastic differential game (4.1)-(4.3) are equivalent problems.

**Remark 4.2.** In Proposition 4.1 we call the infinite-horizon exponential-of-integral optimal control problem and the infinite-horizon stochastic differential game ‘equivalent’ problems based on the equivalence of the sufficient conditions in Theorem 3.2 and equation (4.7). This equivalence was directly established in [12] using large-deviations theory. In particular, the ‘origin’ of the maximizing player \(v\) is explained in [12].

**5. Conclusions**

In this paper, we have presented a simple approach, based on the theory of positive semigroups, for the analysis of the infinite-horizon exponential-of-integral optimal control problem. We derived sufficient conditions for the existence of admissible (stabilizing) controls and optimality. These conditions closely resemble the corresponding conditions for the additive cost optimal stochastic control problem. We also derived a infinite-horizon stochastic differential game that is equivalent to the infinite-horizon exponential-of-integral optimal control problem.

**References**