Dividend Dynamics

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Abstract

As a consequence of firms maintaining stationary leverage ratios, shareholders are being forced to divest (invest) when the firm does well (poorly). This dynamic conceals the leveraged nature of dividends in that, even though returns to dividends (i.e., equity) are more volatile and have higher expected values than returns to EBIT (i.e., debt plus equity), dividends and EBIT are cointegrated in the long run. Accounting for stationary leverage ratios generates stock returns that are more volatile than dividends (Shiller (1981), LeRoy and Porter (1981)), and generates a downward sloping term structure of expected returns and volatilities for dividend strips (van Binsbergen, Brandt and Koijen (2011)).
1 Introduction

It is common in the asset pricing literature to assume that consumption and dividends are cointegrated in the long run. Sometimes this relation is specified explicitly.\footnote{See, for example Menzly, Santos and Veronesi (2004), Santos and Veronesi (2005).} Other times this feature is approximated by specifying consumption and dividend dynamics to have the same long-run expected growth rate. Recently, this second approach has been used by, among others, Campbell and Cochrane (CC, 1999) and Bansal and Yaron (BY, 2004), in an attempt to explain the so-called equity premium puzzle (Mehra and Prescott (1985)).\footnote{The BY model specifies different conditional expected growth rates for consumption and dividends due to a different weighting on a small but persistent state variable $x$ that drives changes in expected growth. However, since this state variable has a stationary distribution centered about zero, consumption and dividends share the same unconditional expected growth rates.}

However, specifying consumption and dividend processes with the same expected growth rate seems to be in conflict with the corporate finance literature, where dividends are viewed as a ‘leveraged cash flow’ and hence are associated with a higher expected growth rate. In particular, in this literature it is common to exogenously specify an unleveraged cash flow (i.e., EBIT) process and a particular capital structure strategy, and then \textit{endogenously} obtain a ‘leveraged’ dividend process that is internally consistent with the EBIT process. Related, several papers in the literature have attempted to account for leverage by using a ‘reduced-form’ approach. For example, Abel (1999, 2005) and Lettau, Ludvigson and Wachter (2008) attempt to capture the leveraged nature of dividends by specifying cash flows of the form $y^\lambda$, where $\lambda = 0$ for fixed income securities, $\lambda = 1$ for unleveraged equity, and $\lambda > 1$ for leveraged equity.

In this paper, we argue that the asset pricing literature should follow the lead of the corporate finance literature in specifying a dividend process that is internally consistent with the EBIT process, because doing so helps explain certain “puzzling” properties of asset prices. First, we show that this approach generates stock return volatility that is significantly higher than long-term dividend volatility, consistent with Shiller (1981) and LeRoy and Porter (1981). Second, we show that properly accounting for the leveraged nature of dividend dynamics automatically generates a term structure of expected returns and volatilities for dividend strips that are decreasing in maturity, consistent with the empirical findings of van Binsbergen, Brandt and Kojien (BBK, 2011) and van Binsbergen, Hueskes, Kojien and Vrugt (2011).

The intuition for why it is important to jointly model EBIT and dividend dynamics in an internally consistent manner stems from the fact that when a firm rebalances its debt levels over time to maintain a stationary leverage process, shareholders are being forced to divest
(invest) when the firm does well (poorly). This is because a good shock to firm’s EBIT leads to an increase in firm value, which allows managers to increase debt levels further. Funds raised by this debt issuance allows the firm to pay out high levels of dividends today, but reduces the fraction of the firm owned by equityholders, and thus, reduces future dividends. The opposite interaction occurs in the event of a bad shock. Thus, even if investors follow a “static” strategy of holding a fixed supply of stock, their position is effectively being ‘managed’ by the capital structure decisions of the firm. Below, we show that these imposed investments/divestments conceal the ‘leveraged nature’ of dividends in that, even though instantaneously dividends are leveraged (in the sense that returns on equity are more volatile and have higher expected values than returns on (equity + debt)), over the long run, EBIT and dividends are cointegrated, and therefore have the same long run expected growth rate and volatility.

This intuition allows us to explain the two asset pricing puzzles mentioned above: First, we demonstrate that when dividend dynamics are specified to exhibit stationary leverage ratios, the so-called “excess volatility puzzle” can be captured in that long-run dividend volatility is automatically lower than stock return volatility. Intuitively, since dividends are cointegrated with EBIT, its long-term volatility is shown to be equal to the volatility of EBIT. In contrast, stock return volatility is pushed up by a “leverage factor” \( \left( \frac{1}{1-L} \right) \). For typical leverage ratios of about 40%, our model generates stock return volatility to approximately 60% higher than long run dividend volatility.

Second, due to the implicit divestments in good times (and reinvestments in bad times) that the firm imposes on stockholders due to their capital structure decisions, long-maturity dividend strips are not as risky as typically imagined – rather, they are about as risky as long-maturity EBIT strips due to the fact that dividends and EBIT are cointegrated. However, in the short run, dividends are more correctly thought of as a leveraged, risky cash flow. Together, this generates a downward sloping term structure of expected returns and volatilities for dividend strips. One implication is that, while BBK interpret their empirical findings as evidence against the preference and output (i.e., consumption) dynamics proposed by CC and BY, our results imply that all that needs to be altered in the CC and BY models for them to be consistent with observed properties of dividend strip returns is their dividend process. Indeed, below we show that if we specify EBIT (which can be reinterpreted to be consumption in an endowment economy) and pricing kernel dynamics similar to the long-run risk model of BY, but replace their dividend process with one that is internally consistent with stationary leverage ratios, we generate dividend strip return characteristics qualitatively similar to those of BBK.
There is a large related literature on the time variation of corporate cash flows and discount rates. While firms can choose to smooth dividends in the short run (Marsh and Merton (1986)), it is more difficult to explain why dividends are smooth in the long run (Shiller (1986)). Our paper attempts to provide an explanation for this.

Other related papers include Campbell and Shiller (1988), who find that variation in dividend yield is driven mostly by changes in discount rates. However, others have questioned the power of return predictability (Stambaugh (1999), Campbell and Yogo (2006)). Further, Larrain and Yogo (2008) find that discount rates do not need to be so volatile when focusing on the overall cash flows of the firm rather than just dividends. Our paper adds to this literature by pointing out long-run variations in dividends are significantly impacted by the capital structure decisions of the firm.

The rest of the paper is as follows. In Section 2 we consider a benchmark “Gordon growth model” framework that possesses analytic solutions, allowing us to provide the intuition for our results. In Section 3 we investigate a model that is reminiscent of the long-run risk framework of Bansal and Yaron (2004). Here, even though the term structure of excess returns for EBIT-strips are upward sloping, the term structure of excess returns for dividend-strips are downward sloping, consistent with the empirical evidence. We conclude in Section 4. Proofs are found in the Appendix.

2 Benchmark Model

Here we consider a simple benchmark model which affords analytic solutions in order to clearly identify which properties of the model generate our main results. Rather than exogenously specifying dividend dynamics, as is typically done in the asset pricing literature, we first model EBIT dynamics, and then impose capital structure dynamics in order to endogenously determine an internally consistent dividend process. In the next section, we generalize the benchmark model by investigating EBIT and pricing kernel dynamics that are better suited for explaining the equity premium puzzle (Mehra and Prescott (1985)).

2.1 EBIT Dynamics

Both EBIT and pricing kernel dynamics are specified to be log-normal:

\[
\frac{dy}{y} = g \, dt + \sigma \, dz \\
\frac{d\Lambda}{\Lambda} = -r \, dt - \theta \, dz.
\]
Hence, risk neutral dynamics follow

\[ \frac{dy}{y} = g^Q dt + \sigma d z^Q, \]  

(3)

where we have defined \( g^Q \equiv (g - \theta \sigma) \). The date-\( t \) price \( P^T(t, y_t) \) of an EBIT strip, whose payoff is the date-\( T \) EBIT flow \( y_T \) is:

\[ P^T(t, y_t) = e^{-r(T-t)} E^Q_t[y_T], \]
\[ = y_t e^{(r-g^Q)(T-t)}. \] 

(4)

Price dynamics are obtained via Ito’s lemma:

\[ \frac{dP^T(t, y_t)}{P^T(t, y_t)} = (r + \sigma \theta) dt + \sigma dz. \] 

(5)

Hence, expected excess return \((\sigma \theta)\), volatility \((\sigma)\) and Sharpe ratio \((\theta)\) are constant across all EBIT strip maturities.

The enterprise value can be determined as the claim to all EBIT strips. The price takes the form as in the Gordon growth model formula:

\[ P(y_t) = \int_t^\infty P^T(y_t) \]
\[ = \frac{y_t}{r - g^Q}. \] 

(6)

From Ito’s lemma, enterprise value dynamics follow:

\[ \frac{dP(y_t) + y_t dt}{P(y_t)} = (r + \sigma \theta) dt + \sigma dz. \] 

(7)

Hence, expected excess return \((\sigma \theta)\), volatility \((\sigma)\) and Sharpe ratio \((\theta)\) are all constant. Moreover, as is well known for the Gordon growth model, return volatility of the claim to EBIT \((\sigma)\) in equation (7) is equal to EBIT volatility in equation (1). In contrast, we will demonstrate below that the volatility of the claim to dividends (i.e., stock volatility) will be higher than dividend volatility by a factor associated with the amount of leverage taken on by the firm.

### 2.2 Dividend Dynamics

We now build upon the framework above by specifying both capital structure and cash dynamics in order to endogenously obtain dividend dynamics. We assume that at all dates-\( t \), the firm issues one-period riskless debt with face value equal to \( L P(y_t) \), implying that at date-(\( t + dt \)) it owes \( L P(y_t)(1 + r dt) \). To make this payment, the firm rolls over its debt, issuing new bonds
with present value equal to \( \bar{LP}(y(t + dt)) = \bar{LP}(y_t + dy(t)) \). Hence, after including EBIT revenues of \( y_t dt \), the firm has as available free cash flow:

\[
dF = \bar{LP}(y_t + dy(t)) + y_t dt - \bar{LP}(y_t)(1 + r dt)
\]

\[
= \left( \frac{\bar{L}}{r - g^Q} \right) \{ y_t + y_t \left[ g^Q dt + \sigma dz^Q \right] \} + y_t dt - \left( \frac{\bar{L}}{r - g^Q} \right) y_t (1 + r dt)
\]

\[
= y_t \left( 1 - \bar{L} \right) dt + \left( \frac{\sigma \bar{L}}{r - g^Q} \right) y_t dz^Q.
\]

(8)

Now, we could choose to specify that the firm pays out this free cash flow as dividend. But note that it is typical in the asset pricing literature to assume that dividend cash flows are locally deterministic, that is, not of the form \( dF = (\cdot) dt + (\cdot) dz \) in equation (8) but rather of the form \( D(t) dt \). In order to generate a “smoother” dividend process, we assume that the firm also maintains a money market account \( M_t \) where it parks its extra cash. Each period, the free cash flow is placed into the account. The account balances grow at the risk free rate. Finally, dividends are paid out from this account, and are set to equal a constant fraction \( \delta \) of cash balances: \( D dt = \delta M_t dt \). Hence, money market dynamics follow

\[
dM + \delta M dt = rM dt + dF.
\]

(9)

In summary, then, over an interval \( dt \), the amount of dividends paid is \( D_t dt = \delta M_t dt \). The level of dividends is determined from the joint-Markov processes \( (M_t, y_t) \), whose risk-neutral dynamics follow:

\[
\frac{dy}{y} = g^Q dt + \sigma dz^Q
\]

\[
dM = \left[ (r - \delta) M_t + y_t \left( 1 - \bar{L} \right) \right] dt + \left( \frac{\sigma \bar{L}}{r - g^Q} \right) y_t dz^Q.
\]

(10)

Under the historical measure, these dynamics are:

\[
\frac{dy}{y} = g dt + \sigma dz
\]

\[
dM = \left[ (r - \delta) M_t + y_t \left( 1 - \bar{L} \right) + \frac{\sigma \theta \bar{L}}{r - g^Q} \right] dt + \left( \frac{\sigma \bar{L}}{r - g^Q} \right) y_t dz.
\]

(11)

As demonstrated in the Appendix, the date-\( t \) price \( V^T(t, M_t, y_t) \) of a dividend strip that has claim to the date-\( T \) dividend payment \( (\delta M_T) \) equals

\[
V^T(t, M_t, y_t) = e^{-(T-t)} \mathbb{E}_t^Q [\delta M_T]
\]

\[
= \delta M_t e^{-\delta(T-t)} + \delta(1 - \bar{L}) y_t e^{-\delta(T-t)} \left( \frac{e^{(T-t)(\delta + g^Q - r)} - 1}{\delta + g^Q - r} \right).
\]

(12)
Expected excess returns are:

\[
\frac{1}{dt} \mathbb{E}_t \left[ \frac{dV^T(t, M_t, y_t)}{V^T(t, M_t, y_t)} - r \right] dt = -\frac{1}{dt} \mathbb{E}_t \left[ \frac{dV^T(t, M_t, y_t)}{V^T(t, M_t, y_t)} \right] dt \\
= \theta \sigma \left[ \frac{\theta \rho + (1 - \rho)}{\rho - g^Q} + \left( \frac{\rho + (1 - L)}{\rho - g^Q} \right) \right].
\]  

(13)

Hence, expected returns depend on the current value of \( \left( \frac{M_t}{y_t} \right) \). To get a sense of a typical value, we use Ito’s lemma to identify the dynamics of \( \left( \frac{M_t}{y_t} \right) \):

\[
d \left( \frac{M_t}{y_t} \right) = \left( \frac{M_t}{y_t} \right) \left[ (r - \delta - g + \sigma^2) + 1 - \rho + \frac{\sigma \rho}{\rho - g^Q} (\theta - \sigma) \right] dt + \sigma \left[ \frac{\rho}{\rho - g^Q} - \frac{M_t}{y_t} \right] dz.
\]

Since the coefficient \( (r - \delta - g + \sigma^2) \) multiplying \( \left( \frac{M_t}{y_t} \right) \) in the drift is negative in our calibrations, this process is mean-reverting toward its long-run average. By setting the expected change to zero, we find the long-run average cash-to-earnings ratio to be

\[
\overline{M_y} = \left( \frac{1 - \rho + \sigma \rho (\theta - \sigma)}{(r - g^Q)} \right).
\]

(14)

From Ito’s Lemma, we find strip return volatility to be

\[
\sqrt{\frac{1}{dt} \mathbb{E}_t \left[ \frac{(dV^T(t, M_t, y_t))^2}{V^T(t, M_t, y_t)} \right]} = \sigma \left[ \frac{\rho}{\rho - g^Q} + \left( \frac{\rho + (1 - L)}{\rho - g^Q} \right) \right].
\]

(15)

Dividing equation (13) by equation (15), we find that the Sharpe ratio is a constant \( \theta \) across all maturities.

To calibrate, we choose \( r = 0.06, \sigma = 0.08, \theta = 0.42, g^Q = 0.02, \rho = 0.33 \) and \( \delta = 1 \). Note that this generates a reasonable “price-dividend ratio” of \( \left( \frac{1}{r - g^Q} \right) = 25 \), and a long-term average cash-to-EBIT ratio of \( \overline{M_y} \approx 0.9 \), which implies a cash position of about 3.5% of firm value. As shown in Figure 1, this calibration generates a downward sloping term structure of dividend strip excess returns, consistent with the empirical findings of van Binsbergen, Brandt and Koijen (2011) and van Binsbergen, Hueskes, Koijen and Vrugt (2011). We find this result to be robust across a wide range of parameter estimates.
Figure 1: Expected excess returns for dividend strips as a function of maturity in the benchmark model. Parameter values are $r = 0.06$, $\sigma = 0.08$, $\theta = 0.42$, $g^Q = 0.02$, $\mathcal{L} = 0.33$ and $\delta = 1$.

2.3 Equity Dynamics

While the prices of dividend strips are quite sensitive to the payout rate $\delta$, the price of the equity claim is independent of $\delta$:

$$ V(M_t, y_t) = \int_t^\infty dTV^T(t, M_t, y_t) $$

$$ = M_t + (1 - \mathcal{L}) \left( \frac{y_t}{r - g^Q} \right). \quad (16) $$

This captures the intuitive notion that the present value of equity is the sum of cash position $M_t$ and the fraction $(1 - \mathcal{L})$ of enterprise value $\left( \frac{y_t}{r - g^Q} \right)$, with debtholders owning the remaining fraction $\mathcal{L} \left( \frac{y_t}{r - g^Q} \right)$. 
Expected excess returns on the stock are:

\[
\frac{1}{dt}E_t \left[ \frac{dV(M_t, y_t) + \delta M_t \ dt}{V(M_t, y_t)} - r \ dt \right] = -\frac{1}{dt}E_t \left[ \frac{dV(M_t, y_t) \ d\Lambda}{V(M_t, y_t) \ \Lambda} \right] = \theta \sigma \left[ \frac{\left( \frac{1}{r - g^Q} \right)}{M_t \ y_t + \left( \frac{1}{r - g^Q} \right)} \right] \quad (17)
\]

From Ito’s Lemma, we find stock return volatility to be

\[
\sqrt{\frac{1}{dt}E_t \left[ \left( \frac{dV(M_t, y_t)}{V(M_t, y_t)} \right)^2 \right]} = \sigma \left[ \frac{\left( \frac{1}{r - g^Q} \right)}{M_t \ y_t + \left( \frac{1}{r - g^Q} \right)} \right]. \quad (18)
\]

Dividing equation (17) by equation (18), we find that the Sharpe ratio is a constant \( \theta \) independent of the ratio \( \frac{M_t}{y_t} \).

We introduce the notation \((\mu^V - r)\) for the excess return in equation (17) and \(\sigma^V\) for return volatility in equation (18). Note that for the benchmark calibration, \(\left( \frac{M_t}{y_t} \right) = 0.9\) and \(\left( \frac{1}{r - g^Q} \right) = 25\). This implies excess returns and stock volatility are approximately

\[
\mu^V - r \approx \frac{\sigma \theta}{1 - L}, \quad (19)
\]
\[
\sigma^V \approx \frac{\sigma}{1 - L}. \quad (20)
\]

The notion of leverage is transparent in equations (19) and (20): As \( L \rightarrow 0 \) (and the cash position set to zero), excess returns and volatilities revert to the unleveraged returns on EBIT. However, as \( L \) approaches unity, excess stock returns and stock volatility become very large.

### 2.4 Leverage Ratios

While it is more standard to define leverage as the ratio of debt to (debt plus equity), in our framework both outstanding debt and the money market account are risk-free. Thus, it is more natural here to define leverage as the ratio of net-debt to (net-debt plus equity), where net-debt is defined as market debt \(\left( \frac{L y_t}{r - g^Q} \right)\) minus cash position \(M_t\). Hence, using equity as in equation (16), we define date-t leverage to be:

\[
\text{Lev}_t = \frac{L \left( \frac{y_t}{r - g^Q} \right) - M_t}{\frac{y_t}{r - g^Q}} = L - (r - g^Q) \left( \frac{M_t}{y_t} \right). \quad (21)
\]

Thus, leverage dynamics basically follow the same stationary process as does the cash-to-EBIT process in equation (14).
2.5 Stock Volatility vs. Dividend Volatility

One well-documented empirical fact is that stock return volatility is significantly higher than dividend volatility. (Shiller (1981), LeRoy and Porter (1981)). While firms can choose to smooth dividends in the short run (Marsh and Merton (1986), it is more difficult to do so in the long run (Shiller (1986)). Here we demonstrate that when one specifies dividend dynamics consistent with stationary leverage ratios, long-run dividend volatility is automatically lower than stock return volatility.

To see this, we define long-run expected dividend growth rate and growth rate volatility as

\[
g_D \equiv \lim_{T \to \infty} \left( \frac{1}{T} \right) \log \left[ \frac{E_0[D_T]}{D_0} \right] \quad (22)
\]

\[
\sigma_D \equiv \sqrt{\lim_{T \to \infty} \left( \frac{1}{T} \right) \log \left[ \frac{\text{Var}_0[D_T]}{(E_0[D_T])^2} \right]} \quad (23)
\]

where we recall that \( D_T = \delta M_T \). We find that

\[
g_D = g \quad (24)
\]

\[
\sigma_D = \sigma. \quad (25)
\]

That is, due to the fact that dividends and EBIT are cointegrated, they have the same long run behavior, as can be seen by comparing equation (1) with equations (24) and (25). Even so, note that the expected return and volatility of the the claim of dividends (i.e., stock returns) are higher than the expected return and volatility for the return on EBIT by a leverage factor \( \left( \frac{1}{1 - L} \right) \), as can be seen by comparing equation (7) with equations (19) and (20).

The intuition for this result is that at any date-\( t \), equity is a leveraged claim, and thus is compensated for its riskiness. However, the long-term equity holder effectively has her portfolio position “managed” by the firm in that they dilute her position when the firm does well by issuing additional debt and paying out a portion of those funds as current dividends (and vice versa if the firm does poorly.)

2.6 General Equilibrium Implications

While this paper focuses on partial equilibrium analysis in that the pricing kernel is specified exogenously, it is straightforward to reinterpret the ‘unleveraged’ EBIT process in equation (1)
as an exogenous output process in an endowment economy where the representative agent owns the stock, the corporate bond, and is responsible for the liability implied in the money market account. With this interpretation, consumption and dividends are cointegrated in the long run, and thus share the same long run expected growth rate as in BY and CC. However, the results of this section demonstrate the importance of specifying the dividend process in a manner quite different than that assumed by BY and CC.

3 Long Run Risk Model

Here we investigate EBIT and pricing kernel dynamics that are closely related to those considered by Bansal and Yaron (BY, 2004) in their one-channel “long-run risk model”. BY demonstrate that this model can capture high expected returns, volatility and Sharpe ratios of stocks even with moderate levels of risk aversion. However, rather than exogenously specifying dividend dynamics as BY did, here we follow the approach of the previous section and determine dividend dynamics endogenously. We acknowledge that there are few new qualitative intuitions to be gained here compared to those in the benchmark model. Therefore, the results will be given with little discussion. However, the model’s quantitative implications are interesting. In particular, the model generates a term structure of expected returns, volatilities and Sharpe ratios that are upward sloping for EBIT-strips, but downward sloping for dividend strips.

3.1 EBIT Dynamics

We specify EBIT dynamics \( \frac{d y}{y} \) to have a small but persistent shock to its expected growth \( x \). Shocks to \( x \) are associated with a large market price of risk \( \theta_2 \). In particular, EBIT dynamics and the pricing kernel follow:

\[
\begin{align*}
\frac{d y}{y} & = (g + x) \, dt + \sigma_y \, dz_1 \\
\frac{d x}{x} & = -\kappa x \, dt + \sigma_{x_1} \, dz_1 + \sigma_{x_2} \, dz_2 \\
\frac{d \Lambda}{\Lambda} & = -r \, dt - \theta_1 \, dz_1 - \theta_2 \, dz_2 
\end{align*}
\]

(26) \hfill (27) \hfill (28)

Hence, risk neutral dynamics follow

\[
\begin{align*}
\frac{d y}{y} & = (g^Q + x) \, dt + \sigma_y^Q \, dz_1^Q \\
\frac{d x}{x} & = \kappa (\bar{x}^Q - x) \, dt + \sigma_{x_1}^Q \, dz_1^Q + \sigma_{x_2}^Q \, dz_2^Q. 
\end{align*}
\]

(29) \hfill (30)

where we have defined \( g^Q \equiv (g - \sigma_y \theta_1), \bar{x}^Q \equiv -\left( \frac{\theta_1 \sigma_{x_1} + \theta_2 \sigma_{x_2}}{\kappa} \right) \).
The date-\( t \) price \( P^T(t, x_t, y_t) \) of the security whose payoff is the date-\( T \) EBIT flow \( y_T \) is:

\[
P^T(t, x_t, y_t) = e^{-r(T-t)}E^Q_t[y_T].
\] (31)

The solution takes the exponential affine form:

\[
P^T(t, x_t, y_t) = y_t e^{A(T-t)+B(T-t)x_t}.
\] (32)

where the deterministic functions \((A(\tau), B(\tau))\) are derived in the Appendix.

Expected excess returns on the EBIT strips satisfy

\[
\frac{1}{dt}E \left[ \frac{dP^T(t, x_t, y_t)}{P^T(t, x_t, y_t)} - r \right] = -\frac{1}{dt}E \left[ \frac{d\Lambda \, dP^T(t, x_t, y_t)}{\Lambda \, P^T(t, x_t, y_t)} \right] = \theta_1 [\sigma_y + B(T-t)\sigma_{s1}] + \theta_2 [B(T-t)\sigma_{s2}]
\] (33)

EBIT strip volatility is

\[
\sigma^P = \sqrt{\frac{1}{dt} \left( \frac{dP^T(t, x_t, y_t)}{P^T(t, x_t, y_t)} \right)^2} = \sqrt{(\sigma_y + B(T-t)\sigma_{s1})^2 + (B(T-t)\sigma_{s2})^2}.
\] (34)

Hence, Sharpe ratios follow

\[
\text{Sh}^P = \frac{\theta_1 [\sigma_y + B(T-t)\sigma_{s1}] + \theta_2 [B(T-t)\sigma_{s2}]}{\sqrt{(\sigma_y + B(T-t)\sigma_{s1})^2 + (B(T-t)\sigma_{s2})^2}}.
\] (35)

We calibrate this model using the parameter values \( g = 0.05, \sigma_y = 0.05, \kappa = 0.3, \sigma_{s1} = 0.01, \sigma_{s2} = 0.02, r = 0.06, \theta_1 = 0.1, \theta_2 = 0.5, \delta = 1, \overline{T} = 0.4 \). As noted in van Binsbergen, Brandt and Koijen (2011), if one models cash flows as above, which is very similar to that used by BY, then this long-run risk model generates an upward sloping term structure of expected returns and volatilities, as shown in Figure 2.

The enterprise value of the firm is equal to the present value of the claim to all EBIT strips:

\[
P(x_t, y_t) = y_t \int_t^\infty dT \, e^{A(T-t)+B(T-t)x_t}.
\] (36)

As noted by Bansal and Yaron (2004), this can be well-approximated by a log-linear approximation, leading to

\[
P(x_t, y_t) \approx y_t \, e^{A+B \, x_t},
\] (37)
where the coefficients \((A, B)\) are determined in the Appendix. In this “one-channel” model, the expected return, volatility, and Sharpe ratio of the claim to EBIT are all constant using this approximation:

\[
\begin{align*}
(\mu^P - r)_{BY} & \approx \theta_1 (\sigma_y + B\sigma_{x_1}) + \theta_2 (B\sigma_{x_2}). \\
\sigma^P_{BY} & \approx \sqrt{(\sigma_y + B\sigma_{x_1})^2 + (B\sigma_{x_2})^2}. \\
Sh^P_{BY} & \approx \frac{\theta_1 (\sigma_y + B\sigma_{x_1}) + \theta_2 (B\sigma_{x_2})}{\sqrt{(\sigma_y + B\sigma_{x_1})^2 + (B\sigma_{x_2})^2}}.
\end{align*}
\]

\[(38) \quad (39) \quad (40)\]

### 3.2 Dividend Dynamics

Assume that at all dates-\(t\), the firm issues one-period riskless debt with face value equal to \(L y_t e^{A+Bx_t}\), implying that at date-(\(t + dt\)) it owes \(L y_t e^{A+Bx_t} (1 + r dt)\). To make this payment, the firm rolls over its debt, issuing new bonds with present value equal to \(L y_{t+dt} e^{A+Bx_{t+dt}} = L(y_t + dy_t) e^{A+B(x_{t} + dx_t)}\). Hence, after including EBIT revenues of \(y_t dt\), the firm has as available...
Figure 3: Volatility of EBIT strip returns in the long-run risk model.

free cash flow:

\[
\begin{align*}
    dF &= \mathcal{L}(y_t + dy_t)e^{A + B(x_t + dx_t)} + y_t \, dt - \mathcal{L}y_t e^{A + Bx_t}(1 + r \, dt) \\
    &= y_t \left[ 1 + \mathcal{L}e^{A + Bx_t} \left( B\kappa(xQ - x) + \frac{1}{2}B^2(\sigma_{x_1}^2 + \sigma_{x_2}^2) + gQ + x + B\sigma_y \sigma_{x_1} - r \right) \right] \, dt \\
    &\quad + \mathcal{L}y_t e^{A + Bx_t} \left[ (B\sigma_{x_1} + \sigma_y) \, dz_Q + B\sigma_{x_2} \, dx_Q \right] \\
\end{align*}
\] (41)

We assume that the firm places the free cash flow into a money market account \( M_t \), and these balances grow at the risk free rate. Finally, dividends are paid out from this account at the constant rate of \( D \, dt = \delta M_t \, dt \). Hence, money market dynamics follow

\[
\begin{align*}
    dM + \delta M \, dt &= rM \, dt + dF. \\
\end{align*}
\] (42)
Figure 4: Sharpe ratios on EBIT strip returns in the long-run risk model.

In sum, then, risk-neutral firm dynamics follow:

\[ \frac{dy}{y} = (g^Q + x) \, dt + \sigma_y \, dz_1^Q \]
\[ dx = \kappa(x^Q - x) \, dt + \sigma_{x_1} \, dz_1^Q + \sigma_{x_2} \, dz_2^Q. \]
\[ dM = y_t \left[ 1 + \mathcal{L}_y e^{A^t + Bx_t} \left( B\kappa(x^Q - x) + \frac{1}{2} B^2 (\sigma_{x_1}^2 + \sigma_{x_2}^2) + g^Q + x + B\sigma_y \sigma_{x_1} - r \right) \right] \, dt \]
\[ + (r - \delta) M_t \, dt + \mathcal{L}_y e^{A^t + Bx_t} \left[ (B\sigma_{x_1} + \sigma_y) \, dz_1^Q + B\sigma_{x_2} \, dz_2^Q \right]. \]

The date-\( t \) claim to the dividend paid out at date-\( T \) satisfies

\[ V^T(t, M_t, x_t, y_t) = e^{-r(T-t)} E_t^Q [\delta M_T] \]
\[ = \delta M_t e^{-\delta(T-t)} + \delta y_t e^{-\delta(T-t)} N(T-t, x_t), \]

where the function \( N(\tau, x) \) satisfies the partial differential equation

\[ N_\tau = (\delta + g^Q + x - r) N + (\kappa x^Q - \kappa x + \sigma_y \sigma_{x_1}) N_x + \frac{1}{2} (\sigma_{x_1}^2 + \sigma_{x_2}^2) N_{xx} + 1 \]
\[ + \mathcal{L}_y e^{A^t + Bx_t} \left( B\kappa(x^Q - x) + \frac{1}{2} B^2 (\sigma_{x_1}^2 + \sigma_{x_2}^2) + g^Q + x + B\sigma_y \sigma_{x_1} - r \right). \]
The boundary condition is \( N(\tau = 0, x) = 0 \). While we were unable to find an analytic solution, the function \( N(\tau, x) \) is well-approximated using a log-linear type of approximation:

\[
N(\tau, x) \approx \alpha_4 \left[ e^{Jx} - e^{G(\tau) + H(\tau)x} \right],
\]

where the constants \((\alpha_4, J)\) and deterministic functions \(G(\tau), H(\tau)\) are given in the Appendix.

It is worth noting that \( G(\tau = 0) = 0 \) and \( H(\tau = 0) = J \), which guarantees that the boundary condition \( N(\tau = 0, x) = 0 \) is satisfied. As a check on the accuracy of the approximation, we note that summing up the value of dividend strips across maturities provides a near perfect estimate of the value of the stock (determined below).

Expected excess returns for the dividend strips satisfy:

\[
(\mu^V - r) \equiv \frac{1}{dt} \mathbb{E}_t \left[ \frac{dV^T(t, M_t, x_t, y_t)}{V^T(t, M_t, x_t, y_t)} - r dt \right] = -\frac{1}{dt} \mathbb{E}_t \left[ \frac{dV^T(t, M_t, x_t, y_t)}{V^T(t, M_t, x(T), y_t)} \Lambda \right] = \theta_1 \left[ \mathbb{Le}^{A+Bx_t}(B\sigma_{x_1} + \sigma_y) + N(\tau, x)\sigma_y + N_x(\tau, x)\sigma_{x_1} \right] + \theta_2 \left[ \mathbb{Le}^{A+Bx_t}(B\sigma_{x_2} + N_x(\tau, x)\sigma_{x_2}) \right].
\]

Hence, expected returns depend on the current value of \( \frac{M_t}{y_t} \). To get a sense of a typical value, we use Ito’s lemma to identify the P-dynamics of \( \frac{M_t}{y_t} \):

\[
d \left( \frac{M_t}{y_t} \right) = \left[ \frac{M_t}{y_t} \right] \left( r - \delta - g - x + \sigma_y^2 \right) + 1 + \mathbb{Le}^{A+Bx_t} \left[ -B\kappa x_t - \frac{1}{2}B^2(\sigma_{x_2}^2 + \sigma_y^2) + g + x - \sigma_y^2\sigma_{x_1} - r \right] dt + \mathbb{Le}^{A+Bx_t} \left[ (B\sigma_{x_1} + \sigma_y)dz_1 + B\sigma_{x_2} dz_2 \right] - \frac{M_t}{y_t} \sigma_y dz_1.
\]

Setting the expected change to zero, we find the average cash-to-earnings ratio to be

\[
\left( \frac{M}{y} \right) = 1 + \mathbb{Le}^{A+Bx_t} \left[ x_t(1-B\kappa) - \frac{1}{2}B^2(\sigma_{x_1}^2 + \sigma_y^2) + g - \sigma_y^2 - r \right].
\]

From Ito’s Lemma, we find strip return volatility to be

\[
\sigma^V \equiv \sqrt{\frac{1}{dt} \mathbb{E}_t \left[ \left( \frac{dV^T(y_t)}{V^T(y_t)} \right)^2 \right]} = \sqrt{\left[ \mathbb{Le}^{A+Bx_t}(B\sigma_{x_1} + \sigma_y) + N(\tau, x)\sigma_y + N_x(\tau, x)\sigma_{x_1} \right]^2 + \left( \mathbb{Le}^{A+Bx_t}(B\sigma_{x_2} + N_x(\tau, x)\sigma_{x_2}) \right)^2}.\]
Dividing equation (49) by equation (52), we find that the Sharpe ratio depends on both maturity and $x$:

$$\text{Sh}^V = \frac{\mu^V - r}{\sigma^V}$$

$$= \theta_1 \left[ \mathcal{L}e^{A+Bx_1}(B\sigma_{x_1} + \sigma_y) + N(\tau, x)\sigma_y + N_x(\tau, x)\sigma_{x_1} \right] + \theta_2 \left[ \mathcal{L}e^{A+Bx_2}B\sigma_{x_2} + N_x(\tau, x)\sigma_{x_2} \right] \sqrt{\left( \mathcal{L}e^{A+Bx_1}(B\sigma_{x_1} + \sigma_y) + N(\tau, x)\sigma_y + N_x(\tau, x)\sigma_{x_1} \right)^2 + \left( \mathcal{L}e^{A+Bx_2}B\sigma_{x_2} + N_x(\tau, x)\sigma_{x_2} \right)^2}.$$  

Figure 5: Excess returns on EBIT strips in the long-run risk model.

Using the calibration discussed above, we find for a wide range of values of $\delta$, the term structure of expected returns and volatility is downward sloping, but Sharpe ratios are relatively flat.

4 Conclusion

In the long run, EBIT and dividends must be cointegrated, and thus must share long run growth rates and volatility. But in the short run, it is more accurate to think of dividends as a leveraged security. We explain this puzzle by endogenously deriving dividend dynamics
when the firm maintains stationary leverage ratios. We argue that when a firm rebalances its
debt levels over time to maintain a stationary leverage process, shareholders are being forced
to divest (invest) when the firm does well (poorly). In turn, this explains two “puzzling”
properties of asset prices. First, we show that this approach generates stock return volatility
that is significantly higher than long-term dividend volatility, consistent with Shiller (1981)
and LeRoy and Porter (1981). Second, we show that properly accounting for the leveraged
nature of dividend dynamics automatically generates a term structure of expected returns and
volatilities for dividend strips that are decreasing in maturity, consistent with the empirical
findings of van Binsbergen, Brandt and Koijen (BBK, 2011) and van Binsbergen, Hueskes,
Koijen and Vrugt (2011).

Our ‘structural approach’ to dividend dynamics is able to explain the peculiar fact that
dividends are “leveraged” in the short run but cointegrated with EBIT in the long run. In
contrast, the “reduced form” approach of Abel (1999) ignores the fact that dividends and
consumption are cointegrated in the long run. We can show that this approach generates a

Figure 6: Volatility of EBIT strip returns in the long-run risk model.
significant bias in the price dividend ratio.\footnote{Further, Abel’s (1999) approach makes it difficult to simultaneously calibrate dividend and consumption dynamics to historical values since in his model their growth rates differ.} This can be intuitively understood as follows. Above, we have emphasized that dividends have two components – one component due to EBIT minus interest, and one component due to changes in capital structure. These changes in capital structure imply that management is diluting/increasing the shareholder’s claim even if the shareholder follows a passive no-trade strategy. We can show that Abel’s reduced-form approach is basically equivalent to considering an agent that reinvests that portion of the dividend that is due to capital structure changes back into the equity. In this case, her dividend growth rate will be higher than the consumption growth rate, as Abel (1999) specifies. Further, both the excess expected return and return volatility are the same for the actual equity process and this managed process. However, the price-dividend is biased upward precisely because a portion of the dividend is being reinvested, whereas the price is unaffected.

Figure 7: Sharpe ratios on EBIT strip returns in the long-run risk model.
5 Appendix

5.1 Proof of Equation (12)

The date-t price \( V^T(t, M_t, y_t) \) of a dividend strip that has claim to the date-T dividend payment \( (\delta M_T) \) equals

\[
V^T(t, M_t, y_t) = e^{-r(T-t)}E_t^Q [\delta M_T].
\]  

(54)

Define \( Q_t \equiv e^{-(\gamma-\delta)t}M_t \). Then, using Itô’s lemma and equation (10) we find

\[
dQ = e^{-(\gamma-\delta)t} \left[ y_t (1 - L) \ dt + \left( \frac{\sigma L}{r - \delta} \right) y_t \ dz^Q \right].
\]  

(55)

Formally integrating this equation, and then substituting back for \( M_t \), we find

\[
M_T = M_te^{(\gamma-\delta)(T-t)} + \int_t^T e^{(\gamma-\delta)(T-s)}y_s(1 - L) \ ds + \int_t^T e^{(\gamma-\delta)(T-s)} \left( \frac{\sigma L}{r - \delta} \right) y_s \ dz^Q.
\]  

(56)

Then using \( E_t^Q \left[ dz^Q \right] = 0 \) for \( s > t \) and \( E_t^Q [y_s] = y_te^{\delta Q(s-t)} \), we find

\[
E_t^Q [M_T] = M_te^{(\gamma-\delta)(T-t)} + \int_t^T e^{(\gamma-\delta)(T-s)}y_te^{\delta Q(s-t)}(1 - L) \ ds
\]

\[= M_te^{(\gamma-\delta)(T-t)} + (1 - L)y_te^{(\gamma-\delta)(T-t)} \left( e^{(\delta + \delta Q - \gamma)(T-t)} - 1 \right). \]

(57)

Plugging this result into equation (54) gives equation (12).

5.2 Proof of Equation (32)

From equation (31), we see that \( e^{-rt}P^T(t, x_t, y_t) \) is a \( Q \)-martingale, implying that

\[
0 = E_t^Q \left[ d \left( e^{-rt}P^T(t, x_t, y_t) \right) \right]
\]

\[= -rP^t + P_t + (g^Q + x)yP_y + \kappa(x^Q - x)Px + \frac{1}{2}y^2 \sigma_y^2 y_\sigma^2 + \frac{1}{2} \left( \sigma_{x_1}^2 + \sigma_{x_2}^2 \right)^2 P_{xx} + \sigma_y \sigma_{x_1} yP_{x_1} \]

Since the state vector dynamics are affine, it is well known (see, for example, Duffie and Kan (1996)) that the solution takes the exponential affine form:

\[
P^T(t, x_t, y_t) = y_t e^{A(T-t) + B(T-t)x_t}.
\]  

(59)

Plugging this functional form into equation (??) and then collecting terms linear and independent of \( x \), we find that the deterministic functions \( A(\tau) \) and \( B(\tau) \) (where \( \tau \equiv (T - t) \)) satisfy the Riccati equations:

\[
B_\tau = 1 - \kappa B
\]

(60)

\[
A_\tau = -r + B(\sigma_y \sigma_{x_1} + \kappa x^Q) + g^Q + \frac{1}{2} B^2 \left( \sigma_{x_1}^2 + \sigma_{x_2}^2 \right),
\]

(61)
with boundary conditions $B(0) = 0, A(0) = 0$. The solutions are

$$B(\tau) = \frac{1}{\kappa} (1 - e^{-\kappa \tau})$$

$$A(\tau) = (g^Q - r) \tau + (\sigma_y \sigma_{x_1} + \kappa \overline{v}^Q) B(\tau) + \frac{1}{2} (\sigma_{x_1}^2 + \sigma_{x_2}^2) (\tau - B(\tau) - \frac{\kappa}{2} B^2(\tau)) \quad (62)$$

### 5.3 Enterprise Value in Long Run Risk Economy

Here we derive the constants $A$ and $B$ used in equation (37). Enterprise value can be determined via

$$P(x_t, y_t) = E_t^Q \left[ \int_t^\infty ds e^{-r(s-t)} y_s \right]. \quad (63)$$

This implies that $(e^{-rt} P(x_t, y_t) + \int_t^t ds e^{-rs} y_s)$ is a $Q$-martingale. Therefore

$$0 = -r + g^Q + x + B\kappa(\overline{v}^Q - x) + \frac{1}{2} B^2 (\sigma_{x_1}^2 + \sigma_{x_2}^2) + B\sigma_y \sigma_{x_1} + e^{-A-Bx} \quad (64)$$

Taylor expanding $e^{-A-Bx} \approx e^{-A-B\overline{v}^Q} (1 - B (x - \overline{v}^Q))$, and then collecting terms linear and independent of $x$ gives two equations:

$$0 = 1 - \kappa B - Be^{-A-B\overline{v}^Q} \quad (65)$$

$$0 = -r + g^Q + x + B\kappa\overline{v}^Q + \frac{1}{2} B^2 (\sigma_{x_1}^2 + \sigma_{x_2}^2) + B\sigma_y \sigma_{x_1} + e^{-A-B\overline{v}^Q} (1 + B\overline{v}^Q). \quad (66)$$

We can rewrite equation (65) as

$$e^{-A-B\overline{v}^Q} = \frac{1 - \kappa B}{B}, \quad (67)$$

which we can use in equation (66) to eliminate its dependence on $A$, making it easy to identify the value of $B$, and in turn, $A$.

### 5.4 Proof of Equation (48)

First let us reexpress equation (47) as

$$N_r = \alpha_0 N + (\alpha_1 - \kappa x) N_s + \frac{1}{2} N_{ss} + 1 + \mathcal{L} e^{A+Bx} (\alpha_2 + x(1 - B\kappa)), \quad (68)$$

where we have defined

$$\alpha_0 \equiv \delta + g^Q + x - r$$

$$\alpha_1 \equiv (\kappa \overline{v}^Q + \sigma_y \sigma_{x_1})$$

$$\sigma_{x}^2 \equiv (\sigma_{x_1}^2 + \sigma_{x_2}^2)$$

$$\alpha_2 \equiv B\kappa\overline{v}^Q + \frac{1}{2} B^2 (\sigma_{x_1}^2 + \sigma_{x_2}^2) + g^Q + B\sigma_y \sigma_{x_1} - r. \quad (69)$$
First let us look at the “general solution” to
\[ N_\tau = \alpha_0 N + (\alpha_1 - \kappa x) N_x + \frac{1}{2} N_{xx}. \]  
(70)

It is well known from Duffie and Lando (1996) that the solution is exponential affine:
\[ N_{GS}(x,\tau) = e^{G(\tau) + xH(\tau)}. \]  
(71)

Plugging this form into equation (70), collecting terms linear and independent of \( x \), and specifying initial conditions as \( G(\tau = 0) = 0, H(\tau = 0) = J \) (\( J \) will be determined judiciously below), we find
\[ H(\tau) = \frac{1}{\kappa} + (J - \frac{1}{\kappa}) e^{-\kappa \tau}. \]
\[ G(\tau) = \alpha_0 \tau + \alpha_1 \left[ \tau + (J - \frac{1}{\kappa})(1 - e^{-\kappa \tau}) \right] + \frac{\sigma^2}{2\kappa^2} \left[ \tau + 2(J - \frac{1}{\kappa})(1 - e^{-\kappa \tau}) + \frac{\kappa}{2} (J - \frac{1}{\kappa})^2 (1 - e^{-2\kappa \tau}) \right]. \]

Now, we look for a “particular solution” to an approximate, linearized version of equation (47):
\[ N_\tau \approx \alpha_0 N + (\alpha_1 - \kappa x) N_x + \frac{1}{2} N_{xx} + 1 + L e^A (1 + Bx) (\alpha_2 + x(1 - B\kappa)), \]
\[ \approx \alpha_0 N + (\alpha_1 - \kappa x) N_x + \frac{1}{2} N_{xx} + 1 + L e^A ((1 + Bx)\alpha_2 + x(1 - B\kappa)). \]

(72)

We look for a solution of the form \( N(\tau, x) = \alpha_4 + \alpha_5 x \). Collecting terms linear and independent of \( x \) we find
\[ 0 = \alpha_4 + \alpha_6 \alpha_5 - \alpha_5 \kappa + L e^A [(1 - \kappa B) + \alpha_2 B] \]
\[ 0 = \alpha_4 \alpha_5 + \alpha_4 \alpha_1 + 1 + L e^A \alpha_2. \]

These two equations uniquely determine \( (\alpha_4, \alpha_5) \).

We then combine the general solution and particular solution and look for a solution of the form
\[ N(\tau, x) = \alpha_6 \left[ e^{Jx} - e^{G(\tau) + xH(\tau)} \right]. \]  
(73)

Using a linear approximation, \( \alpha_6 e^{Jx} \approx \alpha_6 (1 + Jx) = \alpha_6 + \alpha_6 Jx \), we choose \( \alpha_6, J \) so that \( \alpha_6 + \alpha_6 Jx = \alpha_4 + \alpha_5 x \). Hence, we choose
\[ J = \frac{\alpha_5}{\alpha_4} \]
\[ \alpha_6 = \alpha_4. \]  
(74)