**Affine Models for Credit Risk Analysis**

C. GOURIEROUX  
CREST, CEPREMAP, and University of Toronto  

A. MONFORT  
CNAM and CREST  

V. POLIMENIS  
University of California, Riverside  

**ABSTRACT**  
Continuous-time affine models have been recently introduced in the theoretical financial literature on credit risk. They provide a coherent modeling, rather easy to implement, but have not yet encountered the expected success among practitioners and regulators. This is likely due to a lack of flexibility of these models, which often implied poor fit, especially compared to more ad hoc approaches proposed by the industry. The aim of this article is to explain that this lack of flexibility is mainly due to the continuous-time assumption. We develop a discrete-time affine analysis of credit risk, explain how different types of factors can be introduced to capture separately the term structure of default correlation, default heterogeneity, correlation between default, and loss-given-default; we also explain why the factor dynamics are less constrained in discrete time and are able to reproduce complicated cycle effects. These models are finally used to derive a credit-VaR and various decompositions of the spreads for corporate bonds or first-to-default basket.  

**KEYWORDS:** affine model, affine process, CaR process, credit risk, loss-given-default, stochastic discount factor, term structure, through-the-cycle, WAR process

**1 INTRODUCTION**  
In the general strategy followed by the Basle Committee for monitoring the risk included in their financial investments, the banks have to compute a credit-VaR both for regulation and internal control [Basel Committee on Banking Supervision (2001)]. This Value-at-Risk (VaR) defines the amount of reserve required to hedge...
the risk of their credit portfolios, including retail credits (consumer credits, mortgages, revolving credits, over-the-counter corporate loans), corporate bonds and credit derivatives, like mortgage-backed securities and CDOs. This article focuses on portfolios of corporate credits and credit derivatives that are traded on bond markets, and on the introduction of appropriate models for describing the associated risks. These models have to incorporate borrowers’ heterogeneity with both industry-wide and firm-specific effects. This heterogeneity is relevant for the default intensity (loss-given-default) patterns at various maturities, but also for the so-called default correlation (loss-given-default correlation) which accounts for what regulators call concentration risk.

In order to deal with this heterogeneity problem we propose a large class of models, which is flexible to capture time dependence, age dependence, and sector dependence, for default intensity, loss-given-default, and various correlations. Another important feature of the models proposed here is that their tractability in terms of computation of the Treasury bonds, corporate bonds, first-to-default baskets term structures, spread decomposition, credit-VaR and derivative prices. Finally, we adopt discrete-time affine specifications, which are becoming more and more popular in the context of credit risk, compared to the continuous-time affine literature, recently elaborated in [Duffie, Filipovic, and Schachermayer (2003)].

The discrete-time approach has three main advantages. The regulator and some models (for retail credit and analysis of workout loss-given-default on corporates) prefer to use a daily or monthly discrete time framework. A discrete-time approach is numerically easier to implement, since it requires the solution of recursive equations with a daily (or monthly) time unit. By comparison, the continuous-time affine specifications require the solution of multidimensional differential Riccati equations, which is computationally more complicated. Moreover, contrary to the initial continuous-time model, the approximate recursions are generally not compatible with no-arbitrage opportunity. The set of continuous-time affine dynamics is rather restricted, as a consequence of a time coherency condition. Loosely speaking, the continuous-time dynamics are built from Ornstein-Uhlenbeck processes, Cox-Ingersoll-Ross processes (or their multivariate extensions called Wishart processes), and special bifurcation processes. On the contrary, the discrete-time specifications do not require time coherency for periods within the day (or the month). This increases considerably the type of admissible dynamics, including recursive systems, long memory, or sophisticated nonlinear effects. This is especially important in the context of default, where default probability and expected loss-given-default are procyclical.

1 See, e.g., Bertholon, Monfort, and Pegoraro (2003), Gourieroux and Monfort (2006), Singleton (2006), or in the context of Treasury bonds term structure, see, e.g., Polimenis (2001), Dai, Singleton, and Yang (2005), Gourieroux, Sufana (2003), and Dai, Le, and Singleton (2006), and Monfort, and Pegoraro (2005a,b).
In Section 2 we introduce the default arrival model. The survivor intensity rate depends on both systematic and firm-specific factors. The model is compared with alternative specifications introduced in the literature to capture default correlation and with models written in continuous time. By assuming independent individual defaults, conditional on state variables, and by employing affine dynamics for the factors, we recursively compute the conditional joint survivor function in Section 3. The formulas are simplified when the effect of the state factors on the default intensity rate is time independent. The aim of Section 4 is to specify the pricing model. For this purpose we assume an exponential affine pricing kernel (stochastic discount factor or sdf), depending on general factors affecting default. Then, the term structure of Treasury bonds, corporate bonds, and first-to-default baskets can be derived recursively. In particular, we discuss the decomposition and interpretation of the term structure of the spread. Numerical examples are presented in Section 5, where we also discuss the pattern and evolution of the term structures through the cycle. The extension to loss-given-default is presented in Section 6. Section 7 concludes.

2 THE DEFAULT ARRIVAL MODEL

2.1 The Basic Assumptions

Let us consider a cohort of \(n\) obligor firms with the same birth date fixed by convention at \(t = 0\). The birth date can be defined in different ways according to the problem of interest. It may correspond to the date of creation of the firms, if we focus on new industrial sectors, or to the date of the first rating by agencies such as Moody’s or Standard & Poor’s, (S&P) if we consider the introduction on bond markets, or simply to an initial date corresponding to the period of analysis. We denote by \(t_i\), \(i = 1, \ldots, n\) the failure date for corporation \(i\), that is, the lifetime of this corporation. The aim of this section is to specify a joint historical distribution of the lifetimes compatible with various patterns of the term structure of default correlation.

The previous approaches to this problem have several drawbacks. In a first approach [see, e.g., Van den Berg (1997), Li (2000), Gogliardini, Gourieroux (2005a)], the joint distribution of times-to-default is introduced through a parametric family of copulas, but, in general, such a family is not compatible with various patterns of the term structure of default and, moreover, the conditional joint distribution of the times-to-default of the obligors still alive at a given date \(t\), given the information at this date, is difficult to derive and without tractable form. The second main approach is based on the remark that, in continuous time, we have

\[
P(\tau_i > t) = P\left[ \int_0^t \tilde{\lambda}_u du < E_i \right],
\]

where \(\tilde{\lambda}_u\) is the (stochastic) infinitesimal default intensity and \(E_i\) is an exponential random variable independent of \(\tilde{\lambda}_u\) [see Bremaud (1981)]. Then, assuming that
the \( \tilde{\lambda}_i \) are independent across individuals, a default correlation is introduced through the specification of a joint distribution of the \( E_i \)'s. In other words, the correlation is derived from factor \( E_i \)'s, which have the very specific feature of time independence.

In our approach, the distribution of the set of lifetimes is defined in two steps. First, we assume that the lifetimes are independent conditional on the past, present, and future values of a set of systematic and corporate-specific factors. Then, the future realization of the factors is integrated out in order to derive the joint distribution of lifetimes conditional on information available at time \( t \) and in order to create default dependence.

**Assumption 1** There exist general (systematic) and corporate-specific factors,\(^2\) denoted by \( (Z_t), (Z_i^t) \), \( i = 1, \ldots, n \), respectively. These factors are independent, Markovian, and their transitions are such that

\[
E[\exp(u'Z_{t+1}|Z_t)] = \exp[a_g(u)'Z_t + b_g(u)],
\]

\[
E[\exp(u'Z_{i,t+1}^t|Z_i^t)] = \exp[a_c(u)'Z_i^t + b_c(u)], \quad i = 1, \ldots, n,
\]

where the above relations hold for all arguments \( u \) for which the expectation is well defined. Moreover, we assume that the marginal distributions of \( Z_i^0 \), \( i = 1, \ldots, n \) are identical.

Thus the factors satisfy a compound autoregressive (CaR) process [see Darolles, Gourieroux, and Jasiak (2006)]. The conditional distributions are defined by means of the conditional Laplace transform, or moment generating function, restricted to real arguments \( u \).\(^3\) Since the functions \( a_g, b_g, a_c, b_c \) are not highly constrained a priori, the CaR dynamics can be used to represent a large pattern of nonlinear serial dependence, including long memory, cycles, or default clustering in recession periods [see Jarrow and Yu (2001)].

By Assumption 1, the population is assumed homogeneous, that is, the distributions of the corporate-specific factor processes are independent of the firm. Thus the cohort is both homogeneous with respect to the birthdate and to individual characteristics such as the industrial sector, or the initial rating. Finally, the general factor \( Z \) is defined for any date \( t \), whereas the firm-specific factor \( Z_i^t \) only exists until the default date \( \tau_i \) of the \( i \)th firm. This explains why the independence between idiosyncratic and systematic factors is assumed; otherwise, complicated effects have to be taken into account at any firm’s failure time.

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\(^2\) The model can be extended to also include sector-specific factors. These additional factors are not introduced for expository purpose.

\(^3\) We assume in the sequel that the real Laplace transform characterizes the factor distribution. This condition is satisfied for nonnegative or bounded variables [see Feller (1997)]. In a general case, it may require power conditional moments at any order and the possibility to get a series expansion of the Laplace transform in a neighborhood of zero.
[see, e.g., Jarrow and Yu (2001), Gagliardini and Gourieroux (2003) for this extension in the case of two borrowers].

In this article, instantaneous default correlation arises only because of the common risk factors that drive individual firms’ default intensities. Equivalently, given those common factors, default arrivals of different firms become independent:

**Assumption 2** Conditional on the realization path of the factors, \( Z, Z'_j, j = 1, ..., n \), default arrival times \( \tau_i, i = 1, ..., n \) are independent. Moreover, the conditional survivor intensities are such that

\[
P[\tau_i > t + 1 | \tau_j > t, Z, Z'_j, j = 1, ..., n] = P[\tau_i > t + 1 | \tau_j > t, Z_{t+1}, Z'_{t+1}]
\]

\[
= \exp[-(x_{i+1} + \beta'_t Z_{t+1} + \gamma'_t Z'_{t+1})]
\]

\[
= \exp(-\lambda_{i+1}), \text{ say, } \forall t,
\]

where \( \lambda_{i+1}, \beta'_t, \gamma'_t \) are functions of the information included in the current and lagged factor values \( Z_t, Z'_t \).

Since the conditional survivor probability is smaller than one, we have \( \lambda'_t = x_t + \beta'_t Z_t + \gamma'_t Z'_t \geq 0, \forall t \). These restrictions imply conditions on both the sensitivity parameters and the factor distribution. For instance, they are satisfied if both factors and sensitivity coefficients are nonnegative. They can also be satisfied in a more general framework [see Gourieroux and Sufana (2003), Gourieroux, Jasiak, and Sufana (2004)]. Indeed, it has been argued recently that some factors can be represented by the elements of a positive definite symmetric matrix \( \Sigma_t \), say, \( Z_t = \text{vech} (\Sigma_t) \), where vech denotes the operator stacking the different elements of \( \Sigma_t \). In this case, a linear combination of components of \( Z \) can be written as \( \beta'_t Z_t = \text{Tr}(B_t \Sigma_t) \), where \( B_t \) is a symmetric matrix and the trace operator \( \text{Tr} \) computes the sum of diagonal elements. \( \text{Tr}(B_t \Sigma_t) \) is nonnegative, if matrix \( B_t \) is positive definite. Moreover, it is known that the Wishart autoregressive (WAR) process is a special case of the CaR process, valued in the set of positive definite symmetric matrices [see, e.g., Gourieroux (2006) for a survey on the WAR process].

The survivor intensity depends on time by means of factors \( Z_{t+1}, Z'^{i+1}_t \) and sensitivities \( x_{t+1}, \beta'_t, \gamma'_t \). This double time dependence can be interpreted in the following way. Let us assume for a while time-independent factors \( Z_{t+1} = z, Z'^{i+1}_t = z'^i, \text{ say,} \) meaning that the general environment is stable and

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4 \( Z = (Z_t, \forall t), Z'_t = (Z'_t, \forall t), \text{ and, } Z_h = (Z_t, t \leq h), Z'_h = (Z'_t, t \leq h) \).

5 When both factors and sensitivities are age independent, we get the so-called multivariate mixed proportional hazard (MMPH) model described in Van den Berg (1997, 2001). An exponential affine factor representation of the individual heterogeneity in the MMPH framework is usually assumed in the applications to labor economics [see, e.g., Flinn and Heckman (1982), Chamberlain (1985), Heckman and Walker (1990), Bonnal, Fougere, and Serandon (1997), Horowitz (1999)]. In labor economics, the model of Section 2.1 allows for both a term structure of individual heterogeneity, and a description of the latent dynamic effort variable underlying moral hazard.
the characteristics of the corporation such as size and financial ratios stay the same. Even with constant factors, the survivor rate is not the same for a young corporation and an old one. This age effect is captured by the age-dependent sensitivities. The factor $Z_t$ will capture for instance the calendar time effect and is able to create various term structures of default correlation.

Assumption 2 can easily be weakened to allow for bond-dependent sensitivities. In the extended framework, if $Z_t$ includes current and lagged values of a basic factor $Z_t = (\tilde{z}_t, \tilde{z}_{t-1})$, say, and, if the set of bonds is partitioned into two subsets with sensitivities $(\beta_{1, t+1}, 0), (0, \beta_{2, t+1})$, respectively, the model allows us to distinguish primary and secondary bonds [see Jarrow and Yu (2001) IV,B for another constrained specification].

2.2 The Link to Continuous Time

The survival model is defined above in discrete time. However, the main stream of the theoretical literature on credit derivatives considers continuous-time specifications, and it is useful to see which approach is the most flexible in practice for credit risk analysis.

Let us first recall the usual continuous-time modeling for default arrivals [see, e.g., Lando (1994, 1998), Duffie and Singleton (1999)]. In continuous time, the factors $Z, Z'$ are continuous-time affine processes, and time-to-default can take a priori any positive real value.

(i) The continuous-time factor is affine, if the conditional Laplace transform is an exponential affine function of the current value for any real horizon

$$E[\exp(u'Z_{t+h})|Z_t] = \exp[a_g(u,h)'Z_t + b_g(u,h)], \forall h \in (0, \infty).$$  

These restrictions imply the condition of Assumption 1. Thus any time discretized continuous-time affine process is CaR, but there exist a lot of CaR processes without a continuous-time counterpart. This point is developed below.

(ii) The distribution of the default time conditional on the factors path is defined through the (stochastic) infinitesimal default intensity,$$
\tilde{\lambda}_t = \lim_{dt \to 0} \frac{1}{dt} P[t < \tau_t < t + dt | \tau_t > t, Z, Z'].
$$

The associated survivor probability at horizon 1 is

$$P[\tau_t > t + 1 | \tau_t > t, Z, Z'] = \exp\left[-\int_t^{t+1} \tilde{\lambda}_u du\right].$$

If the time unit is small and the infinitesimal default intensity admits continuous path, this survivor probability can be approximated by $\exp(-\tilde{\lambda}_{t+1})$. Thus
the discrete-time specification introduced in the section above is the counterpart of a continuous-time model with affine stochastic default intensity [see Lando (1994, 1998), Duffie and Singleton (1999)]:

\[ \tilde{\lambda}_t = \tilde{x}_t + \tilde{\beta}'_t Z_t + \tilde{\gamma}'_t Z_t. \]

Let us now discuss the flexibility of continuous-time affine models and the modeling of default distribution by means of infinitesimal default intensity.

2.2.1 Lack of Flexibility of Continuous-Time Affine Processes

As already mentioned, the admissible affine dynamics are very restrictive in a continuous-time framework.

(i) Let us first consider the favorable case of Gaussian processes. In continuous time, the affine Gaussian processes are the multidimensional Ornstein-Uhlenbeck processes. Their time discretized versions are the Gaussian VAR(1) processes:

\[ Z_t = \Phi Z_{t-1} + \mu + \epsilon_t, \epsilon_t \sim \text{IN}(0, \Omega), \]

where the autoregressive matrix has an exponential representation \( \Phi = \exp A. \) On the other hand, all Gaussian VAR(1) processes are Car processes.

The condition \( \Phi = \exp(A) \) is restrictive. It implies that the eigenvalues \( \lambda_j, j = 1, \ldots, J \) of the autoregressive matrix are real, strictly positive, and that the autocorrelation function of any linear combination of factors is

\[ \rho(h) = \sum_{j=1}^J \lambda_j^h P_j(h), \]

where \( P_j(h) \) are polynomials in \( h. \)

For instance, the following Gaussian processes are Car processes without continuous-time counterpart: Gaussian white noise, Gaussian recursive system such as \( Z_{1,t} = Z_{2,t-1} + \epsilon_{1,t}, Z_{2,t} = \epsilon_{2,t} \), with \( \Phi = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \), Gaussian process with complex eigenvalues of the autoregressive matrix, that is, with autocorrelograms featuring amortized waves.

(ii) The Car processes are easily extended to account for any autoregressive lag. For instance, a Car (p) process is such that

\[ E[\exp(u'Z_{t+1})|Z_t] = \exp[d_{0g}(u)Z_t + \ldots + d'_{p-1, g}(u)Z_{t-p-1} + b_g(u)], \]

whereas such an extension is not possible in the continuous-time framework. A Gaussian VAR(p) process, or an autoregressive gamma process ARG(p), which extends the CIR process to several lags, are also Car processes [Gourieroux and Jasiak (2006)].

It is often advocated that higher-order dynamics can also be deduced from continuous-time specification, by considering a linear combination \( \beta' Z_t \), say, of a multivariate affine process. This is true, but
In the Gaussian case, the pattern of the autocorrelation function of \( Z_t \), where \( Z_t \) corresponds to an Ornstein-Uhlenbeck process, is highly restricted.

Moreover, an autoregressive gamma [ARG(p)] process has no continuous-time counterpart, and, in particular, its transition density is very different from the transition density of a combination of independent CIR processes for instance.

(iii) More generally, the class of CaR processes offers many more possibilities than the class of time discretized continuous affine models for capturing nonlinear dynamics. This is a consequence of the embedability restriction [called the infinitely decomposability condition in Duffie, Filipovic, and Schachermayer (2003: 10)] introduced to ensure time coherency even for small time units. The conditions are similar to the restrictions exhibited in the linear Gaussian framework, and are easy to understand from the example of a Markov chain with transition matrix \( P \). Let us denote by \( Z_t \) the random vector, whose components are the state indicators of the chain at date \( t \). We get

\[
E[\exp(u'Z_{t+1}|Z_t] = \exp[\sum_{k=1}^{K} \log(E \exp(u'Z_{t+1}|Z_{kt}=1)Z_{kt}].
\]

Thus a Markov chain is a CaR process. But this chain can be considered as a time discretized continuous-time process if, and only if, the transition matrix has an exponential form \( P = \exp(A) \).

In order to be more concrete, let us consider the problem of finding a specification of the dynamics of a factor which has, at the same time, the property of positivity (in order to get positive \( \lambda_i \)) and of cyclical behavior (in order to capture “through the cycle” features) [see Nickell et al. (2000), Bangia et al. (2002), Gagliardini, Gourieroux (2005b), Feug et al. (2006)]. These conditions are easily satisfied by members of the CaR family, for instance, multilag autoregressive gamma processes with a weak (or second order) linear autoregressive representation with complex roots. This point will be developed in Section 5.3.

### 2.2.2 Lack of Flexibility of the Specification by Default Intensity

The continuous-time approach implicitly assumes the existence of an infinitesimal default intensity, and therefore time-to-default with continuous distribution. This assumption is not compatible with available data. Indeed, times-to-default often correspond to predetermined dates of repayment of the debt (as in Merton’s model [Merton (1974)]), implying duration distributions with point mass.

### 2.2.3 Artificial Introduction of Jump Processes

Under domain assumptions, the continuous-time affine models are essentially built from Ornstein-Uhlenbeck and CIR diffusion processes plus jump processes [see Duffie, Filipovic, and Schachermayer (2003)]. In practice, the estimation of diffusion affine processes
often reveals poor fit, and additional jumps are introduced without clear interpretation. An advantage of the discrete-time approach is to avoid this distinction. In discrete time, “every movement is a jump.”

3 THE CONDITIONAL JOINT TIME-TO-DEFAULT DISTRIBUTIONS

Let us study the joint distribution of time-to-default and its variation in time. We first consider the case of path-dependent factor sensitivities, and then we particularize the results to constant sensitivities.

3.1 General Case

Under Assumptions 1 and 2, the joint conditional survivor function can be computed explicitly. This function depends on the information available at time \( t \). When this information is complete, that is, includes the current and lagged values of the factors together with the information on corporate defaults, the conditional survivor function can be defined for any subset \( S \) of the Population-at-Risk, that is, the set \( \text{PaR}_t \) of firms, which are still operating at time \( t \):

\[
S^*_t(h_i, i \in S) = P[\tau_i > t + h_i, i \in S|\text{PaR}_t, S \subset \text{PaR}_t, (\tau_j | j \in \overline{\text{PaR}}_t), Z_t, Z_t', j = 1, \ldots, n],
\]

where \( \text{PaR}_t = \{ i : \tau_i > t \} \) is the Population-at-Risk and \( \overline{\text{PaR}}_t \) denotes its complement [The terminology PaR has been introduced by Cox (1972)]. The property below is proved in Appendix 2.

**Proposition 1** The conditional survivor function is given by

\[
S^*_t(h_i, i \in S) = \exp\left[-\sum_{k=1}^{\tilde{h}} n_{t+k} a_{t+k} + B^{-[\gamma]}(t, t + \tilde{h}) + A^{-[\gamma]}(t, t + \tilde{h})' Z_t + \sum_{i \in S} B^{-[\gamma]}(t, t + h_i) + \sum_{i \in S} A^{-[\gamma]}(t, t + h_i)' Z_t', j \right],
\]

where, for any deterministic sequence\(^6\) \([u] = (u_t)\), operators \( A^{[u]}, B^{[u]} \) are recursively defined by

\[
A^{[u]}(t, t + h) = a[u_{t+1} + A^{[u]}(t + 1, t + h)],
\]

\[
B^{[u]}(t, t + h) = b[u_{t+1} + A^{[u]}(t + 1, t + h)] + B^{[u]}(t + 1, t + h),
\]

for \( h > 0 \), with terminal conditions

\[
A^{[u]}(t, t) = 0, B^{[u]}(t, t) = 0, \forall t,
\]

\( \quad \)

\( ^6 \) The \([u]\) sequence may be infinite or not, whenever it is defined for the interval \( t + 1 \) to \( t + h \), corresponding to the values included in the recursive computation.
and where \( \bar{h} = \max_{i \in S} h_i \), the deterministic sequence \([n]\) is defined by
\[
n_{t+k} = \text{Card} \{ h_i \geq k, i \in S \},
\]
and the product sequence \([n][\lambda]\) by \(n_i \lambda_i\).

Therefore, given the current and past values of the factors, the default history is not informative as a consequence of conditional independence:
\[
S'_i(h_i, i \in S) = P[\tau_i > t + h_i, i \in S | \text{PaR}_t, S \subset \text{PaR}_t, Z_t, Z'_t].
\]

Proposition 1 shows that the conditional survivor function is easily computed numerically by means of discrete-time recursive equations. This numerical implementation has to be compared with the practice of continuous time affine models. In continuous time, it is necessary to solve numerically differential Riccati equations, also by means of discrete-time recursion. But these equations are solved numerically, often by considering approximate recursive systems at a small time unit, 5 mn, say. This approximation implies more computations, 288 times more than in a daily discrete-time model, due to the smaller time unit needed to approximate continuous time. Moreover, these recursive systems are approximations, and are not compatible in general with no-arbitrage opportunity conditions. This is not the case of a coherent discrete-time approach.

Proposition 1 can be used to derive the distribution of a given time-to-default. It can also be used to derive the distribution of the time to first failure in a basket of securities. These computations are needed for credit derivatives currently traded on the market (see Sections 4.4 and 4.5). The credit default swaps (CDSs) are options on a corporate default occurrence during a given period, whereas other options concern the occurrence of a default in a basket of securities (first-to-default basket security).

Let us consider a given obligor \(i\). The conditional survivor function specific to this firm is \(P[\tau_i > t + h_i, Z_t, Z'_t]\), and corresponds to the joint survivor function with \(S = \{i\}, h_i = h\), and \(n_{t+k} = 1\) for \(1 \leq k \leq h\). We get the following result:

**Corollary 1**

\[
P(\tau_i > t + h | \tau_i > t, Z_t, Z'_t) = \exp[-\sum_{k=1}^{h} a_{t+k} + B_g^{-[\beta]}(t, t + h) + A_g^{-[\beta]}(t, t + h)^t Z_t
\]

\[
+ B_c^{-[\gamma]}(t, t + h) + A_c^{-[\gamma]}(t, t + h)^t Z'_t],
\]

where functions \(A_g^{[u]}, A_c^{[u]}, B_g^{[u]}, B_c^{[u]}\) are recursively defined in Proposition 1.

Let us now consider the first-to-default time for a set \(S\) of still operating obligors, \(\tau^*_S = \min_{i \in S \subseteq \text{PaR}_t} \tau_i\). The survivor function of \(\tau^*_S\) is defined by
\[
P(\tau^*_S > t + h | \text{PaR}_t, S \subset \text{PaR}_t, Z_t, Z'_t, i \in S],
\]
and is equal to the joint survivor function evaluated at \(h_i = h, \forall i \in S\). Thus, for all \(1 \leq k \leq h\), we have \(n_{t+k} = n = \text{Card} (S)\), and we get the following result:
Corollary 2

\[ P[\tau_S^i > t + h | \text{PaR}_{i}, S \subseteq \text{PaR}_t, Z_t, Z_{t^i}, i \in S] \]

\[ = \exp[-n \sum_{k=1}^{h} \alpha_{t+k} + B_g^{-n[\beta]}(t, t + h) + A_g^{-n[\beta]}(t, t + h)'Z_t + nB_c^{-[\gamma]}(t, t + h) + A_c^{-[\gamma]}(t, t + h)' \sum_{i \in S} Z_{t^i}], \]

where \( n[\beta] \) denotes the sequence \( n\beta_{t+k} \).

Even if the expressions of the survivor functions given in Proposition 1 and Corollaries 1 and 2 seem rather cumbersome, they are easily implemented from recursive equations. The distribution of the first-to-default time does not depend on the state of individual obligors since the average state \( \frac{1}{2} \sum_{i \in S} Z_{t^i} \) is a sufficient statistic. That is, the first-to-default probability of a security basket that contains some severely distressed corporations is identical to that of another basket where none of the corporations is distressed, but the average state is the same. This is a direct consequence of the homogeneity assumption.

3.2 Constant Factor Sensitivities

In the general framework of Assumption 2, the conditional survivor probability for a given obligor \( i \) (for instance) varies in time because of the stochastic evolution of the factors \( Z_t, Z_{t^i} \), but also because of the deterministic aging of sensitivity coefficients \( \alpha_t, \beta_t, \gamma_t \). A special case arises when sensitivity functions \( \alpha, \beta, \gamma \) are constant. In this framework, Corollaries 1 and 2 provide closed-form expressions for the conditional survivor probabilities.

Let us first consider the slope operator \( A^{[u]}(t, t + h) \), when all the terms of the deterministic sequence \( [u] \) are equal to a same vector \( u \). The recursive equation of Proposition 1 becomes

\[ A^{[u]}(t, t + h) = a[u + A^{[u]}(t + 1, t + h)] = a_u[A^{[u]}(t + 1, t + h)], h > 0, \]

where \( a_u(s) = a(u + s) \) denotes a shifted version of function \( a \) and \( A^{[u]}(t + h, t + h) = 0 \). We deduce that

\[ A^{[u]}(t, t + h) = a_u^{[0]}(0), \]

where \( a_u^{[h]} \) denotes function \( a(, h) \) compounded \( h \) times. Similarly, for a constant sequence \( u_t = u \), the intercept operator \( B^{[u]} \) is given by

\[ B^{[u]}(t, t + h) = \sum_{j=0}^{h-1} b_u(a_u^{[j]}(0)). \]

Therefore we derive the following corollary:
Corollary 3 When the factor sensitivities are constant, 
\[
P[t_i > t + h | \tau_i > t, Z_t, Z_i^t] = \exp\{-h \sigma + \sum_{j=0}^{h-1} b_{g,-\beta} (d_{g,-\beta}^{u,j}(0)) + d_{g,-\beta}^{u,h}(0)' Z_t + \sum_{j=0}^{h-1} b_{c,-\gamma} (d_{c,-\gamma}^{u,j}(0)) + d_{c,-\gamma}^{u,h}(0)' \sum_{i \in S} Z_i^t\},
\]
where the shifted functions are 
\[
a_{g,-\beta}(u) = a_g(u - n\beta), a_{c,-\gamma}(u) = a_c(u - \gamma), \text{ and so on.}
\]

The expression of \( \tau_S^* \) shows the effects of the systematic and corporate-specific factors. The survivor probability depends on the number \( n \) of firms in the basket, and on their average state \( \sum_{i \in S} Z_i^t \). An interesting point is the specific way the number of corporations \( n \) appears in the shift of the \( a_g \) and \( b_g \) functions.

4 AFFINE TERM STRUCTURE AND CREDIT RISK

The joint historical distribution of factors and times-to-default is important for prediction purposes. For Credit-VaR analysis, it has to be completed by specifying the risk-neutral distribution, or equivalently a stochastic discount factor (sdf). The sdf is used for pricing both future money value and individual defaults. Indeed, it is not realistic to study independently the term structure of risk-free interest rates and default risk, which are both related to business cycles. For instance, in a period of high activity, we expect both an increase of the difference between the long- and short-term risk-free rates, and an improvement of credit quality [see, e.g., the study by Duffee (1998)]. For this reason, we assume that systematic factors \( Z \) appearing in the conditional survivor probability can also influence the sdf. Moreover, we select a stochastic discount factor, which is an exponential affine function of the general factors. As a consequence, the benchmark term structure of risk-free interest rates is affine.

4.1 Specification of a Stochastic Discount Factor

The pricing model is completed by specifying a stochastic discount factor \( M_{t,t+1} \) for period \( (t, t+1) \). The sdf is the basis for pricing any derivative written on underlying factors and on default times. Typically the price at \( t \) of a European derivative paying \( g_{t+h} \) at date \( t + h \) is

\[
C_t(g, h) = E_t[M_{t,t+1} \ldots M_{t+h-1,t+h} g_{t+h}] = E_t[M_{t,t+h} g_{t+h}], \text{ say.}
\]

(2)
where $E_t$ denotes the historical expectation, conditional on information including the current and lagged values of the state variables, the knowledge of the population at risk at date $t$, and the default dates of the other firms. Equation (2) is first used to price the zero-coupon bonds and corporate zero-coupon corporate bonds with zero recovery rate. For Treasury bonds, we get

$$B(t, t+h) = E_t(M_{t,t+h}), \forall t, h. \quad (3)$$

For the $h$-year zero-coupon corporate bond with zero recovery rate corresponding to corporate $i \in PaR_t$, we get

$$C_i(t, t+h) = E_t[M_{t,t+h}1_{i > t+h}], \forall t, h. \quad (4)$$

For a first-to-default basket, which pays zero in case of at least one default, and one money unit otherwise, we get

$$C_S(t, t+h) = E_t[M_{t,t+h}1_{\exists i > t+h}], \forall t, h. \quad (5)$$

The case where the recovery rate is nonzero will be treated in Section 6. To restrict the set of admissible risk corrected distributions and get closed-form pricing formulas, we select a sdf, which is exponential affine in the systematic factors.

**Assumption 3**

$$M_{t,t+1} = \exp[v_o + v^tZ_{t+1}]. \quad (6)$$

It would have been possible to also introduce corporate-specific factors in the expression of the sdf. However, the interpretation would become more complicated, since the set of alive corporate-specific factors depends on the date. When a corporation fails, its specific factor ceases to exist. By introducing the effect of individual factors $Z_i$, we also introduce risk corrections for the number and structure of corporations, which is beyond the scope of the present article.\(^7\)

Also note that some components of $v$ or $\beta_i$ can be zero. Therefore general factors can influence the sdf, the default intensity, or both.

---

\(^7\) The number of corporations can have an effect on default correlation. If we consider an industrial sector with two firms only, the default of a firm will increase the monopolistic power of the remaining one, and likely diminish its default probability.
4.2 Zero-Coupon Treasury Bonds, and Zero-Coupon Corporate Bonds with Zero Recovery Rate

The prices of the zero-coupon Treasury bonds are given by

$$B(t, t+h) = E_t[M_{t,t+1} \ldots M_{t+h-1,t+h}]$$

$$= \exp(v_t h)E_t[\exp[v_{t+1}Z_{t+1} + \ldots + v_{t+h}Z_{t+h}]].$$

The closed-form expression of the price follows from Appendix 1.

**Property 2** The price of a zero-coupon Treasury bond is

$$B(t, t+h) = \exp[v_t h + \sum_{j=0}^{h-1} b_{g,y}(a_{g,y}^{oj}(0)) + a_{g,y}^{oh}(0) Z_t].$$

In particular, the geometric yields defined by

$$r(t, t+h) = -\frac{1}{h} \ln B(t, t+h)$$

$$= -v_t - \frac{1}{h} \sum_{j=0}^{h-1} b_{g,y}(a_{g,y}^{oj}(0)) - \frac{1}{h} a_{g,y}^{oh}(0) Z_t,$$

(7)

generate an affine space driven by the general factors. Thus we get an affine term structure of risk-free interest rates [Duffie and Kan (1996)].

The price of a zero-coupon corporate bond with zero recovery rate is given by

$$C_i(t, t+h) = E_t[M_{t,t+1}M_{t+h-1,t+h}1_{t+1>t+h}]$$

$$= E_t[\exp(v_t h + v \sum_{j=1}^{h} Z_{t+j}) \exp(-\sum_{j=1}^{h} [\alpha_{t+j} + \beta_{t+j}Z_{t+j} + \gamma_{t+j}Z_{t+j}^2])]$$

$$= E_t[\exp(v_t h - \sum_{j=1}^{h} \alpha_{t+j} + \sum_{j=1}^{h} [v - \beta_{t+j}Z_{t+j} - \sum_{j=1}^{h} \gamma_{t+j}Z_{t+j}^2])].$$

By applying Lemma 1 in Appendix 1, we derive the term structure of corporate bonds with zero recovery rate.

**Property 3** The price of the zero-coupon corporate bond with zero recovery rate is

$$C_i(t, t+h) = \exp \left[ v_t h - \sum_{j=1}^{h} \alpha_{t+j} + B_g^{\alpha_j}[\beta_j](t, t+h) + A_g^{\alpha_j}[\beta_j](t, t+h) Z_t \
+ B_c^{\alpha_j}[\beta_j](t, t+h) + A_c^{\alpha_j}[\beta_j](t, t+h) Z_t \right].$$

The geometric yields $y_i(t, t+h) = -\frac{1}{h} \log C_i(t, t+h) = 1, 2, \ldots$ generate an affine term structure, now driven by both the general and specific factors. The spread of the corporate bond with zero recovery rate is given by
\[ s_i(t, t + h) = y_i(t, t + h) - r(t, t + h) = \frac{1}{h} \sum_{j=1}^{h} \alpha_{t+j} - \frac{1}{h} B_{g, y}^{-[\beta]}(t, t + h) - \left( \frac{1}{h} A_{g, y}^{-[\beta]}(t, t + h) - a_{g, y}^{oh}(0) \right)^{\prime} Z_t - \frac{1}{h} B_{c}^{-[\gamma]}(t, t + h) \]

\[ + \frac{1}{h} \sum_{j=0}^{h-1} b_{g, y} (a_{g, y}^{aj}(0)) - \frac{1}{h} A_{c}^{-[\gamma]}(t, t + h)^{\prime} Z_t. \]

From Equations (3) and (4), we know that

\[ s_i(t, t + h) = -\frac{1}{h} \log C_i(t, t + h) = -\frac{1}{h} \log E_i^{\prime} [1_{t > t+}] \]

where \( E_i^{\prime} \) denotes the forward risk-neutral measure for term \( h \) [see Merton (1973), Pedersen and Shiu (1994), Geman, El Karoui, and Rochet (1995), for a definition and use of the forward risk-neutral measure]. Property 3 shows that the forward risk-neutral measure is easy to use in the affine framework.

When the sensitivities \( \alpha_{t}, \beta_{t}, \gamma_{t} \) are constant, we get

\[ C_i(t, t + h) = \exp \left\{ v_o h - \alpha h + \sum_{j=1}^{h-1} b_{g, y}^{-[\beta]}(a_{g, y}^{aj}(0)) + a_{g, y}^{oh}(0)^{\prime} Z_t \right\} \]

\[ + \sum_{j=1}^{h-1} b_{c, y} (a_{c, y}^{aj}(0)) + a_{c, y}^{oh}(0)^{\prime} Z_t, \]

\[ y_i(t, t + h) = -v_o + \alpha - \frac{1}{h} \sum_{j=0}^{h-1} b_{g, y}^{-[\beta]}(a_{g, y}^{aj}(0)) - \frac{1}{h} a_{g, y}^{oh}(0)^{\prime} Z_t \]

\[ - \frac{1}{h} \sum_{j=0}^{h-1} b_{c, y} (a_{c, y}^{aj}(0)) - \frac{1}{h} a_{c, y}^{oh}(0)^{\prime} Z_t, \]

\[ s_i(t, t + h) = \alpha + \frac{1}{h} \sum_{j=0}^{h-1} \left[ b_{g, y} (a_{g, y}^{aj}(0)) - b_{g, y}^{-[\beta]}(a_{g, y}^{aj}(0)) - b_{c, y} (a_{c, y}^{aj}(0)) \right] \]

\[ + \frac{1}{h} (a_{g, y}^{oh}(0) - a_{g, y}^{oh}(0)^{\prime} Z_t - \frac{1}{h} a_{c, y}^{oh}(0)^{\prime} Z_t. \]

### 4.3 Decomposition of the Spread for a Zero-Coupon Corporate Bond with Zero Recovery Rate

In the standard actuarial approach, default is assumed independent of risk-free rates, and is priced according to the historical probability [Fons (1994)]. Thus the actuarial value of a zero-coupon corporate bond with zero recovery rate is
The associated actuarial yields are

\[ y_i^h(t, t + h) = r(t, t + h) + \pi_i(t, t + h), \]

where

\[
\pi_i(t, t + h) = -\frac{1}{h} \log P[\tau_i > t + h | \tau_i > t, Z_t, Z_i^t] \\
= -\frac{1}{h} \sum_{k=1}^{h} \log P[\tau_i > t + k | \tau_i > t + k - 1, Z_t, Z_i^t] \\
= \frac{1}{h} \sum_{k=1}^{h} \lambda_{i+k}^{\text{f}}, \text{say,}
\]

can be interpreted as an average forward default intensity. The forward intensity \( \lambda_{i+k}^{\text{f}} = -\log P(\tau_i > t + k | \tau_i > t + k - 1, Z_t, Z_i^t) \) differs from the spot intensity \( \lambda_i^t = -\log P(\tau_i > t + k | \tau_i > t + k - 1, Z_{t+k-1}, Z_i^{t+k-1}) \) due to the time index of factor values. However, Equation (8), which is frequently used by markets to estimate the default probabilities from the spreads \( s_i(t, t + h) \) is not valid in a general framework. The aim of this subsection is to derive a more accurate decomposition of the spread.

From Corollary 1, the average default intensity is given by

\[
\pi_i(t, t + h) = \frac{1}{h} \sum_{j=1}^{h} \lambda_{t+j}^{\text{f}} - \frac{1}{h} B_{g}^{-[\beta]}(t, t + h) - \frac{1}{h} A_{g}^{-[\beta]}(t, t + h)'Z_t \\
- \frac{1}{h} B_{c}^{-[\gamma]}(t, t + h) - \frac{1}{h} A_{c}^{-[\gamma]}(t, t + h)'Z_i^t.
\]

We deduce the following proposition:

**Proposition 4**

\[
s_i(t, t + h) - \pi_i(t, t + h) \\
= -\frac{1}{h} [A_{g}^{-[\beta]}(t, t + h) - A_{g}^{-[\beta]}(t, t + h) - a_{g,v}^{oh}(0)]'Z_t \\
- \frac{1}{h} [B_{g}^{-[\beta]}(t, t + h) - B_{g}^{-[\beta]}(t, t + h) - \sum_{j=0}^{h-1} b_{g,v}[a_{g,v}^{oj}(0)]].
\]
The average default intensity absorbs all idiosyncratic variability in spreads. Even if both the spread term structure $s_i(t, t+h)$, as well as the term structure of average default intensity $\pi_i(t, t+h)$ depend on the stochastic state of the $i$th corporation, their difference does not, and is the same for all corporations of this industry.

The correcting term in the decomposition of the spread given in Proposition 4 measures the effect of the dependence between default and sdf, due to common factors. For instance, at short-term horizon, we have

$$s_i(t, t+1) - \pi_i(t, t+1) = -\log \left\{ \frac{E_t[M_{t,t+1}1_{t> t+1}]}{E_t(M_{t,t+1})E_t(1_{t> t+1})} \right\}$$

$$= -\log \left\{ 1 + \frac{\text{cov}_t(M_{t,t+1}, 1_{t> t+1})}{E_t(M_{t,t+1})E_t(1_{t> t+1})} \right\}.$$

The correcting term can be of any sign. This term is positive (negative), if the sdf and the default indicator are negatively correlated (positively correlated). However, it is difficult to guess the sign of this dependence. Indeed, this sign depends on the conditioning set and the sign of the correlation can change (from negative to positive, or conversely), when this information set increases.

To summarize, Proposition 4 provides the following decomposition of the term structure of corporate bonds with zero recovery rate:

Term structure of corporate bonds with zero recovery rate
\[ = \text{term structure of Treasury bonds} \]
\[ + \text{term structure of average default intensity} \]
\[ + \text{term structure of dependence between sdf and default}. \]

The correcting term takes a simplified form when the sensitivities are time independent.

**Corollary 4** When the sensitivities $\alpha_t, \beta_t, \gamma_t$ are time independent, we get

$$s_i(t, t+h) - \pi_i(t, t+h)$$

$$= -\frac{1}{h} \left[ a_{g,v}^{\alpha} - a_{g,v}^{\beta}(0) - a_{g,v}^{\alpha}(0) \right] Z_t$$

$$- \frac{1}{h} \sum_{j=0}^{h-1} \left[ b_{g,v} - b_{g,v}^{\beta}(0) - b_{g,v}^{\alpha}(0) \right] Z_t.$$

The correcting term is zero, whenever $Z_t$ is partitioned into two independent subvectors $Z_{1t}, Z_{2t}$, and the conformable partitionings of $v$ and $\beta$ are $(v_1,0)^	op$ and
(0, β2)', respectively. Thus the correcting term vanishes when default and sdf are influenced by independent factors.

### 4.4 Term Structures of Yield and Decomposition of the Spread for a First-to-Default Basket

A first-to-default basket with time-to-maturity $h$, written on $n$ firms provides at $t + h$ a payoff of one money unit, if no firms default before $t + h$, and nothing, otherwise. The price at $t$ of this basket is

$$C(t, t + h) = E_t[M_{t,t+h}1_{\tau^* > t+h}] , \forall t, h,$$

where $\tau^* = \min_{i=1,...,n} \tau_i$.

Computations similar to those presented in the previous section give the following results.

**Proposition 5**

$$C(t, t + h) = \exp[v_h t - n \sum_{j=1}^{h} \alpha_{t+j} + B_g^{-n[\beta]}(t, t + h) + A_g^{-n[\beta]}(t, t + h)]'Z_t$$

$$+ nB_c^{-[\gamma]}(t, t + h) + A_c^{-[\gamma]}(t, t + h)'[Z_t^1 + \ldots + Z_t^n],$$

$$s(t, t + h) = \frac{n}{h} \sum_{j=1}^{h} \alpha_{t+j} - \frac{1}{h} B_g^{-n[\beta]}(t, t + h) - \frac{1}{h} \left[ A_g^{-n[\beta]}(t, t + h) - d^{oh}(0) \right]'Z_t$$

$$- \frac{1}{h} A_c^{-[\gamma]}(t, t + h)'(Z_t^1 + \ldots + Z_t^n),$$

$$\pi(t, t + h) = \frac{n}{h} \sum_{j=1}^{h} \alpha_{t+j} - \frac{1}{h} B_g^{-n[\beta]}(t, t + h) - \frac{1}{h} A_g^{-n[\beta]}(t, t + h)'Z_t$$

$$- \frac{1}{h} B_c^{-[\gamma]}(t, t + h) - \frac{1}{h} A_c^{-[\gamma]}(t, t + h)(Z_t^1 + \ldots + Z_t^n),$$

$$s(t, t + h) - \pi(t, t + h) = - \frac{1}{h} \left[ A_g^{-n[\beta]}(t, t + h) - A_g^{-n[\beta]}(t, t + h) - d^{oh}(0) \right]'Z_t$$

$$- \frac{1}{h} [B_g^{-n[\beta]}(t, t + h) - B_c^{-[\beta]}(t, t + h) - \sum_{j=1}^{h} b_{g,v}(a^{oh}_{g,v}(0))].$$

We still get affine term structures for first-to-default basket. The results are simplified if the sensitivities are constant.
Corollary 6 If the sensitivities are constant, we get

\[ C(t, t + h) = \exp\left[\nu_t h - n\alpha h + \sum_{j=0}^{h-1} b_{g,v,n} (a_{g,v,n}^{ij}(0)) + a_{g,v,n}^{ih}(0) Z_t \right] \]

\[ + n \sum_{j=1}^{h-1} b_{c,-\gamma}(a_{c,-\gamma}^{ij}(0)) + a_{c,-\gamma}^{ih}(0) [Z_t^1 + \ldots + Z_t^n], \]

\[ s(t, t + h) = n\alpha - \frac{1}{h} \left[ \sum_{j=0}^{h-1} b_{g,v}(a_{g,v}^{ij}(0)) - b_{g,v,n\beta}(a_{g,v,n\beta}^{ij}(0)) \right] \]

\[ - nb_{c,-\gamma}(a_{c,-\gamma}^{ij}(0)) - \frac{1}{h} [a_{g,v,n\beta}^{ih}(0) - a_{g,v,n\beta}^{ih}(0)] Z_t \]

\[ - \frac{1}{h} a_{g,v,n\beta}^{ih}(0) [Z_t^1 + \ldots + Z_t^n], \]

\[ \pi(t, t + h) = n\alpha - \frac{1}{h} \left[ \sum_{j=0}^{h-1} b_{g,-n\beta}(a_{g,-n\beta}^{ij}(0)) + nb_{c,-\gamma}(a_{c,-\gamma}^{ij}(0)) \right] \]

\[ - \frac{1}{h} a_{g,-n\beta}^{ih}(0) Z_t - \frac{1}{h} a_{c,-\gamma}^{ih}(0) [Z_t^1 + \ldots + Z_t^n], \]

\[ s(t, t + h) - \pi(t, t + h) = \frac{1}{h} \left[ \sum_{j=0}^{h-1} b_{g,v}(a_{g,v}^{ij}(0)) + b_{g,-n\beta}(a_{g,-n\beta}^{ij}(0)) \right] \]

\[ - b_{g,v,n\beta}(a_{g,v,n\beta}^{ij}(0))] \]

\[ + \frac{1}{h} [a_{g,v}^{ih}(0) + a_{g,-n\beta}^{ih}(0) - a_{g,-n\beta}^{ih}(0)] Z_t. \]

Moreover, the term structure of survivor default intensity \( \pi(t, t + h) \) can be decomposed into two parts. The first component corresponds to the marginal effect of individual default risks, obtained under the independence assumption: 8

\[ \pi^\circ(t, t + h) = \sum_{i=1}^{n} \pi_i(t, t + h). \]

The second part corresponds to the residual \( \pi(t, t + h) - \pi^\circ(t, t + h) \). The sign of the residual term depends on the type of dependence between corporate lifetimes. For illustration, let us consider two firms \( n = 2 \). We get

\[ \pi(t, t + h) - \pi^\circ(t, t + h) = -\frac{1}{h} \log P[\tau_1 > t + h, \tau_2 > t + h | \tau_1 > t, \tau_2 > t] \]

\[ + \frac{1}{h} \log P[\tau_1 > t + h | \tau_1 > t, \tau_2 > t] + \frac{1}{h} \log P[\tau_2 > t + h | \tau_1 > t, \tau_2 > t]. \]

This quantity is nonnegative if and only if

8 Assuming the same marginal risks.
\[
P[\tau_1 > t + h, \tau_2 > t + h | \tau_1 > t, \tau_2 > t] \leq P[\tau_1 > t + h | \tau_1 > t, \tau_2 > t] \\
P[\tau_2 > t + h | \tau_1 > t, \tau_2 > t] \Leftrightarrow \text{cov}[\mathbf{1}_{\tau_1 > t + h}, \mathbf{1}_{\tau_2 > t + h} | \tau_1 > t, \tau_2 > t] \leq 0.
\]

Thus \( \pi(t, t + h) \) is larger than \( \pi^*(t, t + h) \), when lifetimes \( \tau_1 \) and \( \tau_2 \) feature negative dependence.

To summarize, we have the following proposition:

**Proposition 6** The term structure of first-to-default basket yields can be decomposed as

\[
y(t, t + h) = r(t, t + h) + \pi^*(t, t + h) + \left[ \pi(t, t + h) - \pi^*(t, t + h) \right] + \left[ s(t, t + h) - \pi(t, t + h) \right]
\]

(10)

corresponding, respectively, to the term structure of Treasury bond yields, the term structure of marginal defaults, the term structure of default correlation, the term structure of dependence between stochastic discount factor and default.

### 4.5 Factor Observability

Up to now we have not discussed the interpretation of the general and corporate-specific factors, and especially their observability. It is well-known that the general factors included in the sdf \( Z^* \) (say) can be recovered from the observed Treasury bond prices, due to the affine structure [see Duffie and Kan (1996)]. Equivalently, factors \( Z^* \) can be replaced by mimicking factors with yield interpretations [see Equation (7)].

A similar argument applies to general factors that influence default only, and to corporate-specific factors. They can be recovered from the corporate yields, taking into account the observability of \( Z^* \). Thus the discussion of factor observability is equivalent to the discussion of observability of corporate term structure of yields, that is, the number and design of the liquid corporate bonds. More precisely, let us assume \( L_0 \) systematic factors and \( L_1 \) idiosyncratic factors per firm. Let us denote by \( H \) (\( H_i \)), the number of liquid Treasury bonds (corporate bonds corresponding to firm \( i \)) at date \( t \). The factor values are identifiable at date \( t \) under the order conditions

\[
H_i \geq L_1, i = 1, \ldots, n,
\]

\[
H + \sum_{i=1}^{n} (H_i - L_1) \geq L_0.
\]

### 4.6 Determination of the Credit-VaR

The results above can be directly used to compute the Credit-VaR of a portfolio of Treasury bonds and corporate bonds with zero recovery rate. Let us consider a portfolio involving a set \( S \) of firms at time \( t \). For corporation \( i \), the portfolio includes a quantity \( x_{i,t}(h) \) of corporate bonds with time to maturity \( h, h = 1, \ldots, H \). The current value of the credit portfolio is
\[ W_t = \sum_{i \in S} \sum_{h=1}^{H} x_{i,t}(h) C_i(t, t + h), \]

whereas its future value is

\[ W_{t+1} = \sum_{i \in S \cap \text{PaR}_{t+1}} \sum_{h=1}^{H} x_{i,t}(h) C_i(t + 1, t + h), \]

where \( S \cap \text{PaR}_{t+1} \) is the set of firms in \( S \), which are still alive at \( t + 1 \). The future portfolio value is stochastic for two reasons: first, the population at risk at date \( t + 1 \) is unknown; second, the future term structure of corporate bonds is also unknown.

Due to the affine structure of the model, this future value can be written in terms of the factor values corresponding to date \( t + 1 \):

\[ W_{t+1} = \sum_{i \in S \cap \text{PaR}_{t+1}} \sum_{h=1}^{H} x_{i,t}(h) \exp[\hat{a}(h - 1) + \hat{b}(h - 1)Z_{t+1} + \hat{c}(h - 1)Z_{t+1}^s], \]

say, where the coefficients \( \hat{a}, \hat{b}, \hat{c} \) are deduced from the pricing formulas. The Credit-VaR is a quantile of the conditional distribution of \( W_{t+1} \) given the information available at time \( t \), that is, \( Z_t, Z_{t+1}^i, i \in S \cap \text{PaR}_t \) (factor values, which are deduced from the observed term structures; see Section 4.6). This distribution can be approximated by Monte Carlo, as follows:

(i) First, draw the future value of the factor \( Z_{t+1}^i \), [respectively, \( Z_{t+1}^i, s \)] in the conditional distribution of \( Z_{t+1} \) given \( Z_t \) [respectively, \( Z_{t+1}^i \) given \( Z_t^i \)].

(ii) Second, simulate independently the default occurrence between \( t \) and \( t + 1 \) for the different firms in \( S \cap \text{PaR}_t \) and the drawn values of the factors. \( S \cap \text{PaR}_{t+1} \) denotes the simulated set of surviving firms.

(iii) Deduce the simulated future value of the portfolio by

\[ W_{t+1}^s = \sum_{i \in S \cap \text{PaR}_{t+1}} \sum_{h=1}^{H} x_{i,t}(h) \exp[\hat{a}(h - 1) + \hat{b}(h - 1)Z_{t+1}^s + \hat{c}(h - 1)Z_{t+1}^{i,s}], \]

(iv) Replicate the procedure for \( s = 1, \ldots, \bar{s} \), where \( \bar{s} \) is the total number of replications.

(v) Approximate the Credit-VaR by the associated empirical quantile of the sample distribution of \( W_{t+1}^1, \ldots, W_{t+1}^\bar{s} \).

This procedure uses in a coherent way the historical distribution of factors and defaults, and the risk correction by sdf for the closed-form expression of
future prices. It requires conditional drawings of the future factor values in step 1. What about such simulations when the conditional probability distributions are only specified through the Laplace transform? There is no general answer, but fortunately a large number of CaR processes admit compound interpretations appropriate for simulation purpose [see Darolles, Gourieroux, and Jasiak (2006)]. In such cases, it is not necessary to compute numerically the conditional cdf by inversion of Laplace transform in order to simulate. This compound interpretation is used to create artificial data from CIR process in the numerical experiment of Section 5.

5 NUMERICAL EXPERIMENTS

Two numerical experiments are performed in this section. First, we illustrate the decomposition of the spreads for a corporate bond with zero recovery rate and a first-to-default basket. Second, we analyze the effect of a cyclical factor (which cannot be captured in a continuous-time model) on both the risk-free term structure and default probability.

5.1 Decomposition of the Spread of a Corporate Bond with Zero Recovery Rate

Let us consider a model with one systematic factor and one specific factor, [Gourieroux and Jasiak (2006)] and let us suppose that both factors follow autoregressive gamma processes, where functions $a_g, b_g, a_c, b_c$ have the following expressions:

\[
\begin{align*}
    a_g(u) &= \frac{\rho_g u}{1 - ud_g}, \rho_g > 0, d_g > 0, u > 1/d_g, \\
    b_g(u) &= -\lambda_g \log(1 - ud_g), \lambda_g > 0, \\
    a_c(u) &= \frac{\rho_c u}{1 - ud_c}, \rho_c > 0, d_c > 0, u < 1/d_c, \\
    b_c(u) &= -\lambda_c \log(1 - ud_c), \lambda_c > 0.
\end{align*}
\]

We assume constant sensitivities with the following values: sensitivities: $\alpha = 0.01, \beta = 2, \gamma = .1$; sdf: $\nu_0 = -.01, \nu = -.2$; initial factor values: $Z_0 = .003, Z_0^i = .3$; factor dynamics: $\rho_g = .9, d_g = .1, \lambda_g = .1, \rho_c = .9, d_c = .1, \lambda_c = .1$.

Figure 1 displays the term structures of the corporate yield, the Treasury bond yield, and the spread up to $h = 40$. Bump shapes of the corporate yield and of the spread, often observed on bond markets, can easily be reproduced even with fixed sensitivities.

Figure 2 displays the decomposition of the spread into a default effect and a default-sdf correlation effect. In the experiment, the latter effect is negative and rather small in absolute value.
Figure 1 One-firm case. Corporate bond yield (solid), Treasury bond yield (dashes), spread (short dashes).

Figure 2 One-firm case. Components of the spread (solid) default effect (dashes), default–sdf correlation effect (short dashes).
5.2 Decomposition of the First-to-Default Basket Spread

Let us now consider a first-to-default basket on \( n \) firms. We still assume a general factor and univariate-specific factors, following independent autoregressive gamma processes. We also assume fixed sensitivities. The values of the parameters are portfolio size: \( n = 3 \); sensitivities: \( \alpha = .01, \beta = .05, \gamma = .01 \); sdf: \( v_o = -.15, v = .05 \); initial values: \( Z_0 = 1, Z_i^0 = 1, i = 1, 2, 3 \); factor dynamics: \( \rho_g = .9, d_g = .1, \rho_c = .9, d_c = .1, \lambda_c = 1 \).

Figure 3 displays the term structures of the first-to-default yield of the Treasury bond yield and the spread. In the example, both yields are decreasing, whereas the spread is first decreasing, and then increasing.

The decomposition of the spread is provided in Figure 4. The main component corresponds to the marginal default effect. The default correlation effect is negative (since the durations \( \tau_1, \tau_2, \tau_3 \) feature positive quadrant dependence [see Joe (1997)]), whereas the default sdf correlation effect is positive.

5.3 The Term Structure Through the Cycle

In this section we assume that the general factor follows an autoregressive gamma (ARG) process of order four. In this case, using the vector \( Z_t = (\bar{Z}_t, \bar{Z}_{t-1}, \bar{Z}_{t-2}, \bar{Z}_{t-3})' \) in order to be in the one-lag general setting, we have
Figure 4 Portfolio case. Components of the spread (solid) marginal default effect (dashes), defaults correlation effect (short dashes), default-sdf correlation effect (dots and dashes).

\[ a_g(u) = \left[ \frac{\varphi_1}{1-u_1 d_g} + u_2, \frac{\varphi_2}{1-u_1 d_g} + u_3, \frac{\varphi_3}{1-u_1 d_g} + u_4, \frac{\varphi_4}{1-u_1 d_g} \right] \]
\[ b_g(u) = -\lambda_g \log(1-u_1 d_g), \]

where \( \varphi_i > 0, d_g > 0, \lambda_g > 0, u_1 > 1/d_g. \)

The stationarity condition is \( \sum_{i=1}^{4} \varphi_i < 1. \) The process \( \tilde{Z}_t \) is positive and admits a weak AR(4) representation, which is

\[ \tilde{Z}_t = d_g \lambda_g + \sum_{i=1}^{4} \varphi_i \tilde{Z}_{t-i} + \epsilon_t. \]

We adopt the following values for the parameters

\[ \varphi_1 = 0.2, \varphi_2 = \varphi_3 = \varphi_4 = 0.1, \]
\[ d_g = 0.1, \lambda_g = 0.3. \]

The numerical values of the parameters in the sdf are \( v_0 = -0.05, v_i = -0.2, (i = 1, \ldots, 4). \) The unconditional mean of the factor process is \( d_g \lambda_g / (1 - \sum_{i=1}^{4} \varphi_i) = 0.06. \) The autoregressive dynamics has been selected to contain a cycle effect of period 4, as seen from the spectral density of the process reported on Figure 5. Such an ARG process with complex roots has no equivalent in continuous time. Moreover it is worth noting that multilag ARG processes are positive and therefore may be used to impose survivor intensities \( \lambda_i^t \) smaller than 1.
There can exist a double effect of a cyclical factor on the term structures. On the one hand, the cyclical component of the factor can influence the pattern of the term structure. Intuitively this effect is larger in the short run than in the long run, since the long-run rate is an average of forward short-run rates, and the cycle effect is smoothed by time averaging. On the other hand, the level and pattern of the term structure can also depend on the current situation, that is, if we are currently in a recession, or in an expansion period. To capture this effect we consider four environments \( (\tilde{Z}_t, \tilde{Z}_{t-1}, \tilde{Z}_{t-2}, \tilde{Z}_{t-3}) \) with mean 6%. These environments are

- **HMLM** = (10%, 6%, 2%, 6%), **MLMH** = (6%, 2%, 6%, 10 %),
- **LMHM** = (2%, 6%, 10%, 6%), **MHML** = (6%, 10%, 6%, 2 %), respectively with H = high, M = medium, L = low.

Figure 6 provides the term structures for the successive situations in the cycle. The short-run impacts can be in opposite directions, whereas the long-run interest rates stay identical.

### 6 LOSS-GIVEN-DEFAULT

#### 6.1 Extension of Affine Model for Credit Risk

The affine specification for credit risk can be extended to account for nonzero recovery rates. Let us consider a given period \( (t, t+1) \). For a default occurring in this period, the loss-given-default (LGD) is equal to one minus the recovery rate,
and denoted by $LGD_{t, t+1}$. Before default, the LGD variable and the associated recovery rate are stochastic variables with values between zero and one.

For derivative pricing, the introduction of LGD allows for distinguishing between corporate bonds with different seniorities: junior subordinated, subordinated, senior subordinated, for instance. More precisely, let us consider zero-coupon corporate bonds with time to maturity $h$. The zero-coupon corporate bond with zero recovery rate is the derivative with payoff 1 at $t + h$, if the corporation is still alive at this date, and 0 otherwise. Similarly, the zero-coupon corporate bond with recovery corresponds to a payoff 1, if the corporation is still alive at $t + h$, and to a payoff $1 - LGD_{t+k, t+k+1}$ received at $t + k + 1$, if default occurs between $t + k$ and $t + k + 1$, where the definition of LGD depends on the seniority level.

We have chosen specifications in which the recovery is paid at default time, and not at the contractual maturity [see, e.g., Duffie and Singleton (2003) and Baho and Bernasconi (2003), for other specifications]. Moreover, we have specified the payoff as a fraction of the face value and not as a fraction of the market value at default. Indeed, the level of the debt at default (i.e., the so-called Exposure-at-Default) is equal to the face value, not the market value. This definition is mandatory for the regulation and is also followed for explicit market LGD (resp. workout LGD) in the

---

9 In practice, observed LGD can be negative, or larger than one, as a consequence of penalties and recovery costs. Following a suggestion of the Basle Committee, they are truncated to (0,1) for regulatory purpose.
databases by Moody’s and S&P (resp. Fitch and S&P). However our methodology clearly applies to both cases. The prices of corporate bonds without recovery and with recovery are, respectively,

\[ C(t, t + h) = E_t \left\{ \prod_{j=0}^{h-1} [M_{t+j,t+j+1} (1 - D_{t+j,t+j+1})] \right\}, \]

and

\[ C^*(t, t + h) = \sum_{k=1}^{h} E_t \left\{ \prod_{j=0}^{k-2} [M_{t+j,t+j+1} (1 - D_{t+j,t+j+1})] M_{t+k-1,t+k} D_{t+k-1,t+k} [1 - LDG_{t+k-1,t+k}] \right\} + E_t \left\{ \prod_{j=0}^{k-1} [M_{t+j,t+j+1} (1 - D_{t+j,t+j+1})] \right\}, \]

where \( M_{t,t+1} \) denotes the sdf and \( D_{t,t+1} \) the default indicator for period \((t, t+1)\).

As above a factor representation can be introduced. This representation assumes that the sdf depends on factor values only, and that default and loss-given-default are independent given the factor value (but generally dependent, when the factor is integrated out). Thus this approach can be considered as an extension of the approach of Brennan and Schwartz (1980) and Duffee (1998), in which the creditor receives an endogenous random fraction of face value immediately upon-default. This approach allows for a symmetric treatment of probability of default and loss-given-default. It avoids additional assumptions, such as the so-called recovery of market value [Duffee and Singleton (1999)], in which “the expected risk-neutral recovery rate is a predetermined fraction of the risk-neutral expected survival market value.” In fact, such a hypothesis can easily be tested in our framework.

The derivative prices can be rewritten in terms of survivor intensity and expected loss-given-default, computed conditional on factor values. We get

\[
C(t, t + h) = E_t \left\{ \prod_{j=0}^{h-1} [M_{t+j,t+j+1} (1 - PD_{t+j,t+j+1})] \right\}
\]

= \( E_t \left\{ \prod_{j=0}^{h-1} [M_{t+j,t+j+1} \pi_{t+j,t+j+1}] \right\}, \]

and

\[
C^*(t, t + h) = \sum_{k=1}^{h} E_t \left\{ \prod_{j=0}^{k-2} [M_{t+j,t+j+1} (1 - PD_{t+j,t+j+1})] M_{t+k-1,t+k} PD_{t+k-1,t+k} (1 - ELGD_{t+k-1,t+k}) \right\} 
\]

+ \( E_t \left\{ \prod_{j=0}^{k-1} [M_{t+j,t+j+1} (1 - PD_{t+j,t+j+1})] \right\}
\]

= \( \sum_{k=1}^{h} E_t \left\{ \prod_{j=0}^{k-2} [M_{t+j,t+j+1} \pi_{t+j,t+j+1}] M_{t+k-1,t+k} (1 - \pi_{t+k-1,t+k}) (1 - ELGD_{t+k-1,t+k}) \right\} 
\]

+ \( E_t \left\{ \prod_{j=0}^{k-1} [M_{t+j,t+j+1} \pi_{t+j,t+j+1}] \right\}, \)

where PD and ELGD denote the default probability and the expected loss-given-default, respectively.

The model is completed by introducing exponential affine specifications, and factors with CaR dynamics in order to use the closed-form expressions of the conditional Laplace transform of the factors. At this step, the exponential affine specification can be written either for the expected loss-given-default, or for the
expected recovery rate. In this model the “risk-neutral” expected loss-given-default and the risk-neutral expected survival market value are both exponential affine functions of the current values of the factors. The price of corporate bonds with recovery are available in a quasi-explicit form, using straightforward generalizations of the recursive methods presented above [see Baho and Bernasconi (2003) for details]. Moreover, the recovery at market value assumption is easily tested in this framework by checking if the coefficients of the factors are the same and just the intercepts differ.

Finally, more detailed spread decomposition can be derived for corporate bonds to highlight the term structure of loss-given-default, the term structure of dependence between default probability and expected loss-given-default, and so on.

6.2 Correlations

Any joint modeling of default and loss-given-default must reproduce the adverse correlations observed on historical data [Altman et al. (2003), Basel Committee on Banking Supervision (2005)]. The recovery rates are on average lower and the default probabilities higher in recessions, explaining the positive (negative) correlations observed between default and loss-given-default (default and recovery rates). These marginal correlations can coexist with conditional dependence of opposite sign at the moment of downturn of the cycle. Therefore any exponential affine intensity model has to allow for this dependence feature. Let us discuss this point on the example of a three-factor model, where

\[
PD = \exp(x_0 + x_1Z_1 + x_2Z_2 + x_3Z_3),
\]

\[
ELGD = \exp(\beta_0 + \beta_1Z_1 + \beta_2Z_2 + \beta_3Z_3),
\]

Standard affine dynamics are as follows:

(i) a Gaussian VAR(1) (Ornstein-Uhlenbeck) process of \((Z_1, Z_2, Z_3)\);

(ii) independent autoregressive gamma (CIR) processes;

(iii) a WAR process for the stochastic matrix \(\begin{pmatrix} Z_1 & Z_2 \\ Z_2 & Z_3 \end{pmatrix}\).

The choice of a Gaussian factor process does not ensure PD and ELGD between zero and one. The choice of independent CIR process ensures PD and ELGD in \((0,1)\) if and only if \(x_0 = \beta_0, x_1 < 0, x_2 < 0, x_3 < 0, \beta_1 < 0, \beta_2 < 0, \beta_3 < 0\). Thus a shock on a factor, \(Z_1\), say, the other types of factors being fixed, has the same type of effect on PD and ELGD. Automatically we get conditional and unconditional positive links between the two exponential affine functions of the CIR factors.

Let us now discuss positive and negative dependence for models with Wishart factors. Since Wishart processes are valued in the set of volatility matrices, we introduce matrix notations:

\[
PD = \exp[-\text{Tr}(AZ)], \quad ELGD = \exp[-\text{Tr}(BZ)],
\]
where $Z = \begin{pmatrix} Z_{11} & Z_{12} \\ Z_{12} & Z_{22} \end{pmatrix}$, $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix}$, $B = \begin{pmatrix} b_{11} & b_{12} \\ b_{12} & b_{22} \end{pmatrix}$, are symmetric positive definite matrices. The stochastic matrix factor $Z$ follows marginally a standard Wishart distribution with $K$ degrees of freedom and Laplace transform [see Gourieroux (2006)]:

$$E \exp[-Tr(UZ)] = \det(Id + 2U)^{-K/2},$$

where $U$ is a symmetric matrix.

Since $A$ and $B$ are symmetric positive definite matrices, PD and ELGD are between 0 and 1. Let us now consider their explicit expressions:

$$PD = \exp[-(a_{11}Z_{11} + a_{22}Z_{22} + 2a_{12}Z_{12})],$$

$$ELGD = \exp[-(b_{11}Z_{11} + b_{22}Z_{22} + 2b_{12}Z_{12}).$$

The sensitivity factor coefficients are constrained by

$$a_{11} > 0, a_{11}a_{22} - a_{12}^2 > 0, b_{11} > 0, b_{11}b_{22} - b_{12}^2 > 0.$$

These constraints are compatible with opposite signs for $a_{12}$ and $b_{12}$. Then, a shock on the stochastic volatility $Z_{12}$ (or equivalently on the stochastic correlation), given the stochastic volatilities $Z_{11}$, $Z_{22}$, implies opposite effects on $\exp[-Tr(AZ)]$ and $\exp[-Tr(BZ)]$. Thus we get a negative correlation for given volatilities. Let us now show that, historically, the effects of shocks on stochastic covolatility cannot dominate the effects due to shocks on volatilities $Z_{11}$ and $Z_{22}$. More precisely, the negative dependence given stochastic volatilities is compatible with a positive unconditional dependence. Let us consider

$$\text{cov}[DP, LGD] = \text{cov}(\exp[-Tr(AZ)], \exp(-Tr(BZ))] = \frac{1}{\det(Id + 2(A + B))} - \frac{1}{\det(Id + 2A)} \frac{1}{\det(Id + 2B)}.$$

The covariance is positive if and only if

$$\Delta = \det(Id + 2A)\det(Id + 2B) - \det(Id + 2(A + B)) \geq 0.$$

Without loss of generality, we can assume $a_{12} = 0$. We get

$$\Delta = 8(b_{11}b_{22} - b_{12}^2)(a_{11} + a_{12} + 2a_{11}a_{22}) + 4[a_{11}b_{11} + a_{22}b_{22} + 2a_{11}a_{22}(b_{11} + b_{22})] \geq 0$$

by the positiveness condition on matrices $A$ and $B$.

---

10 The standard Wishart distribution is the marginal (invariant) distribution of a Wishart autoregressive process.
6.3 Marginal Distribution of LGD

Finally, it is interesting to mention that, in the exponential affine specification with ARG or Wishart factors, the marginal and conditional distributions of LGD can admit a variety of patterns similar to the standard pattern of the beta distribution. More precisely, this distribution corresponds to the distribution of $Y = \exp(-bZ)$, where $Z$ follows a gamma distribution with $v$ degrees of freedom. The associated density function is

$$f(y) = \frac{1}{\Gamma(v)} \frac{1}{b^v} (-\log y)^{v-1} y^{1/b-1} 1_{(0,1)}(y).$$

These densities are either bell-shaped (if $v < 1$, $b < 1$), monotonic with a mode on one boundary (if $v < 1$, $b > 1$, or $v > 1$, $b < 1$), or $U$-pattern with infinite values at both zero and one (if $v > 1$ and $b > 1$). Thus this family is able to provide good fit to observed LGD distributions.

7 CONCLUSION

We have described in this article the affine specification for the analysis of credit risk in a discrete-time framework. This framework assumes a stochastic discount factor, survivor intensities, and loss-given-default (or recovery rates), which are exponential affine functions of CaR factor processes. The affine framework offers a coherent description of the Treasury bond, corporate bond prices with or without recovery, and first-to-default basket, and tractable methods for predicting the risk included in a portfolio of corporate bonds, while taking into account any possible dependence between default and loss-given-default.

The discrete-time affine specification is more flexible than the continuous-time affine specification.

(i) By considering a much larger set of affine dynamics, it provides better data fit, can take into account the procyclical effect existing in default probabilities, loss-given-default, and risk-free rates, or allow for lagged causality and recursive effects.

(ii) From a numerical point of view, the discrete-time affine specification is easy to implement and avoids the numerical approximation of the standard Riccati equations. This allows a diminution of the number of iterations needed to derive the credit derivative prices, while being still compatible with the no-arbitrage opportunity.

(iii) It seems more appropriate for estimation purposes, since the data on failure are available in discrete time (monthly), and for simulation purpose, since the Credit-VaR has to be evaluated in discrete time too.
The discrete-time affine model can also be used to analyze other events, such as the up or downgrades by rating agencies. Such an analysis has to be done before using the so-called migration models [Gagliardini and Gourieroux (2005b), Feug et al. (2006)].

After having surveyed practitioners and academic research, the LGD working group of the Basel Committee pointed out the following three findings (Basel Committee on Banking Supervision (2005)):

1. “The potential for recovery rates to be lower than average during times of high default rates.”
2. “Data limitation posing an important challenge.”
3. “The little consensus for incorporating downturn conditions in LGD.”

As seen in this article, the affine model in discrete time is sufficiently flexible to get the expected sign of conditional dependency between LGD and default (point 1), or to incorporate and predict downturns by the introduction of unobservable factors with cyclical dynamics (point 3).

Nevertheless, data limitation is still a challenge. If databases on rating migrations are sufficient to estimate nonlinear dynamic factor models, and get reliable results on default [see, e.g., Gagliardini and Gourieroux (2005b), Feug et al. (2006)], the lack of a sufficient number of data on workout LGD is clearly a problem. Currently, available microeconomic data on LGD can mainly be used to get unconditional information on the link between LGD and default, or on the type of unconditional LGD distribution, but not to get reliable results after conditioning on observable or unobservable factors. To circumvent this difficulty, it has been proposed to use market data on corporate bonds. But a new technical difficulty arises. The credit derivative returns are submitted to parametric domain restrictions, which may have to be taken into account to avoid inconsistent, or inefficient, calibration procedures [Gourieroux and Monfort (2005)]. This question is clearly beyond the scope of the present article.

APPENDIX 1: CONDITIONAL MULTIVARIATE LAPLACE TRANSFORM FOR A CaR PROCESS

Let us consider a multivariate CaR process, which satisfies

$$E[\exp(\mathbf{u}' Z_{t+1})|Z_t] = \exp[\mathbf{a}(\mathbf{u}') Z_t + b(\mathbf{u})].$$

For any deterministic sequence [\mathbf{u}] of vectors \{\mathbf{u}_s, s = 1, \ldots\}, let us define the transformation

$$E[\exp(\mathbf{u}'_{t+1} Z_{t+1} + \ldots + \mathbf{u}'_{t+h} Z_{t+h})|Z_t],$$

which provides the conditional joint Laplace transform of \(Z_{t+1}, \ldots, Z_{t+h}\). The following lemma holds:
Lemma 1 For any deterministic sequence \([u]\) of vectors \(\{u_s, s = 1, \ldots\}\), we have
\[
E[\exp(u'_{t+1} Z_{t+1} + \ldots + u'_{t+h} Z_{t+h})|Z_t] = \exp[A^{[u]}(t, t+h)Z_t + B^{[u]}(t, t+h)],
\]
where the operators \(A^{[u]}\) and \(B^{[u]}\) depend on functions \(a(.)\), \(b(.)\) as well as on sequence \([u]\), and satisfy the backward recursion
\[
\begin{align*}
A^{[u]}(t, t+h) &= a[u_{t+1} + A^{[u]}(t+1, t+h)], \\
B^{[u]}(t, t+h) &= b[u_{t+1} + A^{[u]}(t+1, t+h)] + B^{[u]}(t+1, t+h),
\end{align*}
\]
for \(h > 0\), with terminal conditions
\[
A^{[u]}(t, t) = 0, B^{[u]}(t, t) = 0, \forall t.
\]

Proof We have
\[
E[\exp(u'_{t+1} Z_{t+1} + \ldots + u'_{t+h} Z_{t+h})|Z_t]
\]
\[
= E\left(E[\exp(u'_{t+1} Z_{t+1} + \ldots + u'_{t+h} Z_{t+h})|Z_{t+1}]|Z_t\right)
\]
\[
= E(\exp[u'_{t+1} Z_{t+1} + A^{[u]}(t+1, t+h)' Z_{t+1} + B^{[u]}(t+1, t+h)]|Z_t)
\]
\[
= E(\exp([u_{t+1} + A^{[u]}(t+1, t+h)]' Z_{t+1})|Z_t) \exp(B^{[u]}(t+1, t+h))
\]
\[
= \exp\{a[u_{t+1} + A^{[u]}(t+1, t+h)]' Z_t + B^{[u]}(t+1, t+h) + b[u_{t+1} + A^{[u]}(t+1, t+h)]\}.
\]
The recursion follows by identification.
Moreover, the terminal conditions are satisfied, since
\[
A^{[u]}(t, t+1) = a(u_{t+1}), B^{[u]}(t, t+1) = b(u_{t+1})
\]
are deduced from the recursive equations applied with \(A^{[u]}(t+1, t+1) = 0, B^{[u]}(t+1, t+1) = 0\).

APPENDIX 2: CONDITIONAL SURVIVOR FUNCTIONS

(i) Let us first compute the survivor function of one firm \(i\), given \(Z\) and \(Z_i\). We deduce from Assumption 2 that
\[
P[\tau_i > h|Z, Z_i] = \Pi_{t=0}^{h} \exp\{-(\alpha_i + \beta_i Z_t + \gamma_i' Z_i')\},
\]
where \(Z = (Z_t, \forall t \geq 0), Z_i = (Z_i', \forall t \geq 0)\).
In particular, \(P[\tau_i > h|Z, Z_i'] = P(\tau_i > h|Z_h, Z_i')\),
where \(Z_h = (Z_t, t \leq h), Z_i' = (Z_i', t \leq h)\). We also deduce that
\[
\begin{align*}
  P[\tau_i \leq h_t | Z, Z_t'] &= P[\tau_i \leq h_t | Z_h, Z_h'], \\
  P[\tau_i = h_t | Z, Z_t'] &= P[\tau_i = h_t | Z_h, Z_h'].
\end{align*}
\]

(ii) Then, the joint survivor function given the entire realization of the factor path is

\[
\begin{align*}
  P[\tau_i > h_t, i \in S | Z, Z_t', j = 1, \ldots, n] &= P[\tau_i > h_t, i \in S | Z, Z_t', j \in S] \\
  &= \prod_{i \in S} P[\tau_i > h_t | Z, Z_t'] \\
  &= \prod_{i \in S} \prod_{j=1}^{h_i} \exp\{-\langle \gamma_j + \beta_j Z_i, \gamma_j' Z_{t,j} \rangle\}.
\end{align*}
\]

(iii) We deduce that

\[
\begin{align*}
  P[\tau_i > t + h_t, i \in S | PaR_t, S \subseteq PaR_t, \tau_j, j \in PaR_t, Z_t, Z_t', j = 1, \ldots, n] &= \frac{P[\tau_i > t + h_t, i \in S, \tau_k > t, k \in PaR_t - S | Z_t, Z_t', j \in PaR_t]}{P[\tau_i > t, i \in PaR_t | Z_t, Z_t', j \in PaR_t]} \\
  &= \frac{E\left[\prod_{i \in S} \prod_{j=1}^{h_i} \exp\left(-\langle \gamma_j + \beta_j Z_i, \gamma_j' Z_{t,j} \rangle\right) | Z_t, Z_t', j \in S\right]}{\prod_{i \in S} \prod_{j=1}^{h_i} \exp\left(-\langle \gamma_j + \beta_j Z_i, \gamma_j' Z_{t,j} \rangle\right)} \\
  &= E\left[\prod_{i \in S} \prod_{k=1}^{h_i} \exp\left(-\langle \gamma_{t+k}^{i} + \beta_{t+k}^{i} Z_{t+k} - \gamma_{t+k}^{i}, \gamma_{t+k}^{i} Z_t \rangle\right) | Z_t\right] \\
  &= E\left[\prod_{k=1}^{h_i} \exp\left(-\langle \gamma_{t+k}^{i} Z_{t+k} - \gamma_{t+k}^{i}, \gamma_{t+k}^{i} Z_t \rangle\right) | Z_t\right] \\
  &= E\left[\prod_{k=1}^{h_i} \exp\left(-\langle \gamma_{t+k}^{i} Z_{t+k} - \gamma_{t+k}^{i}, \gamma_{t+k}^{i} Z_t \rangle\right) | Z_t\right] \\
  &= E[\prod_{k=1}^{h_i} \exp\left(-n_{t+k} \gamma_{t+k}^{i} Z_{t+k} \right) | Z_t] \\
  &= E[\prod_{k=1}^{h_i} \exp\left(-n_{t+k} \gamma_{t+k}^{i} Z_{t+k} \right) | Z_t] \\
  &= \prod_{k=1}^{h_i} \exp\left(-n_{t+k} \gamma_{t+k}^{i} Z_{t+k} \right).
\end{align*}
\]

where \( h = \max_{j \in S} h_i \) and \( n_{t+k} \) denote the number of firms in set \( S \) with \( h_i > k \). Therefore the conditional survivor function can be deduced from the conditional Laplace transform of CaR process \( Z \) (see Appendix 1, Lemma 1).

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