Econometric specification of stochastic discount factor models

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Abstract

We consider the problem of derivative pricing when the stochastic discount factors are exponential-affine functions of underlying state variable. In particular we discuss the conditionally Gaussian framework and introduce semi-parametric pricing methods for models with path dependent drift and volatility. This approach is also applied to more complicated frameworks, such as pricing of a derivative written on an index, when the interest rate is stochastic.

Résumé

Nous considérons le problème de valorisation de produits dérivés, lorsque les facteurs d’escompte stochastiques sont fonctions exponentielles-affines de facteurs sous-jacents. En particulier nous discutons le cas conditionnellement gaussien et développons des méthodes de valorisation semi-paramétriques, pour des modèles où les paramètres d’échelle et de position dépendent de l’historique. Cette approche est également appliquée à des contextes plus complexes, tels que la valorisation d’un dérivé sur indice en présence de taux d’intérêt stochastique.

JEL classification: C1; C5; G1

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Mots clés: Valorisation; Transformée d’Esscher; facteur d’escompte stochastique; Modèle variance-gamma; valorisation semi-paramétrique
1. Introduction

Derivative pricing is generally based on continuous time models, which may include latent stochastic factors, or jumps. This approach provides formulas for derivative prices, expressed either as conditional expectations, or solutions of partial differential equations. However the formulas are often difficult to implement, since both data and hedging strategies are in discrete time. This explains why standard continuous time models have a rather simple structure (see e.g. Black and Scholes, 1973; Hull and White, 1987; Melino and Turnbull, 1990; Heston, 1993). The number of underlying factors is generally small, the risk premia associated with these factors are assumed constant or even equal to zero, and the term structure of interest rates is often treated independently of the index derivatives. However these restrictions, that simplify the implementation of the derivative pricing formulas, induce a poor fit when time series data, or cross-sectional data on derivative prices are considered.

This paper addresses the problem of deriving pricing using the stochastic discount factor introduced in Harrison and Kreps (1979), Garman and Ohlson (1980), Hansen and Richard (1987), Hansen and Jagannathan (1991) and Bansal and Viswanathan (1993). It is known that, if agents make their investments at date $t$ ($t \in \mathbb{N}$) based on an information set $J_t$, and under the absence of arbitrage, the prices of actively traded assets satisfy a linear valuation formula. More precisely there exists a (nonnegative) stochastic discount factor (sdf) $M_{t,t+1}$ function of the updated information set $J_{t+1}$, such that the price of an asset that provides payoff $g_{t+1}$ at date $t+1$ is

$$C_t(g) = E[M_{t,t+1}g_{t+1}|J_t].$$

(1.1)

Since the market is incomplete in discrete time, there exists a multiplicity of Sdfs that are compatible with the valuation formula (1.1) for the actively traded assets (Breeden and Litzenberger, 1978).

In this situation we can simply note the multiplicity of prices for illiquid (derivative) assets and refuse to price them (or propose very large admissible price intervals) (see the discussion by Delbaen et al., 1994). We can also select the (martingale) minimal price at the risk of underestimating the value. However these strategies are generally not followed in accounting, economics and finance. Let us for instance consider accounting. The banks and financial institutions report regularly their balance sheets. These balance sheets include a number of physical or financial assets, which are not liquid, but nevertheless have to be priced. This is typically the case of illiquid derivatives, but also of patents, machines used for productions, or stock of unsold outputs. Let us focus on machines. Generally there does not exist an active second hand market for such machines and second hand market prices are not observed. The accounting practice consists in introducing a convention for explaining how the value of a machine evolves with its age, that is an amortizing scheme. This convention is only one value evolution among an infinite number of admissible ones. For practical purpose two or three alternative conventions have been retained for accounting, with for instance the possibility to choose the amortizing time.

There is also a need for such conventions whenever “prices” are used for comparing illiquid assets.1 This occurs if we have to construct a balance sheet, to determine the capital

1Another solution proposed in the theoretical literature, but also by financial practitioners, is to create new financial markets, and then to price marked-to-market. Typically, since individual corporate default is difficult to
required to balance the risk included in a portfolio of illiquid derivative assets (the so-called Value-at-Risk), to compare competing investment strategies on the basis of future profit...

In our framework the conventions consist in restricting a priori the set of admissible stochastic discount factors. These conventions have to be compatible with some available price data, have to include conventions previously used in the incomplete market framework, have to provide rather simple pricing formula and have to be easy to understand. In this paper we consider a class of sdf, that are exponential functions of an affine combination of factors. This specification corresponds to the Esscher transform used in insurance (see Esscher, 1932), and for derivative pricing (see the pioneer paper by Gerber and Shiu 1994a, b, c and applications in Buhlman et al., 1996, 1998; Shyraev, 1999; Gourieroux and Monfort, 2006; Yao, 2001). Next we discuss the restrictions on the sdf implied by valuation formula (1.1). In Section 2, we review standard pricing formulas derived from either an equilibrium condition (such as the consumption based CAPM, or the model with recursive utility), or the condition of no arbitrage in a continuous time framework (see Hull and White, 1987; Heston, 1993). The aim is to justify the exponential-affine specification of the sdf and to give examples of possible factors. In Section 3, we consider a simple framework in which the riskfree rate is constant (equal to zero) and the information set includes the returns on actively traded assets only. We prove that there exists a single sdf compatible with the exponential-affine specification. Then we discuss in Section 4 the conditionally Gaussian framework and the semi-parametric models with path dependent drift and volatility. These examples are used to derive a semi-parametric pricing method. Section 5 extends the basic approach to the framework of stochastic interest rates and unobservable factors. As an illustration we discuss the conditionally Gaussian factors framework and explain how to price a derivative written on an index when the interest rate is stochastic. Section 6 concludes.

2. Examples of stochastic discount factors

In financial theory the expressions of stochastic discount factors are generally derived under either an equilibrium condition, or arbitrage free restrictions in a continuous time framework. We provide below some examples to show that stochastic discount factors considered in the literature are usually exponential-affine functions of underlying state variables.

Example 1. Consumption based CAPM (CCAPM): In the standard CCAPM, an agent maximizes his expected intertemporal utility expressed in terms of a physical good (Lucas, 1978). The intertemporal transfers are ensured by means of investments on a financial
market. Let us denote by \( U \) the utility function, by \( \delta \) the intertemporal psychological discount rate, and assume the existence of a representative agent. At equilibrium we get the relation:

\[
p_t = E_t \left[ p_{t+1} \frac{q_{t+1}}{q_t} \delta \frac{dU}{dc} \left( \frac{C_{t+1}}{C_t} \right) \right],
\]

where \( p_t \) is the vector of prices of financial assets, \( q_t \) the price of the consumption good and \( C_t \) the quantity consumed at date \( t \).

(i) Thus, for a power utility function, the sdf associated with the consumption based CAPM is

\[
M_{t,t+1} = \frac{q_t}{q_{t+1}} \delta \left( \frac{C_{t+1}}{C_t} \right) = \exp \left[ \log \delta - \log \frac{q_{t+1}}{q_t} + \gamma \log \frac{C_{t+1}}{C_t} \right].
\]

It is an exponential-affine function of two state variables,\(^3\) which are the inflation rate and the rate of change in consumption. It does not depend on asset returns. However the asset returns, that can be considered as additional state variables, are generally included in the available information set, and thus also influence the derivative prices.

(ii) For a CARA utility function we get

\[
\log \left[ \frac{dU}{dc} \left( \frac{C_{t+1}}{C_t} \right) \right] \frac{dU}{dc} \left( C_t \right) = -A (C_{t+1} - C_t).
\]

The sdf is

\[
M_{t,t+1} = \exp \left[ \log \delta - \log \frac{q_{t+1}}{q_t} - A (C_{t+1} - C_t) \right],
\]

which is also exponential-affine.

Thus the choice of a power or CARA utility function is equivalent to the selection of an appropriate (parameter free) transformation of the consumption as state variable.

**Example 2.** Recursive utility: Similar results can be derived, when the representative agent maximizes a recursive utility (see Weil, 1989; Epstein and Zin, 1991). If the utility is a power function and the aggregator admits a Cobb-Douglas form, the sdf is given by

\[
M_{t,t+1} = \delta^{x/\rho} \left( \frac{C_{t+1}}{C_t} \right)^{-\frac{(1-\rho)\alpha}{\rho}} \left( \frac{W_{t+1}}{W_t} \right)^{-\frac{1-x}{\rho}} \left( \frac{q_{t+1}}{q_t} \right)^{-\frac{1}{\rho}}
\]

\[
= \exp \left[ \frac{1}{\rho} \log \delta - \frac{(1-\rho)\alpha}{\rho} \log \frac{C_{t+1}}{C_t} - (1-\frac{x}{\rho}) \log \frac{W_{t+1}}{W_t} - \frac{1}{\rho} \log \frac{q_{t+1}}{q_t} \right].
\]

The exponential-affine function now involves a third state variable \( \log(W_{t+1}/W_t) \), that represents the return on market portfolio.

\(^3\)This is also an exponential-affine function of parameters \( \gamma, \log \delta \). This latter property is useful for estimation purpose.
Example 3. Stochastic volatility model: Standard stochastic volatility models (Hull and White, 1987; Heston, 1993) are

\[
\begin{align*}
    dS_t &= \mu S_t dt + \sigma_t S_t dW^S_t, \\
    df(\sigma_t) &= a(\sigma_t) dt + b(\sigma_t) dW^\sigma_t,
\end{align*}
\]

where \( S_t \) is the asset price, \( \sigma_t \) the stochastic volatility and \((W^S_t), (W^\sigma_t)\) two independent Brownian motions. For a constant riskfree rate \( r \), the discount factor is deduced from Girsanov’s theorem and given by

\[
M_{t,t+1} = \exp\left(-r\right) \exp\left\{ -\left(\mu - r\right) \int_t^{t+1} \frac{dW^S_\tau}{\sigma_\tau} - \frac{1}{2} \left(\mu - r\right)^2 \int_t^{t+1} \frac{d\tau}{\sigma_\tau^2} \right\} \\
\times \exp\left\{ -\int_t^{t+1} \nu_\tau dW^\sigma_\tau - \frac{1}{2} \int_t^{t+1} \nu_\tau^2 d\tau \right\},
\]

where \( \nu_\tau \) is the risk premium associated with stochastic volatility. This premium can be selected arbitrarily as a function of the past due to the incomplete market framework. By approximating the integrals by their discrete time counterpart, we note that

\[
M_{t,t+1} \approx \exp\left(-r\right) \exp\left\{ -\left(\mu - r\right) \frac{\nu_{t+1}^S}{\sigma_t} - \frac{1}{2} \left(\mu - r\right)^2 \frac{1}{\sigma_t^2} - \nu_{t+1}^\sigma - \frac{1}{2} \frac{\nu_t^2}{\sigma_t^2} \right\}.
\]

It is an exponential-affine function of innovations \( \nu_{t+1}^S, \nu_{t+1}^\sigma \) corresponding to both return and volatility processes, with path dependent coefficients.

The examples above show:

(i) the convenience of exponential-affine specifications, which ensure both the positivity and tractability of the sdf;\(^4\)
(ii) the various candidates for state variables, including financial returns, innovations on financial returns or volatilities, rate of change in consumption, inflation rate, etc; these state variables can be observable or latent, financial or physical;
(iii) the multiplicity of specifications in the incomplete market framework due to the arbitrary choice of the risk premium for non traded factors.

In the next section, we adopt an approach in which admissible forms of the sdf are specified a priori to avoid the identification problem. Then this set is structured by taking into account the arbitrage free restrictions.

3. SDF modelling: the principle

To present the modelling principle, let us first consider the framework of a riskfree asset with zero riskfree rate\(^5\) and several risky assets with prices \( p_{j,t}, j = 1, \ldots, J \), and (geometric)

\(^4\)Exponential affine specifications also arise when the risk neutral distribution is at the minimum Kullback–Leibler distance of the historical distribution and satisfies the constraints inferred by option prices (see e.g. Stutzer, 1995, 1996; Buchen and Kelly, 1996). More precisely in this approach the sdf is an exponential-affine function of the derivative payoffs.

\(^5\)If the future evolution of the riskfree rate is known at date \( t \), it is possible to get a zero riskfree rate by a deterministic change of numeraire. The case of a predetermined stochastic interest rate is considered in Section 7.
returns: \( r_{j,t+1} = \log(p_{j,t+1}/p_{j,t}) \). We assume that these assets are actively traded on the markets, and that the different prices are observable by the investor.

### 3.1. The historical distribution

The (conditional) historical distribution of the return \( r_{t+1} = (r_{1,t+1}, \ldots, r_{J,t+1})' \) is defined by means of its (conditional) Laplace transform (also called moment generating function) supposed to belong to a parametric set:

\[
E[\exp(u' r_{t+1}) | r_t] = \exp \psi_j(u; \theta) \quad \text{(say)},
\]

where \( r_t = (r_t, r_{t-1}, \ldots) \) and \( \theta \) denotes the parameters. The Laplace transform is defined on a convex set, that depends on the tails of the conditional distribution. We assume below that this convex set is not reduced to the origin.

### 3.2. The stochastic discount factor

We assume a priori that the sdf can be written under an exponential-affine form:

\[
M_{t,t+1} = \exp(\beta_t' r_{t+1} + \beta_t),
\]

(coefficients \( \alpha_t \) and \( \beta_t \), that can be path dependent, that is function of the return history:6 \( r_t = (r_t, r_{t-1}, \ldots) \).

By writing the pricing formula for the riskfree asset and the \( J \) risky assets, we get \( J + 1 \) restrictions on the sdf and the historical distribution. More precisely the constraints induced by the arbitrage free conditions are

\[
\begin{align*}
E(M_{t,t+1} | r_t) &= 1, \\
E \left[ M_{t,t+1} \frac{p_{j,t+1}}{p_{j,t}} | r_t \right] &= E[M_{t,t+1} \exp r_{j,t+1} | r_t] = 1, \quad j = 1, \ldots, J,
\end{align*}
\]

\[
\iff \begin{align*}
E[\exp(\alpha_t' r_{t+1} + \beta_t) | r_t] &= 1, \\
E[\exp(\alpha_t' r_{t+1} + e_j' r_{t+1} + \beta_t) | r_t] &= 1, \quad j = 1, \ldots, J,
\end{align*}
\]

where \( e_j = (0, \ldots, 0, 1, 0, \ldots, 0)' \), with 1 as component of order \( j \).

\[
\iff \begin{align*}
\exp[\psi_j(\alpha_t; \theta) + \beta_t] &= 1, \\
\exp[\psi_j(\alpha_t + e_j; \theta) + \beta_t] &= 1, \quad \forall j = 1, \ldots, J, \\
\beta_t &= -\psi_j(\alpha_t; \theta), \\
\psi_j(\alpha_t + e_j; \theta) - \psi_j(\alpha_t; \theta) &= 0, \quad \forall j = 1, \ldots, J.
\end{align*}
\]

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6In a framework without derivatives, the entropy based approach (Stutzer, 1995, 1996; Buchen and Kelly, 1996) leads to an exponential-affine sdf where the returns are computed as \( r_{j,t+1} = p_{j,t+1}/p_{j,t} - 1 \). We have retained geometric returns instead of arithmetic returns to get tractable arbitrage restrictions and to include the standard Black–Scholes as a special case.
This system of \( J + 1 \) equations for \( J + 1 \) unknown parameters generally admits a unique solution:
\[
\begin{align*}
\alpha_t &= \alpha(r_t; \theta), \\
\beta_t &= \beta(r_t; \theta) = -\psi_t[\alpha(r_t; \theta), \theta], \text{ say.}
\end{align*}
\]

Then we deduce a unique form of the exponential-affine sdf that satisfies the condition of no arbitrage. The associated risk neutral distribution \( Q \) admits the conditional Laplace transform:
\[
\begin{align*}
\mathbb{E}_Q[\exp(u'r_{t+1})|r_t] &= \mathbb{E}[M_{t,t+1} \exp(u'r_{t+1})|r_t]/\mathbb{E}[M_{t,t+1}|r_t] \\
&= \mathbb{E}[(\alpha_t + u; \theta) + \beta_t] \\
&= \exp[\psi_t(x_t + u; \theta) - \psi_t(x_t; \theta)].
\end{align*}
\]

This result is summarized in the proposition below.

**Proposition 1.** For a zero riskfree rate and an exponential-affine sdf with state variable \( r_{t+1} \), there exists in general a unique admissible risk-neutral distribution. Its conditional Laplace transform is given by
\[
\begin{align*}
\mathbb{E}_Q[\exp(u'r_{t+1})|r_t] &= \exp[\psi_t(x_t + u; \theta) - \psi_t(x_t; \theta)],
\end{align*}
\]

where \( \alpha_t \) is the solution of:
\[
\psi_t(x_t + e_j; \theta) = \psi_t(x_t; \theta), \quad j = 1, \ldots, J,
\]

and \( \psi_t \) denotes the conditional historical Log-Laplace transform.

In discrete time the market is incomplete. Therefore, there exists an infinity of admissible risk neutral distributions. The uniqueness condition in Proposition 1 is due to the convention of exponential-affine sdf.\(^7\)\(^8\) We will explain in the next section how to introduce a multiplicity of sdf by means of unobservable stochastic factors with parameterized risk premia.

By using the sdf \( M_{t,t+1} = M_{t,t+1}(r_{t+1}; \theta) \), we can propose a price for any derivative written on \( r_t \).\(^9\) Let us denote by \( g(r_{t+h}; t+h, h = 1, \ldots, H \) the payoffs provided

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\(^7\)Some other constraints could have been imposed on \( M_{t,t+1} \). For instance (Hansen and Jagannathan, 1997; Cochrane, 2001) assume an affine specification \( M_{t,t+1} = a_0 r_t + b_t \). This convention does not ensure the positivity of the underlying state prices, is not compatible with the absence of arbitrage opportunity, and does not extend the standard specifications seen in Section 2.

\(^8\)The dynamics of the return under the historical and risk neutral distributions can be very different. For instance they can feature stationarity or nonstationarity under the risk neutral distribution whereas the historical world is stationary. This question is difficult to study in the general framework of this paper (see the discussion in Gourieroux et al., 2002b for affine models of interest rates).

\(^9\)The derivatives are usually written on prices and not directly on returns. However, let us consider a European call for instance, with payoff \( (S_{t+1} - K)^+ \) at date \( t+1 \). This derivative is \( S_t \) times the derivative with payoff \( (exp r_{t+1} - k_t)^+ \), where \( k_t = K/S_t \) is the moneyness strike. Thus we are just assuming a preliminary transformation of the payoff.
at $t + h, h = 1, \ldots, H$. The derivative price is

$$C_t(g) = \sum_{h=1}^{H} E[M_{t,t+1}(r_{t+1}; \theta) \ldots M_{t+h-1,t+h}(r_{t+h}; \theta)g(r_{t+h}; t + h)| r_t].$$

(3.2)

In general this price cannot be computed analytically and is approximated by simulations. The simulations can be performed under the historical probability, or the RN distribution, or under another modified probability (see Gourieroux and Jasiak, 2001, Chapter 13, Section 5.2, for a discussion).

3.3. Information and time aggregation

The assumption of exponential-affine sdf depends on the selected information set and time horizon. To illustrate this dependence let us consider below two examples.

(i) If we are interested on derivatives based on a subset of risky assets with returns $r^a_{t+1}$, the basic pricing formula (1.1) applied to $g(r^a_{t+1})$ can be written as

$$C_t(g) = E[M_{t,t+1}g(r^a_{t+1})| r_t] = E[M^*_t,g(r^a_{t+1})| r_t],$$

where $M^*_t,g = E[M_{t,t+1}g(r^a_{t+1})| r_t]$. This modified sdf does not admit in general an exponential-affine expression $M^*_t,g = \exp(\alpha^a_t r^a_{t+1} + \beta^a_t)$, even if the initial sdf does. Thus the information set matters.

(ii) Moreover the assumption of exponential-affine sdf is not stable by a change of time unit. Let us consider pricing at horizon 2 of a European derivative written on the short rate. The price is given by

$$C_t(g) = E[M_{t,t+1}M_{t+1,t+2}g(r_{t+2})| r_t] = E[E[M_{t,t+1}, M_{t+1,t+2}| r_{t+2}, r_t]g(r_{t+2})| r_t].$$

$E[M_{t,t+1}M_{t+1,t+2}| r_{t+2}, r_t]$ is not exponential-affine w.r.t. $r_{t+2}$ even if $M_{t,t+1}$ is.

Therefore, in practice it is necessary to check for the right information set and horizon, before applying the approach of exponential-affine sdf, or to fix by convention the horizon and the information set. (This latter strategy is typically followed by the Basle Committee for credit portfolios.)

4. Examples

The aim of this section is to illustrate the approach of exponential-affine sdf (see Gerber and Shiu, 1994a; Yao, 2001 for other examples with i.i.d. returns). We first describe the conditionally Gaussian framework and the variance-gamma model, previously introduced in the literature. Then we consider a semi-parametric specification with path dependent drift and volatility.
4.1. The conditionally Gaussian framework

If the conditional historical distribution of $r_{t+1}$ given $r_t$ is Gaussian, with mean $m_t$ and variance–covariance matrix $\Sigma_t$, the conditional log-Laplace transform is given by

$$\psi_t(u; \theta) = u'm_t(\theta) + \frac{1}{2} u'\Sigma_t(\theta)u. \quad (4.1)$$

The risk correction term $\alpha_t$ satisfies

$$(\alpha'_t + e'_j)m_t(\theta) + \frac{1}{2}(\alpha'_t + e'_j)\Sigma_t(\theta)(x_t + e_j) - \alpha'_t m_t(\theta) - \frac{1}{2} \alpha'_t \Sigma_t(\theta)x_t = 0, \quad \forall j = 1, \ldots, J,$$

$$\iff e'_t m_t(\theta) + e'_j \Sigma_t(\theta)x_t + \frac{1}{2} e'_j \Sigma_t(\theta)e_j = 0, \quad \forall j = 1, \ldots, J,$$

$$\iff m_t(\theta) + \Sigma_t(\theta)x_t + \frac{1}{2} vdiag \Sigma_t(\theta) = 0,$$

where $vdiag \Sigma_t(\theta)$ is the vector, whose components are the diagonal elements of $\Sigma_t(\theta)$. We deduce that\(^{10}\)

$$\alpha_t = -\Sigma_t(\theta)^{-1} [m_t(\theta) + \frac{1}{2} vdiag \Sigma_t(\theta)]. \quad (4.2)$$

Then the conditional log-Laplace transform of the risk-neutral distribution given in Proposition 1 is

$$\psi_t(x_t + u; \theta) - \psi_t(x_t; \theta) = u'[m_t(\theta) + \Sigma_t(\theta)x_t] + \frac{1}{2} u'\Sigma_t(\theta)u$$

$$= -\frac{1}{2} u' vdiag \Sigma_t(\theta) + \frac{1}{2} u' \Sigma_t(\theta)u. \quad (4.3)$$

The risk-neutral distribution is also conditionally Gaussian, with the same variance–covariance matrix as the conditional historical distribution, and with a conditional mean function of $\Sigma_t(\theta)$, but not of $m_t(\theta)$. These formulas can be applied to a large class of models including for instance the conditionally Gaussian multivariate ARCH models (see e.g. Duan, 1995) and of course the Black–Scholes model (Black and Scholes, 1973). In particular they do not require a Markov condition for $(r_{t+1})$, that is the fact that $m_t$ and $\Sigma_t$ depend on the past through the current return $r_t$ only.

4.2. Variance-gamma model

This model has been introduced in Madan and Seneta (1990), Madan and Milne (1991), Madan et al. (1998). There is only one risky asset and the historical log-Laplace transform of its return is:

$$\psi_t(u) = v_t \log \left( 1 - um_t - u^2 \frac{\sigma^2_t}{2} \right),$$

where $v_t > 0$; it corresponds to a time deformed Gaussian model. The correcting factor is

$$\alpha_t = -\frac{1}{\sigma^2_t} \left( m_t + \frac{\sigma^2_t}{2} \right).$$

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\(^{10}\)The correction differs from the approach by Hansen and Jagannathan (1997) due to the introduction of the Jensen term $\frac{1}{2} vdiag \Sigma_t(\theta)$.
whereas the risk-neutral log-Laplace transform is

\[ \psi^Q_t(u) = v_t \log \left( 1 - u m^*_t - u^2 \frac{\sigma^2_t}{2} \right), \]

where

\[ m^*_t = \frac{m_t + \alpha_t \sigma^2_t}{1 - \alpha_t m_t - \alpha^2_t (\sigma^2_t/2)}, \quad \sigma^2_t = \frac{\sigma^2_t \alpha^2_t}{1 - \alpha_t m_t - \alpha^2_t \sigma^2_t/2}. \]

Thus the (conditional) parameter \( v_t \), that determines the time deformation is unchanged, whereas the drift \( m_t \) and volatility \( \sigma^2_t \) are modified.

### 4.3. Path dependent drift and volatility

In this subsection we consider a single risky asset \((J = 1)\) and assume that the return satisfies:

\[ r_{t+1} = m_t + \sigma_t \epsilon_{t+1}, \quad \sigma_t > 0, \quad (4.4) \]

where \( m_t \) and \( \sigma_t \) are the drift and volatility, respectively, that may depend on lagged returns, and \((\epsilon_t)\) is a sequence of i.i.d. variables with Laplace transform:

\[ \mathbb{E}[\exp(u \epsilon_{t+1})] = \exp \psi(u) \quad \text{(say)}. \quad (4.5) \]

This model is convenient to show the consequence of the conditional normality assumption. Indeed it includes the case of Gaussian errors, but may also allow for other types of conditional distributions with heavier tails, such as stable or Laplace distribution.

#### 4.3.1. The stochastic discount factor and the risk neutral distribution

The conditional log-Laplace transform of \( r_{t+1} \) is given by

\[ \psi_t(u) = m_t u + \psi(\sigma_t u). \]

From Proposition 1, the log-Laplace transform of the risk-neutral distribution is

\[ \psi^Q_t(u) = \psi_t(\alpha_t + u) - \psi_t(\alpha_t) = m_t u + \psi[\sigma_t(\alpha_t + u)] - \psi[\sigma_t \alpha_t], \]

where \( \alpha_t \) is solution of:

\[ \psi_t(\alpha_t + 1) = \psi_t(\alpha_t) \iff -m_t = \psi[\sigma_t(\alpha_t + 1)] - \psi[\sigma_t \alpha_t]. \quad (4.6) \]

**Proposition 2.** There exists a unique solution \( \alpha_t \) to Eq. (4.6), if the log-Laplace transform \( \psi \) is strictly convex and tends to infinity at boundaries of its domain.

**Proof.** Indeed the mapping \( \alpha \to \psi[\sigma_t(\alpha + 1)] - \psi(\sigma_t \alpha) \) is continuous, strictly increasing, with range \((-\infty, +\infty)\). The result follows directly. \( \square \)

Various examples are provided in Table 1. They include standard models such as the Black–Scholes model, the binomial tree with dichotomous errors, or the Laplace model (Gourieroux and Monfort, 2006) and a number of other specifications. In general the approach will be applied in a parametric framework where the drift, volatility and error distribution are parametrized: \( m_t = m(\underline{r}_t; \theta), \sigma_t = \sigma(\underline{r}_t, \theta), \psi(\ ) = \psi(\ ; \theta). \)
4.3.2. Semi-parametric pricing

The approach above can also be applied in a semi-parametric framework with parametric specifications of the drift and volatility:

\[ m_t = m(r_t; \theta), \quad \sigma_t = \sigma(r_t; \theta) \quad \text{(say)}, \]

and unrestricted distribution of the error term. The available observations are returns denoted by \( r_1, \ldots, r_T \). A semi-parametric pricing approach can be developed along the following steps.

Step 1: Calibration. Parameter \( \theta \) is consistently estimated from historical data by quasi-maximum likelihood; the estimator is given by

\[
\hat{\theta}_T = \arg \max_{\theta} \sum_{t=1}^{T} \left\{ -\log \sigma(r_{t-1}; \theta) - \frac{1}{2} \frac{[r_t - m(r_{t-1}; \theta)]^2}{\sigma^2(r_{t-1}; \theta)} \right\}
\]
Step 2: Estimation of the error distribution. Then we compute the residuals:
\[ \hat{e}_t = \frac{r_t - m(r_{t-1}; \hat{\theta}_T)}{\sigma(r_{t-1}; \hat{\theta}_T)}, \quad \tau = 2, \ldots, T. \]
The empirical distribution of the residuals:
\[ \hat{P} = \frac{1}{T-1} \sum_{t=2}^{T} \delta_{e_t}, \]
where \( \delta_{e} \) denotes the point mass at \( e \), is a consistent estimator of the distribution of the error term. The sample log-Laplace transform is given by
\[ \hat{\psi}_T(u) = \log \left( \frac{1}{T-1} \sum_{t=2}^{T} \exp(u \hat{e}_t) \right), \quad u \text{ varying,} \]
and is a consistent estimator of the unknown log-Laplace transform \( \psi \).

Step 3: Estimation of the risk correction factor at date \( t \). The correcting term \( \alpha_t \) associated with true distribution \( \psi \) and parameter \( \theta \) is consistently approximated by the correcting term \( \hat{\alpha}_t \) associated with \( \hat{\psi}_T, \hat{\theta}_T \). The correcting term \( \hat{\alpha}_t \) is the solution of
\[ \sum_{t=2}^{T} \exp[\sigma(r_{t}; \hat{\theta}_T)\hat{\alpha}_t]\left[\exp[m(r_{t}; \hat{\theta}_T) + \sigma(r_{t}; \hat{\theta}_T)\hat{e}_t]\right] = 0. \]

Step 4: Estimation of the sdf for period \( t, t+1 \). The true underlying sdf is estimated by
\[ \hat{M}_{t,t+1} = \text{exp}\left[\hat{\alpha}_t[r_{t+1} - m(r_{t}; \hat{\theta}_T)] - \hat{\psi}_T(\hat{\sigma}_t\hat{\alpha}_t)\right] \]
\[ = \text{exp}\hat{\alpha}_t[r_{t+1} - m(r_{t}; \hat{\theta}_T)] \left[\frac{1}{T-1} \sum_{t=2}^{T} \exp \hat{\sigma}_t\hat{e}_t\right]^{-1} \]
\[ = \hat{m}_{t,t+1}(r_{t+1}, r_t) \quad \text{(say).} \]

Step 5: Pricing. Finally the price of a derivative can easily be estimated. For instance the price of a derivative with residual maturity 1 and payoff \( g(r_{t+1}) \) is estimated by
\[ \hat{C}_T(g) = \hat{E}[\hat{m}_{T,T+1}(r_{T+1}, r_T)g(r_{T+1})] \]
\[ = \hat{E}[g[m(r_T; \hat{\theta}_T) + \sigma(r_T; \hat{\theta}_T)e_{T+1}]\exp[\hat{\alpha}_T\sigma(r_T; \hat{\theta}_T)e_{T+1}]] \]
\[ \times \left[\frac{1}{T-1} \sum_{t=2}^{T} \exp(\hat{\sigma}_T\hat{e}_t)\right]^{-1} \]
\[ = \frac{\sum_{t=2}^{T} \exp(\hat{\alpha}_T\hat{\sigma}_T\hat{e}_t)g(\hat{m}_T + \hat{\sigma}_T\hat{e}_t)}{\sum_{t=2}^{T} \exp(\hat{\alpha}_T\hat{\sigma}_T\hat{e}_t)}. \]

5. General specification

In this section the basic approach to sdf modelling is extended in two directions. First, the information used by the agents to fix their asset demand does not only include asset returns, but also variables from the real sector of the economy and additional factors. These factors are not directly observed by the econometrician. Second, we introduce a time
varying riskfree rate \( r_{t+1}^f \). This rate is predetermined, that is known at time \( t \), but stochastic and therefore unknown at time \( t - 1 \). This allows for a joint analysis of derivatives written on risky assets and on short term interest rate. For instance we consider the analysis of the term structure of interest rates and the effect of a stochastic interest rate on the price of a European call with maturity \( H \) written on a stock index.

5.1. The investors’ information and the sdf

The investors’ information at date \( t \) is denoted by \( J_t \). It includes the data used by investors, when they rebalance their portfolios, submit their orders and decide the volumes to be traded. This information contains:

(i) The current riskfree rate and its lagged values, \( r_{t+1}^f \);
(ii) The lagged returns on \( J \) risky assets, \( r_j \);
(iii) The lagged changes of real economic variables, \( x_t \), that may be macrovariables as GNP, retail price index, ..;
(iv) The values of additional factors, \( f_t \), say, which are known by the investor, but not observed by the econometrician.

At this stage the additional factors cannot be interpreted. However they can be completely or partly recovered through their effect on prices of basic and derivative assets. Thus the investor’s information is

\[
J_t = (r_{t+1}^f, r_j, x_t, f_t).
\] (5.1)

The sizes of the different vectors are \( 1, J, L \) and \( K \), respectively.

In the sequel we are interested in pricing derivatives written on the basic riskfree and risky assets. \(^{11}\) If the derivative provides a payoff \( g(r_{t+2}^f, r_{t+1}) \) at date \( t + 1 \), its price at date \( t \) is given by

\[
C_t(g) = E[M_{t,t+1} g(r_{t+2}^f, r_{t+1}) | J_t],
\] (5.2)

where the stochastic discount factor \( M_{t,t+1} \) depends on \( J_{t+1} \). As before the sdf is constrained to be exponential-affine:

\[
M_{t,t+1} = \exp(\alpha_{o,t} r_{t+2}^f + \gamma_t r_{t+1} + \delta_t \gamma_{t+1}^f + \beta_t),
\] (5.3)

where the coefficients depend on the investors’ information \( J_t \). Let us denote:

\[
F_{t+1} = [r_{t+2}^f, \gamma_{t+1}^f, f_{t+1}^f], \quad \Delta_t = (\alpha_{o,t}, \gamma_t, \delta_t)^t,
\]

the sdf becomes

\[
M_{t,t+1} = \exp[\alpha^t r_{t+1} + \Delta^t F_{t+1} + \beta_t],
\] (5.4)

\(^{11}\)It is also possible to include real variables in the contractual payoff as for French Treasury Bonds indexed on inflation.
5.2. The historical distribution

This distribution has to be considered for all variables in the investors’ information set. The conditional Laplace transform of \((r'_{t+1}, F'_{t+1})'\) given \(J_t\) is

\[
E[\exp(u' r_{t+1} + v' F_{t+1})|J_t] = \exp \psi_j(u, v; \theta). \quad (5.5)
\]

It depends on lagged values of asset returns \(r_t\), real variables \(x_t\), unobservable factors \(f_t\) and on current and lagged values of the riskfree rate \(r'_{t+1}\). \(\theta\) is a vector of parameters that characterizes the conditional distribution.

The riskfree rate has a particular status. It is not only predetermined, but also constrained to be positive, whereas \(r_t\) and \(x_t\) are generally of any sign due to their interpretations as changes (see the example of CCAPM for physical variables).

5.3. Constraints on the sdf

By writing the pricing conditions for the riskfree asset and the \(J\) risky assets, we get the equations:

\[
\begin{align*}
E[M_{t,t+1} \exp r'_{t+1}|J_t] &= 1, \\
E[M_{t,t+1} \exp r_{j,t+1}|J_t] &= 1, \quad \forall j = 1, \ldots, J, \\
\iff \\
\psi_j(x_t, A_t; \theta) + \beta_t + r'_{t+1} &= 0, \\
\psi_j(x_t + e_j, A_t; \theta) + \beta_t &= 0, \quad \forall j = 1, \ldots, J.
\end{align*}
\]

We get a system of \(J + 1\) equations for \(J + L + K + 2\) unknown risk adjustment parameters, with solutions:

\[
\begin{align*}
x_t &= \alpha(A_t, r_t, F_t; \theta), \\
\beta_t &= \beta(A_t, r_t, F_t; \theta). \quad (5.6)
\end{align*}
\]

Since the additional variables do not correspond to returns on assets traded at \(t\), the sensitivity coefficient \(A_t\) can be fixed arbitrarily as function of the information set. Thus the undeterminacy of the risk correction \(A_t\) still exists, despite the a priori convention of exponential-affine sdf. The results are summarized in the proposition below.

**Proposition 3.** For an exponential-affine sdf with state variables \(r_{t+1}, F_{t+1}\), there exists a multiplicity of admissible risk-neutral distributions. Their conditional Laplace transforms are given by

\[
\begin{align*}
E[\exp(u' r_{t+1} + v' F_{t+1})|J_t] &= \frac{E[M_{t,t+1} \exp(u' r_{t+1} + v' F_{t+1})|J_t]}{E[M_{t,t+1}|J_t]} \\
&= \frac{E[M_{t,t+1} \exp r'_{t+1} \exp(u' r_{t+1} + v' F_{t+1})|J_t]}{E[M_{t,t+1}|J_t]} \\
&= \exp(\psi_j(x_t + u, A_t + v; \theta) - \psi_j(x_t, A_t; \theta)),
\end{align*}
\]
where \( z_t \) is the solution of
\[
\psi_j(z_t + e_j \Delta_t; \theta) - \psi_j(z_t, \Delta_t; \theta) - r_{t+1}^f = 0, \quad \forall j = 1, \ldots, J,
\]
and the risk correction \( \Delta_t \) can be chosen arbitrarily.

The dimension of residual incompleteness is equal to the dimension \( K + L + 1 \) of \( \Delta_t \). It corresponds to the number of real economy state variables \( (x_t) \) and factors \( (f_t) \) plus the future riskfree rate \( (r_{t+2}^f) \) introduced in the sdf.

5.4. The conditionally Gaussian framework

Since the riskfree interest rate \( r_{t+1}^f \) admits positive values, the conditionally Gaussian framework can only be applied, if the future path of the riskfree rate is completely known at date \( t \). Thus we assume a deterministic riskfree rate and this subsection extends Section 4.1, when there are additional unobservable factors.

When \( (r_{t+1}^f, F_{t+1}^f)' \) are conditionally Gaussian under the historical distribution, the log-Laplace transform is
\[
\psi_j(u, v; \theta) = (u', v') m_t(\theta) + \frac{1}{2} (u', v') \Sigma_t(\theta) \begin{pmatrix} u \\ v \end{pmatrix}.
\]
The log-Laplace transforms of the risk-neutral distributions are
\[
\psi_j^Q(u, v; \theta) = \psi_j(z_t + u, \Delta_t + v; \theta) - \psi_j(z_t, \Delta_t; \theta)
\]
\[
= (u', v') \left[ m_t(\theta) + \Sigma_t(\theta) \begin{pmatrix} z_t \\ \Delta_t \end{pmatrix} \right] + \frac{1}{2} (u', v') \Sigma_t(\theta) \begin{pmatrix} u \\ v \end{pmatrix},
\]
where \( \Delta_t \) can be chosen arbitrarily. They correspond to Gaussian distributions with the same conditional variance-covariance matrix as the historical distribution and a conditional mean adjusted for risk. The expression of the mean depends on the solution \( z_t \) of the pricing restrictions.

Let us block decompose the conditional mean and variance as
\[
m_t = \begin{pmatrix} m_{r,t} \\ m_{F,t} \end{pmatrix}, \quad \Sigma_t = \begin{pmatrix} \Sigma_{rr,t} & \Sigma_{rF,t} \\ \Sigma_{Fr,t} & \Sigma_{FF,t} \end{pmatrix}.
\]
The restrictions of Proposition 3 are
\[
(\epsilon_j', 0) \left[ m_t + \Sigma_t \begin{pmatrix} z_t \\ \Delta_t \end{pmatrix} \right] + \frac{1}{2} (\epsilon_j', 0) \Sigma_t \begin{pmatrix} \epsilon_j \\ 0 \end{pmatrix} - r_{t+1}^f = 0, \quad \forall j = 1, \ldots, J,
\]
\[
\iff \epsilon_j' m_{r,t} + \epsilon_j' \Sigma_{rr,t} z_t + \epsilon_j' \Sigma_{rF,t} \Delta_t + \frac{1}{2} \epsilon_j' \Sigma_{rr,t} \epsilon_j - r_{t+1}^f = 0, \quad \forall j,
\]
\[
\iff m_{r,t} + \Sigma_{rr,t} z_t + \Sigma_{rF,t} \Delta_t + \frac{1}{2} \text{diag} \Sigma_{rr,t} - r_{t+1}^f \epsilon = 0,
\]
where \( \epsilon = (1, \ldots, 1)' \).

We deduce that
\[
z_t = -\Sigma_{rr,t}^{-1} [m_{r,t} - r_{t+1}^f \epsilon + \frac{1}{2} \text{diag} \Sigma_{rr,t} + \Sigma_{rF,t} \Delta_t], \quad (5.7)
\]
which extends Eq. (4.2). Thus the conditional mean of the risk-neutral distribution is

\[ m_t + \Sigma_t \left( \begin{array}{c} \alpha_t \\ \Delta_t \end{array} \right) \]

\[ = \left( \begin{array}{c} m_{r,t} + \Sigma_{rr,t} \alpha_t + \Sigma_{r,F,t} \Delta_t \\ m_{F,t} + \Sigma_{Fr,t} \alpha_t + \Sigma_{FF,t} \Delta_t \end{array} \right) \]

\[ = \left[ r'_{t+1} e - \frac{1}{2} v \text{diag} \Sigma_{rr,t} \right. \]

\[ \left. m_{F,t} - \Sigma_{Fr,t} \Sigma_{rr,t}^{-1} (m_{r,t} - r'_{t+1} e + \frac{1}{2} v \text{diag} \Sigma_{rr,t}) + (\Sigma_{FF,t} - \Sigma_{Fr,t} \Sigma_{rr,t}^{-1} \Sigma_{r,F,t}) \Delta_t \right] . \]

It is easy to recognize the residual conditional mean \( m_{F,t} - \Sigma_{Fr,t} \Sigma_{rr,t}^{-1} m_{r,t} \) and variance \( \Sigma_{FF,t} - \Sigma_{Fr,t} \Sigma_{rr,t}^{-1} \Sigma_{r,F,t} \). If \( \Delta_t = 0 \), the correction for risk is performed on the returns of traded assets, and there are no correction at all on the residual part, that is on the components of \( F_{t+1} \) which are not conditionally linked with asset returns. This result is a direct extension of the pricing formula derived by Stapleton and Subrahmanyam (1984).

5.5. Models with stochastic interest rate

The pricing approach can be extended to account for stochastic interest rate (see Yao, 2001 for another specification). We first consider a model without unobservable factor, where the short term zero-coupon bond is the only tradable asset. By selecting an autoregressive gamma model for the historical distribution of the riskfree rate, we get a direct extension of the Cox–Ingersoll–Ross model (Cox et al., 1985). Then we discuss the pricing of an index derivative, when the interest rate is stochastic.

5.5.1. Model with a stochastic interest rate only

5.5.1.1. Stochastic discount factor. When the short term zero-coupon bond is the only tradable asset and the sdf is given by

\[ M_{t+1} = \exp(\alpha t_{t+2} + \beta_t), \]

the arbitrage free condition becomes

\[ \beta_t = -r'_{t+1} - \psi_t(\alpha_0), \]

where \( \psi_t(u) = E[\exp(\alpha t_{t+2})|t_{t+1}] \), and \( \alpha_0 \) is a time independent parameter.

As an illustration let us assume that the conditional distribution of \( r'_{t+2} \) is a noncentered gamma distribution \(^1^2\) with log-Laplace transform:

\[ \psi_t(u) = -v \log(1 - uc) + \frac{\rho u}{1 - uc} r'_{t+1} , \]

where \( v, c, \rho \) are positive parameters. \( v \) gives the degree of freedom, \( c \) is a scale parameter, whereas the noncentrality parameter of the conditional distribution is: \( \rho r'_{t+1}/c \).

\(^1^2\)The variable \( r'_{t+2} \) follows the gamma distribution with degree of freedom \( v \), scale parameter \( c \) and noncentrality parameter \( \rho r'_{t+1}/c \), if and only if: \( r'_{t+2}/c \) follows a gamma distribution \( \gamma(v + Z_t) \), where \( Z_t \) is drawn in the Poisson distribution \( \mathcal{P} \rho r'_{t+1}/c \). It is easily checked that this conditional distribution corresponds to a time discretized Cox, Ingersoll, Ross model, if \( 0 < \rho < 1 \) (see Gourieroux and Jasiak, 2000; Darolles et al., 2006), and that its log-Laplace transform is given by (5.10).
We deduce that:
\[
\beta_t = -r_{t+1}^f - \psi_t(x_o) = -r_{t+1}^f + \psi_t(x_o) = -r_{t+1}^f + v \log(1 - x_o c) - \frac{\rho^2_o}{1 - x_o c} r_{t+1}^f,
\]
\[
M_{t,t+1} = \exp \left[ x_o r_{t+2}^f - \left( 1 + \frac{\rho^2_o}{1 - x_o c} \right) r_{t+1}^f + v \log(1 - x_o c) \right].
\]

5.5.1.2. Risk neutral distribution. The log-Laplace transform of the risk-neutral distribution is given by

\[
\mathcal{E}_t[\exp u r_{t+2}^f] = \exp[\psi_t(u + x_o) - \psi_t(x_o)]
\]
\[
= \exp \left[ -v \log \left( \frac{1 - (u + x_o)c}{1 - x_o c} \right) + \frac{\rho(u + x_o)}{1 - (u + x_o)c} r_{t+1}^f - \frac{\rho x_o}{1 - x_o c} r_{t+1}^f \right]
\]
\[
= \exp \left[ -v \log(1 - u c^*) + \frac{\rho^* u}{1 - u c^*} r_{t+1}^f \right],
\]
where \( c^* = c/(1 - x_o c), \rho^* = \rho/(1 - x_o c)^2. \)

For the risk neutral distribution to be defined, the risk correcting parameter \( x_o \) has to be chosen such that \( 1 - x_o c > 0 \Leftrightarrow x_o < 1/c. \)

Thus the risk-neutral conditional distributions belong to the same family of noncentered gamma distributions as the historical conditional distribution. The degrees of freedom are identical for the historical and risk-neutral distributions, whereas parameters \( c^* \) and \( \rho^* \) depend on the risk correction \( x_o \), associated with the future short term interest rate. The standard Cox, Ingersoll, Ross model (Cox et al., 1985) corresponds to the special case of a zero risk premium \( x_o = 0. \)

5.5.1.3. Derivative prices. Let us now consider the derivatives with exponential payoffs \( \exp(u r_{t+2}^f), u \) varying. Since the Laplace transform characterizes the distribution, these derivatives define a generating system for European derivatives (see the discussions in Duffie et al., 2000; Polimenis, 2001).

The price \( C_t(u, 1) \) of the European derivative with payoff \( \exp(u r_{t+2}^f) \) at date \( t + 1 \) is

\[
C_t(u, 1) = \exp(-r_{t+1}^f) \mathcal{E}_t[\exp u r_{t+2}^f].
\]

This price depends on \( r_{t+1}^f \) and \( x_o \):

\[
C_t(u, 1) = \gamma(x_o, r_{t+1}^f; u) \quad \text{(say)}.
\]

We get a semi-interval of admissible prices, when \( x_o \) varies, that is: \( (\gamma^u_t(u), +\infty) \), where \( \gamma^u_t(u) = \min_{x_o < 1/c} \gamma(x_o, r_{t+1}^f; u) \).

Closed form formulas can also be derived for pricing European derivatives at any horizon. We still consider the generating system of derivatives with exponential payoffs.

**Proposition 4.** Let us denote by \( C_t(u, h) \) the price at \( t \) of the European derivative, that provides payoff \( \exp(u r_{t+h+1}^f) \) at \( t + h \). We get

\[
C_t(u, h) = \exp[a(h, u) r_{t+1}^f + b(h, u)].
\]
where functions $a$ and $b$ satisfy the recursive equations:

\[
\begin{align*}
  a(h, u) &= -1 - \frac{\rho z_0}{1 - z_0 c} + \rho \frac{z_0 + a(h - 1, u)}{1 - (z_0 + a(h - 1, u)) c}, \\
  b(h, u) &= v \log(1 - z_0 c) - v \log[1 - (z_0 + a(h - 1, u)) c] + b(h - 1, u),
\end{align*}
\]

for $h \geq 2$.

**Proof.** We get

\[
C_t(u, h) = E_t[M_{t,t+1} C_{t+1}(u, h - 1)]
\]

\[
= E_t \left\{ \exp \left[ z_0 r_{t+2} - \left( 1 + \frac{\rho z_0}{1 - z_0 c} \right) r_{t+1}^f + v \log(1 - z_0 c) \right] \\
+ a(h - 1, u)r_{t+2}^f + b(h - 1, u) \right\}
\]

\[
= \exp \left\{ - \left( 1 + \frac{\rho z_0}{1 - z_0 c} \right) r_{t+1}^f + v \log(1 - z_0 c) + b(h - 1, u) \right\}
\]

\[
\times E_t \{ \exp[z_0 + a(h - 1, u)]r_{t+2}^f \}
\]

\[
= \exp \left\{ - \left( 1 + \frac{\rho z_0}{1 - z_0 c} \right) r_{t+1}^f + v \log(1 - z_0 c) - v \log[1 - (z_0 + a(h - 1, u)) c] \right\}
\]

\[
+ b(h - 1, u) + \rho \frac{z_0 + a(h - 1, u)}{1 - (z_0 + a(h - 1, u)) c} r_{t+1}^f \right\}.
\]

The result follows by identification. □

The nonlinear recursive equation does not depend on argument $u$, that is on the payoff to be priced.\(^{13}\) This argument has an effect through the initial condition only, since: $a(1, u) = -1 + \rho^* u/(1 - uc^*)$, $b(1, u) = -v \log(1 - uc^*)$. Once functions $a$ and $b$ are known, it is easy to derive the joint conditional forward risk neutral distribution $Q_t$ between $t + 1$ and $t + h$ (see El Karoui et al., 1995). Indeed this distribution has a pdf with respect to the historical pdf equal to

\[
M_{t,t+h}/E_t M_{t,t+1} \ldots M_{t+h-1,t+h}/E_t[M_{t,t+1} \ldots M_{t+h-1,t+h}].
\]

Therefore we get

\[
C_t(u, h) = E_t[M_{t,t+h} \exp(\omega r_{t+h+1})],
\]

\[
= E_t(M_{t,t+h}) E [\exp(\omega r_{t+h+1})]
\]

\[
= C_t(0, h) \exp \psi_{t,h}^Q(u),
\]

where $\psi_{t,h}^Q$ is the log-Laplace transform of $Q_t$. The log-Laplace transform is given by

\[
\psi_{t,h}^Q(u) = \log C_t(u, h) - \log C_t(0, h)
\]

\[
= [a(h, u) - a(h, 0)]r_{t+1}^f + b(h, u) - b(h, 0).
\]

\(^{13}\)This property is analogous to a standard property for continuous time models, saying that the price of a European derivative satisfies a partial differential equation, which is independent of the payoff.
It is possible to get closed form expressions of coefficient $a(h, u)$ (and $b(h, u)$). Indeed the series $a(h, u)$ satisfies a rational recursive equation, which is equivalent to

$$\frac{a(h, u) - \gamma_1}{a(h, u) - \gamma_2} = \frac{1 + \gamma_1}{\gamma_1} \frac{\gamma_2}{1 + \gamma_2} \frac{a(h - 1, u) - \gamma_1}{a(h - 1, u) - \gamma_2},$$

where $\gamma_1$ and $\gamma_2$ are distinct real roots of the second degree polynomial: $c^*\gamma_1^2 + \gamma[p^* + c^* - 1] - 1 = 0$. Thus we get

$$\frac{a(h, u) - \gamma_1}{a(h, u) - \gamma_2} = \left[\frac{1 + \gamma_1}{\gamma_1} \frac{\gamma_2}{1 + \gamma_2}\right]^{h-1} \frac{a(1, u) - \gamma_1}{a(1, u) - \gamma_2}.$$ 

This approach can be extended to multifactor models of interest rates and derivation of affine term structure models (see Gourieroux et al., 2002a).

5.5.2. Risky asset and stochastic interest rate

The approach above can be extended to a joint analysis of a risky asset with return $r_{t+1}$ and a stochastic interest rate $r_{t+1}^f$. A simple specification of the historical distribution assumes independence between the riskfree rate process $(r_t^f)$ and the excess return process $(r_t^e = r_{t+1} - r_{t+1}^f)$. Then by selecting appropriate distributions for both components, we can take into account the positivity constraint on the riskfree rate.

As an illustration, let us consider a Gaussian autoregressive model for $(r_{t+1}^f)$ and an autoregressive gamma model for $(r_{t+1}^e)$. Thus the joint conditional log-Laplace transform is

$$\psi_{1,2}(u, v) = \log E[\exp(u r_{t+1}^f + v r_{t+1}^e)] = \log E[\exp(u r_{t+1}^f + v r_{t+1}^e)] = u \exp(\gamma^*_t + b) + \frac{\sigma^2 u^2}{2} + \exp(v r_{t+1}^e) - v \log(1 - \nu) + \frac{\rho v}{1 - \nu} r_{t+1}^e = \psi_1(u) + \psi_2(v)$$ (say),

where $\exp(\gamma^*_t + b)$ denotes the one-step ahead prediction of excess return.

The stochastic discount factor is set as

$$M_{t+1} = \exp[x_o r_{t+2}^f + x_t r_{t+1} + \beta_t],$$

$$M_{t+1} = \exp[x_o r_{t+2}^f + x_t (r_{t+1} - r_{t+1}^f) + \beta_t].$$

The arbitrage free conditions imply:

$$\begin{align*}
E_t(M_{t+1} \exp r_{t+1}^f) &= 1, \\
E_t[M_{t+1} \exp(r_{t+1}^e + r_{t+1}^f)] &= 1,
\end{align*}$$

$$\iff \begin{align*}
\psi_{1,2}(x_t) + \psi_{2,1}(x_o) + r_{t+1}^f + \beta_t &= 0, \\
\psi_{1,2}(x_t + 1) + \psi_{2,1}(x_o) + r_{t+1}^e + \beta_t &= 0.
\end{align*}$$

Thus the risk correcting factor $x_t$ satisfies:

$$\psi_{1,2}(x_t + 1) = \psi_{1,2}(x_t) \iff x_t = -\frac{1}{2} \frac{ar_t^* + b}{\sigma^2}.$$
The log-Laplace transforms of the conditional risk neutral distributions at horizon 1 are given by

\[
\psi_t^Q(u, v) = \psi_{1t}(u + \alpha_t) - \psi_{1t}(\alpha_t) + u\psi_{2t}(v + \alpha_o) - \psi_{2t}(\alpha_o),
\]

where \(\alpha_o\) can be chosen arbitrarily.

The processes \((r_{t+1}^r)\) and \((r_{t+1}^f)\) are still independent in the risk neutral world, for any value of \(\alpha_o\). However the joint distribution of \((r_{t+1}, r_{t+1}^f)\) has to be used for pricing a standard European call written on \(r_{t+1}\), since the payoff depends jointly on \(r_{t+1}^r\) and \(r_{t+1}^f\):

\[
(\exp r_{t+1} - k)^+ = \exp (r_{t+1}^r + r_{t+1}^f - k)^+.
\]

This standard call can be considered as a quanto-option, for which the riskfree rate provides the exchange rate between the money units of dates \(t\) and \(t + 1\), respectively.

6. Concluding remarks

The aim of this paper was to analyze pricing models with exponential-affine stochastic discount factors and to derive the joint specification of the historical and risk neutral distributions. This approach underlies different pricing methods introduced in the literature, such as the pricing with ARCH models, the affine term structure model, or the variance-gamma model. Its general use is promising as illustrated by the various examples given in the paper, that are the semi-parametric pricing for model with path dependent drift and volatility, or the pricing models for stochastic riskfree interest rate. In particular the exponential-affine sdf underlies the affine models currently introduced for credit risk, that is for a coherent analysis of the prices of \(T\)-bonds and corporate bonds (Duffie and Lando (2001); Gourieroux et al., 2002b). This specification of the stochastic discount factor can lead to parametric, or semi-parametric models for asset prices. This specification step is necessary before considering statistical inference, that is estimation of both the historical distribution and the stochastic discount factor from prices data on underlying assets and derivatives (Gagliardini et al., 2004).

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References


Further Reading