Sources of Entropy in Representative Agent Models

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Q1. What does the pricing kernel look like?

- **Dispersion**: entropy
- **Dynamics**: multiperiod entropy and horizon dependence
- **Disasters**: entropy and high-order cumulants
- Illustration: Vasicek model

Q2. How do these pricing kernels compare?

- Power utility
- Recursive preferences
- Habits
- Jumps and disasters
Questions

Q1. What does the pricing kernel look like?

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Q1. What does the pricing kernel look like?
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- Dynamics: multiperiod entropy and horizon dependence ("small")
- Disasters: entropy and high-order cumulants
- Illustration: Vasicek model

Q2. How do these pricing kernels compare?
- Power utility
- Recursive preferences
- Habits
- Jumps and disasters
## Facts about excess returns (monthly)

<table>
<thead>
<tr>
<th>Asset</th>
<th>Mean</th>
<th>Standard Deviation</th>
<th>Skewness</th>
<th>Kurtosis</th>
</tr>
</thead>
<tbody>
<tr>
<td>S&amp;P 500</td>
<td>0.0040</td>
<td>0.0556</td>
<td>−0.40</td>
<td>7.90</td>
</tr>
<tr>
<td>Fama-French (small, low)</td>
<td>−0.0030</td>
<td>0.1140</td>
<td>0.28</td>
<td>9.40</td>
</tr>
<tr>
<td>Fama-French (small, high)</td>
<td>0.0090</td>
<td>0.0894</td>
<td>1.00</td>
<td>12.80</td>
</tr>
<tr>
<td>S&amp;P 500 6% OTM puts</td>
<td>−0.0184</td>
<td>0.0538</td>
<td>2.77</td>
<td>16.64</td>
</tr>
<tr>
<td>Pound Sterling</td>
<td>0.0035</td>
<td>0.0316</td>
<td>−0.50</td>
<td>1.50</td>
</tr>
<tr>
<td>5-year bond</td>
<td>0.0015</td>
<td>0.0190</td>
<td>0.10</td>
<td>4.87</td>
</tr>
</tbody>
</table>

Also: nominal yield spread at 60 months is about 0.001 a month
Facts: summary

Facts

- “Big” excess returns $\gg$ equity premium
- “Small” bond premiums
- Skewness and kurtosis evident

Each tells us something about the pricing kernel
Facts: summary

Facts

- “Big” excess returns $\gg$ equity premium (dispersion)
- “Small” bond premiums (dynamics)
- Skewness and kurtosis evident (disasters)

Each tells us something about the pricing kernel
Dispersion: entropy

“Entropy” defined: for $m_{t+1} > 0$, (conditional) entropy is

$$L_t(m_{t+1}) = \log E_t m_{t+1} - E_t \log m_{t+1}$$

Cumulant expansion

$$L_t(m_{t+1}) = \kappa_{2t}(\log m_{t+1})/2! \underbrace{}_{\text{normal term}}$$

$$+ \kappa_{3t}(\log m_{t+1})/3! + \kappa_{4t}(\log m_{t+1})/4! + \cdots \underbrace{}_{\text{high-order cumulants}}$$
Dispersion: entropy bound

Asset pricing theory: there exists $m_{t+1} > 0$ satisfying

$$E_t(m_{t+1} r_{t+1}) = 1$$

Entropy bound

$$L_t(m_{t+1}) = \log E_t m_{t+1} - E_t \log m_{t+1}$$

$$\geq E_t (\log r_{t+1} - \log r^1_{t+1})$$

$$\Rightarrow E L_t(m_{t+1}) \geq E (\log r_{t+1} - \log r^1_t)$$

Conditional and unconditional entropy

$$L(m_{t+1}) = E L_t(m_{t+1}) + L(E_t m_{t+1})$$
Dynamics: horizon dependence

Multiperiod entropy

\[ m_{t,t+n} = m_{t+1} m_{t+2} \cdots m_{t+n} \]

\[ L_t(m_{t,t+n}) = \log E_t m_{t,t+n} - E_t \log m_{t,t+n} \]

\[ \log q_t^n = -n y_t^n \]

\[ n^{-1} EL_t(m_{t,t+n}) = -E y_t^n - E \log m_{t+1} \]

Horizon dependence

\[ H(n) = n^{-1} EL_t(m_{t,t+n}) - EL_t(m_{t+1}) = -E(y_t^n - y_t^1) \]
What the pricing kernel looks like: summary

Dispersion
- Entropy $\geq 0.01 = 1\%$ a month

Dynamics
- $|\text{Horizon dependence}| \leq 0.001 = 0.1\%$ a month (@ 60 months)

Disasters
- Log pricing kernel unlikely to be normal
Vasicek model: a loglinear example

Pricing kernel has loglinear dynamics

\[ \log m_t = \log m + a(B)w_t \]
\[ = \log m + a_0 w_t + a_1 w_{t-1} + a_2 w_{t-2} + \cdots \]
\[ \{w_t\} \sim \text{NID}(0, 1) \]

Examples

- White noise (iid): \( a_0 \) arbitrary, \( a_j = 0 \) for \( j \geq 1 \)
- AR(1): \( a_0, a_{j+1} = \varphi a_j \) for \( j \geq 0 \)
- ARMA(1,1): \( (a_0, a_1) \) arbitrary, \( a_{j+1} = \varphi a_j \) for \( j \geq 1 \)
Vasicek model: entropy and horizon dependence

Partial sums

\[ A_n = a_0 + a_1 + a_2 + \cdots + a_n \]

Entropy

\[ EL_t(m_{t+1}) = \frac{a_0^2}{2} = A_0^2/2 \Rightarrow a_0 \quad (“big”) \]

\[ EL_t(m_{t,t+n}) = \sum_{j=1}^{n} A_{j-1}^2 / 2 \]

Horizon dependence

\[ H(n) = n^{-1} \sum_{j=1}^{n} (A_{j-1}^2 - A_0^2) / 2 \Rightarrow a_j \quad (“small”) \]
Vasicek model: interest rates

Short rate

\[ \log r_{t+1}^1 = y_t^1 \]
\[ = - \log m - A_0^2/2 - [a(B)/B]^+ w_t. \]

Forward rates

\[ f_t^n = - \log m - A_n^2/2 - [a(B)/B^n]^+ w_t \]

Typical term in mean spread \( f^n - f^0 \) and horizon dependence

\[ A_0^2 - A_n^2 = A_0^2 - [A_0 + (A_n - A_0)]^2 \]
\[ = -2A_0(A_n - A_0) - (A_n - A_0)^2 \]
Vasicek model: estimation strategy

Short rate governed by \((a_1, a_2, \ldots)\)

- ARMA(1,1) pricing kernel \(\Rightarrow\) AR(1) short rate
- Choose \((a_1, \varphi)\) to match variance and autocorrelation

Mean yield spread governed by \(a_0\) ("\(\lambda\)" in affine models)

- Choose \(a_0\) to match mean yield spread for maturity 60 months
What does the pricing kernel look like?

Vasicek example

Vasicek model: ARMA(1,1) review

show AR1, ARMA11
Vasicek model: moving average coefficients

[Graph showing the moving average coefficients for Positive and Negative Yield Spread over different orders, with the x-axis labeled 'Order j' and the y-axis labeled 'a_j'.]
Vasicek model: multiperiod entropy

Backus, Chernov, & Zin (NYU & LSE)
Vasicek model: horizon dependence

Entropy and Horizon Dependence

- Entropy per period
- Negative yield spread
- Positive yield spread
- Entropy lower bound
- Horizon dependence upper bound
- Horizon dependence lower bound

Backus, Chernov, & Zin (NYU & LSE)
Sources of Entropy
Representative-agent models

Power utility

Recursive preferences

- Bansal-Yaron with persistent consumption growth
- ... and stochastic volatility

Habits

- Ratio habits
- Difference habits
- Campbell-Cochrane

Jumps and disasters
How do representative-agent models compare?

Model summary

[Graph showing entropy lower bound and horizon dependence upper and lower bounds for different models.

Sources of Entropy]
Additive power utility

Consumption growth process \((g_t = c_t/c_{t-1})\)

\[
\log g_t = \log g + \gamma(B)v^{1/2}w_t
\]

\(\{w_t\} \sim \text{NID}(0,1)\)

Pricing kernel

\[
\log m_t = \log \beta + (\rho - 1) \log g_t
\]

\[
= \text{constants} + (\rho - 1)\gamma(B)v^{1/2}w_t
\]
Recursive preferences

Preferences

\[ U_t = [(1 - \beta)c^\rho_t + \beta \mu_t(U_{t+1})^\rho]^{1/\rho} \]

\[ \mu_t(U_{t+1}) = (E_t U_{t+1}^\alpha)^{1/\alpha} \]

\[ \alpha, \rho \leq 1 \]

Interpretation

\[ IES = 1/(1 - \rho) \]

\[ CRRA = 1 - \alpha \]

\[ \alpha = \rho \Rightarrow \text{additive power utility} \]
Recursive preferences: basic analytics

Scale everything by $c_t$ ($u_t = U_t/c_t$, $g_{t+1} = c_{t+1}/c_t$)

$$u_t = [(1 - \beta) + \beta \mu_t (g_{t+1} u_{t+1})^\rho]^{1/\rho}$$

Loglinear approximation (exact if $\rho = 0$: $b_0 = 0$, $b_1 = \beta$)

$$\log u_t \approx b_0 + b_1 \log \mu_t (g_{t+1} u_{t+1})$$

Pricing kernel

$$m_{t+1} = \beta \left( \frac{c_{t+1}}{c_t} \right)^{\rho-1} \frac{U_{t+1}/\mu_t (U_{t+1})^{\alpha-\rho}}{\text{short-run risk}} \frac{g_{t+1}^{\rho-1} [g_{t+1} u_{t+1}/\mu_t (g_{t+1} u_{t+1})]^{\alpha-\rho}}{\text{long-run risk}}$$
Recursive preferences: consumption dynamics

Consumption growth

\[
\log g_t = g + \gamma(B) v^{1/2} w_t \\
\{w_t\} \sim \text{NID}(0, 1)
\]

Pricing kernel

\[
\log m_{t+1} = \text{constants} \\
+ \left[ (\rho - 1) \gamma_0 + (\alpha - \rho) \gamma(b_1) \right] v^{1/2} w_{t+1} \\
+ (\rho - 1) \gamma_1 v^{1/2} w_t + (\rho - 1) \gamma_1 v^{1/2} w_{t-1} + \cdots
\]

Critical term: \( \gamma(b_1) = \gamma_0 + b_1 \gamma_1 + b_1^2 \gamma_2 + \cdots \)
Recursive preferences: moving average coefficients
Recursive preferences: entropy and horizon dependence

[Graph showing entropy and horizon dependence over time.]

Entropy and Horizon Dependence

Time Horizon in Months
Recursive preferences: and volatility dynamics

Consumption growth

\[
\begin{align*}
\log g_t &= g + \gamma(B)v_{t-1}^{1/2}w_{gt} \\
v_t &= v + \nu(B)w_{vt} \\
\{w_{gt}, w_{vt}\} &\sim \text{NID}(0, I)
\end{align*}
\]

Pricing kernel

\[
\log m_{t+1} = \text{constants} + [(\rho - 1)\gamma_0 + (\alpha - \rho)\gamma(b_1)]v_{t}^{1/2}w_{gt+1} \\
+ (\alpha - \rho)(\alpha/2)\gamma(b_1)^2b_1\nu(b_1)w_{vt+1} \\
+ (\rho - 1)[\gamma(B)/B] + v_{t-1}^{1/2}w_{gt} \\
- (\alpha - \rho)(\alpha/2)\gamma(b_1)^2\nu(B)w_{vt}
\]
Recursive preferences: moving average coefficients

- Consumption Growth
- Volatility

Order j

Backus, Chernov, & Zin (NYU & LSE)
### Recursive preferences: numerical examples

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Power Utility (1)</th>
<th>Bansal-Yaron Version 1 (2)</th>
<th>Bansal-Yaron Version 2 (3)</th>
<th>Bansal-Yaron Version 3 (4)</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Preferences</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\alpha$</td>
<td>$-9$</td>
<td>$-9$</td>
<td>$-9$</td>
<td>$-9$</td>
</tr>
<tr>
<td>$\rho$</td>
<td>$-9$</td>
<td>$1/3$</td>
<td>$1/3$</td>
<td>$1/3$</td>
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<td><strong>Consumption growth</strong></td>
<td></td>
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<td></td>
</tr>
<tr>
<td>$\gamma_1$</td>
<td>$0.1246$</td>
<td>$0.1246$</td>
<td>$0.1246$</td>
<td>$0.0250$</td>
</tr>
<tr>
<td>$\varphi_g$</td>
<td>$0.9750$</td>
<td>$0.9750$</td>
<td>$0.9750$</td>
<td>$0.9750$</td>
</tr>
<tr>
<td>$\nu_0$</td>
<td>$0$</td>
<td>$0$</td>
<td>$0.28 \times 10^{-5}$</td>
<td>$0.28 \times 10^{-5}$</td>
</tr>
<tr>
<td>$\varphi_v$</td>
<td></td>
<td></td>
<td>$0.9990$</td>
<td>$0.9990$</td>
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<tr>
<td><strong>Entropy and horizon dependence</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$EL_t(m_{t+1})$</td>
<td>$0.0026$</td>
<td>$0.0631$</td>
<td>$0.1319$</td>
<td>$0.0248$</td>
</tr>
<tr>
<td>$H(60)$</td>
<td>$0.0305$</td>
<td>$0.0042$</td>
<td>$0.0077$</td>
<td>$0.0009$</td>
</tr>
</tbody>
</table>
Habits

Preferences

\[ U_t = f(c_t, x_t) + \beta E_t U_{t+1} \]
\[ x_t = "external\ habit" \]

Examples

- Ratio habit: \( c_t / x_t \)
- Difference habit: \( c_t - x_t \)
- Campbell-Cochrane: P2C2E

Standard inputs

\[ \log g_t = \log g + \gamma(B) v^{1/2} w_t \quad [\gamma(B) = 1?] \]
\[ \log x_t = \log x + \chi(B) \log c_{t-1} \quad [\chi(1) = 1?] \]
Ratio habit

Preferences

\[ f(c_t, x_t) = \frac{(c_t/x_t)^\rho}{\rho}, \quad \rho \leq 1 \]

Pricing kernel

\[ m_{t+1} = \beta \left( \frac{c_{t+1}}{c_t} \right)^{\rho-1} \left( \frac{x_{t+1}}{x_t} \right)^{-\rho} \]

\[ \log m_{t+1} = \text{constants} \]

\[ + \left[ (\rho - 1) - \rho B \chi(B) \right] \gamma(B) \nu^{1/2} w_{t+1} \]
How do representative-agent models compare?

Habits

Ratio habit: moving average coefficients

![Graph showing moving average coefficients for Power Utility and Ratio Habit]

Backus, Chernov, & Zin (NYU & LSE)
Ratio habit: entropy and horizon dependence

![Graph showing entropy and horizon dependence over time in months. The x-axis represents time in months ranging from 0 to 60, while the y-axis shows entropy per period ranging from 0 to 0.012. The graph includes lines for entropy, entropy lower bound, and horizon dependence (diff from black line).]
Difference habit

Preferences

\[ f(c_t, x_t) = \frac{(c_t - x_t)^{\rho}}{\rho}, \quad \rho \leq 1 \]

Define surplus

\[ s_t = \frac{(c_t - x_t)}{c_t} = 1 - \frac{x_t}{c_t} \]

Pricing kernel

\[ m_{t+1} = \beta g_{t+1} \left( \frac{s_{t+1}}{s_t} \right)^{\rho - 1} \]

\[ \log m_{t+1} = \text{constants} \]

\[ + (\rho - 1)(1/s)[1 - (1 - s)\chi(B)B]\gamma(B)\nu^{1/2}w_{t+1} \]
Difference habit: moving average coefficients

Backus, Chernov, & Zin (NYU & LSE)
Difference habit: entropy and horizon dependence

![Graph showing entropy and horizon dependence over time.](image)
Campbell-Cochrane model

Surplus follows

\[
\log s_{t+1} - \log s_t = (\varphi_s - 1)(\log s_t - \log s) + \lambda(\log s_t) \nu^{1/2} w_t
\]

\[
1 + \lambda(\log s_t) = \nu^{-1/2} \left( \frac{(1 - \rho)(1 - \varphi_s) - b}{(1 - \rho)^2} \right)^{1/2}
\times (1 - 2[\log s_t - \log s])^{1/2}
\]

Designed to control horizon dependence
## Habits: numerical examples

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Power Utility (1)</th>
<th>Ratio Habit (2)</th>
<th>Difference Habit (3)</th>
<th>Campbell-Cochrane (4)</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Preferences</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \rho )</td>
<td>(-9)</td>
<td>(-9)</td>
<td>(-9)</td>
<td>(-1)</td>
</tr>
<tr>
<td>Consumption growth</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( v^{1/2} )</td>
<td>0.0135</td>
<td>0.0135</td>
<td>0.0135</td>
<td>0.0135</td>
</tr>
<tr>
<td><strong>Habit</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \chi_0 )</td>
<td>0.25</td>
<td>0.25</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \varphi_x ) or ( \varphi_s )</td>
<td>0.75</td>
<td>0.75</td>
<td>0.9885</td>
<td></td>
</tr>
<tr>
<td>( s )</td>
<td>0.5</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Entropy and horizon dependence</strong></td>
<td></td>
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<td></td>
<td></td>
</tr>
<tr>
<td>( EL_t(m_{t+1}) )</td>
<td>0.0091</td>
<td>0.0091</td>
<td>0.0365</td>
<td>0.0231</td>
</tr>
<tr>
<td>( H(60) )</td>
<td>0</td>
<td>(-0.0086)</td>
<td>(-0.0258)</td>
<td>0</td>
</tr>
</tbody>
</table>
Recursive preferences: and jump risk

Consumption growth

\[
\log g_t = g + \nu^{1/2} w_{gt} + z_t \\
h_t = h + \eta(B) w_{ht} \\
(w_{gt}, w_{ht}) \sim \text{NID}(0, I) \\
\nu^{1/2} w_{gt+1} + z_{gt+1} \\
\nu^{1/2} w_{ht+1} - \eta(B) w_{ht}
\]

Pricing kernel

\[
\log m_{t+1} = \text{constants} \\
+ (\alpha - 1)(\nu^{1/2} w_{gt+1} + z_{gt+1}) \\
+ (\alpha - \rho)[(e^{\alpha \theta + (\alpha \delta)^2/2} - 1)/\alpha][b_1 \eta(b_1) w_{ht+1} - \eta(B) w_{ht}]
\]
## Jump risk: numerical examples

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Preferences</th>
<th>Consumption growth</th>
<th>Entropy and horizon dependence</th>
</tr>
</thead>
<tbody>
<tr>
<td><em>α</em></td>
<td>-4 (1)</td>
<td>0.0100 (2)</td>
<td>0.0023 (3)</td>
</tr>
<tr>
<td><em>ρ</em></td>
<td>-4 (1)</td>
<td>-0.3000 (2)</td>
<td>0.1500 (3)</td>
</tr>
<tr>
<td><em>θ</em></td>
<td>-0.3000 (2)</td>
<td>-0.3000 (2)</td>
<td>0.0033 (3)</td>
</tr>
<tr>
<td><em>δ</em></td>
<td>0.1500 (2)</td>
<td>0.1500 (2)</td>
<td>0.9934 (3)</td>
</tr>
<tr>
<td><em>η₀</em></td>
<td>0.0033 (3)</td>
<td></td>
<td></td>
</tr>
<tr>
<td><em>φₜ</em></td>
<td>0.9934 (3)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

### Parameter

#### Preferences

- **α**: Parameter controlling the utility function
- **ρ**: Parameter controlling the persistence of consumption growth

#### Consumption growth

- **h × 12**: Consumption growth rate
- **θ**: Parameter controlling the consumption growth rate
- **δ**: Parameter controlling the depreciation rate

#### Entropy and horizon dependence

- **ELₜ(mₜ₊₁)**: Entropy measure
- **H(60)**: Entropy measure for horizon 60
Model summary

![Entropy and Horizon Dependence Graphs]

- Entropy
- Horizon Dependence

Backus, Chernov, & Zin (NYU & LSE)
Answers — and more questions

Q1. What does the pricing kernel look like?
   ▶ Excess returns suggest \textit{entropy} \( \geq 1\% \) monthly
   ▶ Yield spreads suggest \textit{horizon dependence} \( \leq 0.1\% \) monthly
   ▶ Probably not normal
   ▶ Useful diagnostics for any asset pricing model

Q2. How do representative-agent models compare?
   ▶ Most generate lots of entropy
   ▶ Several generate too much horizon dependence as a result
   ▶ Persistent jump risk helps
   ▶ All this conditional on parameter values

Q3. What’s next?
   ▶ New parameter values?
   ▶ Heterogeneous agents?
   ▶ Imbed in business cycle models?
Related work (some of it)

Bounds
- Alvarez-Jermann, Bansal-Lehmann, Hansen-Jagannathan

Recursive preferences
- Preferences: Epstein-Zin, Kreps-Porteus, Weil
- Asset pricing: Bansal-Yaron, Campbell, Hansen-Heaton-Li

Habits
- Abel, Campbell-Cochrane, Chan-Kogan, Constantinides, Heaton, Sundaresan

Jumps and disasters
What is entropy?

Humpy Dumpty (in “Through the Looking Glass”)

“When I use a word,” Humpty Dumpty said, “it means just what I choose it to mean — neither more nor less.”

Hans-Otto Georgii (quoted by Hansen and Sargent):

When Shannon had invented his quantity and consulted von Neumann on what to call it, von Neumann replied: “Call it entropy. It is already in use under that name and, besides, nobody knows what entropy is anyway.”

Paul Samuelson (“Gibbs in economics,” 1989):

I have limited tolerance for the perpetual attempts to fabricate for economics concepts of “entropy.”
What is entropy?

Reminder

\[ L_t(m_{t+1}) = \log E_t m_{t+1} - E_t \log m_{t+1} \]

Note (with apologies for the notation)

\[ L_t(k_t m_{t+1}) = L_t(m_{t+1}) \text{ for } k_t > 0 \]
\[ m_{t+1} = q_t p_{t+1}^*/p_{t+1} \]

Therefore: entropy = relative entropy = Kullback-Leibler divergence

\[ L_t(m_{t+1}) = L_t(p_{t+1}^*/p_{t+1}) = -E_t \log(p_{t+1}^*/p_{t+1}) \]