Abstract—The discrete wavelet decomposition of second-order harmonizable random processes is considered. The deterministic wavelet decomposition of a complex exponential is examined, where its pointwise and bounded convergence to the function is proved. This result is then used for establishing the stochastic wavelet decomposition of harmonizable processes. The similarities and differences between the wavelet decompositions of general harmonizable processes and a subclass of processes having no spectral mass at zero frequency, e.g., those that are wide-sense stationary and have continuous power spectral densities, are also investigated. The relationships between the harmonizability of a process and that of its wavelet decomposition are examined. Finally, certain linear operations such as addition, differentiation, and linear filtering on stochastic wavelet decompositions are considered. It is shown that certain linear operations can be performed term by term with the decomposition.

Index Terms—Wavelet transform, wavelet decomposition, multiresolution analysis, second-order random processes, harmonizable random processes, spectral representations.

I. INTRODUCTION

CONTINUOUS wavelet transform is a technique for decomposing a signal into components that have good localization properties both in time and frequency [11]–[3]. Because of these time and frequency localization properties, wavelet transform can provide local information on a signal that cannot be obtained using traditional methods such as the Fourier transform. Mallat considered the wavelet transform on a grid in the transform domain (discrete wavelet decomposition), and introduced a multiresolution analysis technique for square integrable signals, i.e., those that are in $L_2$. The discrete wavelet decomposition is remarkable not only because it provides a tool for multiresolution analysis, but it also provides an orthonormal decomposition of $L_2$. Mathematically, a discrete wavelet decomposition is an orthogonal decomposition of functions on a Hilbert space where the basis functions are formed by translations and dilations of a single wavelet basis function [5]. Many wavelet basis functions have already been found, including those with compact support [3], [6], [7].

There are two different but equivalent forms for representing the wavelet decompositions of functions in $L_2$. In the first one, a square integrable function $x(t)$ is represented as a weighted sum of the dilated and translated versions of a wavelet function $\psi(.)$. This can be thought of as a superposition of bandpass components because the Fourier transform of $\psi(.)$ has a bandpass characteristic. In the second form, $x(t)$ is represented as a superposition of the dilated and translated versions of $\psi(.)$ at high frequencies, plus a weighted sum of shifted versions of a scaling function $\phi(.)$. Since the Fourier transform of $\phi(.)$ has a low-pass characteristic, this representation can be interpreted as a decomposition using both low-pass and bandpass components.

The discrete wavelet decomposition can also be used for analyzing nonsquare integrable deterministic functions [8]. In such case, it is necessary to impose certain additional conditions on the scaling and wavelet functions so that the wavelet coefficients are well defined. Furthermore, the two different forms, that are equivalent for representing $L_2$ functions, are no longer equivalent. Such functions outside of $L_2$ must be represented by the second form, viz., as a superposition of low-pass and bandpass components. The reason is roughly that non-square integrable functions can have discrete spectral components at zero frequency, and hence cannot be completely described by scaled and translated versions of $\psi(.)$.

Since many useful and interesting signals encountered in practice are best modeled as random processes, a theory of wavelet decomposition for general random processes is needed. In [9], the wavelet decomposition for nearly 1/f processes are considered. Certain assumptions are made on the correlation of the wavelet coefficients to derive the wavelet decomposition of these processes. A related work is the investigation of fractional Brownian motion using wavelet decompositions [10], [11]. In [12], the wavelet decomposition of finite energy and periodically correlated processes are considered. In addition, truncated finite power processes, i.e., on a finite subset of the real line, are also considered.

In many engineering problems, one frequently encounters random processes that do not have square integrable sample functions. For example, any process that has periodic components falls under this category. The concept of frequency is very important in many practical problems involving filtering, spectral analysis, estimation, prediction, etc. Random processes with an emphasis on frequency can be studied under the theory of harmonizable processes introduced by Loève [13, Section 37.4]. Loosely speaking, a harmonizable process is a second-order process that can be represented as a superposition of complex exponentials (A precise definition will follow in a later section). In general, a harmonizable...
process can either be stationary, wide-sense stationary, or nonstationary. In the special case where it is wide-sense stationary, the harmonic decomposition is identical to the well known spectral representation theorem of Bochner [14, Sections 20–21]. Since the spectral measure may or may not be absolutely continuous, the power spectral density need not exist even if the process is wide-sense stationary. In particular, the spectrum might consist of discrete components, i.e., distinct frequency components with a finite amount of power. Hence, one suspects that a theory of the wavelet decomposition for general harmonizable processes would require a close examination of the two different forms, viz., the representation using bandpass components versus the representation using both low-pass and bandpass components.

It is interesting to note that a spectral representation of cyclostationary processes has been studied [15] also using the harmonizability concept of Loève [13]. A cyclostationary process is decomposed in [15] so that the frequency supports of the component processes are of a uniform width. The wavelet decomposition, on the other hand, partitions the frequency scale dyadically [3, 16, 17] so that high-frequency components have good time resolution, while low-frequency components have good frequency resolution.

In this paper, we consider and establish a theory for the wavelet decompositions of general harmonizable processes. We shall see that as in the deterministic case [8], the wavelet decompositions of harmonizable processes can be represented in two different forms. These two forms are equivalent for processes where there is no spectral mass at zero frequency. An important example of these processes are those having continuous power spectral densities. Only one of these forms, however, is valid for the decomposition of general harmonizable processes. This is analogous to the deterministic wavelet decomposition of functions outside the space $L_2$ [8].

In Section II, we briefly review the two forms of discrete wavelet decompositions for deterministic functions. Section III establishes the bounded convergence of the wavelet decomposition of a complex exponential function. This result is then applied in Section IV to establish the wavelet decompositions of harmonizable processes. The harmonizability of wavelet decompositions will also be considered. Section V considers certain linear operations on wavelet decompositions of harmonizable processes. Finally, the results of this paper are summarized in Section VI.

II. DETERMINISTIC WAVELET DECOMPOSITIONS

The discrete wavelet decomposition of a deterministic square integrable function $x(t)$, denoted by $x(t) \in L_2$, is [4], [5], [7]

$$x(t) = \sum_{m=\infty}^{\infty} \sum_{n=-\infty}^{\infty} X_{m,n} \psi_{m,n}(t),$$

where

$$X_{m,n} = \int_{-\infty}^{\infty} x(t) \overline{\psi}_{m,n}(t) \, dt,$$

the overbar denotes complex conjugate, and $\psi_{m,n}(t)$ is a dilated and translated version of a wavelet function $\psi(t) \in L_2$, viz.,

$$\psi_{m,n}(t) = 2^{m/2} \psi(2^m t - n).$$

Associated with each $\psi(t)$, there is a scaling function $\phi(t)$, also in $L_2$, that satisfies the relationship

$$S_L(t) \triangleq \sum_{n=-\infty}^{\infty} \phi_{L,n}(t) = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{L-1} X_{m,n} \psi_{m,n}(t),$$

where $\phi_{m,n}(t)$ is defined similarly as $\psi_{m,n}(t)$, and

$$S_L(t) = \int_{-\infty}^{\infty} x(t) \overline{\phi}_{L,n}(t) \, dt.$$

Let $\hat{\psi}(f)$ and $\hat{\phi}(f)$ be the Fourier transforms of $\psi(t)$ and $\phi(t)$, respectively. They satisfy [5]

$$\hat{\phi}(n) = \begin{cases} 1, & n = 0, \\ 0, & n = \pm 1, \pm 2, \ldots, \text{integer } n \end{cases}$$

and

$$\hat{\psi}(0) = 0.$$  

Many scaling and wavelet functions, including those with finite support, have already been found [3], [4], [5], [6], [7]. In the language of multiresolution analysis [4], $s_L(t)$ can be interpreted as the approximation of $x(t)$ up to the resolution level $L$. For any $x(t) \in L_2$, $s_L(t)$ satisfies

$$\lim_{L \to \infty} s_L(t) = 0$$

and

$$\lim_{L \to -\infty} s_L(t) = x(t),$$

where the limits are in the $L_2$ sense. Because of (3), $x(t)$ can also be written as

$$x(t) = \sum_{m=K}^{\infty} \sum_{n=-\infty}^{\infty} X_{m,n} \psi_{m,n}(t) + \sum_{n=-\infty}^{\infty} S_{K,n} \phi_{K,n}(t)$$

any finite $K$.  

Furthermore, a Parseval equality

$$\int_{-\infty}^{\infty} x^2(t) \, dt = \sum_{m=\infty}^{\infty} \sum_{n=\infty}^{\infty} |X_{m,n}|^2$$

$$= \sum_{m=\infty}^{\infty} \sum_{n=\infty}^{\infty} |X_{m,n}|^2 + \sum_{n=\infty}^{\infty} |S_{K,n}|^2$$

holds for all finite integer $K$.

Mallat [4] developed a recursive algorithm for computing the wavelet coefficients $X_{m,n}$ and $S_{m,n}$ based on a digital filtering approach. This forms the basis of a multiresolution signal processing framework. Here we only write down those

1The wavelet and scaling functions must be regular (to be defined later) to guarantee that they form a multiresolution framework [5].
results that will be useful in our proofs later. In particular, we have

\[ \phi_{L+1,m}(t) = \sum_{k=-\infty}^{\infty} h_{n-2k}\phi_{L,k}(t) + \sum_{k=-\infty}^{\infty} g_{n-2k}\psi_{L,k}(t), \]

where

\[ h_k = \sqrt{2} \int_{-\infty}^{\infty} \phi(2t-k)\phi(t) \, dt \]

and

\[ g_k = \sqrt{2} \int_{-\infty}^{\infty} \phi(2t-k)\psi(t) \, dt. \]

The discrete-time Fourier transforms of \( h_k \) and \( g_k \), denoted by \( H(f) \) and \( G(f) \), respectively, are periodic functions of \( f \) with period one, and satisfy

\[ \sqrt{2} \hat{\phi}(2f) = H(f)\hat{\phi}(f) \]

and

\[ \sqrt{2} \hat{\psi}(2f) = G(f)\hat{\phi}(f). \]

For any \( x(t) \in L_2 \), the representations (1) and (8) are completely equivalent in the sense that the right hand sides of both equations converge to \( x(t) \). As noted by Meyer [8], this is not the case for functions outside of \( L_2 \). To see this, we first describe the wavelet decomposition of functions that are not in \( L_2 \).

In order to define the wavelet decomposition for nonsquare integrable functions, we need to impose more restrictive conditions on both \( \psi(\cdot) \) and \( \phi(\cdot) \) so that the coefficients \( X_{m,n} \)'s and \( S_{K,n} \)'s in (2) and (4) are well defined, i.e., so that the integrals converge. This is commonly referred to as a regularity condition, which is defined by Meyer as follows.

**Definition:** A wavelet function \( \psi(t) \) is said to be regular of order \( r \) [8] if

\[ \left| \frac{d^m\psi(t)}{dt^m} \right| \leq \frac{B_m}{(1+|t|)^m} \quad \forall t, m \quad q = 0, 1, \ldots, r, \]

for some constants \( B_m \)'s. The regularity of \( \phi(\cdot) \) is defined in the same manner.

Under such condition, it can be shown [8] using the theory of generalized functions [18] that the discrete wavelet decomposition of the form (8) holds for any generalized function of order less than \( r \). The form of (1), however, fails to hold for these functions. For example, if \( x(t) = 1 \) for all \( t \), then \( X_{m,n} = 0 \) for all \( m \) and \( n \), and hence (1) is false. The reason is that (7) no longer holds for functions outside of \( L_2 \) [8].

The regularity condition of (12) can be relaxed so that the discrete wavelet decomposition of the form (8) still holds for many interesting functions. A less restrictive regularity condition used by Mallat is the following.

**Definition:** A wavelet function \( \psi(\cdot) \) is said to be regular [4], [5], if and only if it is continuously differentiable and also satisfies

\[ \left| \frac{d^q\psi(t)}{dt^q} \right| \leq \frac{D_q}{(1+|t|)^q} \quad \forall t \quad q = 0, 1, \ldots, r, \]

where \( D_0 \) and \( D_1 \) are finite constants. The regularity of \( \phi(\cdot) \) is defined in the same way.

In addition to their usefulness in proving theorems, the regularity conditions are also very important in practice even if we only consider discrete wavelet decompositions of functions in \( L_2 \). For example, they guarantee that the multiresolution components of smooth functions will not exhibit fractal like characteristics [7]. They also control the localization properties of wavelet decompositions because of the specified rate of decay. For these reasons, most wavelet functions used in practice are regular [3], [7], [8].

The smoothness of both \( \phi(f) \) and \( \psi(f) \) can be controlled from the filters \( H(f) \) and \( G(f) \) [7]. In particular, one can choose the filters so that

\[ \hat{\phi}(f) = O(f^{-1-\epsilon}) \]

and

\[ \hat{\psi}(f) = O(f^{-1-\epsilon}) \]

as \( f \) goes to infinity, where \( \epsilon \) is strictly positive. Note that (13) implies both \( \hat{\phi}(f) \) and \( \hat{\psi}(f) \) are \( O(f^{-1}) \) as \( f \to \infty \). Hence, (14), (15) are slightly stronger than (13). In practice, most wavelet basis functions, such as the Lemarie wavelet basis [4] and the Daubechies compactly supported wavelet basis [7], satisfy both (14) and (15). The constant \( \epsilon \) can in fact be made arbitrarily large by suitably choosing \( H(f) \) and \( G(f) \) [7].

In the rest of this paper, we assume that both the wavelet and scaling functions satisfy the regularity conditions (13), (14) and (15).

### III. Boundedness of a Deterministic Wavelet Decomposition

The main objective of this section is to show that the wavelet decomposition of the complex exponential \( e^{j2\pi ft} \) converges to the function *boundedly*. This will be needed in Section IV to prove the wavelet decomposition of harmonizable processes. Since this convergence result does not appear to have been reported, we provide a detailed proof.

We first establish upper bounds for \( \hat{\psi}(\cdot) \) and \( \hat{\phi}(\cdot) \).

**Lemma 1:** Let \( \psi(\cdot) \) be a wavelet function that satisfies the regularity conditions (13) and (15). Then its Fourier transform is bounded by

\[ |\hat{\psi}(f)| \leq \min\left( C_0, \frac{C_1}{|f|^{1+\epsilon}}, C_2|f| \right), \]

where \( C_0, C_1 \) and \( C_2 \) are finite constants, and \( \epsilon > 0 \).

**Comment:** It is evident that the bound \( C_1/|f|^{1+\epsilon} \) is most useful for large \( f \) (in magnitude), while \( C_2|f| \) is particularly useful for small \( f \).
Proof: The first bound $|\tilde{\psi}(f)| \leq C_0$ for all $f$ follows from the absolute integrability of $\psi(t)$ which is a result of (13). The second bound follows directly from (15). Finally, since $\tilde{\psi}(f)$ is uniformly continuous, bounded, and equals zero at $f = 0$, there exist a constant $C_2$ such that $|\tilde{\psi}(f)| \leq C_2|f|$, and the lemma is proved. \hfill \Box

**Lemma 2:** Let $\phi(\cdot)$ be a scaling function that satisfies the regularity conditions (13) and (14). Then its Fourier transform can be bounded by

$$|\tilde{\phi}(f)| \leq \min \left( C_3, \frac{C_4}{|f|^{1+\epsilon}}, 1 + C_5|f| \right),$$

where $C_3$, $C_4$, and $C_5$ are finite constants, and $\epsilon > 0$.

Proof: The proof is almost identical to that of Lemma 1. The only difference is in the bound $1 + C_5|f|$, which is a result of (12).

Next, we define a function $I_{K,M,N}(f,t)$ as

$$I_{K,M,N}(f,t) \triangleq \sum_{m=K}^{M} \sum_{n=-N}^{N} \tilde{\phi}(2^{-m}f) e^{2\pi i f 2^{-m}n} \psi(2^{m}t - n) + \sum_{n=-N}^{N} \tilde{\phi}(2^{-K}f) e^{2\pi i f 2^{-K}n} \phi(2^{K}t - n).$$

The next lemma concerns the uniform boundedness of $I_{K,M,N}(f,t)$. Its convergence to a complex exponential function is addressed in the theorem thereafter. A similar theorem concerning the pointwise convergence of the continuous wavelet transform of a bounded function which oscillates around zero is proved in [19]. Here we use the properties of the scaling and wavelet functions, viz., (9)-(11), to show in Theorem 1 the pointwise convergence of (16) to the complex exponential.

**Lemma 3:** Let $\psi(\cdot)$ be a regular wavelet function and $\phi(\cdot)$ be its corresponding regular scaling function. Given any fixed finite integer $K$, the function $I_{K,M,N}(f,t)$ is uniformly bounded for all $M > K$, and all $N, f, t$.

The proof of Lemma 3 is given in Appendix A.

**Theorem 1:** Let $\psi(\cdot)$ be a regular wavelet function and $\phi(\cdot)$ be its corresponding regular scaling function. The wavelet decomposition of $e^{i2\pi ft}$ is given by

$$e^{i2\pi ft} = \lim_{M \to \infty} \lim_{N \to \infty} I_{K,M,N}(f_0,t),$$

where $K$ is any fixed integer.

Furthermore, the convergence is uniformly bounded for all $f_0$ and $t$, i.e., $I_{K,M,N}(f_0,t)$ is uniformly bounded for all $M > K$ and all $N, f_0, t$.

Comment: It can be verified easily that for $x(t) = e^{-i\pi ft}$, (16) is a truncated version of the right-hand side of (8). If we were to assume both $\phi(\cdot)$ and $\psi(\cdot)$ satisfy (12), then it is well known [8] that $I_{K,M,N}(f_0,t)$ would converge pointwise to $e^{i2\pi ft}$. Since we use a weaker regularity condition, viz., (13), (14) and (15), we give a proof of the pointwise convergence.

Proof: In view of Lemma 3 and the foregoing comments, it remains to show that $I_{K,M,N}(f_0,t)$ converges to $e^{i2\pi ft}$ for any $t$, $f_0$, and any finite integer $K$. To this end, we have from (9) that

$$\sum_{n=-N}^{N} \tilde{\phi}(2^{-L-1}f_0) e^{2\pi i f_0 2^{n}t_0} \phi(2^n t_0 - n) =$$

$$= \sum_{n=-N}^{N} \tilde{\phi}(2^{-L-1}f_0) e^{2\pi i f_0 2^{n}t_0} \phi(2^n t_0 - n) + \sum_{k=-\infty}^{\infty} \tilde{h}_{n-2k} \phi(2^{k}t_0 - k).$$

holds for any finite integer $L$. We emphasize that (9) is a basic property of any valid pair of wavelet and scaling functions, regardless of the function being decomposed (in this case, the complex exponential). Since the summations over $k$ are absolutely convergent, we can interchange the order of summations, and apply (10) and (11) to give

$$\sum_{n=-N}^{N} \tilde{\phi}(2^{-L-1}f_0) e^{2\pi i f_0 2^{n}t_0} \phi(2^n t_0 - n) =$$

$$= \sum_{k=-\infty}^{\infty} \tilde{\phi}(2^{k}t_0 - k) \tilde{\psi}(2^{-L-1}f_0) \tilde{h}_{n-2k}.$$
any $f_0$ and $t$, \( t \), \( (19) \)

\[
\lim_{M \to \infty} \sum_{n=-\infty}^{\infty} \phi(2^{-M}f_0)e^{j2\pi f_02^{-M}n}\phi(2^M t - n) = e^{j2\pi f_0 t},
\]

any $f_0$ and $t$. \( (20) \)

Since the summation over $n$ in \( (19) \) is bounded independently of $M$, $f_0$ and $t$, we can apply the dominated convergence theorem to interchange the limit and the summation. Hence, \( (6) \) implies \( (19) \) is true. To show the validity of \( (20) \), we use the Poisson sum formula to write

\[
\lim_{M \to \infty} \sum_{n=-\infty}^{\infty} \phi(2^{-M}f_0)e^{j2\pi f_02^{-M}n}\phi(2^M f_0 - n) = e^{j2\pi (f_0 - 2^{-M}n)t}.
\]

Because of the second bound in Lemma 2, we can apply the dominated convergence theorem to interchange the limit and summation of the right-hand side. We can then conclude from \( (5) \) that \( (20) \) is true, proving the theorem. \( \square \)

IV. STOCHASTIC WAVELET DECOMPOSITION

We consider in this section the stochastic wavelet decomposition of second-order harmonizable random processes. Since we can write a random process as the sum of a mean function and a corresponding zero-mean random process, we can consider the wavelet decompositions of the deterministic and the random parts separately. We, therefore, assume without loss of generality that all the random processes are of zero mean. A process $z(t)$ is said to be of second-order if $E[x^2(t)] < \infty$ for all $t$ \([13, 20]\). By virtue of the Cauchy–Schwarz inequality, all second-order moments and cross moments of $z(t)$ exist. Note that a second-order random process can either be stationary, wide-sense stationary, or nonstationary. A large class of second-order processes are harmonizable, which, roughly speaking, are those that can be represented as a superposition of complex sinusoids. A precise definition of harmonizable processes due to Loève is given as follows.

**Definition [13, Section 37.4]:** A covariance function $R(t, t')$ is said to be harmonizable if there exists a covariance $\gamma(f, f')$ of bounded variation on $\mathbb{R} \times \mathbb{R}$ such that

\[
R(t, t') = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{j2\pi (f_1-f_2) t} d\gamma(f_1, f_2).
\]

The symbol $\mathbb{R}$ denotes the set of real numbers.

**Definition [13, Section 37.4]:** A second-order random process $x(t)$ is said to be harmonizable if there exists a second-order process $\xi(f)$ with a covariance function $E[\xi(f)\xi(f')] = \gamma(f, f')$ of bounded variation on $\mathbb{R} \times \mathbb{R}$ such that

\[
x(t) = \int_{-\infty}^{\infty} e^{j2\pi ft} d\xi(f) \quad \text{with probability one.}
\]

A necessary condition for a process to be harmonizable is that it is second-order continuous. Loève also showed that a random process is harmonizable, if and only if its covariance function is harmonizable. Observe that the function $\gamma(f, f')$ determines, with suitable normalization at the points of discontinuity, a distribution of mass over the two-dimensional frequency plane. Hence, it can be considered as a two-dimensional spectral distribution of $x(t)$. (See, for example, \([21, Section \ 8.3]\).)

In the special case where $x(t)$ is also wide-sense stationary, the process $\xi(f)$, usually called the spectral process, has orthogonal increments, i.e.,

\[
E[\xi(t) d\xi(s)] = dF(s) \delta_{t-s},
\]

where $\delta_t$ is the Kronecker delta, and $F(\cdot)$, usually called the spectral measure, is a finite measure on the real line. Equation \( (21) \) for wide-sense stationary processes then becomes

\[
E[\xi(t) d\xi(s)] = R_{\xi}(t - s') = \int_{-\infty}^{\infty} e^{j2\pi f(t-s')} dF(f).
\]

This is the well-known theorem of Bochner for the spectral decomposition of mean-square continuous wide-sense stationary random processes \([14, Sections \ 20-21]\).

With these notations established, we can then proceed to develop results concerning the wavelet decomposition of harmonizable processes.

**Theorem 2:** A second-order harmonizable random process $x(t)$ can be decomposed as

\[
x(t) = \text{l.i.m.}_{M,N \to \infty} \left[ \sum_{m=-K}^{N} \sum_{n=-N}^{M} X_{m,n}\psi_{m,n}(t) + \sum_{n=-N}^{N} S_{K,n}\phi_{K,n}(t) \right],
\]

any finite $K$, any $t$, \( (23) \)

where l.i.m. denotes limit in the mean-square,

\[
X_{m,n} \triangleq \int_{-\infty}^{\infty} x(t) \overline{\psi_{m,n}(t)} dt = \int_{-\infty}^{\infty} 2^{-m/2} \overline{\phi(2^{-m} f)} e^{j2\pi f_2^{-m} n} d\xi(f),
\]

where $m = K, K+1, \ldots$, \( (24) \)

and

\[
S_{K,n} \triangleq \int_{-\infty}^{\infty} x(t) \overline{\phi_{K,n}(t)} dt = \int_{-\infty}^{\infty} 2^{-K/2} \overline{\phi(2^{-K} f)} e^{j2\pi f_2^{-K} n} d\xi(f).
\]

Furthermore, the convergence in \( (23) \) is uniformly bounded in the mean-square sense.
Comment: Equation (23) is said to be the discrete wavelet decomposition of second-order harmonizable processes. It is formally identical to (8) except that the equalities in the two cases must be interpreted in different senses.

Proof: First note that if

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left| 2^{-m/2} \hat{\phi}(2^{-m} f') \right| \left| 2^{-m/2} \hat{\phi}(2^{-m} f) \right| \, df \, df' \, \gamma(f, f') < \infty, \quad m = K, K + 1, \ldots,$$

(26)

and

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left| 2^{-K/2} \hat{\phi}(2^{-K} f') \left| 2^{-K/2} (2^{-K} f) \right| \, df \, df' \, \gamma(f, f') < \infty, \quad m = K, K + 1, \ldots,$$

(27)

then the wavelet coefficients $X_{m,n}$’s and $S_{K,n}$’s are well defined [13, Section 37.3]. Using Lemmas 1 and 2, and the fact that $\gamma(\cdot, \cdot)$ is of bounded variation, both (26) and (27) are satisfied, and hence the coefficients in (23) are well defined. Using (22), (24), (25), and (16), we have

$$\begin{align*}
E \left[ x(t) - \sum_{m=K}^{L-1} \sum_{n=-N}^{N} X_{m,n} \psi_{m,n}(t) - \sum_{n=-N}^{N} S_{K,n} \phi_{K,n}(t) \right] \\
= E \left[ \int_{-\infty}^{\infty} \left( e^{2\pi ft} - I_{K,M,N}(f, t) \right) \, df \right]^2 \\
= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left( e^{2\pi ft} - I_{K,M,N}(f, t) \right) \, df \, df' \\
\cdot \left( e^{-2\pi f't} - I_{K,M,N}(f', t) \right) \, \gamma(f, f').
\end{align*}$$

(28)

Since the integrand in (28) is uniformly bounded (Lemma 3), and $\gamma(f, f')$ is of bounded variation, the interchange of integration and taking expectation is valid because of the Fubini’s theorem. Furthermore, we can apply the dominated convergence theorem and Theorem 1 to conclude that (28) converges to zero when $M$ and $N$ goes to infinity, which proves (23). The fact that the convergence is uniformly bounded follows because (28) is uniformly bounded for any $N$, $M$, and $t$.

Note that although the equality (23) is in the mean-square sense over the probability space, the equalities (24) and (25) hold with probability one because of the regularity condition and $E|x(t)| < \infty$ [20, p. 80]. Theorem 2 says that given a harmonizable process $x(t)$, we can represent it without loss of information by the coefficients $\{X_{m,n}, S_{K,n} : m = K, K + 1, \ldots; \text{finite} \, K \}$. For any fixed $m$, $X_{m,n}$ can be considered a discrete-time random process. Similarly, $S_{K,n}$ also constitutes a random process. In the actual computation of the coefficients, one can apply as in the deterministic case the pyramidal algorithm of Mallat [4]. That is, given $S_{L,n}$’s for a certain $L$, we can compute $S_{L-1,n}$’s and $X_{L-1,n}$’s by passing through two digital filters followed by down-samplers. The procedure can be repeated recursively to compute all the wavelet coefficients $X_{m,n}$ for $m < L$. The reconstruction of $S_{L,n}$ from $S_{K,n}$ and $X_{m,n}$ ($m = K, K + 1, \ldots, L - 1$) can also be proceeded as described in [4].

Corollary 1: Let $x(t)$ and its wavelet decomposition be as in Theorem 2. Then, the equality

$$\sum_{n=-\infty}^{\infty} S_{L,n} \phi_{L,n}(t) = \sum_{n=-\infty}^{\infty} S_{L-1,n} \phi_{L-1,n}(t) + \sum_{n=-\infty}^{\infty} X_{L-1,n} \psi_{L-1,n}(t)$$

holds in the mean-square sense for any finite integer $L$ and any $t$.

Proof: Letting $K = L - 1$ in (23), we have

$$x(t) = \sum_{m=K}^{L-1} \sum_{n=-N}^{N} X_{m,n} \psi_{m,n}(t) + \sum_{n=-N}^{N} S_{L-1,n} \phi_{L-1,n}(t) + \sum_{n=-N}^{N} X_{L-1,n} \psi_{L-1,n}(t) + \sum_{n=-N}^{N} S_{L-1,n} \phi_{L-1,n}(t).$$

(29)

Since (23) holds for any finite integer $K$, the second and third sums combined must equal in the mean-square sense to $\sum_{n} S_{L,n} \phi_{L,n}(t)$.

Theorem 3: Let $x(t)$ be a second-order harmonizable process. Assume that its spectral distribution $\gamma(\cdot, \cdot)$ does not have a point mass at zero frequency, then the equality

$$\sum_{n=-\infty}^{\infty} S_{L,n} \phi_{L,n}(t) = \sum_{m=K}^{L-1} \sum_{n=-N}^{N} X_{m,n} \psi_{m,n}(t)$$

holds in the mean-square sense for any finite integer $L$ and any $t$.

Proof: In view of Corollary 1, it suffices to show for any $t$ that

$$\sum_{m=-\infty}^{\infty} S_{K,n} \phi_{K,n}(t) = 0.$$
To do so, consider

\[ E[S_{K,n}\bar{S}_{K,l}] = E\left[ \int_{-\infty}^{\infty} 2^{-K/2} \phi(2^{-K} f) e^{i2\pi f} d\xi(f) \right] \]

\[ \cdot \int_{-\infty}^{\infty} 2^{-K/2} \phi(2^{-K} f') e^{-i2\pi f'} d\xi(f') \]

\[ = 2^{-K} a_{K,n,l}, \]

where

\[ a_{K,n,l} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \phi(2^{-K} f) \hat{\phi}(2^{-K} f') \]

\[ \cdot e^{i2\pi f} (fn-f') \left| d\gamma(f,f') \right|. \] (31)

The foregoing exchange of integration and taking expectation is justified by the Fubini's theorem. Then, we can write

\[ \lim_{K \to -\infty} \lim_{N \to \infty} \left| \sum_{n=-N}^{N} S_{K,n}\bar{S}_{K,l}(t) \right|^2 \]

\[ = \lim_{K \to -\infty} \lim_{N \to \infty} \left| \sum_{n=-N}^{N} E[S_{K,n}\bar{S}_{K,l}] \psi_{K,n}(t) \bar{\psi}_{K,l}(t) \right| \]

\[ = \lim_{K \to -\infty} \sum_{n=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} a_{K,n,l} \psi(2^K t - n) \bar{\psi}(2^K t - l). \] (32)

Note that \( a_{K,n,l} \) is bounded independently of \( K, n, \) and \( l \) because \( \hat{\phi}(\cdot) \) is bounded and \( \gamma(\cdot, \cdot) \) is of bounded variation. We can then apply the dominated convergence theorem to interchange the limit and the infinite sums in (32). Consequently, it remains to show \( a_{K,n,l} \to 0 \) as \( K \to -\infty \) for any \( n \) and \( l \). To this end, we use (31) and the dominated convergence theorem to write

\[ \lim_{K \to -\infty} |a_{K,n,l}| \leq \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left| \phi(2^{-K} f) \hat{\phi}(2^{-K} f') \right| \left| d\gamma(f,f') \right| \]

\[ = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} l(1_{f=0}+1_{f'=0}) \left| d\gamma(f,f') \right| \]

\[ = |\gamma(0^+, 0^+)| + |\gamma(0^-, 0^-)| - |\gamma(0^+, 0^-)| - |\gamma(0^-, 0^+)|. \]

Hence, if \( \gamma(\cdot, \cdot) \) has no point mass at the origin, the limit vanishes and the theorem is proved. \( \Box \)

Theorems 2 and 3 imply an alternative form of wavelet decomposition for a subclass of harmonizable processes as follows.

**Theorem 4:** If \( x(t) \) is a second-order harmonizable process with no spectral point mass at the origin, its wavelet decomposition can be represented as

\[ x(t) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} X_{m,n} \psi_{m,n}(t), \quad \text{any } t, \] (33)

where \( X_{m,n} \) is defined in (24). The equality is in the mean-square sense, and the convergence in (33) is also uniformly bounded in the mean-square sense.

**Proof:** Substitute (29) into (23). \( \Box \)

It is evident that the class of processes in Theorem 4 includes all wide-sense stationary processes having absolutely integrable autocorrelation functions, i.e., those where power spectral densities exist. This is an important class of random processes frequently seen in engineering literatures. It is easy to give examples where (33) fails to be true if the autocorrelation function is not absolutely integrable. To this end, consider a random process \( x(t) = U \) where \( U \) is a binary random variable with probability distribution

\[ P_R(U = 1) = P_R(U = -1) = 0.5. \]

We have for this process \( E[x(t) = 0 \text{ for all } t \text{ and } E[x(t)x(t+\tau)] = 1 \text{ for all } t \text{ and } \tau. \) It can be easily verified that \( X_{m,n} = 0 \) for all \( m \) and \( n \), while \( S_{K,n} = U \) for all \( n \) and for any fixed \( K \), confirming that (33) fails to hold for such a process. In general, if the limit in (30) is nonzero, then Theorem 3 is not valid and hence Theorem 4 also does not hold. If we trace through the proof of Theorem 3, we can observe from (31) that for any process where the spectral distribution \( \gamma(f,f') \) has a point mass at the origin, the sequence \( a_{K,n,l} \) will not go to zero as \( K \) goes to negative infinity. Roughly speaking, this is because \( \psi_{m,n}(f) \) equals zero at \( f = 0 \) for any \( m \) and \( n \). As a result, no linear combination of \( \psi_{m,n}(t) \) can represent a process that has a point mass at zero frequency. In such case, we must use a wavelet decomposition of the form (23).

Next, we consider the wavelet decompositions of the covariance functions of harmonizable processes. The following theorem relates (23) to the wavelet decomposition of a covariance function.

**Theorem 5:** The wavelet decomposition of a second-order process \( x(t) \) exists, if and only if its covariance function has the decomposition

\[ R(t,t') = \sum_{m,m',n,n'} \alpha_{m,m',n,n'} \psi_{m,n}(t) \bar{\psi}_{m',n'}(t') \]

\[ + \sum_{m \geq 2K,n} \eta_{m,K,n} \psi_{m,n}(t) \bar{\psi}_{K,n}(t') \]

\[ + \sum_{m' \geq 2K,n'} \eta_{m',K,n'} \psi_{m',n'}(t') \phi_{K,n}(t) \]

\[ + \sum_{n,n'} \beta_{K,n,n'} \phi_{K,n}(t) \bar{\phi}_{K,n'}(t'). \]

When they exist, the coefficients are related by

\[ \alpha_{m,m',n,n'} = E[X_{m,n} \bar{X}_{m',n'}], \]

\[ \eta_{m,K,n} = E[X_{m,n} \bar{S}_{K,n'}], \]

and

\[ \beta_{K,n,n'} = E[S_{K,n} \bar{S}_{K,n'}]. \]
This theorem is a direct consequence of the convergence in quadratic mean (mean-square) criterion [13, pp. 135–136], and hence its proof is omitted.

In the context of signal processing [4], [17], the wavelet decomposition can be interpreted as a multistep linear filtering and down-sampling procedure on the original process. Hence, if a process is harmonizable, i.e., if it can be represented as a superposition of sinusoids, the output of such a process through linear filters and down-samplers should also be a superposition of sinusoids. The next theorem asserts this precisely.

**Theorem 6:** Let \( x(t) \) be a second-order continuous random process, and let its wavelet decomposition be formally given by (23). Let

\[
x_L(t) = \sum_{m=-\infty}^{\infty} X_{L,m} \tilde{\psi}_{L,m}(t)
\]

and

\[
s_L(t) = \sum_{m=-\infty}^{\infty} S_{L,m} \phi_{L,m}(t).
\]

Then, the following statements are true.

1) If \( x(t) \) is harmonizable, then its wavelet decomposition is valid in the sense of Theorem 2. Furthermore, \( x_L(t) \) and \( s_L(t) \) are also harmonizable for any \( L \).

2) If for any fixed \( K, s_K(t) \) is harmonizable, then \( x_L(t) \) and \( s_L(t) \) for all \( L < K \) are harmonizable.

3) If for any fixed \( K, s_K(t) \) is harmonizable, and \( x_L(t) \)'s are harmonizable for all \( L \geq K \), then \( x(t) \) is harmonizable.

**Proof:** First consider statement 1. If \( x(t) \) is harmonizable, then Theorem 2 implies that its wavelet decomposition is valid. We can use (24) to write

\[
x_L(t) = \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} \tilde{\psi}(2^{-m}f) e^{j2\pi f2^{-m}t} d\xi(f) \psi(2^{-m}t-n).
\]

Note that the equality holds with probability one because both (24) and (25) hold with probability one. It can be verified using the regularity of \( \psi(\cdot) \) and the Fubini's theorem that we can interchange the order of integration and summation. We can then apply the Poisson sum formula to write

\[
x_L(t) = \int_{-\infty}^{\infty} e^{j2\pi f t} \tilde{\psi}(2^{-m}f) \left( \sum_{k=-\infty}^{\infty} \tilde{\psi}(2^{-m}f-k)e^{-j2\pi 2^{-m}k} \right) d\xi(f) \psi(2^{-m}t-n).
\]

To conclude that \( x_L(t) \) is harmonizable, we need to show that the induced spectral measure is of bounded variation. To this end, we write

\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left| \sum_{k=-\infty}^{\infty} \tilde{\psi}(2^{-m}f-k) \tilde{\psi}(2^{-m}f') \right| d\gamma(f+f', t, t')
\]

\[
= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left| \sum_{k=-\infty}^{\infty} \tilde{\psi}(2^{-m}f-k) \tilde{\psi}(2^{-m}f') \right| d\gamma(f+f', t, t').
\]

Lemma 1 implies that the integrand in the double integral is bounded. Since \( \gamma(\cdot, \cdot) \) is of bounded variation, we can conclude that the spectral measure is of bounded variation, and hence \( x_L(t) \) is harmonizable. Similarly, \( s_L(t) \) is also harmonizable. Statement 2 follows from similar steps as before and the fact that

\[
x_{m,n} = \int_{-\infty}^{\infty} x(t) \overline{\tilde{\psi}_{m,n}(t)} dt = \int_{-\infty}^{\infty} s_K(t) \overline{\psi_{m,n}(t)} dt, \quad m < K,
\]

\[
s_{m,n} = \int_{-\infty}^{\infty} x(t) \overline{\phi_{m,n}(t)} dt = \int_{-\infty}^{\infty} s_K(t) \overline{\phi_{m,n}(t)} dt, \quad m < K.
\]

To prove statement 3, let

\[
x_L(t) = \int_{-\infty}^{\infty} e^{j2\pi ft} d\xi_L(f), \quad L = K, K+1, \ldots,
\]

and

\[
s_K(t) = \int_{-\infty}^{\infty} e^{j2\pi ft} d\rho_K(f),
\]

where \( \xi_L(f) \)'s and \( \rho_K(f) \) are the spectral processes. Because of Corollary 1, and the fact that the sum of two harmonizable processes are harmonizable, we can write

\[
s_{L+1}(t) = \int_{-\infty}^{\infty} e^{j2\pi ft} d\rho_{L+1}(f),
\]

where

\[
\rho_{L+1}(f) = \rho_L(f) + \xi_L(f).
\]
Letting $E[PL(f)\tilde{p}_L(f')] = \gamma_L(f,f')$, we want to show that
\[ E[s_L(t)\tilde{s}_L(t')] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i2\pi(f-t'f')} d\gamma_L(f,f') \quad (34) \]
converges to a harmonizable covariance function. From Theorem 2, $s_L(t)$ converges boundedly to $z(t)$ in mean-square. Hence, (34) is uniformly bounded for all $L$ and $t$. Then it follows from a convergence theorem of Stieltjes integrals [14, pp. 322] that
\[ \lim_{L \to \infty} E[s_L(t)\tilde{s}_L(t')] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i2\pi(f-t'f')} d\gamma(f,f'), \]
for some measure $\gamma(f,f')$. Using the harmonizability theorem of Loève [13, p. 142], we can conclude that $z(t)$ is harmonizable.

V. LINEAR OPERATIONS ON STOCHASTIC WAVELET DECOMPOSITIONS

In this section, we consider linear operations on the wavelet decomposition of harmonizable processes. The first result concerns the sum of two harmonizable processes.

**Theorem 7:** Let $z(t)$ and $y(t)$ be harmonizable processes with wavelet decompositions
\[ x(t) = \sum_{m=K}^{\infty} \sum_{n=-\infty}^{\infty} X_{m,n}\psi_{m,n}(t) + \sum_{n=-\infty}^{\infty} S_{K,n}\phi_{K,n}(t) \]
and
\[ y(t) = \sum_{m=K}^{\infty} \sum_{n=-\infty}^{\infty} Y_{m,n}\psi_{m,n}(t) + \sum_{n=-\infty}^{\infty} S_{K,n}\phi_{K,n}(t). \]
Then, the wavelet decomposition of their sum is
\[ x(t) + y(t) = \sum_{m=K}^{\infty} \sum_{n=-\infty}^{\infty} (X_{m,n} + Y_{m,n})\psi_{m,n}(t) + \sum_{n=-\infty}^{\infty} (S_{K,n} + S_{K,n})\phi_{K,n}(t). \]
The proof of Theorem 7 is given in Appendix B.

Theorem 7 says that the wavelet decompositions of two harmonizable processes can be summed term by term. One can see by induction that the same conclusion holds true for a finite sum of harmonizable processes.

Next, we consider the derivative of harmonizable processes. Using the fact that $\gamma(f,f')$ is of bounded variation, it can be shown that the derivative of $z(t)$ exists in the mean-square sense, if and only if
\[ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |f'| \, d\gamma(f,f') < \infty. \quad (35) \]
Its wavelet decomposition is then given by the following theorem.

**Theorem 8:** Let $z(t)$ be a harmonizable process with a spectral measure satisfying (35). Then its derivative $\dot{z}(t)$ exists in the mean-square sense, and its wavelet decomposition is given by
\[ \dot{z}(t) = \frac{1}{M,N \to \infty} \left[ \sum_{m=K}^{M} \sum_{n=-N}^{N} A_{m,n}\psi_{m,n}(t) + \sum_{n=-N}^{N} B_{K,n}\phi_{K,n}(t) \right], \]
any finite $K$, where
\[ A_{m,n} = \int_{-\infty}^{\infty} \dot{z}(t)\overline{\psi}_{m,n}(t) \, dt \]
\[ = \int_{-\infty}^{\infty} j2\pi f2^{-m}j2\pi(2^{-m}f)e^{i2\pi j2^{-m}n} \, d\xi(f), \]
and
\[ B_{K,n} = \int_{-\infty}^{\infty} \dot{z}(t)\phi_{K,n}(t) \, dt \]
\[ = \int_{-\infty}^{\infty} j2\pi f2^{-K}j2\pi(2^{-K}f)e^{i2\pi j2^{-K}n} \, d\xi(f). \]
Furthermore, the convergence in (36) is bounded in the mean-square sense.

The proof of this theorem is almost identical to that of Theorem 2, except that we need to use (35) to guarantee the existence of integrals. This theorem can be generalized in an obvious way for the wavelet decomposition of higher order derivatives of $z(t)$ provided they exist and that (35) is suitably modified.

Finally, we consider the integration of harmonizable processes in the form of linear filtering. In particular, we consider the convolution integral
\[ \int_{-\infty}^{\infty} h(t-\tau)x(\tau) \, d\tau. \quad (37) \]
We assume here that $h(\cdot)$ is absolutely integrable, which is both necessary and sufficient to ensure that the filter is bounded input bounded output stable [22, Section 2.6]. Then, it can be shown [13, pp. 138–140] that (37) exists in the mean-square sense, and the convolution integral can also be written as
\[ \int_{-\infty}^{\infty} e^{i2\pi f\hat{h}(f)} \, d\xi(f), \quad (38) \]
where $\hat{h}(\cdot)$ is the Fourier transform of $h(\cdot)$. The next theorem concerns the convolution of a harmonizable process using the discrete wavelet decomposition.
Theorem 9: Let $h(t)$ be an absolutely integrable function, and $x(t)$ be a harmonizable process with the wavelet decomposition given by (23). Then, the following equality holds in the mean-square sense:

$$\int_{-\infty}^{\infty} h(t-\tau)x(\tau)\,d\tau = \sum_{m=K}^{\infty} \sum_{n=-\infty}^{\infty} X_{m,n} \lambda_{m,n}(t) + \sum_{n=-\infty}^{\infty} S_{K,n} \mu_{K,n}(t),$$

(39)

where

$$\lambda_{m,n}(t) = \int_{-\infty}^{\infty} h(t-\tau)\psi_{m,n}(\tau)\,d\tau$$

and

$$\mu_{K,n}(t) = \int_{-\infty}^{\infty} h(t-\tau)\phi_{K,n}(\tau)\,d\tau.$$

That is, the convolution can be performed term by term using the wavelet decomposition.

The proof of Theorem 9 is given in Appendix C.

VI. CONCLUSION

We have considered in this paper the wavelet decomposition of complex exponential functions, and showed in particular that the convergence is uniformly bounded. This result is then applied to establish the wavelet decomposition of second-order harmonizable processes. This class of processes are those that can be represented as a superposition of complex exponentials. These processes can be stationary, wide-sense stationary, or nonstationary; and their spectral measures can either be continuous, discrete, or a mixture of both. It is shown that there are two different forms for representing the discrete wavelet decompositions of harmonizable processes that do not have point masses at zero frequency. An important example is the class of wide-sense stationary processes having power spectral densities. Only one of the two forms, however, is valid in providing a multiresolution decomposition of general harmonizable processes that has a discrete component at zero frequency.

We have also examined the addition, differentiation, and integration of stochastic wavelet decompositions. In particular, term by term addition of wavelet decompositions of harmonizable processes is justified. In the case of convolutions, it is shown that if the filter impulse response is bounded input bounded output stable, then the output process can be directly obtained from the wavelet decomposition of the input process through an interpolation formula, which is a result of term by term convolution.

APPENDIX A

PROOF OF LEMMA 3

We write

$$\left|J_{K,M,N}(f,t)\right| \leq \sum_{m=K}^{M} \sum_{n=-N}^{N} \left|\tilde{\psi}(2^{-m}f)\right| \left|\psi(2^{m}t-n)\right|$$

$$+ \sum_{n=-N}^{N} \left|\tilde{\phi}(2^{-K}f)\right| \left|\phi(2^{K}t-n)\right|$$

$$\leq \sum_{m=K}^{\infty} \left|\tilde{\psi}(2^{-m}f)\right| \sum_{n=-\infty}^{\infty} \left|\psi(2^{m}t-n)\right|$$

$$+ \left(\sup_{f} \left|\tilde{\phi}(f)\right|\right) \sum_{n=-\infty}^{\infty} \left|\phi(2^{K}t-n)\right|$$

$$\leq \sum_{m=K}^{\infty} \left|\tilde{\psi}(2^{-m}f)\right| \sum_{n=-\infty}^{\infty} \left(1 + \left|2^{m}t-n\right|\right)^{-2}$$

$$+ C \sum_{n=-\infty}^{\infty} \frac{C}{\left(1 + \left|2^{K}t-n\right|\right)^{2}}.$$

The bounds for $\psi(\cdot)$ and $\phi(\cdot)$ follow from the regularity condition. Since the infinite sums over $n$ are bounded independently of $m$, $K$ and $t$, it remains to show that

$$S_{0} = \sum_{m=K}^{\infty} \left|\tilde{\psi}(2^{-m}f)\right|$$

(A.1)

is bounded independently of $f$. First if $f = 0$, then the sum is identically zero because $\psi(0) = 0$. Next suppose $f > 0.$
Pick and fix an arbitrary positive and finite number \( f_1 \). For \( 0 < f \leq f_1 \), we have from (A.1) and Lemma 1 that

\[
S_0 \leq \sum_{m=K}^{\infty} C_22^{-m}f \leq \sum_{m=K}^{\infty} C_22^{-m}f_1 = C_2f_12^{-K+1},
\]

\( 0 < f \leq f_1 \),

(A.2)

which is bounded. For \( f > f_1 \), let \( M_1 = \lceil \log_2 (f/f_1) \rceil \) where \( \lceil \alpha \rceil \) denotes the largest integer no greater than \( \alpha \). Then, we can apply Lemma 1 to give

\[
S_0 = \sum_{m=K}^{M_1} |\tilde{\psi}(2^{-m}f)| + \sum_{m=M_1+1}^{\infty} |\tilde{\psi}(2^{-m}f)| \\
\leq \sum_{m=K}^{M_1} \frac{C_12^{m(1+\epsilon)}}{f^{1+\epsilon}} + \sum_{m=M_1+1}^{\infty} C_22^{-m}f \\
< 2^{1+\epsilon}C_1 \left( \frac{2M_1+1}{f} \right)^{1+\epsilon} + \sum_{k=0}^{\infty} C_22^{-k}f \quad f > f_1.
\]

Note that \( M_1 = \lceil \log_2 (f/f_1) \rceil \) implies \( f/2f_1 < 2M_1 \leq f/f_1 \).

Therefore,

\[
S_0 < \frac{2^{1+\epsilon}C_1}{f^{1+\epsilon}} + \sum_{k=0}^{\infty} C_22^{-k}f_1 < \frac{2^{1+\epsilon}C_1}{f^{1+\epsilon}} + 2C_2f_1,
\]

\( f > f_1 \),

(A.3)

which is again bounded. Equations (A.2) and (A.3) show that \( S_0 \) is uniformly bounded for all \( f > 0 \). A similar argument shows that the same conclusion holds true for all \( f < 0 \), and the lemma is proved. \( \square \)

**APPENDIX B**

**PROOF OF THEOREM 7**

For any \( M \) and \( N \), we write

\[
E \left[ x(t) + y(t) \right] = \sum_{m=M}^{M} \sum_{n=N}^{N} (X_{m,n} + Y_{m,n})\psi_{m,n}(t) \\
- \sum_{n=-N}^{N} (S_{K,n}^{x} + S_{K,n}^{y})\phi_{K,n}(t) \\
= E \left[ \int_{-\infty}^{\infty} (e^{2\pi ft} - I_{K,M,N}(f,t))d\xi_{t}(f) \right] \\
+ \int_{-\infty}^{\infty} \left( e^{2\pi ft} - I_{K,M,N}(f,t) \right)d\xi_{t}(f) \]

\[
\leq E \left[ \int_{-\infty}^{\infty} (e^{2\pi ft} - I_{K,M,N}(f,t))d\xi_{t}(f) \right]^2 + \int_{-\infty}^{\infty} \left( e^{2\pi ft} - I_{K,M,N}(f,t) \right)d\xi_{t}(f) \]

\[
+ \frac{2}{1/2} \left[ \int_{-\infty}^{\infty} (e^{2\pi ft} - I_{K,M,N}(f,t))d\xi_{t}(f) \right]^{1/2}.
\]

Letting \( M \) and \( N \) go to infinity, we obtain the desired result. \( \square \)

**APPENDIX C**

**PROOF OF THEOREM 9**

Let

\[
J_{K,M,N}(t) = \sum_{m=K}^{M} \sum_{n=-N}^{N} X_{m,n}\psi_{m,n}(t) + \sum_{n=-N}^{N} S_{K,n}\phi_{K,n}(t).
\]

Then, for any \( M \) and \( N \), we write

\[
E \left[ \int_{-\infty}^{\infty} h(t - \tau)\tau(t)dt = \sum_{m=K}^{M} \sum_{n=-N}^{N} X_{m,n}\lambda_{m,n}(t) \right] \\
= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(t - \tau)\tau(t)E[\tau(t) - J_{K,M,N}(\tau)] \\
\cdot \{ \tau(t') - J_{K,M,N}(\tau') \} d\tau' \]

\[
\leq \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(t - \tau)\tau(t')E[\tau(t) - J_{K,M,N}(\tau)]^{1/2} \\
\cdot \{ \tau(t') - J_{K,M,N}(\tau') \}^{1/2} d\tau d\tau'.
\]

(C.1)

Since \( h(\cdot) \in L_1 \) and \( J_{K,M,N}(t) \) converges boundedly to \( x(t) \), the Fubini’s theorem implies that the interchange of integration and taking expectation in the first equality is valid. Furthermore, the dominated convergence theorem implies that (C.1) converges to zero as \( M \) and \( N \) goes to infinity. \( \square \)

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