Optimal Economic Growth And Uncertainty:  
The Discounted Case  

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INTRODUCTION  

The cornerstone of one-sector optimal economic growth models is the existence and stability of a steady-state solution for optimal consumption policies. The optimal consumption policy is the stable branch of the saddle point solution of the system of differential equations governing the dynamics of the economy. Examples of this type of behavior can be found in Cass [2] and Koopmans [4]. However, the stable branch solution is a knife-edge policy in the sense that any perturbation, no matter how small, results in instability and eventual annihilation. (This phenomenon is true when the Euler conditions are adhered to after the perturbation). Small perturbations might occur due to observation errors or the lack of knowledge of the exact production functions. It seems reasonable to expect that all sorts of human errors influence decision variables. Hence unless perfect knowledge of all variables, present and future, were known with complete certainty, and unless all decisions were made with exactness, economies of the one-sector deterministic type would lead at best to suboptimal consumption and investment policies.  

One would expect analogous instability results if production due to aggregation error, say, were not known with certainty. Moreover, it should be expected that planners know the future with something less than certainty. Since it seems impossible, in the face of uncertainty of the future,  

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to predict the exact effects of policies which are formulated in the present, it would be comforting to know that it is possible to formulate the model so that the stability properties of optimal capital accumulation paths are preserved even if under the influence of small perturbation (or large perturbations, although perhaps infrequently). In fact, if a model of the the economy took these uncertainties or random fluctuations into consideration at the outset, when the optimal plans are calculated, it would be expected that the system behaved in a manner analogous to the deterministic system of the Cass–Koopmans type.

It seems reasonable that the possibility of error in observation, in output or in aggregation should be accounted for when calculating optimal policies in the present. If errors or random events are indeed incorporated, optimal policies will be affected by the expectation of these events. Under these conditions, it is worthwhile to introduce uncertainty directly into the process. The deterministic theory would then be justified on the grounds that it is indeed an approximation of a more general model. Thus, when the assumptions of the model are loosened to include uncertainty, the qualitative results are not radically different. This last statement is a conclusion which may be deduced from the results of this paper.

In dealing in this uncertain context, it is no longer feasible to maximize discounted integrals of returns since these returns are random variables. The choice of a criterion function is then crucial for the results. The criterion used in this paper is consistent with much of the literature of optimal decisions under uncertainty. We maximize the expected sum of discounted utilities. In this way the possibility of random fluctuations which is inherent in the model is taken into account when calculating optimal policies. This approach leads to results similar to the results obtained in the deterministic model except that the concepts of steady-state and long-run stability must be extended to cover the generalized model.

To be somewhat more precise, we treat a one-sector model of economic growth and introduce uncertainty into the model through a random element in the production function. This random variable might also be thought of as an observation error on the capital stock. The criterion is to maximize the expected sum of discounted utilities. The main emphasis of the paper is on long-run equilibrium of the economy under the influence of optimal policies. This long-run equilibrium is analogous to the modified golden rule in the deterministic theory.

All of the previous authors who have generalized the deterministic growth model in the direction of uncertainty have dealt with the problem of finding optimal consumption or investment policies, not with long-run or asymptotic properties. Since one maximizes the expected value of the
integral of utility, the Euler type equations characterize the expected behavior of the system. In this way, the dynamics of the system are hidden in the background. The long-run analysis of this paper follows somewhat along the lines of [10] which will be discussed below.

The only work which is known to the authors which attempts to generalize deterministic optimal growth models to uncertainty along the lines suggested above is the work of Mirrlees [12]. Works by Phelps [13] and Levhari and Srinivasan [6] are somewhat related to optimal growth under uncertainty. However, they were conceived in the area of optimal investment and savings behavior for the individual. Their approach is different from the approach of this paper since they employ linear production functions, not the usual concave production functions which characterize much of the one-sector growth literature. It seems worthwhile, at this point, to discuss the literature cited above.

Mirrlees optimizes the expected integral of discounted utility over an infinite horizon. His model is of a one-sector economy in which uncertainty is introduced into the production function as labor augmenting technical progress. The differential equation governing the system in the random model is written as

\[ C_t + K_t = F(A_t L_t, K_t), \quad K_t \geq 0, \quad L_t \geq 0. \]  

(1)

Here \( A_t \) is a random variable with known distribution. Also, labor is assumed to grow at a constant exponential rate.

Note that due to the fact that the right-hand side of Eq. (1) is a random variable, the control variable and the state variable are in general random variables.

In order to keep \( A_t \) nonnegative and at the same time to invoke the theory of diffusion processes, which is used since diffusion processes have essentially continuous sample paths, Mirrlees postulates that \( A_t \) is log normally distributed. Hence, his is a stochastic process generated by a Wiener process. (This is the continuous analog of the discrete time process with independent increments.) Thus, Mirrlees treats a continuous-time, continuous-state space (space of capital labor ratios) problem.

Mirrlees finds conditions on the consumption function (assumed to be a time-independent function of present capital stock and the present value of the random variable \( A \)), which characterize the optimal expected behavior of the system over time. These conditions, which are analogous to the Euler equations in the deterministic model are particularly difficult to work with since they are partial differential equations. They arise due to the fact that continuous time diffusion processes are described by partial differential equations.
These partial differential equations are then used in the special case of the utility function \( U(C) = -C^{-n} \), \( n > 0 \). The homogeneity built into the structure of the problem allows Mirrlees to reduce the partial differential equation to an ordinary differential equation. Using this differential equation the qualitative behavior of the system is analysed. Mirrlees also investigates the existence of optimal policies. However, there are numerous difficulties in this problem. For example, for almost all sample paths the Wiener process are not differentiable.

In a recent paper in portfolio analysis, Merton [8] uses techniques similar to those employed by Mirrlees. His results are obtained by reducing the partial differential equation to an ordinary one by using a transformation function which does not seem to work in more general cases.

The work of Phelps represents the first important model along the lines of the theory of growth under uncertainty. He uses the theory of dynamic programming to investigate optimal consumption decisions in a discrete time growth model. Phelps' main concern is the individual. Thus, his model employs a linear production function with only one possible investment opportunity. He investigates the relationship between investment decisions and the length of the planning horizon. Furthermore, several important examples are worked out. The method is to approximate the solution of the infinite horizon problem with the solutions of finite horizon problems as the horizon goes to infinity. These methods and the method of functional equations used by Levhari and Srinivasan will be used below to analyse the model of this paper. Levhari and Srinivasan work explicitly with the dynamic programming formulation. They show, in a nonrigorous way, the important properties of optimal policies, e.g., continuity.

The basic framework for this paper was developed by Mirman in [9, 10], for discrete-time, one-sector stochastic growth models in the positive theory of economic growth. The behavior of the economy under the influence of a rather large class of admissible consumption policies is investigated. These works represent a nontrivial extension of the Solow [14] positive growth model to the stochastic case. In [9] and [10] the random element enters in a more general way than in the Mirrlees' or Phelps' work, i.e., the formulation of the model is completely general in the sense that the production function is of the form \( f(k, A) \). Here \( k \) is the capital labor ratio and \( A \) is the random variable.

Formulating the model in this way allows the production function to have the Harrod neutral form of Mirrlees, the linear form of Phelps, Levhari and Srinivasan, the Hicks neutral form of Mirrlees and Mirman or the random shock form of Leland [5] and Mirman [11] (i.e., \( f(k) + A \)). The long-term or steady-state behavior of the stochastic process for
positive models of economic growth is the main focus in [9] and [10].

The model is essentially a discrete time Markov process defined on the continuum of nonnegative real numbers corresponding to capital labor ratios. At any time $t$ the capital labor ratio or capital stock is a random variable. Moreover, the sequence of capital stocks form a Markov process. The distribution function $F_t$ of $k_t$ is generated by the transition function of the Markov process. The concept of a stationary distribution being analogous to the steady-state concept of deterministic theory plays a major role in the theory. A stationary distribution $F_\infty$ is one which the Markov transition function does not change; i.e., $F_\infty$ holds period after period. The techniques of probability theory are employed to determine the relationship between steady state (in a generalized or statistical sense) and various consumption policies. It was shown that to each admissible consumption policy there corresponds a unique steady-state probability distribution over the set of possible capital–labor ratios. This generalized the usual deterministic results of the existence and uniqueness of steady states corresponding to certain consumption functions.

The existence and uniqueness of a generalized steady state would be uninteresting if the steady state had no stability properties. In [9, 10], it was shown that starting from an arbitrary capital labor ratio there is convergence of the subsequent distribution functions to the unique steady-state distribution function.

Finally, in a review of the literature of this type, it should be pointed out that the inventory theory literature contains examples of problems which in structure are similar to the growth models. An analysis of stochastic stability has been carried out by Karlin [3]. However, the inventory models do not seem directly applicable especially in light of the fact that the functional analytic technique used by Karlin can be greatly simplified in the simple one-sector optimal growth paradigm.

The model used in this paper is analogous to the Mirrlees and Mirman model of a one-sector economy under uncertainty, which is essentially the generalization of the Cass–Koopman model with a random variable in the production function. In fact, our methods unify the structure of growth theory. The dynamic programming formulation makes the Cass–Koopman results somewhat easier to obtain. It is thus seen that this paper represents a nontrivial extension and unification of the work of Cass, Koopman, Mirman and Solow. The dynamic programming techniques are used to construct the necessary conditions which are satisfied by optimal policies. These necessary conditions are the stochastic analog to the well-worn Euler equations. These Euler type conditions are then explored to get at the properties of optimal investment and consumption policies. The consumption policy in turn determines the
behavior of the distribution functions. The usual deterministic type properties are found to hold in the uncertainty case; namely, continuity and monotonicity of investment and consumption as a function of existing capital stock. Further properties characterizing the dynamics of the stochastic growth process are investigated. These properties are analogous in the deterministic theory to the existence of a unique steady-state point for optimal policies. The analogous result in the random theory is the existence of a "stable" set of capital labor ratios over which there exists a unique stationary distribution. It is easily seen that stability of the iterated distribution functions follow from the properties of optimal policies. Finally, a very simple proof, requiring no additional probability theoretic results of the existence and uniqueness of a stationary measure is given. This unique stationary distribution is defined on the "stable" set. The proof depends heavily on the properties of the inverse optimal process which is inverse in the temporal sense.

It should be noted that the techniques used in this paper are quite different from the techniques used in [10]. In [10], probability theory is the main tool, whereas in this paper analytic techniques are mainly used. As a result, the assumptions in [10] on the distribution functions of the random variable representing the random technology are much stronger than the assumptions of this paper. Unfortunately, the techniques employed below do not generalize to the multisector case.

Section 1 of the paper is devoted to setting down the model precisely and in detail. Some of the more standard results on the properties of optimal policies are discussed and proved in Section 1. Section 2 deals with the deterministic model from a dynamic programming point of view. Further properties of optimal policies are proved in Section 3. This is followed by the proof of the main convergence theorem in Section 4. An appendix on the sensitivity of optimal policies for finite time horizons is included. The results contained in the appendix generalize the work of Brock in [1] on deterministic sensitivity. These results are used in deriving some of the main results of this paper.

1. THE MODEL

In this section the assumptions and the motion equation of a one sector growth model of an economy in discrete time with future production uncertainties will be stated and discussed. The production function for the economy is given by

\[ Y_t = F(K_t, L_t; r_t), \]
where \( Y_t, K_t, L_t \), are, respectively, output, capital and labor at time \( t \). The random variable representing uncertainty is given by \( r_t \). As is customary in this type of model, it is assumed that the function \( F(\cdot, \cdot; \cdot) \) is homogeneous of the first degree in its first two arguments. In the usual way, all variables can be put into per capita terms, namely,

\[
y_t = \frac{Y_t}{L_t} = F \left( \frac{K_t}{L_t}, 1; r_t \right) = f(x_t, r_t),
\]

where \( x_t = \frac{K_t}{L_t} \). Another way of thinking about this model is to have \( x_t \) denote the capital stock. In this the labor component of the model is suppressed.

For each possible value \( \rho \) of the random variable \( r_t \), the function \( f(\cdot, \rho) \) is endowed with the well-known properties of production functions. Let \( f'(\cdot, \cdot) = \frac{\partial f(\cdot, \cdot)}{\partial k} \). Then we assume

\[
f(0, \rho) = 0, \quad f'(0, \rho) > 0, \quad f''(0, \rho) < 0, \quad \text{for all values of } \rho. \tag{1.1}
\]

Moreover, the Inada conditions are satisfied, namely,

\[
f'(0, \rho) = +\infty, \quad f'(\infty, \rho) = 0 \tag{1.2}
\]

for all values of \( \rho \). Finally the functions \( f \) and \( f' \) are continuous in both arguments jointly.

The statistics of the random variables \( r_t \) are introduced as follows. Let \( \Omega_t \) be the set of possible "states of the world" which influence the production process at time \( t \). A typical element of \( \Omega_t \) would be \( \omega_t \) which represents the occurrence of, say, an epidemic of a given proportion at time \( t \).

Let \( \mathcal{F}_t \) be a collection of subsets of \( \Omega_t \) on which probabilities are defined. A typical member of \( \mathcal{F}_t \) would be \( \omega_t \in \Omega_t \) which imply that labor participation in the period is less than 90 \( \% \) effective due to an epidemic. The set \( \mathcal{F}_t \) is assumed to be a \( \sigma \)-algebra of subsets of \( \Omega_t \). Probabilistic or measure-theoretic concepts will be used throughout this paper. However, only the most basic of the measure-theoretic concepts will be used. For a discussion of these concepts, see Loève [7]. Let \( P_t \) be a function, a probability measure, defined on \( \mathcal{F}_t \) which assigns a probability to each element of \( \mathcal{F}_t \), i.e., for any set \( F_t \in \mathcal{F}_t \), the probability that the state of the world \( \omega_t \) at time \( t \) is an element of \( F_t \) is given by

\[
P_t(F_t) = \Pr[\omega_t \in F_t].
\]

It is assumed that the probability space \((\Omega_t, \mathcal{F}_t, P_t)\) is the same in each period. Hence, for convenience, time subscripts will be dropped and the
state space will be denoted by \((\Omega, \mathcal{F}, P)\). To translate random happenings into measurable values which are needed for the production function the random variable \(r_t\) is introduced. The function \(r_t\) takes the measure space \((\Omega, \mathcal{F}, P)\) into the real line \((R, \mathcal{B})\) where \(\mathcal{B}\) is the collection of subsets on the real line, on which a probability measure is definable (the set of Borel sets is usually taken, Loève [7]). Hence, \(r_t : \Omega_t \rightarrow R\) such that \(r_t^{-1}(B) \in \mathcal{F}\), \(B \in \mathcal{B}\). Here \(r_t^{-1}(B) = \{\omega \in \Omega : r_t(\omega) \in B\}\). The random variables \(r_t\) are assumed to be independent and identically distributed. For convenience time subscripts will be used only when necessary to explicitly distinguish events in different periods.

The assumptions that the measure space representing states of the world is independent of time and that the random variables are independent and identically distributed are strong simplifying assumptions. They are made for simplicity of analysis. The results of this paper can be generalized to the case where the \(r_t\) are not independent or identically distributed. However, the assumption that the space \((\Omega, \mathcal{F}, P)\) is independent of time seems to remain crucial to the limit theorem of this paper.

The mapping \(r : \Omega \rightarrow R\) generates a measure on the Borel subsets of the real line which is given by \(\nu(\cdot)\). The mapping is as follows:

\[
\nu(r \in S) = \text{Pr}\{\omega \in r^{-1}(S)\},
\]

where

\[
r^{-1}(S) \in \mathcal{F} \quad \text{for all } S \in \mathcal{B}.
\]

and

\[
r^{-1}(S) = \{\omega : r(\omega) \in S\}.
\]

Hence the statistics of the random "states of the world" are represented by the numerical random variable \(r\) with associated measure \(\nu(\cdot)\).

An easy way of looking at the measure \(\nu(\cdot)\) is to consider the case in which \(r\) takes on only two possible values. Suppose

\[
r = 1 \quad \text{with probability } p
\]

and

\[
r = 2 \quad \text{with probability } (1 - p).
\]

Then if \(S\) is any arbitrary set such that \(1 \notin S\), \(2 \notin S\), \(\nu(S) = 0\), if \(1 \in S\), \(2 \notin S\), \(\nu(S) = p\). If \(2 \in S\), \(1 \notin S\), \(\nu(S) = 1 - p\); finally if \(1, 2 \in S\) then \(\nu(S) = 1\).

It is assumed that the random event \(r\) either increases output or decreases output for all values of \(x\) and that the random events are indexed in such a way as to have

\[
\frac{\partial f(x, r)}{\partial r} > 0, \quad \text{for all } x.
\]
Moreover, we assume that the random event can only affect production in a “compact” sense. Namely, there exists numbers \( 0 < \alpha < \beta < \infty \) such that
\[
\rho \in [\alpha, \beta]
\]
and for all \( x > 0 \),
\[
\infty > f(x, \beta) > f(x, \alpha) > 0. \tag{1.3}
\]

In this discussion it is assumed, without loss of generality, that for every \( \rho > \alpha \), \( \nu([\alpha, \rho]) > 0 \). Since, if not, one could just as well choose \( \alpha = \sup\{\eta : \nu((0, \eta]) = 0\} \). Similarly, \( \nu([\rho, \beta]) > 0 \) for each \( \rho < \beta \).

The object of this study is to characterize the long-run behavior of the growth process which is represented by the stochastic difference equation
\[
c_t + x_t = f(x_{t-1}, r_{t-1}), \tag{1.4}
\]
where \( c_t \) is an optimal consumption policy. Here a consumption policy is optimal when consumption is chosen so as to maximize the expected sum of discounted utilities. Furthermore, the analysis is based on the assumption of an infinite time horizon.

The utility function, given by \( u(c_t) = u_t \), is assumed to have all the usual properties, namely,
\[
u' > 0, \quad u'' < 0.
\]

We assume that \( u'(0) = +\infty \) so as to insure interior solutions.

In this framework the maximization problem becomes
\[
\max E \sum_{t=0}^{\infty} \delta^t u(c_t),
\]
subject to the constraints
\[
c_t + x_t = f(x_{t-1}, r_{t-1}), \quad t = 1, 2, 3, ...
\]
and
\[
c_0 + x_0 = s, \quad x_t, c_t \geq 0.
\]

Here \( s > 0 \) is the given initial data of the problem representing the historically given capital stock. Also the subjective discount rate is given by \( 0 < \delta < 1 \). Note it is assumed that investment is reversible. This assumption allows the possibility that \( x_{t+1} < x_t \).

From the definition of \( \beta \) and the conditions imposed on the production function, there exists a \( x_\beta \) such that for all \( x > x_\beta \)
\[
f(x, \beta) < x.
\]
Hence capital cannot be sustained no matter what the state of the world if $x > x_0$.

It is immediately apparent from the above remark that for any initial capital stock $s$ there exists a solution to the maximization problem which by strict concavity is unique. From the usual dynamic programming arguments the optimal consumption policy in the $t$th period may be written in the form $\tilde{c}_t = g(f(\tilde{x}_{t-1}, r_{t-1}))$. The associated consumption policy is given by the equation

$$\tilde{x}_t = f(\tilde{x}_{t-1}, r_{t-1}) - g(f(\tilde{x}_{t-1}, r_{t-1})) = h(f(\tilde{x}_{t-1}, r_{t-1})) = H(\tilde{x}_{t-1}, r_{t-1}).$$  \hfill (1.5)

(Note that $h(\cdot)$ and $g(\cdot)$ are defined only on the positive real line.)

From the above equations it is clear that the values $\tilde{c}_t$, $\tilde{x}_t$ are random variables and the sequence of capital stocks $\tilde{x}_0, \tilde{x}_1, ..., \tilde{x}_n, ...$ is a Markov process.\(^1\) This stochastic process is generated by the difference equation (1.5).

The optimal policy in this model satisfies the functional equation

$$u'[g(f(\tilde{x}_{t-1}, r_{t-1}))] = E_{x_t} \delta u'[g(f(\tilde{x}_t, r_t))] f'(\tilde{x}_t, r_t).$$  \hfill (1.6)

(Note that all measures considered in this paper are defined on the positive real line; hence all integrals will be taken over the positive real line). Here $E_{x_t}$ is the conditional expectation operator, conditioned on the value $\tilde{x}_t$ of the stochastic process at time $t$. Equation (1.6) is easily verified (e.g., see Levhari and Srinivasan [6], Theorem 2) by a standard dynamic programming type argument, which may be summarized as follows. First the infinite horizon maximization problem is truncated to yield a finite horizon maximization problem. The optimal policy in the finite horizon problem is constrained so as to yield no stocks at the end of the horizon. It is then shown that for this finite horizon problem the optimal policy satisfies a functional equation of the type (1.6). Passing to the limit yields equation (1.6) as a necessary condition for the optimal policy in the infinite horizon problem.

The usual properties of optimal policies are easily found from Eq. (1.6). These will be stated as Lemmas 1.1 and 1.2 below. Moreover, Lemma 1.2 implies that all functions are measurable. First consider the $T$-period

\(^1\) A stochastic process $(x_t), t = 1, 2, ...$ is said to be a Markov process if

$$P_r\{x_t | x_1, ..., x_{t-1}\} = P(x_t | x_{t-1}),$$

that is, if the probability distribution of the random event $x_t$, given that the values $x_1, ..., x_{t-1}$ are known, depends only on the last value of the process, namely, $x_{t-1}$. 
maximization problem with target capital stock at time \( T \) equal to zero:

\[
\max E \sum_{t=0}^{T} \delta^t u(c_t)
\]

subject to

\[
x_t^T + c_t^T = f(x_{t-1}^T, r_{t-1})
\]

and

\[
x_0^T + c_0^T = s, \quad x_T^T = 0.
\]

Let

\[
x_t = g_T^{-1}(f(z_t^{-1}, r_{t-1})) \quad \text{and} \quad z_t^T = g_T^{-1}(f(x_t^{-1}, r_{t-1}))
\]

be optimal investment and consumption policies for the \( T \)-period horizon problem. It is an easily verified fact that \( h_T^T(\cdot) \) and \( g_T^T(\cdot) \) are increasing continuous functions. These properties will be used below to show that \( x_t \) and \( c_t \) are strictly increasing continuous functions.

**Lemma 1.1.** The optimal policies

\[
\bar{x}_t = h(f(\bar{x}_{t-1}, r_{t-1})) \quad \text{and} \quad \bar{c}_t = g(f(\bar{x}_{t-1}, r_{t-1}))
\]

are increasing functions with \( h(0) = g(0) = 0 \).

**Proof.** Since \( \bar{c}, \bar{x} \geq 0 \), it follows that \( h(0) = g(0) = 0 \). In the appendix it will be shown that for each initial value \( s \), \( x_t^T < x_{t+1}^T \); hence, \( \bar{x}_t^T \to \bar{x}_t^T \), monotonically as \( T \to \infty \). Similarly, \( \bar{c}_t^T \to \bar{c}_t^T \). From this fact \( \bar{x}_t = h(s_t) \) is at least nondecreasing in \( s_t \) since for all \( T, t, \bar{x}_t^T = h_T^T(s_t) \) are all increasing in \( s_t \).

To show that \( \bar{x}_t \) is increasing in \( s \), consider the functional Eq. (1.6) rewritten as (dropping the index \( t \) on \( s \) for convenience),

\[
u'[g(s)] = \delta \int_{a}^{b} u'[g(f(s - g(s, \rho)))]f'(s - g(s, \rho)) \, \nu(d \rho).
\]

Suppose the initial endowment changes from \( s \) to \( s + \Delta s, \Delta s > 0 \) and \( g(s) = g(s + \Delta s) > 0 \). Then \( u'(g(s)) = u'(g(s + \Delta s)) \). However, for each \( \rho \), \( u'[g(f(s + \Delta s - g(s + \Delta s), \rho)) f'(s + \Delta s - g(s + \Delta s), \rho)] \) cannot remain constant since \( f \) increases and \( u', f' \) decrease. Therefore, \( g(s + \Delta s) = g(s) \) cannot be optimal. Hence \( g(s) \) is strictly increasing. Similarly investment \( h(s) \) is increasing.

**Lemma 1.2.** The functions \( \bar{x}_t = h(\cdot) \) and \( \bar{c}_t = g(\cdot) \) are continuous in \( s \).
Proof. From Lemma 1.1, \( c_0(s) \) and \( x_0(s) \) are increasing in \( s \). Thus for all \( s_0 > 0 \),

\[
\lim_{s \uparrow s_0} c_0(s) = c_0(s_0^-) \leq c_0(s_0^+) = \lim_{s \downarrow s_0} c_0(s).
\]

and

\[
\lim_{s \uparrow s_0} x_0(s) = x_0(s_0^-) \leq x_0(s_0^+) = \lim_{s \downarrow s_0} x_0(s).
\]

However,

\[
c_0(s) + x_0(s) = s.
\]

Hence

\[
c_0(s_0^-) + x_0(s_0^-) = s_0
\]

and

\[
c_0(s_0^+) + x_0(s_0^+) = s_0.
\]

From which we get

\[
[c_0(s_0^-) - c_0(s_0^+)] + [x_0(s_0^-) - x_0(s_0^+)] = 0.
\]

Since both \( c_0(s) \) and \( x_0(s) \) are increasing in \( s \), both terms of the sum must be nonpositive. Hence

\[
c_0(s_0^-) = c_0(s_0^+)
\]

and

\[
x_0(s_0^-) = x_0(s_0^+),
\]

which is continuity.

Let \( \mu_t \) be the measure associated with the possible values of the random variables \( \bar{x}_t \). The measures \( \mu_t \) are defined as

\[
\mu_t(B) = \Pr(\bar{x}_t \in B), \quad t = 0, 1, 2, ...
\]

The value \( \mu_t(B) \) represents the probability that the optimal capital stock at time \( t \), \( \bar{x}_t \) belongs to the set \( B \). Let \( F_t(x) \) be the distribution function associated with the measure \( \mu_t \). The distribution function \( F_t(x) \) and the measure \( \mu_t \) are related as follows:

\[
F_t(x) = \Pr(\bar{x}_t < x) = \mu_t([0, x)).
\]

Let \( P(x, B) \) be the probability transition function of the process. \( P(x, B) \) is the probability that the capital stock is in the set \( B \) one step after it started in state \( x \), i.e., if the capital stock at time \( t \) is \( \bar{x}_t \), then the
probability that $\bar{x}_{t+1}$ belongs to the set $B$ is $P(\bar{x}_t, B)$. Note $P(x, \cdot)$ is a measurable function as a result of Fubini's theorem. Symbolically,

$$P(x, B) = \Pr\{H(x, r) \in B\} = \Pr\{r \in B_x\} = \nu(B_x). \quad (1.8)$$

The function $H(x, r)$ takes capital stocks into capital stocks [cf. Eq. (1.5)]. Here

$$B_x = \{\eta: H(x, \eta) = y, y \in B\},$$

and $B \in \mathcal{B}^+$, the Borel sets on the positive real line.

The distribution function $F_t$ and the measures $\mu_t$ over the capital stocks are generated by the probability transition function as follows,

$$\mu_t(B) = \int P(\xi, B) \mu_{t-1}(d\xi), \quad (1.9)$$

and

$$F_t(x) = \int P(\xi, [0, x)) dF_{t-1}(\xi).$$

Intuitively, this means that the probability of having a capital stock in the set $B$ at time $t$ is the probability of having a capital stock $\xi$ in period $t - 1$ times the probability of going from capital stock $\xi$ to the set of capital stocks $B$. This is then summed over all possible capital stocks at time $t$.

It is useful at this point to introduce the inverse function of $H(x, r)$ with respect to its first argument. This inverse function plays an important role in the proof of the convergence of the sequence of distribution functions which will be discussed subsequently.

The function $q(y, r)$ is defined in terms of the function $H$ by the equation

$$H(q(y, r), r) = y.$$

This function is well-defined since $H$ is strictly increasing in its first argument (cf. Lemma 1.1 above). The function $q$ can be thought of as generating a stochastic process as follows:

$$q(\bar{x}_t, r_{t-1}) = \bar{x}_{t-1}. \quad (1.10)$$

This stochastic process is in a temporal sense inverse to the stochastic process generated by the function $H$. Henceforth, this process will be called the inverse process.

The measures $\mu_t, \mu_t, t = 0, 1, \ldots$ and the transition probability function $P(\cdot, \cdot)$ can be characterized explicitly in terms of the measure $\nu$ and inverse function $q$. Recall $\nu$ is the measure which determines the statistical behavior of the random variable $r$. 
The probability transition function becomes

\[ P(x, B) = P\{H(x, r) \in B\} = \Pr\{x \in q(B, r)\}, \]

where

\[ q(B, r) = \{x = q(y, r); y \in B\}. \]

Here \( q(B, r) \) is the set of possible capital stocks from which one can get to some element in the set \( B \) if \( r \) represents the prevailing "state of the world".

The measures \( \mu_t(B) = \Pr\{\bar{x}_t \in B\}, t = 0, 1, 2,... \) are given by

\[ \mu_t(B) = \int P(\xi, B) \mu_{t-1}(d\xi) - \int \nu(B, \xi) \mu_{t-1}(d\xi) - \int \mu_{t-1}(q(B, \rho)) v(d\rho). \quad (1.11) \]

If a steady-state distribution is defined to be any measure which satisfies the equation

\[ \mu(B) = \int P(x, B) \mu(dx), \]

then for our stochastic process a steady-state distribution \( \mu_\infty \) must satisfy

\[ \mu_\infty(B) = \int \nu(B, \xi) \mu_\infty(d\xi) - \int \mu_\infty(q(B, r)) v(dr). \]

Intuitively, for a measure \( \mu_\infty \) to be a steady-state measure, the probability \( \mu_\infty(B) \) of being in the set \( B \) must remain unchanged from period to period. This concept of steady state is analogous to the deterministic case since, in that case, the steady-state capital stock remain constant from period to period. In the random case, the distribution of capital stocks remains constant from period to period.

2. THE DETERMINISTIC MODEL

The long-run behavior of the optimal Stock in a deterministic one-sector growth model has been explored most notably by Cass [2] and Koopmans [4]. The main result of the Cass-Koopmans work is the existence of a steady-state solution which in the discounted case is known as the modified golden rule. Moreover, optimal policies in the deterministic case possess nice stability properties. More precisely, the optimal capital stock converges to the modified golden rule path from all initial points.

In this section the technique of dynamic programming will be used to
yield the Cass–Koopmans results in a very simple and straightforward way. This exercise is intended to illustrate the technique which is employed in the next two sections. Appeal to this technique yields results in the random theory which are analogous to the Cass–Koopmans results.

Consider the problem

\[
\text{Maximize } \sum_{t=0}^{\infty} \delta^t u(c_t) \tag{2.1}
\]

subject to

\[
c_0 + x_0 = s
\]

and

\[
c_t + x_t = f(x_{t-1}), \quad t = 1, 2, \ldots.
\]

As in the random case discussed above, there exists by strict concavity, for each \(s\), a unique solution to the maximization problem (2.1). This may be written as \(x_0 = h(s)\). Furthermore, by straightforward dynamic programming considerations (e.g., see Levhari and Srinivasan [6]) there exists a policy function \(h(\cdot)\) such that

\[
\bar{x}_t = h[f(\bar{x}_{t-1})].
\]

Moreover \(h(\cdot)\) is an increasing continuous function with \(h(0) = 0\).

The necessary condition for a maximum yields the functional equation

\[
u'(c_t) = \delta u'(c_{t+1}) f'(x_t). \tag{2.2}
\]

Here,

\[
c_t = f(x_{t-1}) - x_t
\]

and

\[
c_{t+1} = f(x_t) - x_{t+1}.
\]

Equation (2.2) determines the steady-state behavior and the dynamics of the growth path. These conditions are analogous to the Euler equations of the calculus of variations. Substituting the optimal values \(\bar{c}_t = g(f(\bar{x}_{t-1}))\) into Eq. (2.2), the functional equation reduces to

\[
u'(g(f(\bar{x}_{t-1}))) = \delta u'(g(f(\bar{x}_t))) f'(\bar{x}_t). \tag{2.3}
\]

Define the functions

\[
a(x) = u'(g(f(x))) \tag{2.4}
\]

and

\[
b(x) = \delta a(x) f'(x). \tag{2.5}
\]
Since $g(\cdot)$ is an increasing function, both $a(x)$ and $b(x)$ are decreasing functions.

Furthermore,

$$d(x) = \frac{b(x)}{a(x)} = \delta f'(x).$$

(2.6)

Hence, $d(x)$ decreases in $x$ and $d(\infty) = 0$, $d(0) = \infty$. Therefore, there exists a unique $\bar{x}$ such that $a(\bar{x}) = b(\bar{x})$. Here $\bar{x}$ is precisely the modified golden rule. Moreover, from Eq. (2.6) it is easily seen that for $x < \bar{x}$,
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If \( b(x) > a(x) \) and for \( x > \bar{x} \), \( b(x) < a(x) \). From this fact, it follows that the process \( \bar{x}_t \), with \( a(\bar{x}_t) = b(\bar{x}_{t+1}) \) is globally stable. For suppose that \( \bar{x}_t > \bar{x} \), then

\[
a(\bar{x}_t) = b(\bar{x}_{t+1}) < a(\bar{x}_{t+1}).
\]

Hence since \( a(x) \) decreases

\[
\bar{x}_{t+1} < \bar{x}_t, \quad \text{i.e., } \bar{x}_t \downarrow \bar{x}, \text{ as } t \to \infty.
\]

Similarly, for \( \bar{x}_t < \bar{x} \). These results are depicted in Figs. 1 and 2.


In this section we make extensive use of the functional Eq. (1.7) in analyzing the properties of optimal policies. These properties will be exploited in the following section to show that the sequence of distribution functions of the optimal capital stocks converges to a unique limiting distribution.

Letting

\[
d(x, r) = u'[g(f(x, r))],
\]

Eq. (1.7) becomes

\[
d(x, r) = \delta \int d(H, \eta)f'(H, \eta)v(d\eta).
\]

Here \( H = H(x, r) \) is the optimal investment policy defined by Eq. (1.5).
Note that since \( f \) and \( g \) are increasing functions in both of their arguments and \( u' \) is a decreasing function, \( d \) is a decreasing function of both arguments.

Define a fixed point \( x_p \), of \( H(x, p) \) corresponding to the realization \( p \) of the random variable \( r \); i.e., \( H(x_p, p) = x_p \). The following lemma shows that for any \( p \in (\alpha, \beta] \) there exists an \( \epsilon_p > 0 \) such that no positive fixed point of the investment function \( H(\cdot, \rho) \) exists in the interval \((0, \epsilon_p)\). Moreover, since \( H(\cdot, \rho) \) is continuous for each \( p \in (\alpha, \beta] \), there exists a minimum positive fixed point of the function \( H(\cdot, \rho) \). The lemma also shows that the minimum positive fixed point of \( H(\cdot, \rho) \) for each \( p \in (\alpha, \beta] \)
is stable. Figures 3 and 4 illustrate two situations which are ruled out by Lemma 1.1. Figure 5 depicts the kind of situation that is still possible. Note that the value \( \alpha \) causes some difficulty since it cannot be said that a minimum positive fixed point of the function \( H(\cdot, \alpha) \) exists. Fortunately, this does not raise any serious difficulties for the proof of the main theorem.

**Lemma 3.1.** Let \( \bar{x}_{t+1} = H(\bar{x}_t, r_t) \), where \( H(x, r) \) is the optimal capital policy. Then for each \( \rho \in (\alpha, \beta] \) there exists an \( \epsilon_\rho > 0 \) such that for all \( x \in (0, \epsilon_\rho) \), \( H(x, \rho) > x \).

**Proof.** Let \( \rho \in (\alpha, \beta] \), and let \( x_0 \) be a fixed point of \( H(\cdot, \rho) \). From Eq. (3.2),

\[
d(x_0, \rho) = \delta \int d(x_0, \eta) f'(x_0, \eta) \nu(d\eta).
\]

Since \( d, f' \) are positive functions and \( d(x, \rho) \) is decreasing in \( \rho \), we may write

\[
d(x_0, \rho) \geq \delta \int_\alpha^\beta d(x_0, \eta) f'(x_0, \eta) \nu(d\eta) \\
\geq \delta \min_{\eta} \{f'(x_0, \eta)\} \nu((\alpha, \beta]) d(x_0, \rho).
\]

Note that \( \alpha, \beta \) have been chosen so that \( \nu((\alpha, \rho]) > 0 \), for \( \rho > \alpha \) and \( \nu((\rho, \beta]) > 0 \) for all \( \rho < \beta \). Hence

\[
1 \geq \delta \min_{\eta \in (\alpha, \beta]} \{f'(x_0, \eta)\} \nu((\alpha, \rho]) > 0. \tag{3.3}
\]

Since \( f' \) is a continuous function, it achieves a minimum over the compact set \([\alpha, \rho]\). Let \( \zeta \) be such that

\[
f'(x_0, \zeta) = \min_{\eta} f'(x_0, \eta).
\]

Then from (3.3)

\[
1 \geq \delta \nu((\alpha, \rho]) f'(x_0, \zeta) > 0.
\]

Suppose that there exists an arbitrarily small positive fixed point of \( H(\cdot, \rho) \). More precisely, suppose that \( \{x_n\} \) is a sequence of fixed points of \( H(\cdot, \rho) \) such that \( x_n \to 0 \), as \( n \to \infty \). From the Inada conditions,

\[
f'(x_n, \zeta) \to \infty, \text{ as } n \to \infty.
\]

*This proof has benefitted from the comments of the referee.*
This contradicts inequality (3.3) which is valid for all fixed points of $H(\cdot, \rho)$. This discussion is valid for all $\rho \in (\alpha, \beta]$. Hence, for each $\rho \in (\alpha, \beta]$ there exists an $\epsilon_\rho > 0$ such that for all $x \in (0, \epsilon_\rho]$ either $H(x, \rho) > x$ or $H(x, \rho) < x$.

Suppose that $H(x, \rho) < x$ for all $x \in (0, \epsilon_\rho]$. Furthermore, let $\bar{x}_0 \in (0, \epsilon_\rho)$. Then from the hypothesis $H(x, \rho) < x$, for $r_t \in [\alpha, \rho]$,

$$H(\bar{x}_t, r_t) = \bar{x}_{t+1} < \bar{x}_t.$$ 

Note that the last expression is valid for the realization $r_t = \alpha$ since $H(x, \alpha) < H(x, \rho) < x$. Hence, for $r_t \in [\alpha, \rho]$, $\bar{x}_t \to 0$, as $t \to \infty$, as a consequence, $f'(\bar{x}_t, \zeta) \to \infty$, as $t \to \infty$, for all $\zeta \in [\alpha, \rho]$, indeed for all $\zeta \in [\alpha, \beta]$.

To show a contradiction, consider the realization $r_t = \rho$. We then have

$$d(\bar{x}_{t+1}, \rho) > \delta \int d(\bar{x}_{t+1}, \eta) f'(\bar{x}_{t+1}, \eta) \nu(d\eta),$$

since $\bar{x}_{t+1} < \bar{x}_t$ and $d(\cdot, \cdot)$ is decreasing in both arguments. Moreover,

$$d(\bar{x}_{t+1}, \rho) \geq \delta \int_\alpha^\rho d(\bar{x}_{t+1}, \rho) f'(\bar{x}_{t+1}, \eta) \nu(d\eta).$$

From which we find

$$1 > \delta \min_{\eta \in [\alpha, \rho]} f'(\bar{x}_{t+1}, \eta) = \nu[\alpha, \rho]$$

(3.4)

Define $\zeta_{t+1}$ to be the value of $\zeta \in [\alpha, \rho]$ which minimizes $f'(\bar{x}_{t+1}, \zeta)$ i.e.,

$$\min_{\zeta \in [\alpha, \rho]} f'(\bar{x}_{t+1}, \zeta) = f'(\bar{x}_{t+1}, \zeta_{t+1}).$$

Letting $r_t = \rho$ for all $t$, the above argument implies that $\bar{x}_t \to 0$ and $f'(\bar{x}_{t+1}, \zeta) \to \infty$, for all $\zeta \in [\alpha, \beta]$. Since $f'(\cdot, \cdot)$ is continuous on the compact set $[0, \epsilon_\rho] \times [\alpha, \beta]$, it is uniformly continuous on this set. Hence it is impossible for $f'(\bar{x}_{t+1}, \zeta_{t+1})$ to remain bounded while $f'(\bar{x}_{t+1}, \zeta) \to \infty$ for all $\zeta \in [\alpha, \beta]$. This contradicts condition (3.4) and completes the proof.

From Lemma 3.1 there exists (by the continuity of $H$) a minimum positive fixed point of $H(\cdot, \rho)$ for all $\rho \in (\alpha, \beta]$. Thus the value $x_{m_\rho}$ may be defined as follows:

$$x_{m_\rho} = \min \{x > 0 \mid H(x, \rho) \leq x\} > 0, \quad \rho \in (\alpha, \beta].$$

(3.5)

Note it is possible that $x_{m_\rho} \to 0$ as $\rho \to \alpha$.

Lemma 3.1 rules out the possibility of an infinite number of positive
fixed points near the origin of the function $H(\cdot, \rho)$ for all $\rho \in (\alpha, \beta)$. However, as stated above, an infinite number of positive fixed points cannot be ruled out for the function $H(\cdot, \alpha)$. On the other hand, it is easily seen that if $\{x_n\}$ is a sequence of positive fixed points of $H(\cdot, \alpha)$ such that $x_n \to 0$, we have

$$v(\rho: H(x, \rho) < x, x_n < x < x_{n+1} \to 0, \text{ as } n \to \infty,$$

i.e., the probability that the process remains unstable tends to zero as $x$ gets closer to the origin. If this were not the case and $v(\alpha) = 0$ there would exist an $\eta > \alpha$ such that $H(\cdot, \eta)$ possesses an infinite number of positive fixed points which is impossible by Lemma 3.1. However, if $v(\alpha) > 0$ then, as will be shown in Lemma 3.2, $H(\cdot, \alpha)$ must possess a minimum positive fixed point. Hence in the case $v(\alpha) > 0$ there exists an $x_{\alpha}$ such that for all $x < x_{\alpha}$, $H(x, \rho) > x$ for all $\rho \in [\alpha, \beta]$.

**Lemma 3.2.** If $v(\alpha) > 0$ then there exists an $\epsilon > 0$ such that for $x \in (0, \epsilon)$, $H(x, \alpha) > x$.

**Proof.** Consider the inequality, with $x_\alpha$ a fixed point of $H(\cdot, \alpha)$,

$$d(x_\alpha, \alpha) \geq d(x_\alpha, \alpha)f'(x_\alpha, \alpha) \nu(\alpha) > 0.$$ 

Hence

$$1 \geq f'(x_\alpha, \alpha) \nu(\alpha).$$

An argument similar to the one of Lemma 3.1 implies the existence of an $\epsilon > 0$ such that no fixed points of $H(\cdot, \alpha)$ are in the interval $(0, \epsilon)$. Moreover, again from Lemma 3.1, $H(x, \alpha) > x$.

Lemma 3.3 establishes the stability of the maximum fixed point of $H(x, \rho)$. Note that Lemma 3.3 implies the existence of a fixed point for $H(x, \cdot)$, indeed, a maximum fixed point always exists since $f'(\infty, \rho) = 0$ for all $\rho \in [\alpha, \beta]$, and $H(x, \rho)$ is continuous in $x$ and $\rho$.

Let

$$x_{M\rho} = \max\{x | H(x, \rho) = x\}. \quad (3.6)$$

**Lemma 3.3.** For each $\rho \in [\alpha, \beta]$, $x_{M\rho}$ exists and for all $x > x_{M\rho}$, $H(x, \rho) < x$.

**Proof.** Since $H(x, \rho) < f(x, \beta)$ and $f(x, \beta)$ has a unique positive fixed point, the continuity of the function $H(\cdot, \beta)$ implies that the maximum fixed point $x_{M\beta}$ exists.

Hence, either $H(x, \rho) > x$ or $H(x, \rho) < x$ for all $x \in (x_{M\rho}, \infty)$. Let us take $\rho = \beta$, and suppose that the last fixed point $x_{M\beta}$ is unstable for all
Let the initial stocks be $s > x_{MB}$, then $x_{t}(\beta^t) \to \infty$. Here $x_{t}(\beta^t) = x_{t}(\beta, \beta, ..., \beta)$; i.e., $x_{t}(\beta^t)$ is investment at period $t$ if the realization of $r$ equals $\beta$, $t$ times. However, from the feasibility conditions

$$x_{t}(\beta^t) \leq f^t(s, \beta),$$

where $f^t(s, \beta)$ is defined inductively by $f^t(s, \beta) = f^{t-1}(f(s, \beta), \beta)$. Moreover, $f^t(s, \beta) \to \hat{x}_{MB}$, where $\hat{x}_{MB}$ is the unique positive fixed point of $f(\cdot, \beta)$. This yields a contradiction. Hence the maximum fixed point of $H(\cdot, \beta)$ is stable. This proof is valid for all $\rho \in [\alpha, \beta]$ since $H(\cdot, \rho) < H(\cdot, \beta)$.

The next lemma establishes a crucial property of paths of optimal capital accumulation. In effect, it establishes the existence of a “stable” set which acts like a set of recurrent states in the theory of Markov Processes. This “stable” set, as will be seen, has the property that each subset of its complement (complement with respect to the positive real line) has zero probability under the stationary measure. Moreover, from any initial value, all probability will eventually flow to this “stable” set. In order to make this idea more precise, the following definition is needed.

**DEFINITION.** The function $H(x, \beta)$ is said to have a stable fixed point configuration if $x_{Ma} < \hat{x}_{MB}$.

Thus the configuration of fixed points of the function $H(x, \beta)$ is said to be stable if the last fixed point of $H(x, \alpha)$ occurs before the first fixed point of $H(x, \beta)$. A configuration of fixed points which is not stable is depicted in Fig. 4.

**LEMMA 3.4.** Let $x_{t+1} = H(x_t, r_t)$ be an optimal investment policy. Then the function $H(x, \beta)$ has a stable fixed point configuration.

**Proof.** Let $x_{\beta}$ be a fixed point of $H(\cdot, \beta)$. Using the functional Eq. (3.2) we get

$$d(x_{\beta}, \beta) = \delta \int d(x_{\beta}, \eta) f'(x_{\beta}, \eta) \nu(d\eta). \tag{3.7}$$

Since $d(\cdot, \cdot)$ is a decreasing function of its second argument, we have

$$d(x_{\beta}, \beta) < d(x_{\beta}, \rho), \quad \rho \in [\alpha, \beta).$$

Thus from Eq. (3.7)

$$d(x_{\beta}, \beta) > \delta \int d(x_{\beta}, \beta) f'(x_{\beta}, \eta) \nu(d\eta).$$
Hence,

\[ 1 > \delta \int f'(x_\beta, \eta) \, v(d\eta). \]  
(3.8)

Similarly, letting \( x_\alpha \) be a fixed point of \( H(\cdot, \alpha) \), we get

\[ d(x_\alpha, \alpha) > d(x_\alpha, \rho), \quad \rho \in (\alpha, \beta) \]

and

\[ 1 < \delta \int f'(x_\alpha, \eta) \, v(d\eta). \]  
(3.9)

Suppose that \( H \) has an unstable fixed-point configuration. In fact, let \( x_\alpha > x_\rho \). Now \( x_\alpha \) must satisfy (3.9) and \( x_\rho \) must satisfy (3.8). Moreover, since \( f' \) decreases in its first argument, we have

\[ 1 < \delta \int f'(x_\alpha, \eta) \, v(d\eta) \leq \delta \int f'(x_\beta, \eta) \, v(d\eta) < 1, \]

which is a contradiction. Therefore the function \( H(x, r) \) has a stable fixed-point configuration, as was to be shown.

Lemma 3.4 implies that optimal policies look at worst like the function depicted in Fig. 5. Moreover, the set \([x_{M\alpha}, x_{m\beta}]\) the stable set of the \( H \) process, has the property that under the stationary measure no probability will lie outside of \([x_{M\alpha}, x_{m\beta}]\). Eventually, all probability evolves to this set no matter what the initial stock. This phenomenon will be discussed in detail in the next section.

4. The Convergence of Distribution Functions

In this section we use the properties of the optimal policy to derive the long-run asymptotic behavior of the optimal capital stock. Consider again the measure \( \mu_t \) associated with the optimal capital stock \( \bar{x}_t \) starting from given initial stocks \( s \). The measures \( \mu_t \) are generated by Eq. (1.11) which gives the statistical motion of the stochastic process \( \bar{x}_1, \ldots, \bar{x}_t, \ldots \). For convenience, we restate Eq. (1.11);

\[ \mu_t(B) = \int \mu_{t-1}(q(B, \rho)) \, v(d\rho). \]  
(4.1)

Here \( q(\cdot, \rho) \) is the function generating the inverse process which is defined by the equation \( H(q(x, \rho), \rho) = x \). Using the measures \( \mu_t \), the distribution function \( F_t(x) \) of \( \bar{x}_t \) is defined by

\[ F_t(x) = P{\bar{x}_t < x} = \mu_t([0, x]). \]
Hence Eq. (4.1) may be written in terms of distribution functions as

$$F_t(x) = \int F_{t-1}(q(x, \rho)) \nu(dp), \quad (4.2)$$

since $q(0, \rho) = 0$.

The proof of the convergence of the distribution functions $F_t(x)$ to a limiting distribution $F(x)$ will be based on the following intuitive ideas. It will be shown that the difference between $F_t(x)$ and $F_{t+h}(x)$ for arbitrary $h > 0$ can be made arbitrarily small, independent of $x$, if $t$ is chosen large enough. This fact implies the uniform convergence of the sequence $F_t(x)$ and hence the existence of a limit distribution function $F(x)$. In order to show uniform convergence the process is iterated backwards. Suppose that the initial value of the sequence is $F_0(x)$ which is a degenerate distribution concentrated at some point $x_0$. We then consider the set of points from which one can get to $x$ in $t$ steps. This set of points has the property that, except for arbitrarily small probability, the $t$-th and $(t + h)$-th inverse iterates are identical with respect to the distribution function $F_0(x)$. Such $t$ and $h$ can be found independent of $x$. Hence, since the proof depends strongly on this temporally inverse reasoning, we first investigate the inverse process

$$\bar{x}_{t+1} = q(\bar{x}_t, r_t).$$

Clearly, the properties of $q$ are related to the properties of $H$ which were derived in the previous section. For example, the continuity of the function $q$ follows directly from the continuity of $H$. Moreover, let $x_{\rho}$ be a fixed point of $H(x, \rho)$; i.e., $H(x_{\rho}, \rho) = x_{\rho}$. Then $q(x_{\rho}, \rho) = x_{\rho}$, since, by definition,

$$H(q(x_{\rho}, \rho), \rho) = x_{\rho}.$$

Hence $H$ and $q$ have the same set of fixed points.

From (3.6), $x_{M_{\rho}}$ is the maximum fixed point of $H(\cdot, \rho)$ and, from (3.5), $x_{m_{\rho}}$ is the minimum fixed point of $H(\cdot, \rho)$. From Lemmas 3.1 and 3.3, it follows that, for $\rho \in (\alpha, \beta]$, $H(x, \rho) < x$, $x \in (x_{M_{\rho}}, \infty)$ and $H(x, \rho) > x$, $x \in (0, x_{m_{\rho}})$. The next lemma establishes a similar property for the inverse function $q$.

**Lemma 4.1.** For all $x \in (x_{M_{\rho}}, \infty)$, $q(x, \rho) > x$. For all $x \in (0, x_{m_{\rho}})$, $q(x, \rho) < x$, for $\rho \in (\alpha, \beta]$.

**Proof.** Suppose that $x \in (x_{M_{\rho}}, \infty)$, then there exists a $y \in (x_{M_{\rho}}, \infty)$ such that $H(y, \rho) = x$. This follows from the facts that $H(\cdot, \rho)$ is an increasing continuous function for each $\rho$, $H(x_{M_{\rho}}, \rho) = x_{M_{\rho}}$, and
$H(\infty, \rho) = \infty$. Moreover, for all $y \in (x_{M_0}, \infty)$ we have $H(y, \rho) < y$. Hence $x < y$. From the definition of the inverse function, we have $q(x, \rho) = y$ and

$$x < q(x, \rho), \hspace{1cm} (4.3)$$

i.e., for all $x \in (x_{M_0}, \infty), (4.3)$ is satisfied. A similar argument shows that for $x \in (0, x_{M_0}), x > q(x, \rho)$. Hence stability properties of the function $q(\cdot, \rho), \rho \in (\alpha, \beta]$ is established. Note that the function $q(\cdot, \alpha)$ is a special case in the same way that $H(\cdot, \alpha)$ is a special case. Namely, it is possible that there exists a sequence $\{x_n\}$ of positive fixed points of $q(\cdot, \alpha)$ such that $x_n \to 0$. In this case, as was pointed out above, if $\nu(\alpha) = 0$, then

$$\nu(\rho: H(x, \rho) < x, x_n \leq x \leq x_{n-1}) \to 0, \text{ as } n \to \infty.$$ 

This implies the equivalent property for the function $q(\cdot, \alpha)$; i.e.,

$$\nu(\rho: q(x, \rho) > x, x_n \leq x \leq x_{n-1}) \to 0, \text{ as } n \to \infty.$$ 

More precisely, for any $\epsilon > 0$, there exists an $x(\epsilon)$ such that the probability that $q(x, \rho) \geq x$, for $x < x(\epsilon)$, is less than $\epsilon$. This turns out to be more than enough to establish the convergence result which we are after.

In the case $\nu(\alpha) > 0$, an even stronger result is possible. As was shown in Lemma 3.2, there exists a minimum positive fixed point $x_{M_0}$ of $H(\cdot, \alpha)$. Moreover, $H(\cdot, \alpha)$ has the property that for all $x < x_{M_0}, \ H(x, \alpha) > x$. A similar property is valid for $q(\cdot, \alpha)$; namely, for $x < x_{M_0}, \ q(x, \alpha) < x$.

We are now in a position to proceed to the proof of the main result. The essence of which is that starting from any $x > 0$ the $q$ process eventually leaves any positive compact set not containing the origin with probability greater than $\epsilon > 0$. In fact, the process leaves in finite time. In order to make this statement precise it is necessary to introduce the concept of a transient set for a stochastic process.

**DEFINITION.** Let $A$ be a nonnull set (i.e., $A$ has positive Lebesque measure). The set $A$ is said to be a transient set for a stochastic process having probability transition function $P(\cdot, \cdot)$ if for each $x \in A$, the expected number of visits to the set $A$ is finite; i.e.,

$$\sum_{n=0}^{\infty} P^n(x, A) < \infty.$$ 

**Note.** For transient sets $P^n(x, A) \to 0$, as $n \to \infty$. However, this condition is not sufficient.

Let us denote the interval $[x_{M_0}, x_{M_0}]$ by the letter $A$. Consider the
sequence \( \{x_n\} \) of positive fixed points of \( q(\cdot, \alpha) \). Define the sets \( A_n = [x_n, x_M] \). The next lemma establishes the fact the sets \( A_n \) are transient. Note the proof is valid whether or not \( x_n \to 0 \). In fact, it is valid whether or not \( q(\cdot, \alpha) \) possesses a finite or an infinite number of positive fixed points \( x_n \).

**Lemma 4.2.** The sets \( A_n \) are transient for the process \( \hat{x}_t = q(\hat{x}_{t-1}, r_t) \), \( \hat{x}_0 = x \).

The proof of this lemma is based on a simple idea. Namely, the interval \( A_n \) is decomposed into two not necessarily disjoint intervals. For every point in each of these intervals a set of possible realizations of the random variable \( r \) having positive probability with respect to the \( \nu \) measure is identified. For realizations in this set the inverse process eventually leaves the interval \( A_n \) with a positive probability after a fixed number of steps. In fact, we show that the expected number of visits to the set \( A_n \) is finite. Hence \( A_n \) is transient. Since this statement is with respect to the inverse process, an equivalent way of stating this conclusion is that, if the process is started outside of the interval \( A_n \) then under the forward process \( H(x, r) \), with positive probability the system enters the interval \( A_n \). Figure 6 may be used as a guide to the proof.
Consider the interval $A_n = [x_n, x_{M0}]$. As a consequence of the continuity of $q$ there exists an $\eta < \beta$ and an $x_n < x_{M0}$ with

$$x_n = \min\{x \mid q(x, \eta) = x\}, \quad (4.4)$$

such that for all $\rho \in (\eta, \beta]$ (cf. Fig. 6),

$$q(x, \rho) < x, \quad \text{for all } x \in (0, x_n]. \quad (4.5)$$

It will be shown that for all $x \in [x_n, x_{M0}]$ the expected number of visits to the set $A_n$ from $x$ is finite.

Let $\zeta \in (\eta, \beta)$, then $\nu([\zeta, \beta]) > 0$. Then by continuity the minimum positive fixed point $x_{\zeta}$ of $q(\cdot, \zeta)$ is such that $x_n < x_\zeta$. Hence, for all $x \in [x_n, x_{M0}]$,

$$q(x, \rho) < x, \quad \text{for } \rho \in [\zeta, \beta],$$

with probability at least $\gamma = \nu([\zeta, \beta]) > 0$.

Suppose that the event $r_t = \zeta$ occurs in each time period. Since $q(x, \zeta) < x$ for each $x \in [x_n, x_{M0}]$ there exits an $M(x)$ such that for all $m \geq M(x)$, $q^m(x, \zeta) < x_n$. Here $q^m(x, \zeta)$ is defined recursively as $q^m(x, \zeta) = q(q^{m-1}(x, \zeta), \zeta)$. Thus with probability at least $\gamma^M$, the process leaves the interval $[x_n, x_{M0}]$ and never returns. Hence the probability of remaining in the set $[x_n, x_{M0}]$ after $M$ steps is less than $1 - \gamma^M$. In fact, the process starting in the set $[x_n, x_{M0}]$ stays in the set $A_n$ with probability less than $1 - \gamma^M$.

With this information the expected number of visits to the set $A_n$ from any point within the set $[x_n, x_{M0}]$ may be calculated. The probability of remaining in the set after $2M$ steps is less than $(1 - \gamma^M)^2$. In other words, the probability of remaining in the set after $2M$ steps is less than the probability of remaining in the set after $M$ steps times the probability of remaining in the set another $M$ steps. In general, the expected number of visits to the set $A_n$ from the set $[x_n, x_{M0}]$ is less than

$$\sum_{j=1}^{\infty} (1 - \gamma^M)^j,$$

which is finite. Hence we have shown that the set $[x_n, x_{M0}]$ is transient.

Using a similar construction there exists an $\tilde{\eta}$ such that $\tilde{\eta} > \alpha$ and $x_\eta > x_{M\alpha}$ with

$$x_\eta = \max\{x \mid q(x, \tilde{\eta}) = x\} \quad (4.6)$$

such that for all $\rho \in [\alpha, \tilde{\eta}]$,

$$q(x, \rho) > x, \quad \text{for all } x \in [x_\eta, \infty). \quad (4.7)$$
It follows, using an argument similar to the one above, that there exists a \( \tilde{\eta} \) and \( \tilde{M} \) such that for each \( x \in [x_{\eta}, x_{MB}] \), the probability of leaving this set after \( \tilde{M} \) steps is greater than \( \tilde{\eta}^{\tilde{M}} \). Hence, the expected number of visits to the set \([x_{\eta}, x_{MB}]\), after starting in this set, is finite. In which case the set \([x_{\eta}, x_{MB}]\) is transient.

In order to complete the proof it is necessary to show that there exists an \( \tilde{\eta} = \eta \) which together with \( x_{\eta} \) satisfies conditions (4.6) and (4.7). When such an \( \tilde{\eta} \) is found, it is easily seen from the above discussion that the set \( A_n \) is transient, since, as has been shown, the expected number of visits to the set \( A_n \) after starting from any point in \( A_n \) is finite. Moreover, as is clear from the properties of the function \( q \), it is impossible to enter the set \( A_n \) from any point not in \( A_n \).

From the continuity of \( q \), it follows that for all \( \tilde{\eta} < \eta \) conditions (4.6) and (4.7) are satisfied. Hence if \( \eta > \eta \) an \( \eta \) exists with the property that \( \tilde{\eta} = \eta \) and \( x_{\eta} = x_n \). However, suppose that \( \tilde{\eta} < \eta \). Then \( x_n \) has the properties of \( x_\eta \); namely,

\[
q(x, \rho) > x, \quad x \in [x_\eta, \infty), \quad \rho \in [\alpha, \tilde{\eta})
\]

since

\[
q(x, \rho) > x, \quad x \in [x_\eta, \infty), \quad \rho \in [\alpha, \eta).
\]

Moreover \( \tilde{\eta} < \eta \) implies that \( x_\eta < x_n \) (and thus \([x_\eta, \infty) \supset [x_n, \infty)\) since

\[
x_\eta = \min\{x \mid q(x, \tilde{\eta}) = x\} < \min\{x \mid q(x, \eta) = x\}
\]

\[
< \max\{x \mid q(x, \eta) = x\} = x_n.
\]

Hence, instead of showing that the set \([x_\eta, x_{MB}]\) is transient, the above discussion can be used to show that the set \([x_n, x_{MB}]\) is transient. In other words, the value \( \eta \) and the point \( x_n \) satisfy conditions analogous to (4.6) and (4.7) which imply that the set \([x_n, x_{MB}]\) is transient. This completes the proof.

Define the \( n \)-step inverse process \( q^n(x, r^n) \) from the inverse process starting from \( x \) as

\[
q^n(x, r^n) = q(\hat{x}_{n-1}, r_n) = \hat{x}_n.
\]

Here \( r^n \in [\alpha, \beta]^n, [\alpha, \beta]^n = [\alpha, \beta] \times [\alpha, \beta] \times \cdots \times [\alpha, \beta] \) (since exponents are not used in this paper, no confusion will arise).

Note that for all \( x \in (x_{MB}, \infty) \), \( q^n(x, r^n) \to \infty \) as \( n \to \infty \) with probability one since for all \( r \in [\alpha, \beta] \), \( q(x, r) > x \), whenever \( x \in (x_{MB}, \infty) \). Also, if \( H(\cdot, \infty) \) has a finite number of positive fixed points there exists a minimum
positive fixed point $x_{ma}$. In this case, $q^n(x, r^n) \to 0$, as $n \to \infty$, for all $x \in (0, x_{ma})$.

The next Lemma establishes a uniform bound on the time of exit of all but at most $\epsilon$ probability from the set $A_n$.

**Lemma 4.3.** Let $\epsilon > 0$ be given. Consider the set $A_n = [x_n, x_{Mn}]$. There exists an $M(A_n) = M_n$ such that for all $m \geq M_n$ and all $x \in A_n$,

$$P\{q^m(x, r^n) \in A_n\} < \epsilon.$$ 

**Proof.** Since the set $A_n$ is transient there exists for each $x$, and for each $\epsilon > 0$, and $M(x)$ such that for all $m \geq M(x)$,

$$P^n(x, An) = P(q^m(x, r^n) \in A_n) < \epsilon.$$ 

To show that there exists an $M$ such that $M \geq M(x)$, for all $x$, suppose the contrary; i.e., suppose that there exists a sequence $x_t \in A_n$ such that $M(x_t) \to \infty$. Since $A_n$ is a compact set, there exists a subsequence $x_j$ (denoting $x_{t_j}$ by $x_j$) which converges to $\bar{x} \in A_n$ and $M(\bar{x}) < \infty$.

Once $q^m(\bar{x}, \rho^n)$ has left the set $A_n$, either $q^m(\bar{x}, \rho^n) \in [x_{Mn}, \infty)$, in which case $q^t(\bar{x}, \rho^t) \to \infty$, as $t \to \infty$, or $q^m(\bar{x}, \rho^n) \in [0, x_n]$. In the latter case, the process will eventually leave the set $A_{n+1} = [x_{n+1}, x_{Mn}]$, here $x_{n+1} < x_n$. In either case, for any $\epsilon > 0$, there exists an $\eta > 0$ such that, for all $t > M(\bar{x})$,

$$\Pr(d(q^t(\bar{x}, \rho^t), A_n) < \eta) < \epsilon,$$

where $d(\cdot, \cdot)$ is the usual distance between a point and a set. Since for each $t$, $q^t(\cdot, \cdot)$ is uniformly continuous on the compact set $A_{m\varepsilon}[\alpha, \beta]^t$, there exists a $\xi > 0$ and a $J > 1$ such that, for all $j \geq J$, $|\bar{x} - x_j| < \xi$ and

$$|q^t(\bar{x}, \rho^t) - q^t(x_j, \rho^t)| \leq \eta.$$ 

Hence, for all $j \geq J$ and all $t \geq M$,

$$\Pr\{q^t(x_j, \rho^t) \notin A_n\} \leq \epsilon.$$ 

This contradicts the definition of $x_j$; namely, it contradicts the assumption that $M(x_j) \to \infty$, and completes the proof.

These two lemmas lead to the main theorem.

**Theorem 4.1.** There exists a distribution function $F(x)$ such that $F_t(x) \to F(x)$, as $t \to \infty$, uniformly for all $x$. Furthermore, $F(x)$ does not depend on the initial stock $s$.

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3 This proof has benefitted from the comments of the referee.
Proof. Equation (4.2) which generates the distribution functions for successive values of capital is stated again for convenience:

\[ F_t(x) = \int F_{t-1}(q(x, \rho)) \nu(d\rho). \quad (4.2) \]

Let \( F_0(x) \) be a degenerate distribution corresponding to the initial value \( x^* \); i.e., \( F_0(x) = 0 \), \( x < x^* \) and \( F_0(x) = 1 \) for \( x > x^* \). We must show that for any \( \epsilon > 0 \) there exist a \( T \) such that for all \( t > T \) and all \( h > 0 \),

\[ |F_{t+h}(x) - F_t(x)| < \epsilon. \]

Here \( T \) may be chosen independent of \( x \).

Consider Eq. (4.2) which relates \( F_t(x) \) to \( F_{t-1}(x) \). The function \( F_{t-1}(x) \) may be found in a similar way using Eq. (4.2). This relates \( F_t(x) \) and \( F_{t-2}(x) \) as

\[ F_t(x) = \int \int F_{t-2}(q(q(x, \rho_1), \rho_2)) \nu(d\rho_1) \nu(d\rho_2). \]

In general, iterating in this way \( t \) times we get,

\[ F_t(x) = \int \cdots \int F_0(q(\cdots q(q(x, \rho_1), \rho_2),\ldots, \rho_i)) \nu(d\rho_1) \cdots \nu(d\rho_i). \]

Written symbolically as

\[ F_t(x) = \int_t F_0(q^t(x, \rho)) \nu_t(d\rho), \]

where the integral is a \( t \)-fold integral and \( \nu_t \) represents the measure over the space \([x, \beta] \)'. In this way the difference \( F_{t+h}(x) - F_t(x) \) may be represented as

\[
F_{t+h}(x) - F_t(x) = \int_{t+h} F_0(q^{t+h}(x, \rho)) \nu_{t+h}(d\rho) \\
- \int_{t+h} F_0(q^t(x, \rho)) \nu_{t+h}(d\rho). \quad (4.8)
\]

In the last integral the extra \( h \) iterates in the integration are added for convenience. Since \( \nu(\cdot) \) is a probability measure the value of the integral is not altered. Hence, rewriting (4.8), we have

\[
F_{t+h}(x) - F_t(x) = \int_{t+h} [F_0(q^{t+h}(x, \rho)) - F_0(q^t(x, \rho))] \nu_{t+h}(d\rho).
\]
Suppose that $q(\cdot, a)$ has an infinite number of positive fixed points $x_n$ such that $x_n \to 0$, as $n \to \infty$. Moreover, suppose that the mass of the degenerate initial distribution $F_0$ is concentrated at $x \leq x_M$. We may choose a positive fixed point $x_n$ such that $x_n \leq x^*$. Recall that for any $\epsilon > 0$ there exists an $M_n$ such that for all $m \geq M_n$ and every $x \in A_n$, $P\{q^m(x, r^m) \in A_n\} < \epsilon$. Moreover, for $x \notin A_n$, $P\{q^t(x, r^t) \in A_n\} = 0$ for all $t > 0$, since for any $\rho$, $q(\cdot, \rho)$ is strictly increasing.

Consider again Eq. (4.8), which may be written as

$$F_{t+h}(x) - F_t(x) = \int_{t+h} \left[ F_0(q^{t+h}(x, \rho)) - F_0(q^t(x, \rho)) \right] \nu_{t+h}(d\rho).$$

Here

$$I_1 = \int_{\{q^t(x, \rho) \in (0, x)\}} \left[ F_0(q^{t+h}(x, \rho)) - F_0(q^t(x, \rho)) \right] \nu_{t+h}(d\rho),$$

$$I_2 = \int_{\{q^t(x, \rho) \in (x_M, \infty)\}} \left[ F_0(q^{t+h}(x, \rho)) - F_0(q^t(x, \rho)) \right] \nu_{t+h}(d\rho),$$

$$I_3 = \int_{\{q^t(x, \rho) \in A_n\}} \left[ F_0(q^{t+h}(x, \rho)) - F_0(q^t(x, \rho)) \right] \nu_{t+h}(d\rho).$$

Thus, we split the integral into three pieces corresponding to the events that the $t$-th iterate of $q$, starting from $x$, belongs to the set $(0, x)$, $(x_M, \infty)$, or $A_n$. Note that $q^t(x, r^t) \in (0, x)$ implies that $q^{t+h}(x, r^t) \in (0, x)$.

Moreover, if $q^t(x, \rho) \in (0, x)$ or $q^t(x, \rho) \in (x_M, \infty)$ we have

$$F_0(q^{t+h}(x, \rho)) = F_0(q^t(x, \rho))$$

since $F_0(x)$ is degenerate with all mass concentrated at $x^*$. Hence, from Lemma 4.3 there exists an $M_n$ such that $I_1 = 0$, $I_2 = 0$, for $t > M_n$.

Thus, (4.9) may be written as

$$| F_{t+h}(x) - F_t(x) | = | I_3 |$$

$$\leq \int_{\{q^t(x, \rho) \in A_n\}} \left| F_0(q^{t+h}(x, \rho)) - F_0(q^t(x, \rho)) \right| \nu_{t+h}(d\rho).$$

However, the probability of being in the set $A_n$ after the $t$-th period is less than $\epsilon$. Moreover, since $F_0$ is a distribution function, $| F_0(x) - F_0(y) | < 2$. Hence

$$| F_{t+h}(x) - F_t(x) | < 2\epsilon.$$
Note that since $x^* \in A_n$ it is possible for the $t$-th iterate of $q$ to be greater than $x^*$ and the $(t + h)$th iterate of $q$ to be less than $x^*$, in which case (4.10) would not be indentically zero.

Finally, suppose that $H(\cdot, \alpha)$ does not have a sequence of positive fixed points tending to zero. In this case, by the continuity of $q$ there exists a minimum positive fixed point $x_{ma}$. If $x^* \geq x_{ma}$ merely choose $x_n = x_{ma}$ and the above proof goes through unchanged. Similarly, if $x^* < x_{ma}$, choose $x_n < x^*$ in which case since $q(x, \alpha) < x$ for all $x < x_{ma}$, the above proof is again valid.

Hence we have shown that the sequence $F_t(x)$ is uniformly convergent. Thus there exists a limit function $F$ such that $F_t(x) \to F(x)$ uniformly. Below we show that $F(x)$ is in fact unique, independent of initial stocks and that $F$ is a distribution function.

Clearly $F(x)$ is 1 for $x \in (x_{Ma}, \infty)$ and 0 for $x \in (0, x_{ma})$. Moreover, continuity from the left follows from uniform convergence. Hence $F$ is a distribution function. To see that $F$ is independent of initial stocks we need only show that there is one distribution that satisfies Eq. (4.2).

Suppose that $F$ and $G$ are two distribution functions satisfying (4.2). Let

$$H(x) = F(x) - G(x).$$

Then

$$H(x) = \int H(q(x, \rho)) \nu(d\rho).$$

Iterating, we get

$$H(x) = \int_t H(q^t(x, \rho^t)) \nu_t(d\rho).$$

Now by an argument similar to that used to show $F_t \to F$, we have $H(x) \equiv 0$ for $x \in [0, x_{Ma}] \cup [x_{Mb}, \infty)$. Moreover, $q^t(x, \rho^t)$ eventually enters the set $[0, x_{Ma}] \cup [x_{Mb}, \infty)$ with probability one; hence $H(x) \equiv 0$. Thus the limit $F$ is unique and therefore independent of initial stocks.

APPENDIX: STOCHASTIC SENSITIVITY

In this appendix, some of the results on sensitivity contained in [1] are extended to the random case.

Consider the maximization problem:

$$\max \mathbb{E} \sum_{t=0}^T u(c_t^{T+1})$$
subject to

\[ c_{t+1}^T + x_{t+1}^T = f(x_{t-1}^T, r_{t-1}) \]

and

\[ c_{0+1}^T + x_{0+1}^T = s. \]

Moreover, we require \( x_{T+1}^T = 0 \). Let \( \tilde{x}_0^{T+1}(s) \) and \( \tilde{c}_0^{T+1}(s) \) be optimal policies.

**Theorem A1.** We have for \( s > 0 \), \( \tilde{x}_0^{T}(s) < \tilde{x}_0^{T+1}(s) \), \( T = 1, 2, \ldots \).

**Proof.** Suppose that \( \tilde{x}_0^T \geq \tilde{x}_0^{T+1} \). Using the usual functional equation, we find that \( \tilde{c}_0^T \leq \tilde{c}_0^{T+1} \) implies that

\[ u'(\tilde{c}_0^T) \geq u'(\tilde{c}_0^{T+1}) = \int u'((\tilde{c}_1^{T+1})f'((\tilde{x}_0^{T+1}), \rho) \nu(d\rho), \tag{A1} \]

while

\[ u'(\tilde{c}_0^T) = \int u'((\tilde{c}_1^T)f'((\tilde{x}_0^T), \rho) \nu(d\rho). \tag{A2} \]

It will now be shown that there exists a set \( A_1 \) of positive \( \nu \) measure such that \( \tilde{x}_1^T(\omega) \geq \tilde{x}_1^{T+1}(\omega) \), \( \omega \in A_1 \).

Suppose that

\[ \tilde{c}_1^T(\omega) > \tilde{c}_1^{T+1}(\omega) \quad \text{a.e.} \tag{4} \]

Then

\[ u'(\tilde{c}_1^T(\omega)) < u'(\tilde{c}_1^{T+1}(\omega)) \quad \text{a.e.} \]

Then, since \( \tilde{x}_0^T \geq \tilde{x}_0^{T+1} \),

\[ f'((\tilde{x}_0^T), r_0) \leq f'((\tilde{x}_0^{T+1}, r_0). \]

Thus

\[ \int u'((\tilde{c}_1^T)f'((\tilde{x}_0^T), \rho) \nu(d\rho) \leq \int u'((\tilde{c}_1^{T+1})f'((\tilde{x}_0^{T+1}), \rho) \nu(d\rho). \]

However, this contradicts (A1) and (A2). Thus, there does exist a set \( A_1 \) of positive measure on which \( \tilde{c}_1^T(\omega) \leq \tilde{c}_1^{T+1}(\omega) \). Moreover, on \( A_1 \),

\[ \tilde{x}_1^T(\omega) - \tilde{x}_1^{T+1}(\omega) = f(\tilde{x}_0^T, r_1(\omega)) - \tilde{c}_1^T(\omega) = f(\tilde{x}_0^{T+1}, r_1(\omega)) + \tilde{c}_1^{T+1}(\omega) \geq 0. \tag{A3} \]

In this way, a sequence of sets \( A_1, \ldots, A_T \) with positive measure can be found. On the other hand for \( A_T \subset \Omega_1 \times \cdots \times \Omega_T \),

\[ 0 = \tilde{x}_r^T(\omega_1, \ldots, \omega_T) \geq \tilde{x}_r^{T+1}(\omega_1, \ldots, \omega_T). \tag{A4} \]

\(^4\) Here a.e. means except for a set of \( \nu \) measure zero.
Using the facts that \( u'(0) = +\infty \) and \( f(0, r) = 0 \), \( \bar{x}_{T+1}^T(\omega_1, \ldots, \omega_T) > 0 \). Hence (A3) contradicts (A4) which proves the theorem.

Consider the case in which the final stocks are constrained to be \( \bar{x}_T^T = b \). Here \( b = b(\omega_1, \ldots, \omega_T), \omega_i \in \Omega_i \). Moreover, \( b \) is chosen to be feasible for each sequence of \( \omega \)'s. By feasibility, it is meant that \( b \) is attainable after \( T + 1 \) steps. Let \( \bar{x}_0^T(b) \) be the optimal capital policy as a function of the final constraint \( b \). The \( T + 1 \)'s will be omitted for convenience.

**Theorem A2.** Let \( b_1(\omega_1, \ldots, \omega_T) < b_0(\omega_1, \ldots, \omega_T) \) for almost all \( (\omega_1, \ldots, \omega_T), \omega_i \in \Omega_i \). Then \( \bar{x}_0^T(b_1) < \bar{x}_0^T(b_2) \).

**Proof.** Suppose that \( \bar{x}_0^T(b_1) \geq \bar{x}_0^T(b_2) \) (the \( T \)'s will be dropped for convenience). Then \( \bar{c}_0(b_1) \leq \bar{c}_0(b_2) \). Here \( \bar{c}_0(b_1) + \bar{x}_0(b_1) = s \). Hence,

\[
1 \leq \frac{u'(\bar{c}_0(b_1))}{u'(\bar{c}_0(b_2))} = \frac{E u'(\bar{c}_1(b_1)) f'(\bar{x}_0(b_1), \omega_1)}{E u'(\bar{c}_1(b_2)) f'(\bar{x}_0(b_2), \omega_1)}.
\]

(A5)

However, \( \bar{x}_0(b_1) \geq \bar{x}_0(b_2) \) implies that \( f'(\bar{x}_0(b_1), \omega) \leq f'(\bar{x}_0(b_2), \omega) \). If \( u'(\bar{c}_1(b_2)) > u'(\bar{c}_1(b_1)) \) for almost all \( \omega \), then the numerator in (A5) would be less than the denominator which would be a contradiction. Hence, there exists a set of positive measure \( A_1 \) such that \( u'(\bar{c}_0(b_1), \omega) < u'(\bar{c}_0(b_2), \omega) \) or, equivalently, \( \bar{c}_0(b_2) > \bar{c}_0(b_1) \). Thus as in Theorem A1, \( \bar{x}_1(b_1) \geq \bar{x}_1(b_2) \) on a set of positive measure. Continuing in this way we find that \( \bar{x}_T(b_2) \leq \bar{x}_T(b_1) \) on a set of positive measure. However, this leads to a contradiction since

\[
b_1 = \bar{x}_T(b_1) < \bar{x}_T(b_2) = b_2.
\]

The next theorem extends Theorem A1. Let \( b = 0 \) in the maximization problem, i.e., the condition \( \bar{x}_{T+1}^T = 0 \) is imposed.

**Theorem A3.** We have \( \bar{x}_T^T(\omega_1, \ldots, \omega_i) \uparrow \bar{x}_T^T(\omega_1, \ldots, \omega_i) \) for all \( (\omega_1, \ldots, \omega_i), \omega_i \in \Omega_i \), as \( T \to \infty \).

**Proof.** Clearly, \( \bar{x}_{T+1}^T(\omega_1, \ldots, \omega_T) > 0 \) for almost all \( (\omega_1, \ldots, \omega_T), \omega_i \in \Omega_i \). If not, then \( f(0, r) = 0 \) implies that \( \bar{x}_{T+1}^T = 0 \) which contradicts the optimality of \( \{\bar{c}_{T+1}^T\} \). In the necessary condition the expectation was with respect to the measure \( \nu \) governing the values of \( \omega_T \). Hence, \( \bar{x}_{T+1}^T > 0 \) almost everywhere. The condition \( b_0 = \bar{x}_{T+1}^T \) on the final stocks of the \( T \) period program induces a related \( T \) period problem. Namely,

\[
\max E \sum_{t=0}^{T} u(\bar{c}_t)
\]
subject to
\[ c_0 + x_0 = s, \]
\[ c_t + x_t = f(x_{t-1}, r_{t-1}), \]
and
\[ x_T(\omega_1, \ldots, \omega_T) = b_2(\omega_1, \ldots, \omega_T). \]
Here
\[ b_2(\omega_1, \ldots, \omega_T) = \bar{x}_T^{T+1} > b_1 \equiv 0. \]
Applying Theorems A1 and A2 implies that \( \bar{x}_T^T < \bar{x}_T^{T+1} \) for almost all \( \omega_1, \ldots, \omega_T, \omega_i \in \Omega_i \). Thus \( \bar{x}_T^T \uparrow \bar{x}_i \) for almost all \( \omega_1, \ldots, \omega_T, \) such that \( \omega_i \in \Omega_i \). Thus proves the theorem.

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