Ambiguity Aversion and Asset Prices in Production Economies

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Abstract

We introduce ambiguity and ambiguity aversion into a standard one-sector production-based real business cycle model where mean productivity growth rates are uncertain. We show that with mild capital adjustment costs and a low coefficient of relative risk aversion, the model can explain several salient features about macroeconomic quantities and asset prices including a high equity premium, a low and smooth risk-free rate, a low consumption growth volatility and a high investment growth volatility relative to output growth volatility. Moreover, the model can generate long horizon predictability of equity returns by price-dividend ratios, investment-capital ratios, Tobin's Q and consumption-wealth ratios. Introducing an unobservable state and Bayesian learning into the model can further account for countercyclical equity premia.

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1 Introduction

The standard rational representative-agent model encounters substantial difficulties in explaining a number of well-known stylized facts in the U.S. financial data. The most prominent puzzle is "equity premium puzzle" of Mehra and Prescott (1985), stating that the equity premium observed in the data is too high to be reconciled with plausible levels of risk aversion in a standard rational representative-agent model. Weil (1989) finds that even if an unreasonably high degree of risk aversion can explain the high equity premium in the data, it will inevitably lead to an excessively high risk-free rate, giving rise to the "risk-free rate puzzle". Shiller (1981) shows that the volatility of equity returns is too high to be explained by smooth movements in fundamentals, resulting in the "equity volatility puzzle". Moreover, the empirical literature documents that the expected equity premium moves countercyclically (Fama and French (1989)), and that excess returns are predictable by valuation ratios with predictability increasing in the horizon of returns, see Fama and French (1988a,b). Schwert (1989) finds persistent and countercyclical movements in conditional volatility of equity returns. Earlier works (e.g., Rouwenhorst (1995), Jermann (1998) and Boldrin et al. (2001)) have shown that reconciling the above mentioned stylized facts about asset prices with salient features of macroeconomic quantities prove to be challenging for the standard dynamic stochastic general equilibrium (DSGE) models, once consumption and dividends are endogenously determined.

In this paper, we introduce ambiguity and ambiguity aversion into a standard DSGE model where mean productivity growth is uncertain. When calibrated to match the second moments of aggregate consumption and output, our model is able to reproduce salient features of asset returns data. With a low relative risk aversion coefficient of 2, our model generates a high equity premium of 6.3% per year (in log units) and a low and smooth risk-free rate. In addition, our model produces predictability of excess returns by investment-capital ratios, Tobin's Q, price-dividend ratios and consumption-wealth ratios. Our model is also able to replicate key empirical regularities of macroeconomic quantities including high volatility of investment growth. Further, when we allow agents in the model to learn about an unobservable state underlying productivity growth, the model can also account for countercyclical variation in equity premium and conditional volatility of returns. These results represent a significant improvement over the existing production-based asset pricing models (for example, Kaltenbrunner and Lochstoer (2010), Campanale et al. (2010), and Croce (2010)), which rely on recursive utility functional forms and capital adjustment costs, in explaining the joint behavior of asset returns and aggregate quantities.

Our model features regime-switching productivity growth, capital adjustment costs, the separation between risk aversion and the intertemporal elasticity of substitution (IES), and most importantly, ambiguity aversion. Specifically, we assume that productivity growth has two distinct regimes, labeled as "good" and "bad" regimes.[1] There is an underlying state that governs growth regimes and follows a Markov chain. This specification of regime shifts in productivity growth builds on the evidence provided by Cagetti et al. (2002) and Edge and Williams (2007).

The information structure of our model distinguishes risk from subjective uncertainty, where the latter is interpreted by Klibanoff et al. (2005, 2009) as ambiguity. In particular, risk is captured by the probability distribution of productivity growth conditioned on the underlying state. Subjective uncertainty is characterized by a probability distribution over the state. In this paper, we consider two separate cases depending on whether the underlying state is observable or not. Since the state evolves according to a Markov chain, in the full information case where the state is observable, the probability distribution over the state is characterized by the transition probabilities governing the Markov chain. In the case where the underlying state is unobservable, agents engage in Bayesian learning and subjective uncertainty is represented by Bayesian posterior beliefs.

We follow Ju and Miao (2012) and adopt a new class of preferences known as "generalized recursive smooth ambiguity preferences". As an extension of the recursive smooth ambiguity model of Klibanoff et al. (2009), this generalized model admits a three-way separation among risk aversion, ambiguity aversion and the IES, and therefore incorporates recursive preferences as a special case when decision makers are ambiguity neutral. Existing works on learning and ambiguity aversion have so far been confined to endowment economies.[2] We extend the contributions of these studies to a production setting and show that learning and ambiguity aversion

[1] Models with regime shifts have been extensively explored in endowment economies. Examples include David (1997), Veronesi (1999), and Cecchetti et al. (2000), among others. Cagetti et al. (2002) study a continuous time production economy with regime switching growth rates. However, the thrust of their paper is not reproducing moments of asset returns and macroeconomic quantities in the data, but rather they study the dynamics of risk-return trade-off and the evolution of dividend-price ratio in presence of demand for robustness in decision rules.

[2] Epstein and Schneider (2008) show that the interaction between ambiguity aversion and information quality generates ambiguity premia and persistent effect on asset prices. Leippold et al. (2008) study asset pricing implications of learning and ambiguity aversion and find that the model can match the key asset-return moments in the data well. Ju and Miao (2012) examine the implications of regime-switching endowment processes, learning and ambiguity aversion, using the same utility preferences as in this paper. Collard et al. (2011) study the impact of state uncertainty, model uncertainty and ambiguity aversion.

nicely augment the standard real business cycle (RBC) model to characterize asset returns well, while our model maintains its merits in explaining aggregate quantities.

In our model, the representative agent is risk averse in the usual sense; i.e., he dislikes any mean-preserving spread of consumption. The agent is ambiguity averse in the sense that he dislikes any mean-preserving spread of expected continuation value induced by the probability distribution over the underlying state. Ambiguity aversion is a product of imposing some concave transformation in the certainty equivalent of future continuation value. The transformation reflects the agent's pessimistic view and effectively makes the agent put more weight on states with low continuation value. The smooth ambiguity model adopted in the paper has the advantage of relaxing the tight link between ambiguity and ambiguity aversion.[3] As a result, we are able to do comparative statics analysis by holding the information structure fixed, while varying the extent of ambiguity aversion.

In our model, capital adjustment is rigid due to adjustment costs. The two-fold role of capital adjustment costs is well recognized in the literature, see Jermann (1998) and Kaltenbrunner and Lochstoer (2010). On one hand, high adjustment costs can make the return on the equity claim greatly risky and thus create the potential to increase equity premium. As noted by Jermann (1998), without adjustment costs, agents can easily smooth consumption by altering production plans. In this case, even with strong risk aversion, both the volatility of the stochastic discount factor (SDF) and equity premium are low. Moreover, Tobin's Q always remains constant at 1, removing most of risks on the equity claim. On the other hand, high adjustment costs imply a counterfactually low volatility of investment growth. Due to substantial capital rigidity, investment becomes much less responsive to productivity shocks. However, the historical data show that the volatility of investment growth is about three times higher than that of output growth.

Given moderate levels of risk aversion, reconciling a high equity premium, a low and smooth risk-free rate, and a high volatility of investment growth - simultaneously - is a challenging task in the standard production-based model.[4] Earlier studies such as Jermann (1998) and Kaltenbrunner and Lochstoer (2010) rely on high adjustment costs and strong aversion toward intertemporal substitution to produce a high equity premium. But these two modeling features

[3] See Klibanoff et al. (2005, 2009) for more discussion. The separation between ambiguity and ambiguity aversion, however, cannot be achieved in the multiple priors utility model such as Chen and Epstein (2002) or Epstein and Schneider (2008).

[4] Mehra and Prescott (1985) set the reasonable range for the coefficient of relative risk aversion between 1 and 10.

together inevitably lead to an excessively high and volatile risk-free rate and unreasonably smooth investment growth. In our model, we consider an IES greater than 1, a low risk aversion of 2 and mild adjustment costs. A high IES implies small variation in risk-free rates since the tight tension between strong aversion toward intertemporal substitution and the capital friction impeding consumption smoothing is relaxed. On the other hand, both low adjustment costs and ambiguity aversion imply a high volatility of investment growth. Facing adjustment costs, the representative agent's precautionary savings motive induced by ambiguity aversion leads to slow capital accumulation. As a result, the marginal product of investment is high, making investment responsive to productivity shocks.

An important question is: with a low risk aversion and small adjustment costs, where does the 6% equity premium implied by our model come from? A key ingredient in generating a high equity premium in our model is ambiguity aversion. The agent's pessimistic view about future continuation value introduces an additional multiplicative term into the stochastic discount factor (SDF). This term substantially increases the volatility of the SDF and also increases equity premium in a similar way as in the endowment economy model of Ju and Miao (2012). We also characterize how ambiguity aversion distorts the agent's subjective beliefs in both the full information model and the model with Bayesian updating. We find that ambiguity aversion imputes endogenous pessimism in the agent's subjective beliefs.

In the model with a hidden state and Bayesian updating, the implied equity premium and the price of risk move countercyclically. The intuition for this result is straightforward. When the agent learns about the unobservable state, fluctuating beliefs induce time-varying evolution of the price of risk and equity premium. In good times, small magnitude of state uncertainty leads to a low price of risk. In bad times when productivity shocks arrive, state beliefs about future growth prospects deteriorate, giving rise to increased uncertainty about the state of economy. Combined with ambiguity aversion, heightened state uncertainty results in a high price of risk, implying its countercyclical variation.

Our paper belongs to the growing, but still limited, production-based asset pricing literature. Tallarini (2000) finds that with an IES of 1, an extremely high level of risk aversion is required to justify the Sharpe ratio observed in the data. However, the implied equity premium is extremely low. Kaltenbrunner and Lochstoer (2010) examine two types of DSGE models with Epstein-Zin preferences, one with permanent shocks and the other with transitory shocks. They find that long run consumption risk can endogenously arise in these models.[5] With a reasonable risk aversion coefficient, a time discount factor greater than 1 and a rather low IES, their transitory-shock model can generate a high equity premium, whereas the volatility of risk-free rates is implausibly high. Campanale et al. (2010) show that disappointment aversion in a DSGE model with "Chew-Dekel" class of preferences and convex capital adjustment costs is important to match the mean equity premium observed in the data. But the risk-free rate is still excessively volatile in their model. Croce (2010) investigates the asset pricing and business cycle implications of long run productivity risk. The model requires a high level of risk aversion to generate a sizable equity premium. Gourio (2012) studies the implications of time varying disaster risk on asset prices and business cycles. His model can match well the post-war data on aggregate quantities and asset prices. Ai et al. (2012) consider the role of investment options as intangible capital in a general equilibrium model with production. They show that the model can explain key features of both macroeconomic quantities and asset returns. Moreover, their model can explain the value premium observed in the data.

The rest of the paper is organized as follows. Section 2 presents the production economy model where the underlying state is assumed to be fully observable. Section 3 discusses the calibration and quantitative results. Section 4 extends the full information model in Section 2 to a model with a hidden state, and examine the quantitative implications of the model. Section 5 concludes. Our numerical algorithm, calibration of the degree of ambiguity aversion and data construction scheme are included in the Appendix.

2 The Production Economy Model

In this section, we present a production economy model with regime-switching productivity growth and capital adjustment costs. There are many infinitely-lived firms and one representative household or agent. We also characterize the representative agent's preferences and the market equilibrium.

[5] In an endowment economy, long run risks are defined as consumption and/or dividend growth having a small but persistent predictable component; see Bansal and Yaron (2004).

2.1 Production Set

A single type of consumption good is produced according to a constant-returns-to-scale Cobb-Douglas production function:

$$Y_t = K_t^{\alpha} \left(A_t N_t\right)^{1-\alpha} = C_t + I_t \tag{1}$$

where Y_t is the output, K_t is the capital stock, N_t is the quantity of labor hours, and A_t is the aggregate productivity level. The second equality in equation (1) follows from the aggregate resource constraint. Labor input is assumed to be exogenous and fixed at $\bar{N}$, which is normalized to 1.[6] Uncertainty in the economy is driven by the stochastic behavior of productivity growth. The growth rate, defined by $\Delta a_{t+1} \equiv \log\left(\frac{A_{t+1}}{A_t}\right)$, follows a Markov-switching process,

$$\Delta a_t = \mu\left(z_t\right) + \sigma \epsilon_t, \qquad \epsilon_t \sim N\left(0,1\right), \tag{2}$$

where z_t evolves according to a Markov chain with two regimes. We denote the high growth regime (good regime) by $z_t = 1$, and low growth regime (or bad regime) by $z_t = 2$. We assume $\mu\left(1\right) \gg \mu\left(2\right)$. The transition probability matrix, $\mathbf{P}$, is given by

$$\mathbf{P} = \begin{bmatrix} p_{11} & 1-p_{11} \\ 1-p_{22} & p_{22} \end{bmatrix}. \tag{3}$$

where p_{11} denotes the "good-to-good" transition probability, and p_{22} denotes the "bad-to-bad" transition probability.[7] By assuming that first differences of log productivity levels follow a nonlinear stationary process with regime shifts, the specification of growth rates can capture the recurrent feature of business cycles, according to Hamilton (1989) and Cagetti et al. (2002).[8]

[6] Tallarini (2000) sets $\bar{N} = 0.23$. Our calibration exercises show that different values of $\bar{N}$ produce similar results.

[7] Using the approach of Garcia et al. (2008), one can show that the process (2) can be obtained as a result of Markov chain discretization of the long-run risk model of Croce (2010) with constant volatility. Although Croce (2010) also explores time variation in conditional variance of productivity growth, we assume constant volatility to keep the model parsimonious.

[8] Hamilton (1989) and Campbell and Mankiw (1987), among others, show that business cycles are associated with a large permanent effect on the long run level of output. See Hamilton (1989) for more discussion.

As in Jermann (1998), the capital stock evolves according to

$$K_{t+1} = (1-\delta_K) K_t + \phi\left(\frac{I_t}{K_t}\right) K_t \tag{4}$$

$$\phi\left(\frac{I_t}{K_t}\right) = a_1 + \frac{a_2}{1-\frac{1}{\xi}}\left(\frac{I_t}{K_t}\right)^{1-\frac{1}{\xi}}, \xi > 0 \tag{5}$$

where I_t is the investment in period t, δ_K is the rate of depreciation of the capital, and $\phi(\cdot)$ is a concave function that allows for convex capital adjustment costs. The adjustment cost is parameterized inversely by the parameter ξ. The constants a_1 and a_2 are set such that there are no adjustment costs in the nonstochastic steady state.

2.2 Preferences

We assume that the agent's preferences are represented by the generalized recursive smooth ambiguity utility model first proposed by Ju and Miao (2012).[9] Here, we adapt their utility function into our production economy setting. Since productivity growth rates, as the main driving force of the economy, follow a Markov-switching process, we assume that the agent has the following utility function

$$V_{z_t,t}(C) = \left[(1-\beta)C_t^{1-\rho} + \beta\left\{\mathcal{R}_{z_t,t}\left(V_{z_{t+1},t+1}(C)\right)\right\}^{1-\rho}\right]^{\frac{1}{1-\rho}}, \tag{6}$$

$$\mathcal{R}_{z_t,t}\left(V_{z_{t+1},t+1}(C)\right) = \left(\mathbb{E}_{z_t}\left[\left(\mathbb{E}_{z_{t+1},t}\left[V_{z_{t+1},t+1}^{1-\gamma}(C)\right]\right)^{\frac{1-\eta}{1-\gamma}}\right]\right)^{\frac{1}{1-\eta}}, \tag{7}$$

where $\beta \in (0,1)$ is the subjective discount factor, ρ is the inverse of the IES parameter ψ ($\rho = 1/\psi$), γ is the coefficient of relative risk aversion, η is the ambiguity aversion parameter and, as we discuss below, must satisfy $\eta \geq \gamma$. $V_{z_t,t}(C)$ is the period-t continuation value of a consumption plan C given period-t productivity regime z_t and all other period-t state variables, and $\mathcal{R}_{z_t,t}\left(V_{z_{t+1},t+1}(C)\right)$ is the certainty equivalent of future continuation value given period-t state. More specifically, when $z_t = 1$, the certainty equivalent becomes

$$\mathcal{R}_{1,t}\left(V_{z_{t+1},t+1}(C)\right) = \left(p_{11}\left(\mathbb{E}_{1,t}\left[V_{z_{t+1},t+1}^{1-\gamma}(C)\right]\right)^{\frac{1-\eta}{1-\gamma}} + (1-p_{11})\left(\mathbb{E}_{2,t}\left[V_{z_{t+1},t+1}^{1-\gamma}(C)\right]\right)^{\frac{1-\eta}{1-\gamma}}\right)^{\frac{1}{1-\eta}}$$

[9] Hayashi and Miao (2011) provide the axiomatic foundation for this class of preferences.

otherwise when $z_t = 2$

$$\mathcal{R}_{2,t}\left(V_{z_{t+1},t+1}(C)\right) = \left((1-p_{22})\left(\mathbb{E}_{1,t}\left[V^{1-\gamma}_{z_{t+1},t+1}(C)\right]\right)^{\frac{1-\eta}{1-\gamma}} + p_{22}\left(\mathbb{E}_{2,t}\left[V^{1-\gamma}_{z_{t+1},t+1}(C)\right]\right)^{\frac{1-\eta}{1-\gamma}}\right)^{\frac{1}{1-\eta}}$$

where $\mathbb{E}_{j,t}[\cdot]$, $j = 1, 2$ denotes the conditional expectation operator given period-$(t+1)$ state $z_{t+1} = j$ and the history up to period-t. We can explicitly write the conditional expectation in the following form

$$\mathbb{E}_{j,t}\left[V^{1-\gamma}_{z_{t+1},t+1}(C)\right] = \int V^{1-\gamma}_{j,t+1}(C)\frac{1}{\sigma\sqrt{2\pi}}\exp\left(-\frac{(\Delta a_{t+1}-\mu_j)^2}{2\sigma^2}\right)d(\Delta a_{t+1}), \quad j = 1, 2.$$

As noted by previous papers (see Klibanoff et al. (2005, 2009)), a key property of the smooth ambiguity model is that it achieves the separation between ambiguity and ambiguity aversion, where the latter is identified as a characteristic of the decision maker's tastes. Ju and Miao (2012) show that given the utility function in equation (6), the agent is ambiguity averse if and only if $\eta > \gamma$. In other words, in the certainty equivalent in equation (7), the transformation induced by the condition $\eta > \gamma$ precludes the compound reduction between the agent's subjective beliefs, which correspond to the transition probabilities, and the conditional distributions of growth rates. Thus, the smooth ambiguity utility model implies that the agent displays different attitudes toward subjective uncertainty, which is captured by uncertain future productivity regimes, and productivity risk, which is characterized by a conditional distribution of productivity growth rates. However, in the special case of recursive utility with $\eta = \gamma$, these two attitudes are intertwined and are both governed by the risk aversion parameter, since we can integrate conditionals over the agent's subjective beliefs to produce a predictive distribution of productivity growth rates based on the current information. In this case, we obtain an otherwise standard production-based model with recursive utility and regime-switching productivity growth rates.

There are three main advantages in using the utility preferences above. First, the smooth ambiguity framework allows us to examine the comparative statics effect of ambiguity aversion. That is, without altering the information structure, we are interested in how the equilibrium allocations and asset prices would change in response to a change in the degree of ambiguity aversion. Second, the utility function is homothetic, providing tractability. Alternative models exist for studying ambiguity, such as Hansen and Sargent's robust control formulation with log-exp specification. But that formulation does not satisfy homogeneity, making numerical analysis difficult due to the curse of dimensionality. Third, unlike the recursive smooth ambiguity model proposed by Klibanoff et al. (2009), the generalized specification further disentangles risk aversion from the intertemporal elasticity of substitution. Thus, the model gains another channel for explaining well the data on aggregate quantities and asset prices.

2.3 Asset Prices

We present the social planner's problem in the Appendix.[10] The social planner chooses consumption and investment to maximize his welfare. Given that the utility function (6) satisfies homogeneity, the problem can be formulated in terms of stationary variables that are defined in the Appendix. After the equilibrium allocations are found, we can solve for asset prices.

The first order condition for labor inputs implies $Y_t - w_t N_t = \alpha Y_t$. As a result, aggregate dividends are given by

$$D_t = Y_t - w_t N_t - I_t = \alpha Y_t - I_t \tag{8}$$

Tobin's Q is

$$q_t = \frac{1}{\phi'\left(\frac{I_t}{K_t}\right)},$$

where ϕ' stands for the partial derivative. Following the standard DSGE literature, the return on investment is

$$R_{t+1} = \frac{1}{q_t}\left\{q_{t+1}\left[1 - \delta_K + \phi\left(\frac{I_{t+1}}{K_{t+1}}\right)\right] + \frac{\alpha Y_{t+1} - I_{t+1}}{K_{t+1}}\right\} \tag{9}$$

The log return is $r_t = \log(R_t)$. Ju and Miao (2012) and Hayashi and Miao (2011) show that the stochastic discount factor for the generalized recursive smooth ambiguity utility is given by

$$M_{z_{t+1},t+1} = \beta\left(\frac{C_{t+1}}{C_t}\right)^{-\rho}\left(\frac{V_{z_{t+1},t+1}}{\mathcal{R}_{z_t,t}\left(V_{z_{t+1},t+1}\right)}\right)^{\rho-\gamma}\left(\frac{\left(\mathbb{E}_{z_{t+1},t}\left[V_{z_{t+1},t+1}^{1-\gamma}\right]\right)^{\frac{1}{1-\gamma}}}{\mathcal{R}_{z_t,t}\left(V_{z_{t+1},t+1}\right)}\right)^{-(\eta-\gamma)}, \quad z_{t+1} = \{1, 2\}, \tag{10}$$

where $\mathbb{E}_{z_{t+1},t}[\cdot]$ denotes the conditional expectation operator given period-$(t+1)$ state z_{t+1} and the history up to period-t. The consensus in the asset pricing literature is that a model aiming to produce a sizable equity premium must generate a sufficiently volatile stochastic discount factor (SDF). In comparison with endowment economy models, production-based asset pricing models

[10] The economy can be decentralized with a complete set of Arrow-Debreu securities using the standard arguments.

encounter more difficulty in generating sizable equity premia. In production-based models, if not dealt with, endogenous consumption smoothing leads to smooth SDFs and low risk exposure of equity claims. In our model, ambiguity aversion implies high curvature in preferences and alters the SDF by introducing an additional multiplicative term in equation (10). Thus, our model generates a sufficiently volatile SDF. However, in and by itself, ambiguity aversion cannot explain the high equity premium observed in the data since the risk exposure of the equity claim would be still too low without capital adjustment costs. Capital adjustment costs can effectively reduce the flexibility of consumption smoothing and create time variation in the marginal product of capital as well as the return on investment. Our numerical analysis below illustrates this argument.

If the agent is ambiguity neutral ($\eta = \gamma$), the last multiplicative term in equation (10) vanishes, and the SDF has the same functional form as in Croce (2010) and Kaltenbrunner and Lochstoer (2010). If the agent further displays constant relative risk aversion ($\gamma = \rho$), then we obtain the familiar SDF for expected utility.

As usual, the return on investment, R_{t+1}, satisfies the Euler equation

$$\mathbb{E}_{z_t,t}\left[M_{z_{t+1},t+1}R_{t+1}\right] = 1. \tag{11}$$

The risk-free rate, $R_{f,t}$, is the reciprocal of the conditional expectation of the SDF:

$$R_{f,t} = \frac{1}{\mathbb{E}_{z_t}\left[M_{z_{t+1},t+1}\right]} \tag{12}$$

The log risk-free rate is defined by $r_{f,t} = \log\left(R_{f,t}\right)$. We introduce constant financial leverage, and the levered excess return in log terms is defined as $r_{E,t+1} - r_{f,t} \equiv \left(r_{t+1} - r_{f,t}\right)\left(1 + B/E\right)$ where B/E is the average debt-equity ratio of the firm. Following Boldrin et al. (1995) and Croce (2010), we keep the leverage constant and assume that $B/E = 2/3$.

Ambiguity aversion can be alternatively interpreted as a distortion to the agent's subjective beliefs. To see this clearly, we can write the Euler equation as

$$0 = p_{11}\mathbb{E}_{1,t}\left[M_{z_{t+1},t+1}\left(R_{t+1} - R_{f,t}\right)\right] + \left(1 - p_{11}\right)\mathbb{E}_{2,t}\left[M_{z_{t+1},t+1}\left(R_{t+1} - R_{f,t}\right)\right]$$

for $z_t = 1$ and

$$0 = (1 - p_{22})\, \mathbb{E}_{1,t}\left[M_{z_{t+1},t+1}\left(R_{t+1} - R_{f,t}\right)\right] + p_{22}\mathbb{E}_{2,t}\left[M_{z_{t+1},t+1}\left(R_{t+1} - R_{f,t}\right)\right]$$

for $z_t = 2$. It is easy to show that the two equations above can be rewritten as the following:

$$\begin{aligned}
0 &= \tilde{p}_{11}\mathbb{E}_{1,t}\left[M^{EZ}_{z_{t+1},t+1}\left(R_{t+1} - R_{f,t}\right)\right] + (1 - \tilde{p}_{11})\, \mathbb{E}_{2,t}\left[M^{EZ}_{z_{t+1},t+1}\left(R_{t+1} - R_{f,t}\right)\right], \\
0 &= (1 - \tilde{p}_{22})\, \mathbb{E}_{1,t}\left[M^{EZ}_{z_{t+1},t+1}\left(R_{t+1} - R_{f,t}\right)\right] + \tilde{p}_{22}\mathbb{E}_{2,t}\left[M^{EZ}_{z_{t+1},t+1}\left(R_{t+1} - R_{f,t}\right)\right],
\end{aligned}$$

where $M^{EZ}_{z_{t+1},t+1}$ can be interpreted as the stochastic discount factor under Epstein-Zin recursive utility[11]:

$$M^{EZ}_{z_{t+1},t+1} = \beta\left(\frac{C_{t+1}}{C_t}\right)^{-\rho}\left(\frac{V_{z_{t+1},t+1}}{\mathcal{R}_{z_t,t}\left(V_{z_{t+1},t+1}\right)}\right)^{\rho-\gamma}.$$

Thus, $\tilde{p}_{11}$ and $\tilde{p}_{22}$ can be interpreted as the distorted transition probabilities and are given by

$$\tilde{p}_{11,t} = \frac{p_{11}}{p_{11} + (1 - p_{11})\left(\dfrac{\mathbb{E}_{2,t}\left[V^{1-\gamma}_{z_{t+1},t+1}\right]}{\mathbb{E}_{1,t}\left[V^{1-\gamma}_{z_{t+1},t+1}\right]}\right)^{-\frac{\eta-\gamma}{1-\gamma}}}, \tag{13}$$

$$\tilde{p}_{22,t} = \frac{p_{22}}{(1 - p_{22})\left(\dfrac{\mathbb{E}_{1,t}\left[V^{1-\gamma}_{z_{t+1},t+1}\right]}{\mathbb{E}_{2,t}\left[V^{1-\gamma}_{z_{t+1},t+1}\right]}\right)^{-\frac{\eta-\gamma}{1-\gamma}} + p_{22}}. \tag{14}$$

As a result, our production economy model with ambiguity aversion can be reinterpreted as a model with recursive utility augmented with the above time-varying distorted transition probabilities.[12] The distortion in beliefs is entirely driven by ambiguity aversion and endogenously determined as an equilibrium outcome. Furthermore, note that

$$\left(\mathbb{E}_{1,t}\left[V^{1-\gamma}_{z_{t+1},t+1}\right]\right)^{\frac{1}{1-\gamma}} > \left(\mathbb{E}_{2,t}\left[V^{1-\gamma}_{z_{t+1},t+1}\right]\right)^{\frac{1}{1-\gamma}},$$

[11] Of course, in the functional form of the SDF in equation (10), both consumption growth C_{t+1}/C_t and the continuation value $V_{z_{t+1},t+1}$ depend on the ambiguity aversion parameter η in an implicit and nonlinear way. As a result, the solution to the SDF $M^{EZ}_{z_{t+1},t+1}$ for $\eta > \gamma$ is not exactly identical to that obtained under recursive utility assuming $\eta = \gamma$. But as for the effect of ambiguity aversion on the SDF, our numerical analysis suggests that the last multiplicative term in equation (10) dominates over other terms. Thus, the derived distorted transition probabilities can account for most of the impact of ambiguity aversion on subjective beliefs.

[12] Cecchetti et al. (2000) study asset prices in an endowment economy with distorted beliefs. The belief distortion in their model is motivated by the difference between the agent's subjective beliefs and empirical estimates. In their model, the distortion is exogenously specified rather than endogenously determined as in our model.

since Regime-1 is assumed to be the high mean growth regime. Thus, under ambiguity aversion ($\eta > \gamma$), equation (13) implies $\tilde{p}_{11} < p_{11}$, and equation (14) implies $\tilde{p}_{22} > p_{22}$. In both cases, an ambiguity-averse agent attaches more weight on the low mean growth state than the ambiguity-neutral agent does, reflecting the pessimism induced by ambiguity aversion.

3 Calibration and Quantitative Results

In this section, we calibrate our model and assess its performance in reproducing salient features of macroeconomic quantities and asset returns observed in the data. Consistent with Bansal and Yaron (2004) and Kaltenbrunner and Lochstoer (2010), we use a long sample of U.S. annual data ranging from 1929 to 1998. Macroeconomic data (consumption and investment) are taken from the Bureau of Economic Analysis (BEA). Data on asset returns are from the Center for Research in Security Prices (CRSP). All nominal quantities are deflated using an appropriate deflator. We calibrate the model at a quarterly frequency and obtain time-aggregated annual statistics. All calibration exercises are done with respect to real and per capita empirical counterparts.

Due to nonlinearities, our model does not admit an analytical solution. We solve the model using the value function iteration algorithm with Chebyshev interpolation. After the model is solved successfully and the equilibrium allocations are found, we simulate the model to compute statistics on aggregate quantities and asset returns. We report time-aggregated annual statistics based on 20,000 Monte Carlo experiments, each with 300 quarters of simulated data. Our numerical algorithm is explained in the Appendix.

3.1 Parameter values

Table 1 reports the parameter values that are held constant throughout our calibration. Following the existing literature on business cycles and asset prices (e.g., Boldrin et al. (2001) and Tallarini (2000)), the capital share α is set to a value of 0.36, and the depreciation rate of capital δ_K is set at 0.021. The volatility of productivity growth is set at $\sigma = 0.037$, similar to that in Kaltenbrunner and Lochstoer (2010). The high and low mean growth rates μ_1 and μ_2 are set, respectively, at 0.012 and -0.029, following Cagetti et al. (2002). The two transition probabilities are set, respectively, at $p_{11} = 0.94$ and $p_{22} = 0.78$, similar to Cagetti et al. (2002). This implies that the average duration of a recession is about 4.5 quarters, and the average duration of a expansion is about 14 quarters, which is consistent with the empirical evidence documented in Rouwenhorst (1995) and Hamilton (1989). These parameter values of the productivity process implies that the stationary level of productivity growth is about 0.3%, comparable to values in Kaltenbrunner and Lochstoer (2010) and Croce (2010).[13]

[Insert Table 1 about here]

We set the coefficient of relative risk aversion to a low value of 2, following Ju and Miao (2012) and Weitzman (2007). We consider values of the IES parameter ψ greater than 1, consistent with the long-run risk literature including Ai (2010) and Ai et al. (2012), among others. Values of of the IES parameter greater than 1 are also in line with the empirical estimates obtained by Attanasio and Vissing-Jorgensen (2003) and Bansal et al. (2007). The subjective discount factor β is set to match the mean risk-free rate across all calibrations, at levels strictly less than 1. Regarding the key parameter in our model, the ambiguity aversion parameter η, there is still no consensus about its level in the empirical literature.[14] Here, we follow Ju and Miao (2012) and Chen et al. (2011) and use thought experiments to calibrate the ambiguity aversion parameter, in a similar way as how the risk aversion parameter is calibrated in the literature.[15] The Appendix contains the details of these experiments. These results show that a plausible range of values for η is between 2 and 80 depending on the size of the ambiguity premium, given that the relative risk aversion parameter is set equal to 2.[16] Finally, we choose values of the capital adjustment cost parameter ξ to be $\xi \in [4.5, 9]$ depending on different calibrations, to match the volatility of consumption growth and consumption volatility relative to output. In the case of no adjustment costs, ξ is set at an extremely large value.[17]

[13] The steady-state probability of the good regime is $(1 - p_{22})/(2 - p_{11} - p_{22})$.

[14] Anderson et al. (2009) interpret ambiguity aversion as a preference for robustness and estimate the robustness parameter to be over 1000.

[15] In the robust control framework, Anderson et al. (2003), Hansen (2007), and Hansen and Sargent (2010) advocate using detection-error probabilities to calibrate "statistical ambiguity". Their approach explicitly assumes a distortion to the probability distribution of state vector. By solving related minimization problems with relative entropy, they can explicitly characterize the worst case distortion to the process of state vector and then calculate detection-error probabilities to assess the difficulty of distinguishing alternative models. It is not straightforward to adapt this approach to our model for several reasons. First, unlike Hansen and Sargent (2010), the information structure in our model is not explicitly perturbed. Second, different from Hansen and Sargent (2010), who consider exp-log specification, our utility preferences are in power form. Third, Hansen (2007) and assume Gaussian innovations, while our adopted regime-switching model implies that the distribution of the state variable is a mixture of normals at each point of time. Thus, an explicit closed-form worst case density of productivity growth rates for our model is not available, which renders the technique of detection-error probabilities inapplicable.

[16] Ambiguity premium in the thought experiments is defined in the Appendix.

[17] Our values of ξ are substantially higher than those values in other studies, which implies much smaller adjustment costs in our model. Kaltenbrunner and Lochstoer (2010) choose $\xi = 0.7$ for their long run risk model with transitory shocks. Jermann (1998) and Boldrin et al. (2001) set ξ at 0.23.

Our model, thus, has two main ingredients: ambiguity aversion and capital adjustment costs. Therefore, we compare three different calibrations for sensitivity analysis: our preferred benchmark model with ambiguity aversion and adjustment costs, Model I with ambiguity neutrality and adjustment costs, and Model II with ambiguity aversion and no adjustment costs. By doing so, we can understand the importance of each ingredient of our model. We calibrate our benchmark model to simultaneously match consumption growth volatility ($\sigma_{\Delta c}$), the ratio between consumption growth volatility and output growth volatility ($\sigma_{\Delta c}/\sigma_{\Delta y}$) and the mean risk-free rate $\mathbb{E}(r_f)$ in the data. Model I is calibrated to produce similar values of these statistics. Our calibration is done by tuning parameter values of β, η and ξ. Model II is different, since it assumes no adjustment costs and thus lacks a mechanism to generate the same level of consumption volatility as in the long sample data. As a result, there are different parameterizations across calibrations, which are presented in Table 2.

3.2 Implications on Quantities

Panel A of Table 2 reports the annual statistics of the key macroeconomic quantities based on the long sample 1929–1998. We observe (1) a low volatility of consumption growth (2.72% per year), (2) a low consumption volatility relative to output ($\sigma_{\Delta c}/\sigma_{\Delta y} = 0.52$), (3) a high investment volatility relative to output ($\sigma_{\Delta i}/\sigma_{\Delta y} = 3.32$), (4) an average of investment-to-output (I/Y) of 0.19, (5) moderate persistence in consumption growth rates at an annual frequency, and (6) high correlations among macroeconomic aggregates. Along these dimensions, our benchmark model produces results that are largely consistent with the data. Importantly, the implied volatility of investment growth (16% per year) is quite high in comparison with other calibrations and three times higher than output volatility.

[Insert Table 2 about here]

Model I differs from the benchmark model in the degree of ambiguity aversion. Assuming ambiguity neutrality, this model features recursive utility, regime-switching growth rates and adjustment costs. Comparing Model I with the benchmark model, we find that adding ambiguity aversion to a DSGE model with recursive utility increases the volatility of investment growth to a more plausible level relative to its empirical counterpart. Our model improves upon the existing production-based asset pricing literature, for example Kaltenbrunner and Lochstoer (2010), in matching investment volatility in the data.

To understand this result, we present in Table 3 conditional moments of quantities for the two productivity regimes. A comparison between Model I and the benchmark model shows that ambiguity aversion enhances investment volatility in both regimes, and this effect is more pronounced when the current regime is the low mean growth state (investment volatility rises from 12.28% to 19.67% per year). An ambiguity-averse agent is pessimistic and attaches more weight to states with low continuation value. This pessimistic evaluation implies a high marginal product of investment, which tends to counterbalance the effect of decreasing marginal product of investment caused by adjustment costs. Thus, ambiguity aversion and adjustments costs affect investment volatility in opposite directions. Further, when the current regime has low growth rates, it is more likely that future marginal product of investment is high since the bad regime is less persistent. As a result, ambiguity aversion increases investment volatility more than in the case when the current regime is good. Model III has ambiguity aversion but no adjustment costs. As a result, the implied consumption growth is smooth, while investment growth is highly volatile (18.46% per year).

[Insert Table 3 about here]

Our model can reproduce the persistence pattern of consumption growth. The 1-year lagged autocorrelation coefficient of consumption growth rates is 0.48 in the data. Our benchmark model and Model I can generate levels of autocorrelation close to what is observed in the data. Assuming no adjustment costs, Model II generates the highest persistence level among all the calibrations. This is an outcome of the regime-switching model for growth rates, which captures persistence in productivity growth. The persistence of productivity then translates into persistence in macroeconomic quantities. This modeling feature has a similar role as the long run productivity risk component assumed by Croce (2010) and Ai et al. (2012), where both papers also generate high persistence in consumption growth. Capital adjustment costs, however, dampen the level of persistence in quantities as consumption becomes more volatile in response to rigidity in capital accumulation. Taken together, these two counterbalancing effects give rise to the persistence level in consumption close to the data, as implied by the benchmark model and Model I.

The cross-correlations among macroeconomic quantities in our benchmark calibration are consistent with the empirical regularities in the data. The contemporaneous correlation between consumption and investment growth rates is about 0.68 in the data, while a model lacking either ambiguity aversion (Model I) or adjustment costs (Model II), produces an excessively high or low correlation. Without adjustment costs, investment responds sharply to productivity fluctuations, whereas consumption reacts slowly due to endogenous consumption smoothing, resulting in a low contemporaneous correlation. In the presence of adjustment costs, it is more costly to adjust capital rapidly than do it gradually. Thus, investment and consumption tend to co-move, generating a high correlation. By increasing investment volatility, ambiguity aversion implies a correlation of consumption and investment growth rates that is closest to the data across all the calibrations.

Our benchmark model also produces an average investment-output ratio consistent with the data, while Model I and II imply counterfactually higher levels. A comparison among the calibrations suggests that both ambiguity aversion and adjustment costs are the crucial elements driving this result. Without either of the two, the average investment-to-output ratio would be too high to be reconciled with the data. To understand this result, we examine the impact of ambiguity aversion on levels of consumption and investment with and without adjustment costs. Figure 1 displays optimal (normalized) consumption and investment in equilibrium as functions of the capital stock, assuming no adjustment costs. In both productivity regimes, we observe that ambiguity aversion reduces consumption (Panel A and B) while increases investment (Panel C and D). The intuition is that in the absence of adjustment costs, the precautionary savings motive driven by ambiguity aversion tends to make the agent invest more and consume less, resulting in faster capital accumulation. This mechanism is also illustrated in Cagetti et al. (2002), where ambiguity aversion is interpreted as a preference for robustness. However, in the presence of adjustment costs, it is no longer feasible to alter production plans to smooth consumption with no costs. Taking into account capital rigidity, the ambiguity-averse agent would instead consume more and invest less than the ambiguity-neutral agent does. This mechanism is illustrated in Figure 2, which shows that capital rigidity reverse the impact of ambiguity aversion on consumption and investment. The low level of investment relative to output implied by the benchmark model gives rise to an average I/Y ratio of about 0.25, which is close to the data at 0.19.

As in a standard RBC model, all of our calibrations generate countercyclical dividends. Table 2 shows that the contemporaneous correlation between the dividend-output ratio and consumption growth is negative in our calibrations, which is consistent with the data. Note that dividends in a production economy are macroeconomic dividends, which are defined by equation (8) in our model.[18] In good states, investment tends to be high, and this makes dividends payout decrease as a result. The opposite scenario occurs in bad states. This cyclical pattern in the data is replicated in our model with regime switching productivity growth. More importantly, the countercyclical nature of dividends makes the equity claim less risky and poses a more stringent challenge to production-based models in producing a high equity premium. We illustrate below how ambiguity aversion can significantly increase the equity premium, while the countercyclical feature of dividends remains intact. Panel A of Figure 5 demonstrates this feature of our model.

3.3 Asset Pricing Implications

In this section, we discuss the implications of our model on asset returns. In particular, we show that our benchmark model featuring ambiguity aversion and adjustment costs can explain several important stylized facts about asset returns in the data, namely, a high equity premium, a low and smooth risk-free rate and predictability of excess returns. Furthermore, we show that alternative models lacking either ambiguity aversion or adjustment costs are unable to simultaneously explain the salient features of asset returns well.

The main stylized facts that general equilibrium models attempt to explain in the asset pricing literature are: (1) the equity premium puzzle of Mehra and Prescott (1985), (2) the equity volatility puzzle of Shiller (1981), (3) the risk-free rate puzzle of Weil (1989), (4) countercyclical equity premia documented by Fama and French (1989)), and (5) predictability of equity returns over long horizons documented by Fama and French (1988a,b) and others.

These puzzles pose a greater challenge to production-based asset pricing models, since consumption and dividends are endogenously determined. DSGE models aiming to solve these puzzles must not only generate macroeconomic quantities consistent with the observed regularities in the data, but also produce sufficiently high price of risk as well as high risk exposure in the equity claim. By introducing ambiguity aversion, our benchmark model can reconcile a

[18] We thank the anonymous reviewer for pointing out important features of macroeconomic dividends in the data and their implications on asset pricing in RBC models. We discuss the construction of macroeconomic dividends data in the Appendix. Macroeconomic dividends can be split into equity payout and debt payout. Jermann and Quadrini (2012) finds that for U.S. firms equity payout is procyclical and debt payout is countercyclical.

high investment volatility with a high equity premium when adjustment costs are assumed to be small. Moreover, the separation between risk aversion and intertemporal elasticity of substitution together with a high IES allow the model to produce a smooth risk-free rate. In addition, with a high IES and ambiguity aversion, the mean risk-free rate can be kept at a low level. Panel B of Table 2 summarizes the implications of our model on asset returns.

3.3.1 Risk-free rate

All the calibrations in Table 2 are able to generate a low and smooth risk-free rate. Compared to the benchmark model, Model I has ambiguity neutrality and requires a higher subjective discount factor β to match the mean risk-free rate. All else being equal, a high degree of ambiguity aversion is associated with a low mean risk-free rate since the agent worries about future states with low continuation value. In equilibrium, the risk-free bond that pays one unit of consumption good in the next period seems valuable, resulting in a low risk-free rate.

Figure 3 displays subjective beliefs and distorted beliefs about the next period being in the good regime (Regime 1), as implied by equations (13) and (14). It is clear that ambiguity aversion strongly distorts subjective beliefs in a pessimistic direction. Further, the size of the distortion is particularly large when the economy is in the good regime and the subjective beliefs indicate a high probability that the next period state continues to be good. In the bad regime (Regime 2), the distortion is smaller in magnitude. This asymmetric distortion leads to a lower mean risk-free rate in Regime 1, as shown in the third column of Panel B, Table 3. This is in contrast to Model I assuming ambiguity neutrality, which implies a higher mean risk-free rate in Regime 1 because expected consumption growth is high in the good regime.

The volatility of the risk-free rate is low in all the calibrations, with its levels even lower than in the data. This is primarily an outcome of high values of the IES parameter. In equilibrium, the risk-free rate depends on expected consumption growth through intertemporal substitution effect, see Bansal and Yaron (2004). Since agents with $\psi > 1$ are willing to substitute consumption intertemporally, time variation in expected consumption growth only creates a small volatility in the risk-free rate.

3.3.2 Price of risk and equity premium

One important tension in the existing production-based asset pricing models is how to reconcile the high equity premium with a smooth risk-free rate and the high volatility of investment observed in the data, see Kaltenbrunner and Lochstoer (2010) and Campanale et al. (2010). Our benchmark model can successfully resolve this difficulty. Table 2 shows that in our benchmark calibration the unconditional mean of the equity premium on the levered equity claim is 6.28% per year, close the data (6.33%). The key insight is that both ambiguity aversion and adjustment costs are important to generate a high equity premium. Our benchmark model is able to generate both a highly volatile stochastic discount factor, due to the introduction of ambiguity aversion, and variation in the return on the equity claim, thanks to adjustment costs.

As noted before, high capital adjustment costs lead to volatile returns on the equity claim. The existing literature relies on this mechanism to generate a sizable equity premium. In our benchmark calibration, the annual volatility of the levered returns on investment is 3.28%. This level is still far below its empirical counterpart in the data (19.42%) since in our model it is important to keep adjustment costs at a low level in order to match the high volatility of investment in the data. However, even with this low volatility of equity returns, our model can still produce a high equity premium of 6.28% per year. The Euler equation in equation (11) implies that conditional equity premium is given by (see Cochrane (2005))

$$\mathbb{E}_t\left(R_{t+1}\right) - R_{f,t} = -\frac{\sigma_t\left(M_{t+1}\right)}{\mathbb{E}_t\left(M_{t+1}\right)}\sigma_t\left(R_{t+1}\right)\rho_t\left(M_{t+1}, R_{t+1}\right).$$

Although the volatility of equity returns is not high enough in the benchmark model, ambiguity aversion can still increase equity premium by substantially magnifying the volatility of the stochastic discount factor and thus the price of risk $\sigma\left(M\right)/\mathbb{E}\left(M\right)$. In comparison with Model I, which has ambiguity neutrality and only implies a price of risk of 0.09, the benchmark model has a price of risk of 2.70. In the benchmark calibration, it is the last multiplicative term in equation (10) that imputes large variation into the stochastic discount factor when $\eta > \gamma$. A comparison between the benchmark model and Model II reveals the importance of adjustment costs in producing a high equity premium. In Model II, although the stochastic discount factor is volatile, $\sigma\left(M\right)/\mathbb{E}\left(M\right) = 2.46$, the implied equity premium is low at 1% per year. This is due to constant Tobin's Q and small variation in consumption in the absence of adjustment costs.

Thus, the risk exposure of the equity claim is extremely low.

Table 3 further reports the price of risk and the first and second moments of equity premium in different productivity regimes. For Model I, equity premium is slightly higher in the bad regime than in the good regime. By contrast, our benchmark model in Table 3 shows that under ambiguity aversion both the price of risk and equity premium are higher in the good regime than in the bad regime. The equity premium in the good regime is 7.26% per year while only 2.60% per year in the bad regime. Thus, the high unconditional equity premium obtained in our benchmark calibration is largely explained by the conditional equity premium in the good regime. A similar result is also found for the price of risk. Figure 4 further illustrates this result. This pattern of variation across regimes is driven by the asymmetric effect of ambiguity aversion on the agent's subjective beliefs in the two distinct regimes. Figure 3 shows that the size of the distortion to the subjective belief is substantially larger in the good regime than in the bad regime. In other words, the model allows more scope for the impact of ambiguity aversion when the good regime is present. High persistence of the good regime therefore magnifies the impact of pessimism induced by ambiguity aversion on the price of risk and equity premium, both of which rise sharply as a result when the current regime is good.

On the other hand, once in the bad regime, the agent holds an unfavorable belief over future states. Lower persistence of the bad regime then implies limited scope for the effect ambiguity aversion. Thus, the price of risk and equity premium rise mildly in the bad regime. Taken together, ambiguity aversion in the full information case leads to a positive relationship between equity premium and productivity regimes. This result seems to have created a tension between matching the high equity premium in the data and generating countercyclical equity premia, which is another important dynamic feature of asset returns. In the next section, we show how this difficulty is resolved by introducing an unobservable state and Bayesian learning into the model.

3.3.3 Predictability

Empirical literature documents the predictability of excess returns by several valuation variables including the price-dividend ratio (e.g., Campbell and Shiller (1988), Fama and French (1988a) and Welch and Goyal (2008)), Tobin's Q, the investment-capital ratio (e.g., Cochrane (1991)), and the *cay* variable (e.g., Lettau and Ludvigson (2001)), which is a proxy for fluctuations in the aggregate consumption-wealth ratio. In particular, empirical evidence suggests that high price-dividend ratios and investment-capital ratios tend to forecast low future excess returns, and high consumption-wealth ratios predict high future excess returns. Here, we show that our asset pricing model with production can reproduce these patterns of predictability. Consistent with the empirical literature, we examine the predictive power of several variables including the investment-capital ratio, Tobin's Q, the price-dividend ratio and consumption-wealth ratio, all of which are endogenously determined as an outcome of equilibrium allocations.

Epstein and Zin (1989) define the wealth-consumption ratio as

$$\frac{W_t}{C_t} = \frac{1}{1-\beta}\left(\frac{V_t}{C_t}\right)^{1-\frac{1}{\psi}}, \tag{15}$$

where W_t is the wealth level in period t.

Our model with regime switching growth rates implies that high current investment-capital ratios, Tobin's Q and price-dividend ratios individually predict low future excess returns, while high consumption-wealth ratios forecast high future excess returns. In good times when the mean growth rate of productivity is high, investment tends to be high and capital accumulates at a fast rate, while the consumption-wealth ratio tends to be low since the substitution effect dominates when the IES parameter ψ is high. In addition, in the presence of adjustment costs, Tobin's Q is procyclical. In the future, when productivity growth switches to the bad regime, low growth rates are realized and as a result, the return on the equity claim decreases. Conversely, in bad times, investment is low and capital accumulation is slow, while consumption tends to be high. After the economy switches to the good regime, high growth rates are realized, and favorable economic conditions result in a significant increase in the return on the equity claim, due to the precedent low capital accumulation in bad times.

[Insert Table 4 about here]

To study these predictive relationships, we regress the 1-quarter 1-, 2- 3-, and 5-year cumulative excess log returns onto investment-capital ratios, Tobin's Q, log price-dividend ratios and log consumption-wealth ratios respectively. We generate 20,000 Monte Carlo simulated data series, each with 200 quarters of data. We run predictive regressions with different regressors on each simulated data set. Table 4 reports the average slope coefficients and the R^2 statistics, for the benchmark model. We observe that when the investment-capital ratio, Tobin's Q and the price-dividend ratio are used as a regressor, the slope coefficients are all negative, suggesting their inverse relationships with future excess returns. When consumption-wealth ratios are used as a regressor, the predicted relationship with future returns becomes positive, as is consistent with the empirical evidence found by Lettau and Ludvigson (2001). The only exception is that when the horizon is 1 quarter, our benchmark model produces, albeit very small, a negative average slope coefficient of -0.0057 (Panel D). This is not a surprising result since ambiguity aversion tends to increase the current consumption under adjustment costs. In all panels of Table 4, the R^2 statistic is increasing in the horizon of returns, suggesting that the explanatory power of the regressors is higher with long-horizon returns. In particular, this feature of long-horizon predictability is pronounced in our benchmark model. Given a horizon of 5 years, the R^2 implied by the benchmark model is about 20% for all the regressors.

These trends obtained for the benchmark model are all in line with empirical results we report in the second and third columns of Table 4, under the title "Data".[19] There is one notable exception. We directly measure consumption-wealth ratio in our model based on equation (15). The most common empirical proxy for this measure in the literature is the *cay* variable in Lettau and Ludvigson (2001). We use the *cay* data in our empirical regressions.[20] However, in a comparison with the data-based results, the results generated from our benchmark model, while demonstrating the same pattern in the sign of the slope parameters and the increasing trend in R^2s, are significantly smaller in magnitude. Nevertheless, this is not detrimental to the intuition of our model regarding the predictability of returns. Our measure of the consumption-wealth ratio in the model and the empirical measure *cay* are not the same. Lettau and Ludvigson (2001) incorporate income wealth in their construction of *cay*. We abstract from evolution of labor force in our model. Thus, the estimates of the slope parameter on the consumption-wealth ratio in the model are bound to be different in magnitude from those based on the *cay* data.

4 Extension: Learning About an Unobservable State

In this section, we consider the case where the underlying state z_t is unobservable. The representative agent can update his beliefs about the hidden state using Bayes' rule. The Bayesian posterior belief is denoted by π_t and defined by $\pi_t = \Pr(z_{t+1} = 1 \mid \Omega_t)$. Suppose that the prior

[19] Details of the construction of empirical measures for I/K and Tobin's Q are provided in the Appendix.
[20] We thank Martin Lettau for providing the *cay* data on his web page.

belief π_0 is known, the posterior beliefs are updated according to Bayes' rule in the following

$$\pi_{t+1} = \frac{p_{11} f\left(\Delta a_{t+1} \mid z=1\right) \pi_t + \left(1-p_{22}\right) f\left(\Delta a_{t+1} \mid z=2\right)\left(1-\pi_t\right)}{f\left(\Delta a_{t+1} \mid z=1\right) \pi_t + f\left(\Delta a_{t+1} \mid z=2\right)\left(1-\pi_t\right)}$$

where $f\left(s|i\right) = \frac{1}{\sqrt{2\pi\sigma}} \exp\left[-\left(s-\mu_i\right)^2 / \left(2\sigma^2\right)\right]$ is the density function of the normal distribution with mean μ_i and variance σ^2.

In this case, the smooth ambiguity utility function has the following form:

$$V_t(C) = \left[(1-\beta)C_t^{1-\rho} + \beta\left\{\mathcal{R}_t\left(V_{t+1}\left(C\right)\right)\right\}^{1-\rho}\right]^{\frac{1}{1-\rho}}, \tag{16}$$

$$\mathcal{R}_t\left(V_{t+1}\left(C\right)\right) = \left(\mathbb{E}_{\pi_t}\left[\left(\mathbb{E}_{z_{t+1},t}\left[V_{t+1}^{1-\gamma}\left(C\right)\right]\right)^{\frac{1-\eta}{1-\gamma}}\right]\right)^{\frac{1}{1-\eta}}. \tag{17}$$

Compared to the utility function in equation (6) in the full information case, the crucial difference lies in the certainty equivalent. Due to non-observability of z_t, the certainty equivalent in equation (16) depends on the state belief π_t rather than being regime-dependent. In particular, the certainty equivalent can be rewritten as

$$\mathcal{R}_t\left(V_{t+1}\left(C\right)\right) = \left(\pi_t\left(\mathbb{E}_{1,t}\left[V_{t+1}^{1-\gamma}\left(C\right)\right]\right)^{\frac{1-\eta}{1-\gamma}} + \left(1-\pi_t\right)\left(\mathbb{E}_{2,t}\left[V_{t+1}^{1-\gamma}\left(C\right)\right]\right)^{\frac{1-\eta}{1-\gamma}}\right)^{\frac{1}{1-\eta}}. \tag{18}$$

Here, the notion of ambiguity aversion can be alternatively interpreted as "irreducibility of compound distributions", which corresponds to the case with $\eta > \gamma$.[21] That is, ambiguity aversion precludes the evaluation of future utility based on the predictive distribution of growth rates. To see this clearly, let us consider an ambiguity-neutral agent who follows the standard Bayesian approach. In such a case, we can first obtain predictive density from combining the posterior and conditional likelihood as

$$p\left(\Delta a_{t+1}|\Delta a^t\right) = \pi_t f\left(\Delta a_{t+1}|z=1\right) + \left(1-\pi_t\right) f\left(\Delta a_{t+1}|z=2\right)$$

and then evaluate the certainty equivalent based on this predictive distribution:

$$\mathcal{R}_t\left(V_{t+1}\right) = \mathbb{E}_t\left[V_{t+1}^{1-\gamma}\right] = \int V_{t+1}^{1-\gamma} p\left(\Delta a_{t+1}|\Delta a^t\right) d\left(\Delta a_{t+1}\right)$$

[21] See Hansen (2007) and Seo (2009) for more discussion on the relationship between irreducibility of compound distributions and the ambiguity-sensitive behavior.

where Δa^t denotes the history of productivity growth rates. However, it is obvious from equation (18) that we are unable to compute the certainty equivalent in the above mentioned way if $\eta > \gamma$. Thus, the smooth ambiguity utility model implies that the agent displays different attitudes toward state uncertainty, which is represented by time varying beliefs, and productivity risk.

The stochastic discount factor for this utility function is given by

$$M_{z_{t+1},t+1} = \beta \left(\frac{C_{t+1}}{C_t}\right)^{-\rho} \left(\frac{V_{t+1}}{\mathcal{R}_t\left(V_{t+1}\right)}\right)^{\rho-\gamma} \left(\frac{\left(\mathbb{E}_{z_{t+1},t}\left[V_{t+1}^{1-\gamma}\right]\right)^{\frac{1}{1-\gamma}}}{\mathcal{R}_t\left(V_{t+1}\right)}\right)^{-(\eta-\gamma)}. \tag{19}$$

The Euler equation is

$$\mathbb{E}_{\pi_t,t}\left[M_{z_{t+1},t+1}\left(R_{t+1}\right)\right] = 1, \tag{20}$$

and the risk-free rate $R_{f,t}$ is given by

$$R_{f,t} = \frac{1}{\mathbb{E}_{\pi_t,t}\left[M_{z_{t+1},t+1}\right]}.$$

The social planner's problem with learning about the hidden state is presented in the Appendix. In this model, the agent's subjective beliefs are characterized by endogenously filtered state beliefs, which determine the equilibrium allocations. Asset returns and their moments can be obtained in the same manner as in the full information model.

To see how ambiguity aversion alters the Bayesian posterior beliefs, we can rewrite equation (20) as

$$0 = \tilde{\pi}_t \mathbb{E}_{1,t}\left[M_{z_{t+1},t+1}^{EZ}\left(R_{t+1} - R_{f,t}\right)\right] + \left(1-\tilde{\pi}_t\right)\mathbb{E}_{2,t}\left[M_{z_{t+1},t+1}^{EZ}\left(R_{t+1} - R_{f,t}\right)\right],$$

where $M_{z_{t+1},t+1}^{EZ}$ accommodates a similar interpretation as in the full information case and is given by

$$M_{z_{t+1},t+1}^{EZ} = \beta \left(\frac{C_{t+1}}{C_t}\right)^{-\rho} \left(\frac{V_{z_{t+1},t+1}}{\mathcal{R}_t\left(V_{z_{t+1},t+1}\right)}\right)^{\rho-\gamma}.$$

It is straightforward to show that the probability $\tilde{\pi}_t$ satisfies

$$\tilde{\pi}_t = \frac{\pi_t\left(\mathbb{E}_{z_{t+1}=1,t}\left[V_{t+1}^{1-\gamma}\right]\right)^{-\frac{\eta-\gamma}{1-\gamma}}}{\pi_t\left(\mathbb{E}_{z_{t+1}=1,t}\left[V_{t+1}^{1-\gamma}\right]\right)^{-\frac{\eta-\gamma}{1-\gamma}} + \left(1-\pi_t\right)\left(\mathbb{E}_{z_{t+1}=2,t}\left[V_{t+1}^{1-\gamma}\right]\right)^{-\frac{\eta-\gamma}{1-\gamma}}}. \tag{21}$$

We can interpret $\tilde{\pi}_t$ as distorted beliefs, which in no case are equivalent to the Bayesian posterior beliefs.[22] The distorted belief depends not only on the Bayesian belief but also on expectations of continuation value. Since our numerical results show that $\left(\mathbb{E}_{1,t}\left[V_{t+1}^{1-\gamma}\right]\right)^{\frac{1}{1-\gamma}} > \left(\mathbb{E}_{2,t}\left[V_{t+1}^{1-\gamma}\right]\right)^{\frac{1}{1-\gamma}}$, it is straightforward to see $\tilde{\pi}_t < \pi_t$ under ambiguity aversion ($\eta > \gamma$).

4.1 Further Results

Since we allow for an unobservable state and Bayesian learning, the implications on quantities and asset returns differ from those in our full information model. To do sensitivity analysis, we describe several calibrations in what follows.

Table 5 displays the results of three calibration exercises. Model III is calibrated to match the volatility of consumption growth, the relative volatility between consumption and output, the mean risk-free rate and equity premium in the data. Compared with the benchmark model in Table 2, Model III implies a smaller investment volatility but a higher correlation between consumption and investment. Relative to the benchmark model, our model with Bayesian learning requires a higher level of adjustment costs to replicate the unconditional mean of equity premium in the data. When the agent learns about the hidden state, the filtering process will generate highly persistent state beliefs. The high persistence in beliefs makes the return on the equity claim less risky than in the full information case as Bayesian updating smooths out the effect of jumps in productivity regimes. Thus, all else being equal, the level of adjustment costs must be increased in order to produce sufficiently volatile equity returns and to match the level of equity premium in our benchmark model. This leads to a larger standard deviation of levered equity returns of 5.33% per year, which is the highest among all the calibrations in this paper. On the other hand, high adjustment costs make investment growth less volatile and consumption growth more volatile. Since consumption and investment tend to co-move with productivity shocks, the correlation between their growth rates implied by Model III is high.

[Insert Table 5 about here]

Model IV assume ambiguity neutrality ($\eta = \gamma = 2$), but has the same level of adjustment costs as Model III ($\xi = 4.5$). Not surprisingly, the model fails to generate a high price of risk

[22] As in the full information case, the distorted belief derived here may not be taken as a change of measure to account for all the impact of ambiguity aversion on the perceived distribution of productivity growth, due to our production economy setting and the separation between risk aversion and the intertemporal elasticity of substitution.

($\sigma(M)/\mathbb{E}(M) = 0.09$) and thus a sizable equity premium ($\mathbb{E}(r_{ep}) = 0.31\%$). Without adjustment costs, it is still difficult to generate a high equity premium even after ambiguity aversion is taken into account. This result is illustrated by the calibration results of Model V in Table 5, where the implied unconditional mean of equity premium is only 0.36%. Model V features Bayesian learning, recursive utility and ambiguity aversion. This model is similar to the model examined by Cagetti et al. (2002), where they study the impact of robustness concerns on the dynamics of risk-return trade-off for agents with power utility. Model V is more general than their model in that we further allow for the separation between risk aversion and the intertemporal elasticity of substitution. There is also an important difference between the information structure of our model and that of Cagetti et al. (2002). Cagetti et al. (2002) assume that agents treat state estimates originating from the nonlinear filter for their regime-switching model like any other observable state variable, and thus agents do not distort the current period state probabilities. Instead, the dynamic evolution equation for the state vector is distorted explicitly. However, in our model with Bayesian learning, the current period state probabilities are distorted in an implicity way.

As in the full information case, we find that both ambiguity aversion and adjustment costs are required to generate reasonable levels of statistics on macroeconomic and financial quantities. Capital rigidity generates sufficient risks in the equity claim and ambiguity aversion induces endogenous pessimism and increases the price of risk. Also as in full information model, we find that the negative correlation between consumption growth and dividend-output ratio, $Corr(\Delta c, D/Y)$, is present for all the three calibrations with learning. Panel B of Figure 5 demonstrates this relationship.

[**Insert Figure 5 about here**]

4.2 Beliefs and asset prices dynamics

In our model with Bayesian learning, the dynamics of Bayesian-updated beliefs are a key determinant of asset returns. In this section, we examine the relationship between state beliefs and conditional moments of asset returns. Our analysis not only sheds light on the mechanism driving our results on unconditional moments of asset returns, but also enhances our understanding of the dynamic behavior of asset-return moments over business cycles. Specifically, we show that the model with Bayesian learning can account for countercyclical variation in equity premium, another important dynamic asset pricing phenomenon that standard DSGE models have difficulty to explain.

We start by investigating the quantitative impact of ambiguity aversion on Bayesian beliefs. Figure 6 displays simulated Bayesian beliefs, π_t, and distorted beliefs, $\tilde{\pi}_t$, implied by a degree of ambiguity aversion of $\eta = 60$. The plotted Bayesian beliefs (dashed line) indicate that a Bayesian agent finds it relatively easy to distinguish the two productivity regimes. The Bayesian belief π_t stays near 0.9 in most of the time, while occasional jumps in the underlying state drive the belief toward low levels. The implied distorted beliefs depicted in Figure 6 suggest that an ambiguity averse agent slants his beliefs towards the low productivity regime, which is associated with low continuation value, as is revealed by the expression (21). In the language of Hansen and Sargent (2010), this endogenous pessimistic distortion leads to time varying price of uncertainty and carries a sizable uncertainty premium.[23]

[Insert Figure 6 about here]

We perform comparative statics experiments to study the effect of ambiguity aversion on the dynamics of asset returns by varying the parameter η, given the parameterization of Model III in Table 5. When agents update beliefs about the hidden state, conditional moments of asset returns depend on the state belief. Moreover, these relationships crucially hinge on the degree of ambiguity aversion. Panel A of Figure 7 shows the conditional price of risk, $\sigma_t(M)/\mathbb{E}_t(M)$, as a function of the state belief π_t for three different values of η ($\eta = 2, 35$ and 60), where capital is set at its long run mean in the simulation. Likewise, Panel B, C and D depict, respectively, the conditional risk-free rate, conditional equity premium and conditional volatility of excess returns as functions of π_t. Under ambiguity neutrality, the price of risk displays a mildly humped shape, where its maximum is attained around $\pi_t = 0.5$. The plotted conditional equity premium and conditional volatility of excess returns exhibit a similar pattern. These results are in line with equilibrium models with Bayesian learning, for example see Veronesi (1999). The intuition is that when agents are most uncertain about the estimate of the unobservable state, which normally occurs for π_t being around 0.5, the price of risk is high, resulting in a high conditional equity premium and volatility of excess returns.

[23] Due to complexity of our problem, it is impossible to derive the uncertainty premium on the equity claim in analytical form.

[**Insert Figure 7 about here**]

Introducing ambiguity aversion into the model alters these results in an important way. It is obvious from Panel A of Figure 7 that ambiguity aversion increases the conditional price of risk for all possible state beliefs, due to the large variation of the stochastic discount factor induced by ambiguity aversion. Further, the magnitude of the increment is the largest when the state belief is near its stationary level. In general, a Bayesian agent takes high levels of the belief π_t as representing an optimistic view about future states. However, favorable beliefs can accommodate a large impact of ambiguity aversion as there is an ample room allowed for ambiguity-aversion-driven distortion to Bayesian beliefs. Figure 6 illustrates this mechanism: high levels of the Bayesian belief π_t tend to be associated with large ambiguity-aversion-driven distortions. In Figure 7, when the state belief is close to 1, the conditional price of risk decreases sharply because uncertainty about the estimate of the unobservable state is low. Together with adjustment costs, ambiguity aversion leads to the relationship between conditional equity premium and the state belief as observed in Panel C.

Panel B of Figure 7 shows that the conditional risk-free rate is increasing in the state belief. When a Bayesian agent believes that future growth rates are more likely to be in the good regime, expected consumption growth is higher. Due to the intertemporal substitution effect, the risk-free rate in equilibrium is high. Ambiguity aversion, however, introduces additional precautionary savings motives. The risk-free bond paying one unit of consumption good in the next period therefore seems more valuable, leading to low risk-free rates for all levels of the state belief. This effect is particularly strong when the belief is around its stationary level and the allowed scope for the ambiguity-aversion-driven distortion is large. Thus, the conditional risk-free rate as a function of the state belief becomes more convex under ambiguity aversion. The convex curvature may potentially create large variation in the risk-free rate. But thanks to the high intertemporal elasticity of substitution, the volatility of the risk-free rate is still low in our model.

Our model with Bayesian learning explains the dynamic behavior of equity premium and the volatility of excess returns observed in the data. Since, due to its high persistence, the high-growth regime is dominant, the distribution of the state belief, π_t, is highly skewed toward 1. Thus, in good times when π_t is close to 1, uncertainty about the estimate of the unobservable state is low, leading to a low equity premium. In bad times, productivity shocks deteriorate the state belief through Bayesian updating. The increased state uncertainty together with ambiguity aversion magnifies conditional equity premium, as is obvious from Panel C of Figure 7. The impact of state uncertainty on the conditional volatility of excess returns follows a similar pattern. This mechanism explain the ability of our model to generate countercyclical equity premia and volatilities of excess returns.

To further investigate the cyclical variation of conditional moments of asset returns, we simulate aggregate consumption growth rates, under the parameterization of Model III in Table 5. In addition, we compute both conditional equity premium and conditional volatility of excess returns given the simulated state variables. Results are depicted in Figure 8. For Model III with $\eta = 60$, Panel A of Figure 8 demonstrates that the correlation between consumption growth and predicted equity premium is -0.18.[24] This level is close to the empirical estimate reported by Ju and Miao (2012) in their endowment economy model. Except for deep recessions when consumption growth falls to substantially low levels, equity premium moves countercyclically.[25] In addition, throughout the simulation period the conditional volatility of excess returns moves in the opposite direction of consumption growth (the coefficient of correlation is about -0.80).

[Insert Figure 8 about here]

5 Conclusion

In this paper, we have studied asset pricing and business cycle implications of a real business cycle model that features regime switching productivity growth and an ambiguity-averse representative agent. Our model can explain well the stylized facts on macroeconomic quantities and asset prices observed in the data. In particular, ambiguity aversion imputes endogenous pessimism into the model and helps to explain the equity premium puzzle while generating a low risk-free rate, even when the degree of risk aversion is low. Our model with a high intertemporal elasticity of substitution also implies smooth risk-free rates. Assuming small capital adjustment costs, the model can match the high volatility of investment growth observed in the data. Our model also generates predictability of excess returns by investment-capital ratios, Tobin's Q, price-dividend

[24] For moderately lower values of η, our numerical simulation shows that conditional equity premium exhibits a much stronger countercyclical pattern. For example, when η is 35 and other parameters remain the same as in Model III, the correlation between consumption growth and the predicted equity premium is as high as -0.62.

[25] The co-movement of consumption growth and conditional equity premium in severe recessions arises because the state belief declines substantially and this may lead to a low conditional equity premium, as shown in Panel C of Figure 7.

ratios, and consumption-wealth ratios, consistent with the empirical literature. When we assume that the underlying state of productivity regimes is unobservable and the agent can engage in Bayesian updating, our model can further account for countercyclical variation in the price of risk and equity premium. Nevertheless, the volatility of equity returns implied by our model is still rather low compared with the data, due to modest risk exposure of the equity claim inherent in the model.

Future research could extend this work in several directions. First, although we have assumed a representative-agent economy, it would be interesting to study a heterogenous-agent economy where agents possess different ambiguity attitudes. Heterogeneity in the degree of ambiguity aversion might generate asset pricing implications that are fundamentally different from those in a representative-agent model. Second, features of learning and ambiguity could also be introduced into general equilibrium models to study the cross section of expected returns (e.g., value premium). For example, in a model with production and growth opportunities, ambiguity may affect the exercising decision of growth options. Finally, we could estimate the model proposed in this paper using structural estimation methods and time-series data on macroeconomic aggregates and asset returns. The estimation results may shed new light on the nature of ambiguity aversion.

6 Appendix

6.1 Numerical algorithm

We solve the model numerically, using the value function iteration algorithm with Chebyshev interpolation as in Kaltenbrunner and Lochstoer (2010).[26] We define the following stationary variables:

$$\left\{\hat{C}_t, \hat{I}_t, \hat{Y}_t, \hat{K}_t, \hat{V}_t\right\} = \left\{\frac{C_t}{A_t}, \frac{I_t}{A_t}, \frac{Y_t}{A_t}, \frac{K_t}{A_t}, \frac{V_t}{A_t}\right\}$$

The social planner's problem in the full information model is given by

$$\hat{V}_t\left(\hat{K}_t, z_t\right) = \max_{\hat{C}_t, \hat{I}_t}\left\{(1-\beta)\,\hat{C}_t^{1-\rho} + \beta\left(\mathbb{E}_{z_t}\left[\left(\mathbb{E}_{z_{t+1},t}\left[\hat{V}_{t+1}^{1-\gamma}\left(\hat{K}_{t+1}, z_{t+1}\right)\left(\frac{A_{t+1}}{A_t}\right)^{1-\gamma}\right]\right)^{\frac{1-\eta}{1-\gamma}}\right]\right)^{\frac{1-\rho}{1-\eta}}\right\}^{\frac{1}{1-\rho}}$$

subject to the following constraints:

$$\begin{aligned}
\hat{C}_t + \hat{I}_t &= \hat{Y}_t \equiv \hat{K}_t^{\alpha}\bar{N}^{1-\alpha} && (22)\\
e^{\Delta a_{t+1}}\hat{K}_{t+1} &= (1-\delta_k)\,\hat{K}_t + \phi\left(\frac{\hat{I}_t}{\hat{K}_t}\right)\hat{K}_t && (23)\\
\Delta a_t &= \mu\left(z_t\right) + \sigma\epsilon_t, \quad \epsilon_t \sim N\left(0,1\right) && (24)\\
\hat{C}_t &\geq 0, \hat{K}_{t+1} \geq 0 && (25)
\end{aligned}$$

The social planner's problem in the incomplete information model is

$$\hat{V}_t\left(\hat{K}_t, \pi_t\right) = \max_{\hat{C}_t, \hat{I}_t}\left\{(1-\beta)\,\hat{C}_t^{1-\rho} + \beta\left(\mathbb{E}_{\pi_t}\left[\left(\mathbb{E}_{\pi_{t+1},t}\left[\hat{V}_{t+1}^{1-\gamma}\left(\hat{K}_{t+1}, \pi_{t+1}\right)\left(\frac{A_{t+1}}{A_t}\right)^{1-\gamma}\right]\right)^{\frac{1-\eta}{1-\gamma}}\right]\right)^{\frac{1-\rho}{1-\eta}}\right\}^{\frac{1}{1-\rho}}$$

where π_{t+1} is updated from π_t according to Bayes rule.

Since the capital adjustment cost function $\phi(\cdot)$ is not defined for negative investment, we restrict $\hat{I}_t > 0$ in the numerical procedure by taking log values on the following variables:

$$\left\{\hat{c}_t, \hat{i}_t, \hat{y}_t, \hat{k}_t, \hat{v}_t\right\} = \left\{\log\left(\hat{C}_t\right), \log\left(\hat{I}_t\right), \log\left(\hat{Y}_t\right), \log\left(\hat{K}_t\right), \log\left(\hat{V}_t\right)\right\}$$

[26] The code is written in Compaq Visual FORTRAN 6.6 and available upon request from the authors.

The maximization problems can be rewritten as

$$\hat{v}_t\left(\hat{k}_t, z_t\right) = \frac{1}{1-\rho}\max_{\hat{c}_t,\hat{\imath}_t}\log\left\{(1-\beta)\,e^{(1-\rho)\hat{c}_t} + \beta\left(\mathbb{E}_{z_t}\left[\left(\mathbb{E}_{z_{t+1},t}\left[e^{(1-\gamma)\left[\hat{v}_{t+1}\left(\hat{k}_{t+1},z_{t+1}\right)+\Delta a_{t+1}\right]}\right]\right)^{\frac{1-\eta}{1-\gamma}}\right]\right)^{\frac{1-\rho}{1-\eta}}\right\} \tag{26}$$

and

$$\hat{v}_t\left(\hat{k}_t, \pi_t\right) = \frac{1}{1-\rho}\max_{\hat{c}_t,\hat{\imath}_t}\log\left\{(1-\beta)\,e^{(1-\rho)\hat{c}_t} + \beta\left(\mathbb{E}_{\pi_t}\left[\left(\mathbb{E}_{\pi_{t+1},t}\left[e^{(1-\gamma)\left[\hat{v}_{t+1}\left(\hat{k}_{t+1},\pi_{t+1}\right)+\Delta a_{t+1}\right]}\right]\right)^{\frac{1-\eta}{1-\gamma}}\right]\right)^{\frac{1-\rho}{1-\eta}}\right\} \tag{27}$$

Solving the full information model

The numerical solution to the maximization problem (26) involves solving two value states, $\hat{v}_t\left(\hat{k}_t, 1\right)$ and $\hat{v}_t\left(\hat{k}_t, 2\right)$. We use a 5th order Chebyshev polynomial to approximate $\hat{v}_t\left(\hat{k}_t, 1\right)$ and $\hat{v}_t\left(\hat{k}_t, 2\right)$ on a Chebyshev grid with 20 collocation points for the state variable $\hat{k}$. Assume that the value function is $\left\{G_n\left(\hat{k}, 1\right), G_n\left(\hat{k}, 2\right)\right\}$ at nth iteration. We use a numerical optimizer (Fortran IMSL routine "DUVMIF") to find the optimal policy $\left\{\hat{\imath}^*\left(\hat{k}, 1\right), \hat{\imath}^*\left(\hat{k}, 2\right)\right\}$ that respectively maximizes

$$\hat{v}^*\left(\hat{k}, 1\right) = \frac{1}{1-\rho}\max_{\hat{\imath}_t}\log\left\{(1-\beta)\,e^{(1-\rho)\hat{c}} + \beta\left(\mathbb{E}_{z=1}\left[\left(\mathbb{E}_{z'}\left[e^{(1-\gamma)\left[G_n\left(\hat{k}',z'\right)+\Delta a'\right]}\right]\right)^{\frac{1-\eta}{1-\gamma}}\right]\right)^{\frac{1-\rho}{1-\eta}}\right\}$$

and

$$\hat{v}^*\left(\hat{k}, 2\right) = \frac{1}{1-\rho}\max_{\hat{\imath}_t}\log\left\{(1-\beta)\,e^{(1-\rho)\hat{c}} + \beta\left(\mathbb{E}_{z=2}\left[\left(\mathbb{E}_{z'}\left[e^{(1-\gamma)\left[G_n\left(\hat{k}',z'\right)+\Delta a'\right]}\right]\right)^{\frac{1-\eta}{1-\gamma}}\right]\right)^{\frac{1-\rho}{1-\eta}}\right\}$$

To compute $\mathbb{E}_{z_t=1}\left[\left(\mathbb{E}_{z_{t+1},t}\left[e^{(1-\gamma)\left[G_n\left(\hat{k}_{t+1},z_{t+1}\right)+\Delta a_{t+1}\right]}\right]\right)^{\frac{1-\eta}{1-\gamma}}\right]$, notice

$$\begin{aligned}\mathbb{E}_{z=1}\left[\left(\mathbb{E}_{z'}\left[e^{(1-\gamma)\left[G_n\left(\hat{k}',z'\right)+\Delta a'\right]}\right]\right)^{\frac{1-\eta}{1-\gamma}}\right] &= p_{11}\left(\mathbb{E}_{z'=1}\left[e^{(1-\gamma)\left[G_n\left(\hat{k}',z'\right)+\Delta a'\right]}\right]\right)^{\frac{1-\eta}{1-\gamma}} \\ &\quad + (1-p_{11})\left(\mathbb{E}_{z'=2}\left[e^{(1-\gamma)\left[G_n\left(\hat{k}',z'\right)+\Delta a'\right]}\right]\right)^{\frac{1-\eta}{1-\gamma}}\end{aligned}$$

$$\begin{aligned}\mathbb{E}_{z=2}\left[\left(\mathbb{E}_{z'}\left[e^{(1-\gamma)\left[G_n\left(\hat{k}',z'\right)+\Delta a'\right]}\right]\right)^{\frac{1-\eta}{1-\gamma}}\right] &= (1-p_{22})\left(\mathbb{E}_{z'=1}\left[e^{(1-\gamma)\left[G_n\left(\hat{k}',z'\right)+\Delta a'\right]}\right]\right)^{\frac{1-\eta}{1-\gamma}} \\ &\quad + p_{22}\left(\mathbb{E}_{z'=2}\left[e^{(1-\gamma)\left[G_n\left(\hat{k}',z'\right)+\Delta a'\right]}\right]\right)^{\frac{1-\eta}{1-\gamma}}\end{aligned}$$

where we use Gauss-Hermite quadrature with 5 nodes to approximate the two conditional expectations. At the end of nth iteration, we use the new set of value function $\left\{\hat{v}^*\left(\hat{k},1\right),\hat{v}^*\left(\hat{k},2\right)\right\}$ to update Chebyshev coefficients and obtain $\left\{G_{n+1}\left(\hat{k},1\right),G_{n+1}\left(\hat{k},0\right)\right\}$.

Solving the Bayesian learning model

To solve the maximization problem (27), we use a 5th×5th order Chebyshev product polynomials over a 20×10 Chebyshev grid for the state variables $\left(\hat{k}_t,\pi_t\right)$. Assume that the value function is $G_{n+1}\left(\hat{k},\pi\right)$ at nth iteration. We numerically solve the recursion

$$\hat{v}^*\left(\hat{k},\pi\right)=\frac{1}{1-\rho}\max_{\hat{\imath}_t}\log\left\{(1-\beta)\,e^{(1-\rho)\hat{c}}+\beta\left(\mathbb{E}_\pi\left[\left(\mathbb{E}_{\pi'}\left[e^{(1-\gamma)\left[G_n\left(\hat{k}',\pi'\right)+\Delta a'\right]}\right]\right)^{\frac{1-\eta}{1-\gamma}}\right]\right)^{\frac{1-\rho}{1-\eta}}\right\}$$

where Gauss-Hermite quadrature with 5 nodes is used to approximate the conditional expectations, and π' is updated according to Bayes' rule. At the end of nth iteration, we update Chebyshev coefficients and obtain $G_{n+1}\left(\hat{k},\pi\right)$.

6.2 The size of the ambiguity aversion parameter

We rely on thought experiments similar to Ellsberg Paradox to elucidate values of the ambiguity aversion parameter η used in the calibration exercises of this paper. Our approach closely follows Ju and Miao (2012) and Chen et al. (2011).[27]

The classic example of Ellsberg Paradox is the static two urns case. Suppose that there are two urns filled with black and white marbles. Subjects are told that one urn has 50 white and 50 black marbles. The second urn contains 100 marbles, either white or black. The exact composition of the second urn is unknown to the subjects. Subjects are asked to place a bet on the color of the ball drawn from each urn. The bet could be on either black or white. Subjects win a prize worth d dollars if a bet on a specific urn is correct, otherwise they do not win or lose anything. Halevy (2007) reports that the majority of subjects prefer a bet on the first urn over the second urn.

The standard expected utility framework fails to explain this behavior, regardless of the level of risk aversion or beliefs held by the subjects. The certainty equivalents of a bet are identical

[27] See Halevy (2007) for further details on using thought experiments to calibrate the ambiguity aversion parameter. Anderson et al. (2009) report empirical estimates of an ambiguity aversion parameter that ranges between -2.922 to 1,540.556. We find this range for the degree of ambiguity aversion to be too large, and hence, follow Ju and Miao (2012) in calibrating the ambiguity aversion parameter.

for the two urns when subjects have expected utility. Ambiguity aversion, however, can create a difference between the certainty equivalents over the two urns, which gives rise to the ambiguity premium. As in Ju and Miao (2012), the ambiguity premium is formally defined as

$$u^{-1}\left(\int_{\Theta}\int_{S} u(c)d\theta d\zeta\left(\theta\right)\right)-v^{-1}\left(\int_{\Theta} v\left(u^{-1}\left(\int_{S} u\left(c\right)d\theta\right)\right)d\zeta\left(\theta\right)\right). \tag{28}$$

In our model, the functional forms of u and v are given by

$$\begin{aligned} u\left(c\right) &= \frac{c^{1-\gamma}}{1-\gamma}, \gamma>0, \neq 1 \\ v\left(x\right) &= \frac{x^{1-\eta}}{1-\eta}, \eta>0, \neq 1 \end{aligned}$$

Following Ju and Miao (2012), we specify the set of probability probability distributions for the bet on the second urn as (0,1) and (1,0) ($\Theta=\{(0,1),(1,0)\}$), and the subjective prior as $\zeta=(0.5,0.5)$. Let w be the subject's wealth level. Then the ambiguity premium is defined as

$$\left(0.5(d+w)^{1-\gamma}+0.5w^{1-\gamma}\right)^{\frac{1}{1-\gamma}}-\left(0.5(d+w)^{1-\eta}+0.5w^{1-\eta}\right)^{\frac{1}{1-\eta}}. \tag{29}$$

for $\eta>\gamma$. Obviously, the size of this premium naturally depends on the size of the bet or the prize-wealth ratio d/w. Table 6 presents the ambiguity premium, expressed as a percentage of the expected value of the bet $d/2$, for various parameter values of η, where the risk aversion parameter γ is set at 2. We consider three different prize-wealth ratios, $d/w=1\%$ (Panel A), $d/w=0.75\%$ (Panel B) and $d/w=0.5\%$ (Panel C). The size of the ambiguity premium depends on the prize-wealth ratio. All else being equal, a smaller bet is associated with a lower ambiguity premium. Researchers often use small bets in experimental studies. In the light of previous experimental evidence presented by Camerer (1999) and Halevy (2007), the ambiguity premium is typically about 10-20 percent of the expected value of a bet. Table 6 shows that the implied ambiguity premium for various values of the ambiguity aversion parameter η is consistent with the experimental results documented in the literature. In our calibration analysis, the maximal value of the parameter η is 60, which is reasonable based on the results of Table 6.

6.3 Construction of I/K and Tobin's Q

- Investment-capital ratio: We follow Cochrane (1991) and use the ratio of investment to capital stock at time t as a predictor for k-step ahead excess returns. In the context of our model, the law of motion of capital includes capital adjustment costs. Since data on the size of capital stock is sparse, we construct the I_t/K_t ratio based on a recursion. The construction of this recursion is as follows:

$$\begin{aligned}
K_{t+1} &= (1-\delta_K)K_t + \phi\left(\frac{I_t}{K_t}\right)K_t, \\
\frac{K_{t+1}}{I_t} &= (1-\delta_K)\frac{K_t}{I_t} + \phi\left(\frac{I_t}{K_t}\right)\frac{K_t}{I_t}, \\
\frac{K_{t+1}}{I_{t+1}} &= (1-\delta_K)\frac{K_t}{I_t} + \phi\left(\frac{I_t}{K_t}\right)\frac{I_t}{I_{t+1}}, \\
\frac{K_{t+1}}{I_{t+1}} &= \underbrace{(1-\delta_K)\frac{K_t}{I_t} + \phi\left(\frac{I_t}{K_t}\right)\frac{1}{1+\triangle i}}_{A}, \\
\frac{I_{t+1}}{K_{t+1}} &= \frac{1}{A},
\end{aligned}$$

  where $\triangle i$ is the growth rate of investment. To initiate this recursion, we impose $K_{1929} = I_{1929}/\delta_K$, following Ai et al. (2012). In the construction of this data, we experimented with the capital adjustment costs function in Campanale et al. (2010) and also with no adjustment costs. When running the predictive regressions of the form $r_{ep,t\to t+h} = \alpha + \beta I_t/K_t + \varepsilon_t$, the empirical results are robust to the choice of functional form of adjustment costs used for construction of I_t/K_t. When we ignore adjustment costs, estimated values and standard errors of the estimated intercept and slope parameters, $\hat{\alpha}$ and $\hat{\beta}$, are somewhat larger in comparison with the cases with adjustment costs, but these changes are not large enough to affect statistical significance or inferences made. Thus, we only report empirical results based on Jermann-type capital adjustment costs.

- Tobin's Q: To the best of our knowledge, there is no consensus on what is the best empirical measure of Tobin's Q. Based on the data available on FRED II data bank maintained by the Federal Reserve Bank of St. Louis, we build a proxy that performs well empirically. Our measure is the ratio of market value of outstanding equity (net worth of equity) of non-farm and non-financial corporate businesses, to the total net worth, based on balance sheets of non-farm and non-financial corporate businesses at the end of each quarter for 1949-2010 period. We aggregate the values to construct annual Tobin's Qs.

- Macroeconomic dividends: Let Y_t be the output, I_t investment, N_t the number of labor hours and w_t the aggregate wages. In a standard RBC model, aggregate macroeconomic dividends can be defined as:

$$D_t = Y_t - I_t - w_t N_t.$$

Divide by Y_t to get:

$$\frac{D_t}{Y_t} = 1 - S_t^I - S_t^N,$$

where S_t^I and S_t^N are share of investment to output and labor income share of output, respectively. In our model, in absence of government expenditure and international trade, S_t^I is equal to $I_t/(C_t + I_t)$.

Aggregate macroeconomic dividends and financial dividends differ in important dimensions. Macroeconomic dividends are the source of interest payments for corporate bonds and also equity payouts. Furthermore, equity payouts need to be split between stock repurchases and financial dividends.

We compute the ratio of aggregate macroeconomic dividends to output in the data. Bureau of Labor Statistics series ID PRS85006173 is an index of labor share of income, based on 2005 values. We follow Gomme and Rupert (2007) and set labor income share of output for 2005 equal to 0.72.

Table 1: **Calibration values of model parameters.**

Parameter	Description	Value
γ	Coefficient of risk aversion	2.00
α	Capital share	0.36
δ	Depreciation rate of capital	0.021
μ_1	Mean growth rate (regime 1)	0.0114
μ_2	Mean growth rate (regime 2)	-0.029
σ	Standard deviation	0.037
p_{11}	Transition probability (regime 1 to regime 1)	0.94
p_{22}	Transition probability (regime 2 to regime 2)	0.78

Table 2: **Full information model: Calibration results**

Statistic	U.S. Data	Benchmark $\beta = 0.989$, $\eta = 60$ $\psi = 2$, $\xi = 9$	Model I $\beta = 0.998$, $\eta = 2$ $\psi = 2$, $\xi = 9$	Model II $\beta = 0.992$, $\eta = 50$ $\psi = 1.5$, $\xi = \infty$
Panel A: Macroeconomic moments				
$\sigma_{\Delta c}$ (%)	2.72	2.72	2.53	1.95
$\sigma_{\Delta i}$ (%)	17.36	15.90	11.00	18.46
$\sigma_{\Delta c}/\sigma_{\Delta y}$	0.52	0.52	0.48	0.37
$\sigma_{\Delta i}/\sigma_{\Delta y}$	3.32	3.00	2.08	3.51
$\rho(\Delta c_t, \Delta c_{t+1})$	0.48	0.56	0.55	0.74
$\rho(\Delta c, \Delta i)$	0.68	0.63	0.85	0.28
$\rho(\Delta y, \Delta i)$	0.89	0.92	0.98	0.90
$\rho(\Delta c, \Delta y)$	0.92	0.86	0.93	0.60
$\mathbb{E}(I/Y)$	0.19	0.25	0.35	0.34
$\rho(\Delta c, D/Y)$	-0.18	-0.62	-0.53	-0.74
Adj. cost/output (%)	n.a.	0.16	0.16	0.00
Panel B: Financial moments				
$\mathbb{E}(r_f)$ (%)	0.86	0.86	0.86	0.82
$\sigma(r_f)$ (%)	0.97	0.60	0.66	0.43
$\mathbb{E}(r_{ep})$ (%)	6.33	6.28	0.25	1.07
$\sigma(r_{ep})$ (%)	19.42	3.28	2.13	0.68
$\sigma(M)/\mathbb{E}(M)$	n.a.	2.70	0.09	2.46

This table reports annualized statistics on macroeconomic quantities and asset returns from calibrating our full information model with different parameterizations. Empirical moments are taken from Kaltenbrunner and Lochstoer (2010) and are computed using US annual data from 1929 to 1998, except for $\rho(\Delta c, D/Y)$ where we use 1947-2011 data. $\sigma_{\Delta x}$ denotes the standard deviation of growth rates of x (in log units). $\rho(\Delta x, \Delta y)$ denotes the contemporaneous correlation of growth rates of x and y. $\rho(\Delta c_t, \Delta c_{t+1})$ denotes the first-order autocorrelation in consumption growth at an annual frequency. $\rho(\Delta c, D/Y)$ denotes correlation between consumption growth and the ratio of macroeconomic dividends to output. $\mathbb{E}(r_{ep})$ measures the unconditional mean of equity premium and is defined by $\mathbb{E}\left(r_{E,t+1} - r_{f,t}\right)$, where the equity claim is a levered claim on aggregate payouts and the leverage ratio is 2/3. $\sigma(r_{ep})$ measures the standard deviation of excess returns and is defined by $\sigma\left(r_{E,t+1} - r_{f,t}\right)$. $\sigma(M)/\mathbb{E}(M)$ denotes the unconditional price of risk. Results are generated from 20,000 simulations each with 200 quarters of simulated data.

Table 3: **Full information model: Conditional moments**

Statistic		Benchmark $\beta = 0.989$, $\eta = 60$ $\psi = 2$, $\xi = 9$	Model I $\beta = 0.998$, $\eta = 2$ $\psi = 2$, $\xi = 9$	Model II $\beta = 0.992$, $\eta = 50$ $\psi = 1.5$, $\xi = \infty$
Panel A: Conditional macroeconomic moments				
$\sigma_{\Delta c}$ (%)	regime 1	2.28	2.34	1.77
	regime 2	2.44	2.47	1.76
$\sigma_{\Delta i}$ (%)	regime 1	14.39	10.01	14.47
	regime 2	19.67	12.28	27.43
$\rho(\Delta c, \Delta i)$	regime 1	0.69	0.90	0.26
	regime 2	0.59	0.86	-0.13
$\rho(\Delta c, \Delta y)$	regime 1	0.85	0.96	0.58
	regime 2	0.77	0.94	0.10
$E(I/Y)$	regime 1	0.26	0.37	0.35
	regime 2	0.21	0.32	0.28
Panel C: Conditional financial moments				
$\mathbb{E}(r_f)$ (%)	regime 1	0.76	1.39	0.73
	regime 2	1.27	-1.09	1.15
$\sigma(r_f)$ (%)	regime 1	0.61	0.42	0.31
	regime 2	0.52	0.45	0.70
$\mathbb{E}(r_{ep})$ (%)	regime 1	7.26	0.25	1.55
	regime 2	2.60	0.28	-0.69
$\sigma(M)/\mathbb{E}(M)$	regime 1	3.03	0.08	2.76
	regime 2	0.52	0.10	0.51
$\sigma(r_{ep})$ (%)	regime 1	2.92	2.04	0.44
	regime 2	3.85	2.45	0.68

This table reports conditional moments on macroeconomic quantities and asset returns for the three calibrated models presented in Table 3. The statistics are computed conditional on the two productivity growth regimes respectively (regime 1 “good”, and regime 2 “bad”).

Table 4: **Long-horizon predictability: Full information model**

	Data		Benchmark	
Horizon(s)	Slope	R^2	Slope	R^2
Panel A: Investment-Capital Ratio (I/K)				
1 Quarter	-0.1563	0.0112	-0.1626	0.0060
1 Year	-1.3800	0.0385	-1.1233	0.0386
2 Years	-1.7644	0.1317	-2.7387	0.0951
3 Years	-2.1142	0.1645	-4.2668	0.1495
5 Years	-5.5868	0.2112	-6.7007	0.2378
Panel B: Tobin's Q				
1 Quarter	-0.0431	0.0429	-0.0331	0.0064
1 Year	-0.4047	0.0788	-0.2266	0.0410
2 Years	-0.7077	0.1267	-0.5453	0.1003
3 Years	-1.0060	0.1754	-0.8410	0.1565
5 Years	-1.0225	0.1925	-1.3026	0.2460
Panel C: Price-Dividend Ratio ($p - d$)				
1 Quarter	-0.0453	0.0123	-0.0015	0.0038
1 Year	-0.1201	0.0832	-0.0132	0.0284
2 Years	-0.3123	0.1418	-0.0343	0.0759
3 Years	-0.3872	0.2005	-0.0548	0.1230
5 Years	-0.5004	0.3208	-0.0875	0.1996
Panel D: Consumption-Wealth Ratio ($c - w$)				
1 Quarter	0.2608	0.0168	-0.0057	0.0033
1 Year	2.0226	0.0821	0.0927	0.0170
2 Years	3.5671	0.1258	0.3206	0.0614
3 Years	4.0023	0.1632	0.5507	0.1120
5 Years	6.6811	0.1892	0.9244	0.1989

This table reports predictive regression results for the benchmark model, with its parameterization given in Table 2. We obtain the regression slopes and R^2s from regressing annualized excess returns (in log units) onto the end-of-year investment-capital ratio, Tobin's Q, log price-dividend ratio and consumption-wealth ratio. The results are computed by taking the average of OLS estimates over 20,000 simulated samples. The horizon of returns includes 1 quarter, 1 year, 2, 3 and 5 years. We obtain the data-based slopes and R^2s from OLS fitting of $r_{ex,t\rightarrow t+h} = a + bx_t + \varepsilon_t$, where h is the horizon of returns, and x_t is one of the predictors discussed above. We use CRSP data and the methodology in Appendix A of Campanale et al. (2010) to construct data on the price-dividend ratio. We use the variable, *cay*, constructed by Lettau and Ludvigson (2001) as an empirical proxy for the consumption-wealth ratio defined in our model. Details of the construction of empirical measures for I/K and Tobin's Q are included in the Appendix.

Table 5: **Bayesian learning model: Calibration results**

Statistic	U.S. Data	Model III $\beta = 0.989$, $\eta = 60$ $\psi = 2$, $\xi = 4.5$	Model IV $\beta = 0.998$, $\eta = 2$ $\psi = 2$, $\xi = 4.5$	Model V $\beta = 0.994$, $\eta = 50$ $\psi = 1.5$, $\xi = \infty$
Panel A: Macroeconomic moments				
$\sigma_{\Delta c}$ (%)	2.72	2.72	2.99	2.03
$\sigma_{\Delta i}$ (%)	17.36	13.77	9.87	19.55
$\sigma_{\Delta c}/\sigma_{\Delta y}$	0.52	0.52	0.56	0.39
$\sigma_{\Delta i}/\sigma_{\Delta y}$	3.32	2.63	1.87	3.73
$\rho(\Delta c_t, \Delta c_{t+1})$	0.48	0.54	0.46	0.75
$\rho(\Delta c, \Delta i)$	0.68	0.85	0.94	0.08
$\rho(\Delta y, \Delta i)$	0.89	0.97	0.99	0.78
$\rho(\Delta c, \Delta y)$	0.92	0.94	0.98	0.52
$\mathbb{E}(I/Y)$	0.19	0.26	0.35	0.35
$\rho(\Delta c, D/Y)$	-0.18	-0.62	-0.47	-0.65
Adj. cost/output (%)	n.a.	0.26	0.28	0.00
Panel B: Financial moments				
$\mathbb{E}(r_f)$ (%)	0.86	0.86	0.86	0.87
$\sigma(r_f)$ (%)	0.97	0.56	0.59	0.50
$\mathbb{E}(r_{ep})$ (%)	6.33	6.31	0.59	0.36
$\sigma(r_{ep})$ (%)	19.42	5.33	3.82	0.53
$\sigma(M)/\mathbb{E}(M)$	n.a.	1.54	0.09	1.32

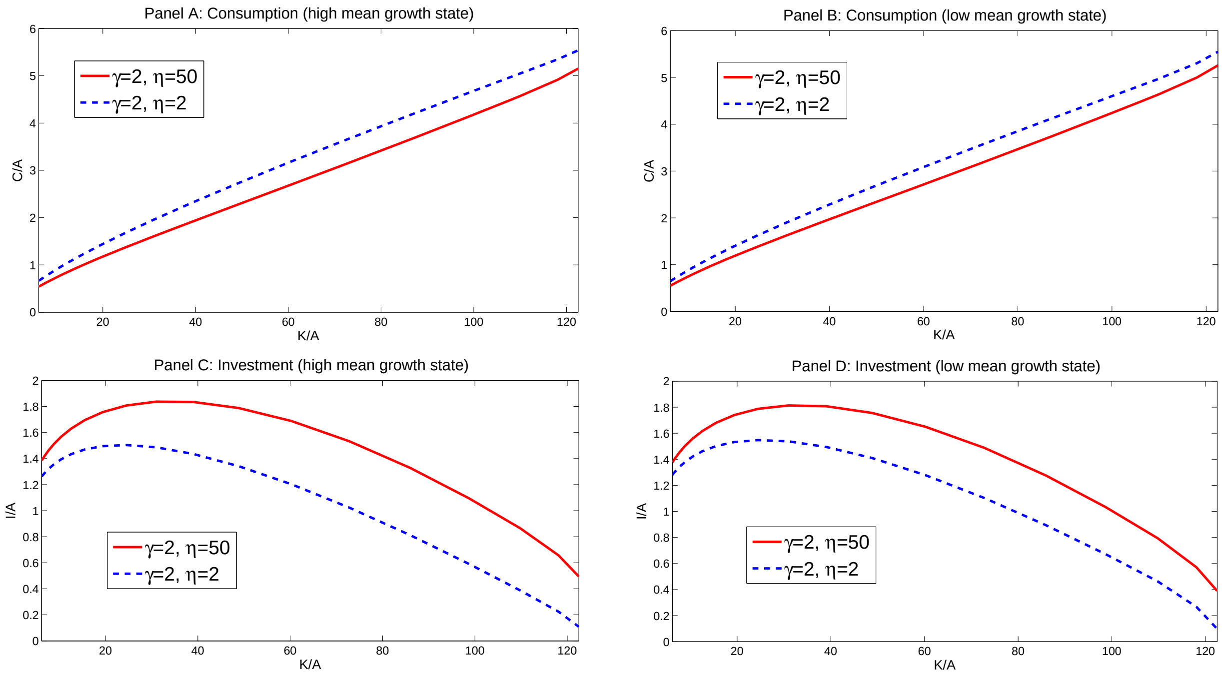
This table reports annualized statistics on macroeconomic quantities and asset returns from calibrating our model presented in Section 4 with different parameterizations (Model III–V). Results are generated from 20,000 simulations each with 200 quarters of simulated data. The quantities are the same as those reported on Table 2.

Table 6: **Ambiguity premium**

$\gamma \diagdown \eta$	20	30	40	50	60	80
Panel A: Prize-wealth ratio (d/w)=1%						
2	4.47	6.94	9.39	11.82	14.21	18.91
Panel B: Prize-wealth ratio (d/w)=0.75%						
2	3.36	5.22	7.07	8.91	10.74	14.36
Panel C: Prize-wealth ratio (d/w)=0.50%						
2	2.22	3.49	5.97	4.73	7.20	9.66

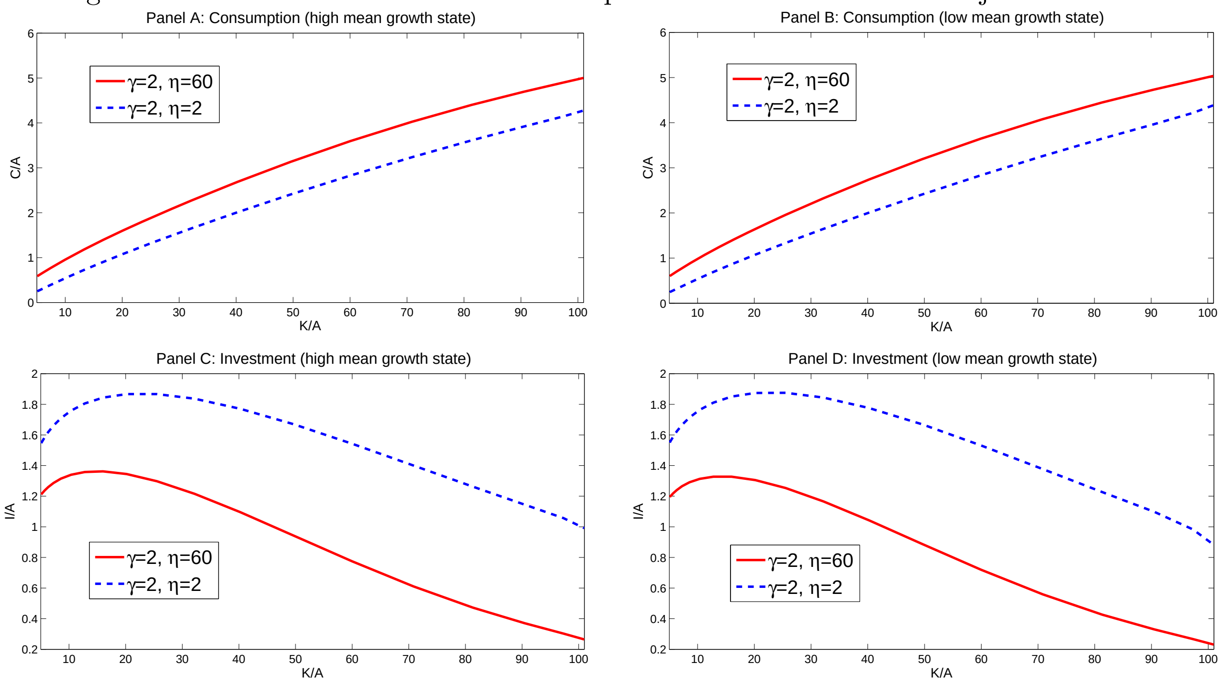
This table reports the ambiguity premium, expressed as a percentage of the expected value of the bet $d/2$, for various values of η. The risk aversion parameter γ is set at 2.

Figure 1: Full information model: Consumption and investment without adjustment costs



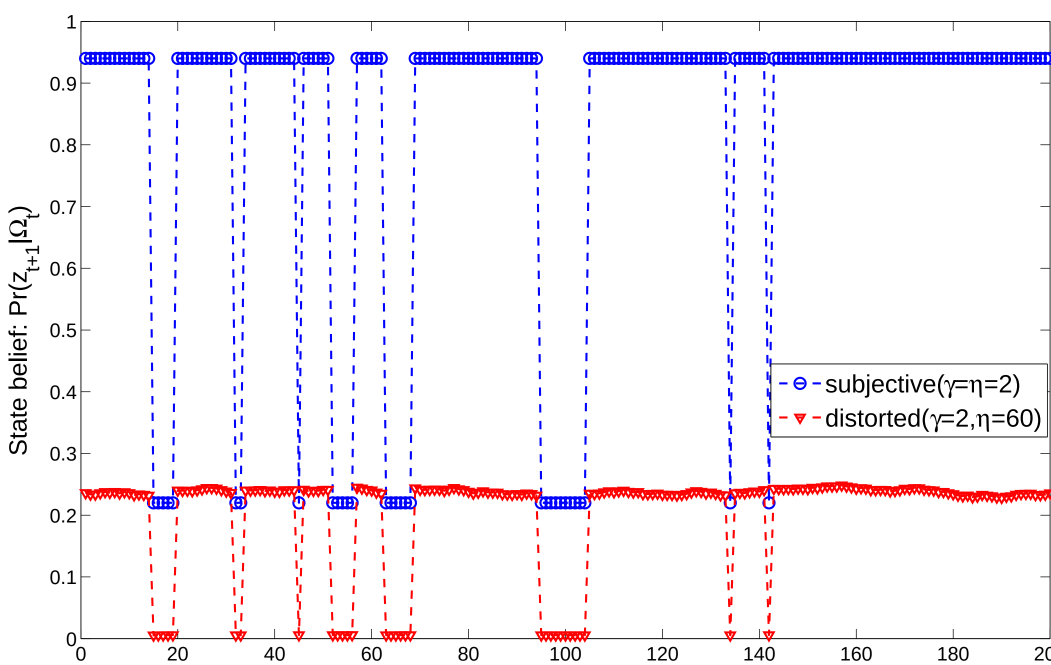
This figure plots (normalized) consumption (C/A) and investment (I/A) as functions of (normalized) capital (K/A) assuming no adjustment costs. The comparative statics results are computed for two cases: ambiguity neutrality ($\eta = \gamma = 2$ and ambiguity aversion $\eta = 50, \gamma = 2$). Other parameter values are the same as in Model II, Table 2.

Figure 2: Full information model: Consumption and investment with adjustment costs



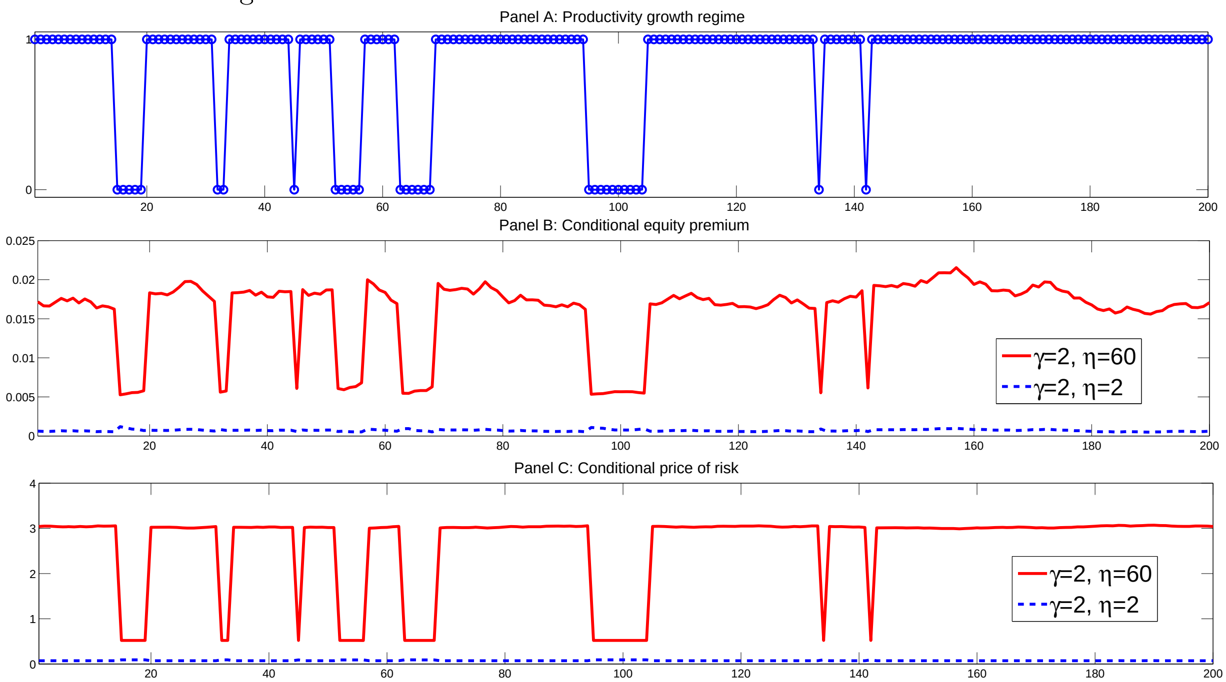
This figure plots (normalized) consumption (C/A) and investment (I/A) as functions of (normalized) capital (K/A) assuming adjustment costs. The comparative statics results are computed for two cases: ambiguity neutrality ($\eta = \gamma = 2$ and ambiguity aversion $\eta = 60, \gamma = 2$). Other parameter values are the same as in the benchmark calibration in Table 2.

Figure 3: Full information model: Subjective and distorted beliefs



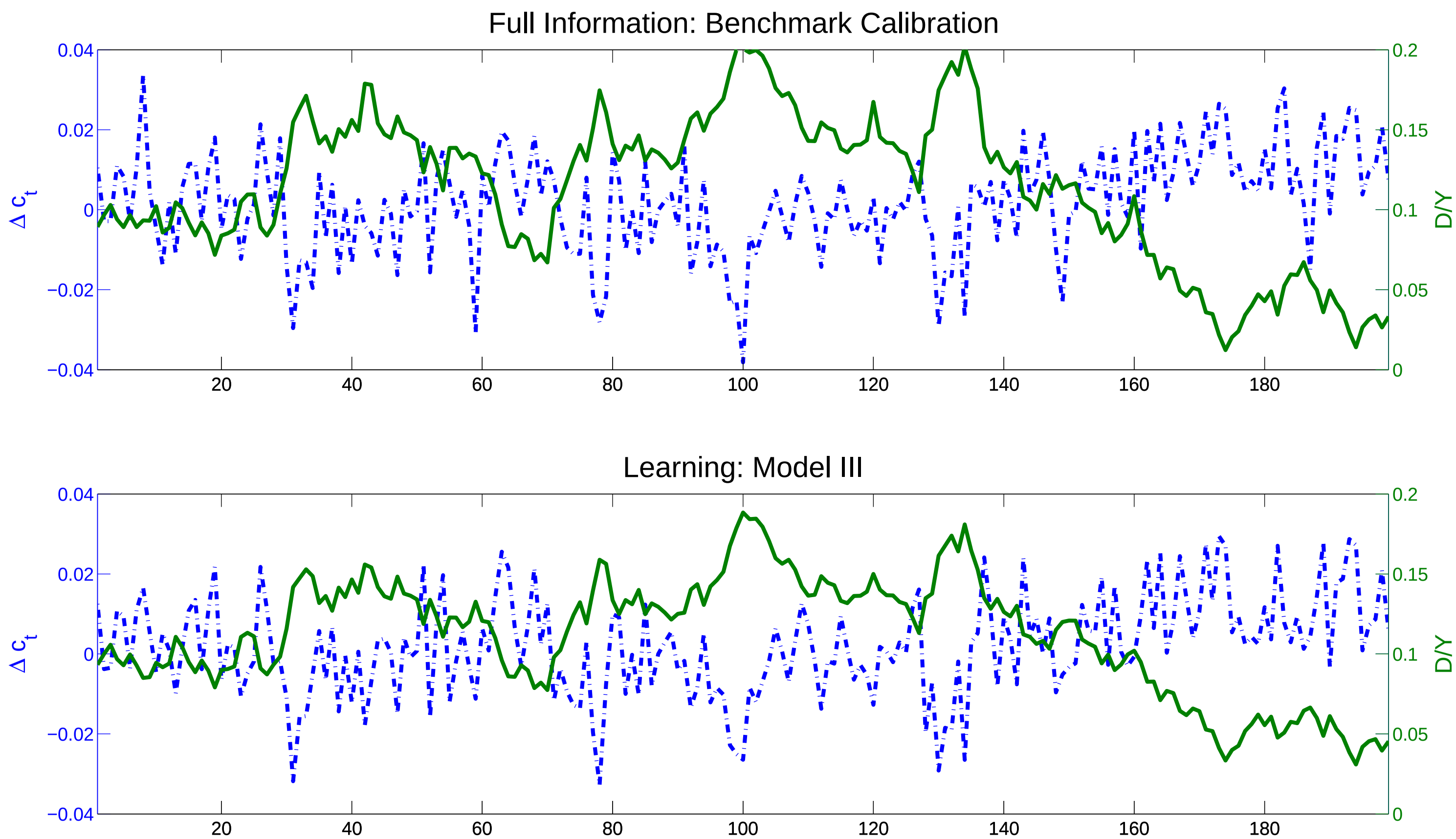
This figure plots simulated beliefs (subjective and distorted) for the full information model. The distorted beliefs are computed based on the parameterization of the benchmark calibration in Table 2.

Figure 4: Full information model: Simulated financial moments



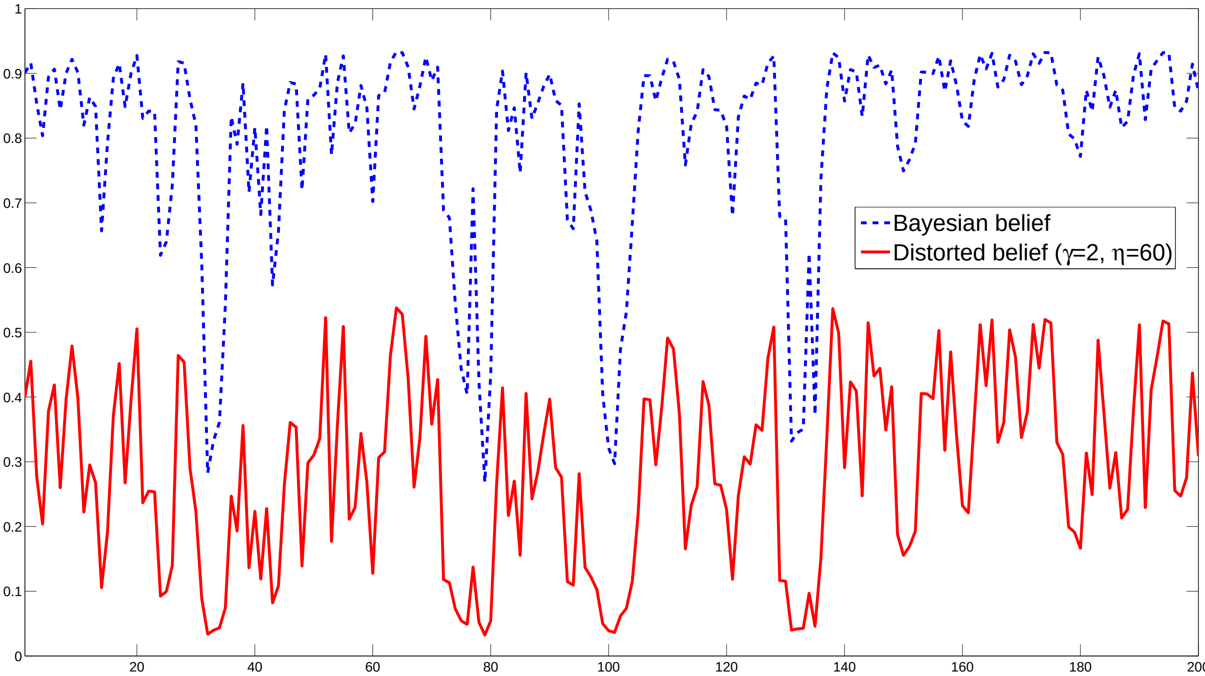
This figure plots simulated productivity growth regimes (Panel A), conditional equity premium (Panel B) and conditional price of risk (Panel C). The comparative statics results are computed for two cases: ambiguity neutrality ($\eta = \gamma = 2$) and ambiguity aversion ($\eta = 60, \gamma = 2$). Other parameter values are the same as in the benchmark calibration in Table 2.

Figure 5: Consumption Growth and Dividend-Output Ratio



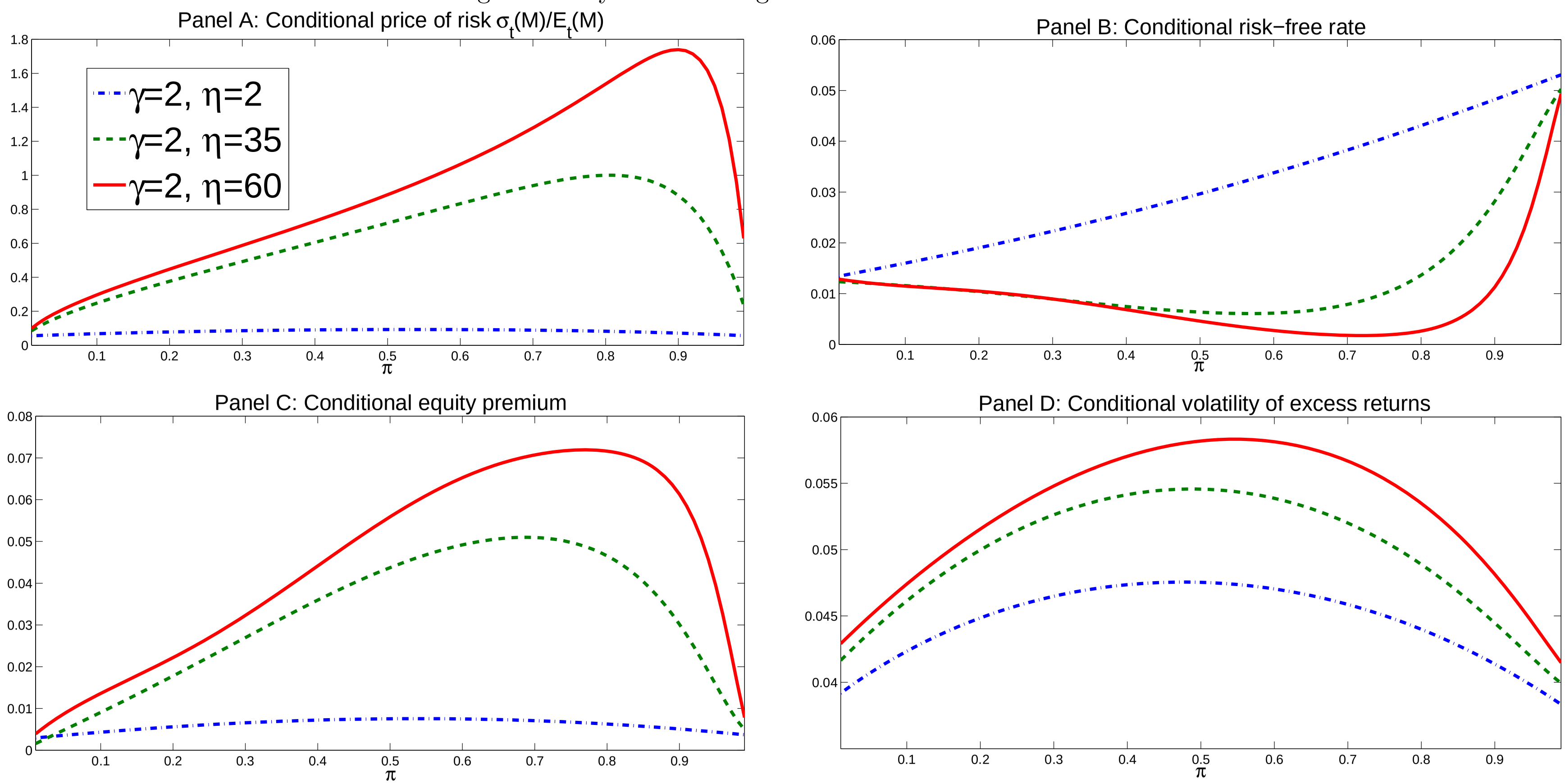
Panel A plots simulated consumption growth (dashed line) and dividend-output ratio (solid line) for the full information model. Parameter values are the same as in the benchmark calibration in Table 2. Panel B plots simulated consumption growth (dashed line) and dividend-output ratio (solid line) for the model with a hidden state and learning. Parameter values are the same as in Model III in Table 5. The model is simulated for 300 quarters, where the last 200 quarters of the simulated data are plotted.

Figure 6: Bayesian learning model: Bayesian and distorted beliefs



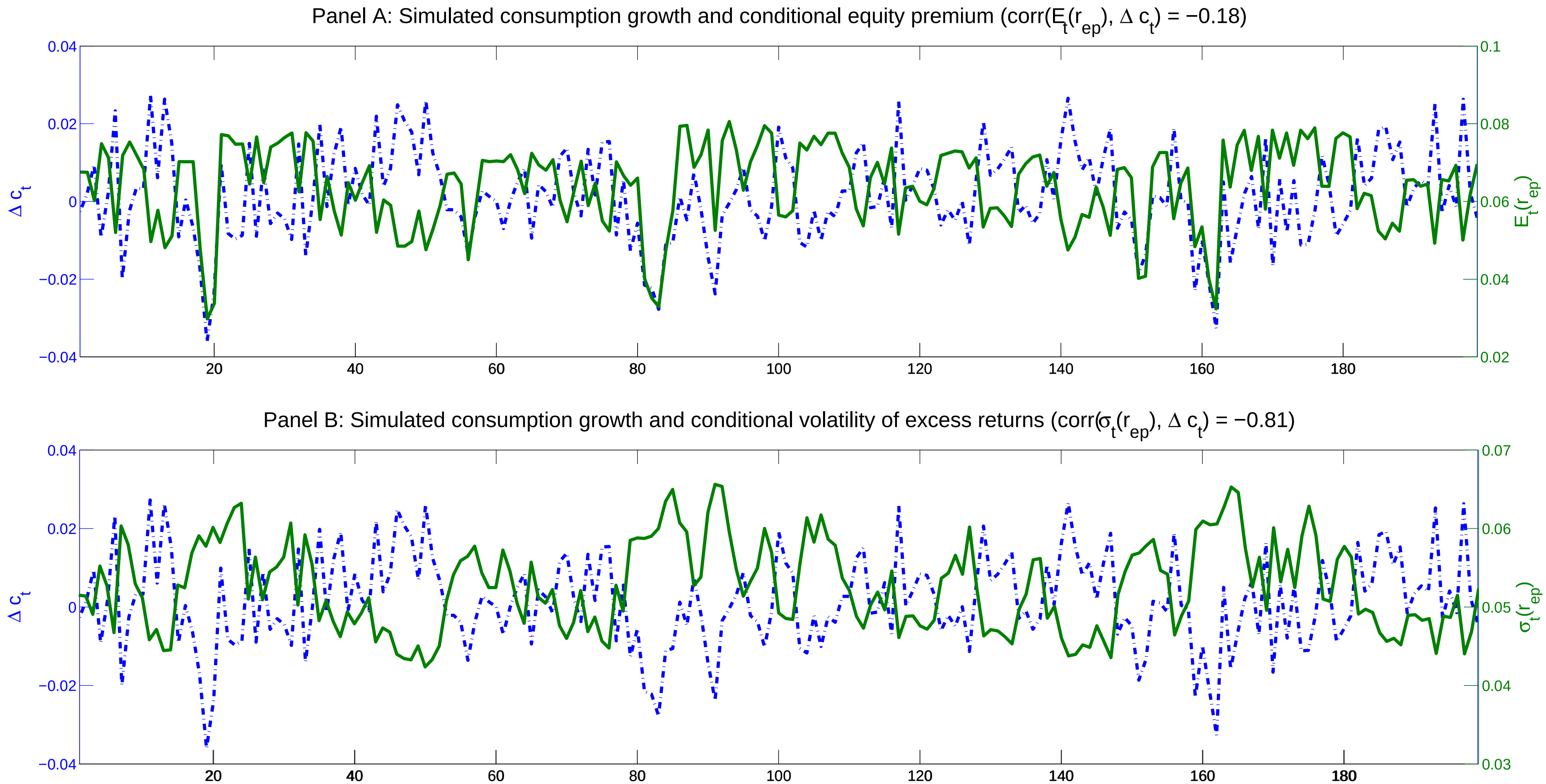
This figure plots simulated Bayesian state beliefs and distorted beliefs for the model with learning. Bayesian beliefs, π_t, are computed using Bayes' rule, given simulated productivity growth rates. The distorted beliefs, $\tilde{\pi}_t$, are computed under the parameterization of Model III in Table 5.

Figure 7: Bayesian learning model: Conditional moments



This figure plots the conditional price of risk (Panel A), the conditional risk-free rate (Panel B), conditional equity premium (Panel C) and the conditional volatility of excess returns (Panel D) as functions of the state belief π_t. The comparative statics results are computed for (1) $(\eta = \gamma = 2)$, (2) $(\eta = 35, \gamma = 2)$ and (3) $(\eta = 60, \gamma = 2)$. Other parameter values are the same as in Model III in Table 5.

Figure 8: Bayesian learning model: Simulation



Panel A plots simulated consumption growth (dashed line) and conditional equity premium (solid line). Panel B plots simulated consumption growth (dashed line) and the conditional volatility of excess returns (solid line). The parameterization is given in Model III in Table 5. The model is simulated for 300 quarters, where the last 200 quarters of the simulated data are plotted. Panel A shows that the correlation between consumption growth and the predicted equity premium is -0.18. Panel B shows that the correlation between consumption growth and the predicted volatility of excess returns is -0.81.

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