Asset Prices in Affine Real Business Cycle Models*

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Abstract

I describe a tractable way to study macroeconomic quantities and asset prices in a large class of dynamic stochastic general equilibrium models. The proposed approximate solution is analytical, log-linear, and adjusted for risk. Therefore, it is well suited to investigate economic mechanisms, describe the time series properties or estimate the model, and deal with stochastic volatility. I explain the pitfalls encountered by previous attempts to use simple approximation techniques, in particular with models featuring recursive preferences. Finally, I show the theoretical relationship between my solution and higher-order perturbation methods.

Keywords: approximation methods, adjustment for risk, recursive preferences, stochastic volatility, asset prices.

JEL Classification: C63, G12, E32.

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1 Introduction

Recursive preferences and time variation in means and volatilities have become important features of consumption-based asset pricing literature. Introducing these features in the real business cycle framework allowed researchers to study the joint behavior of real and financial variables along the business cycle. Because the analysis of asset prices requires computing risk adjustments, simple log-linearization is inadequate. Furthermore, numerical methods such as the value-function iteration are computationally expensive and ill-suited for problems with a large number of state variables.

In this paper I propose a simple alternative. For a large class of models of interest I describe how to compute risk adjustments that accurately characterize asset pricing implications, while retaining the simplicity of log-linearization. My approach exploits the exponential-affine and jointly normal structure of shocks.

I see three main advantages to my method. First, the linear structure of the solution makes it easy to describe the time-series properties of the variables of interest and carry out estimation. Second, the analytical form allows the researcher to inspect the mechanisms behind quantity dynamics and asset prices. Finally, the proposed method can deal with stochastic volatility much more easily compared to other techniques. In particular, with standard perturbation methods at least a third order expansion is typically required to capture the first order dynamic effect of changes in volatility. See for instance Fernandez-Villaverde, Guerron-Quintana, Rubio-Ramirez, and Uribe (2011). The log-linearized solution proposed in this paper directly accounts for those effects.

Methods combining linearization and log-normal risk adjustment similar to the one presented in this paper have been applied heuristically in finance literature. See for example Jermann (1998), Lettau (2003) and Backus, Routledge, and Zin (2007). My
contribution is to describe the general method, formally show its relation to perturbation algorithms, and discuss some of the pitfalls encountered by the previous attempts to use linearization and log-normality, notably in the case of recursive preferences.

My approach can be applied to a large class of models that can feature such elements as Epstein-Zin-Weil (see Epstein and Zin (1989), Epstein and Zin (1991) and Weil (1989)) preferences, habit formation, capital adjustment costs etc. Models with recursive preferences are of particular interest. First, a growing strand in the literature relies on the separation between relative risk aversion and elasticity of inter-temporal substitution to address asset pricing puzzles within the real business cycle framework. See Tallarini (2000), Kaltenbrunner and Lochstoer (2010), Rudebusch and Swanson (2012), and Croce (2013) to mention just a few contributions. Second, solving these models involves using additional equilibrium conditions and computing additional risk adjustments related to the value function. This is because the value function enters directly in the Euler equation. This is done mechanically when this paper’s method is applied, but had appeared somewhat problematic in the previous attempts to use simple approximation techniques with Epstein-Zin-Weil utility. For instance, Backus, Routledge, and Zin (2007) consider a model nested within my general framework (and similar to one of the examples I discuss). I will argue that their choice of equilibrium conditions to linearize leads to an inconsistent approximation of the value function, which is of crucial importance for models with recursive preferences. Uhlig (2010) considers a similar model. The author follows Lettau (2003) to first find the log-linearized dynamics of quantities and then computes risk adjustments from the asset pricing equations only. I show that it is important to consistently compute the risk adjustments resulting from all the forward-looking equations in the model. When preferences are of Epstein-Zin-Weil form, certainty equivalent of
future utility enters the first order conditions. Ignoring the effect of risk on the certainty equivalent results in a significant bias in variables such as the risk free rate.

I illustrate the performance of my method on three examples. The first one is based on Kaltenbrunner and Lochstoer (2010) model of endogenous long-run risks. I use this example to discuss various issues pertaining to recursive preferences. In the second example I assume habit formation and high capital adjustment costs. Finally, the third example introduces a highly persistent stochastic volatility process. For each model I compare key moments obtained using different solution methods. I also implement the Den Haan and Marcet (1994) accuracy of approximation test, and this both for the benchmark and more extreme calibrations. My approach produces accurate solutions for a wide range of parameters, it also compares well with more complex solution methods.

The approach proposed in his paper is related to higher-order perturbation methods, where equilibrium conditions are expanded around a steady state using Taylor series. I show that my solution is equivalent to a perturbation solution where some higher-order terms describing the dynamics of the quantities are omitted while the key terms accounting for risk adjustments and first order effects of volatility shocks are retained. As a result, standard perturbation software can be used to implement my method by simply setting some terms to 0. I show on several examples of interest that the two ways of finding approximate solutions are almost identical numerically.\footnote{I use the perturbation methods framework developed by Schmitt-Grohe and Uribe (2004) and Dynare ++ software.} In this respect, my work is close to Swanson, Anderson, and Levin (2006), van Binsbergen, Fernández-Villaverde, Koijen, and Rubio-Ramírez (2008), Rudebusch and Swanson (2012), and Caldara, Fernandez-Villaverde, Rubio-Ramirez, and Yao (2012), who all address the specific issue of solving models with recursive preferences and stochastic volatility using perturbation methods.
My approach is an easy and tractable substitute for the more complex techniques considered in those papers.

Benigno, Benigno, and Nistico (2010) describe how to adapt Schmitt-Grohe and Uribe (2004) perturbation framework in a way that captures linear dynamic effects of changes in volatility with a second order approximation (as opposed to third order required in a standard perturbation solution). In contrast, I show how those effects can be accurately obtained in the first-order solution.

The affine dynamics of exogenous state variables is central to my solution method. I build on the work by Duffie, Pan, and Singleton (2000), who introduced continuous time affine processes as a powerful modeling tool that allows finding closed-form solutions for a number of problems in finance and economics. For instance, related work by Eraker (2008) and Eraker and Shaliastovich (2008) shows how to solve an endowment economy model where consumption growth is driven by a vector of exogenous state variables that has affine dynamics.

The rest of the paper is organized as follows. Section 2 presents the method and discussed its relation to perturbation. Section 3 presents three applications. Finally, Section 4 concludes.

2Note that my analysis is in discrete time. For the discrete time counterpart to Duffie, Pan, and Singleton (2000) see, for example, Le, Singleton, and Dai (2010).
2 Solution Method

2.1 Method

The set of equilibrium conditions for a number of DSGE models of interest can be written as

\[ E_t \exp (\Theta \hat{x}_t + \Theta_+ \hat{x}_{t+1}) = 1, \]  
\[ F (\hat{x}_t, \hat{x}_{t+1}, \Sigma_t \epsilon_{t+1}) = 0. \]

The vector \( \hat{x}_t \) contains the deviations from non-stochastic steady state of all the variables,\(^3\) both pre-determined and forward-looking. Expectational equations take a special exponential-affine form (1). \( \Theta \) and \( \Theta_+ \) are conforming matrices of coefficients. Finally, innovations are jointly normal \( \epsilon_{t+1} \sim N(0, I_{n_e}) \) and following Du\textit{c}e, Pan, and Singleton (2000) the conditional co-variance matrix of shocks is affine\(^4\) in the variables of the model:\(^5\)

\[ \left( \Sigma_t \Sigma_t^\top \right)_{ij} = (G)_{ij} + (G_x)_{ij} \hat{x}_t. \]

A linear solution to the model

\[ \hat{x}_{t+1} = H + H_x \hat{x}_t + H \Sigma_t \epsilon_{t+1} \]

would imply that \( \hat{x}_t \) is a vector of conditionally jointly normal variables. Making this conjecture and using the properties of log-normal distribution as well as simply linearizing

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\(^3\) The variables in \( x_t \) are typically the logs of the quantities of interest.

\(^4\) In discrete time and with normal innovations to \( x_t \) the affine specification for the variance-covariance matrix does not in general allow me to ensure it is positive-definiteness. I use this specification to derive analytical expressions. When I solve the model numerically I can ensure that the volatility is positive by replacing negative realizations with a very small number.

\(^5\) Note that since \( G_1 \) is an \((n \times n \times n)\) array, \((G_1)_{ij}\) is a row vector of dimension \((1 \times n)\).
all the non-expectational equations (2) I can re-write the equilibrium conditions

\[ \Theta \hat{x}_t + \Theta_+ E_t \hat{x}_{t+1} + \frac{1}{2} \text{diag} (\Theta_+ \text{Var}_t \hat{x}_{t+1} \Theta_+^\top) = 0, \]

\[ \Phi \hat{x}_t + \Phi_+ \hat{x}_{t+1} + \Phi_\epsilon \Sigma_t \epsilon_{t+1} = 0, \]

where \( \Phi, \Phi_+ \) and \( \Phi_\epsilon \) are conforming matrices of coefficients\(^6\). The resulting system can now be solved in one of the two standard ways - undetermined coefficients or linear difference equations, see Cochrane (2001) - to find a solution of the form and thus verify the conjecture.

Compared to the simple log-linearization the system (5) has additional terms in \( \text{Var}_t \hat{x}_{t+1} \). When volatility is constant, these consist of constants accounting for the risk adjustments that will appear in the vector \( H \) of the solution. When volatility is stochastic, these additional terms will also capture the dynamic effects of changes in volatility on the other variables in the model. The assumption of affine dynamics (3) becomes important at this stage. I have the following conjecture-and-verify result. A linear solution of the form (4) implies

\[ \text{Var}_t \hat{x}_{t+1} = H_\epsilon \Sigma_t \Sigma_t^\top H_\epsilon^\top = H_\epsilon G H_\epsilon^\top + \sum_i H_\epsilon G_{x_i} H_\epsilon^\top x_i, \]

which ensures that the system (5) is indeed linear.

### 2.2 Comparison with Perturbation Methods

In this section I will discuss how the solution method presented above can be understood without reference to log-normality and in terms of standard perturbation approach. For

\(^6\text{diag}(X)\) is the vector of elements on the main diagonal of the matrix \( X \).
simplicity let me consider the case with constant volatility $\Sigma_t = \Sigma$\textsuperscript{7} Second order perturbation around the non-stochastic steady state provides us with the following approximation of the policy function\textsuperscript{8}

\[
\hat{x}_{t+1} \approx H_x\hat{x}_t + H_{\epsilon}\Sigma\epsilon_{t+1} + \frac{1}{2} \left( \hat{x}_t^T H_{xx} \hat{x}_t + 2\hat{x}_t^T H_{x\epsilon} \epsilon_{t+1} + \epsilon_{t+1}^T \Sigma^T H_{\epsilon\epsilon} \Sigma \epsilon_{t+1} + \sigma^2 H_{\sigma\sigma} \right),
\]

where $\sigma$ is the perturbation parameter scaling exogenous shocks. See Schmitt-Grohe and Uribe (2004), Juillard (2008) for the derivation of (7). In particular the authors show that the cross-terms $H_{\epsilon\epsilon} = H_{x\epsilon} = 0$, which implies that to the second order only the unconditional means of the variables are affected by risk adjustments. On the other hand, as discussed above, the log-linearized and risk-adjusted solution can be written as

\[
\hat{x}_{t+1} \approx H_x\hat{x}_t + H_{\epsilon}\Sigma\epsilon_{t+1} + H.
\]

By construction the first order coefficients $H_x$ and $H_{\epsilon}$ are exactly the same in both approximations. Again trivially, log-linearization ignores quadratic terms in $H_{\epsilon\epsilon}$, $H_{xx}$, and $H_{x\epsilon}$, and is only a good substitute for higher order perturbation methods if these terms are quantitatively small. The only remaining question is the relation between the risk adjustment implied by the two methods, $\frac{1}{2} H_{\sigma\sigma}$ and $H$.

I argue that $\frac{1}{2} H_{\sigma\sigma} = H$ provided dynamic second-order (quadratic) terms can be ignored: $H_{xx}, H_{\epsilon\epsilon}, H_{x\epsilon} = 0$. To see this, notice that the second derivative of the system (5) with respect to $\sigma$ evaluated at the steady state ($\sigma = 0$)

\textsuperscript{7}The intuition is the same for the case when volatility is stochastic. Benigno, Benigno, and Nistico (2010) show how second-order perturbation approach can be adapted to the case where volatility is stochastic. An argument similar to the one below can be applied in their setting.

\textsuperscript{8}Notice that $H_{xx}$ and $H_{\epsilon\epsilon}$ are three-dimensional arrays and therefore $\hat{x}_t^T H_{xx} \hat{x}_t$ and $\epsilon_{t+1}^T \Sigma^T H_{\epsilon\epsilon} \Sigma \epsilon_{t+1}$ denote $n \times 1$ vectors. Here, I follow Duffie, Pan, and Singleton (2000) notation.
\[ E_t \left( \exp (\Theta \hat{x}_t + \Theta \hat{x}_{t+1}) \circ ((\Theta^\top H_t \Sigma \epsilon_{t+1}) \circ (\Theta^\top H_t \Sigma \epsilon_{t+1}) + \Theta^\top \epsilon_{t+1} \Sigma^\top H_t \Sigma \epsilon_{t+1} + \Theta^\top H_\sigma) \right) = 0 \]

implies that if \( H_\alpha = 0 \) then

\[ \Theta^\top H_\sigma = E_t (\Theta^\top H_t \Sigma \epsilon_{t+1}) \circ (\Theta^\top H_t \Sigma \epsilon_{t+1}) = \text{diag} \left( \Theta^\top \text{Var} \epsilon_{t+1} \Theta^\top \right). \]

In other words, my method allows to compute second-order risk adjustments for the cases where first-order terms accurately describe the dynamic properties of the model, and this without computing the second derivatives of the system:

\[ \hat{x}_{t+1} \approx H_t \hat{x}_t + H_t \Sigma \epsilon_{t+1} + \frac{1}{2} (\hat{x}_t^\top H_x \hat{x}_t + \sigma^2 \epsilon_{t+1} \Sigma^\top H_t \Sigma \epsilon_{t+1} + 2 \sigma \hat{x}_t^\top H_x \Sigma \epsilon_{t+1}) + \frac{1}{2} \sigma^2 H_\sigma, \]

In the next section I will demonstrate the accuracy of the approximation on a number of examples of interest. I will compare the performance of the proposed method against the perturbation solution benchmark. Recent literature has documented the accuracy of higher-order perturbation compared to other numerical techniques for a variety of models that feature recursive preferences, adjustment costs, stochastic volatility etc. See for example Caldara, Fernandez-Villaverde, Rubio-Ramirez, and Yao (2012). The task of this paper is to show that retaining only selected higher order terms in the expansion results in a more tractable, yet as accurate solution.

Ignoring quadratic terms comes with the additional benefit that it automatically alleviates stability issues of higher order expansions as discussed for instance in Kim, Kim,
Finally, the close link between my method and standard perturbation algorithm makes numerical implementation straightforward. In general, I see the analytical tractability as an important advantage of my method. However any second order perturbation software (for example Schmitt-Grohe and Uribe (2004) package) can be adapted to produce a log-linearized and risk-adjusted solution by simply forcing all $H_{ee}, H_{xx}, H_{xe}$ terms to be 0. Computational efficiency is thereby increased as I do not require the software to compute second order derivatives and solve the system of equations for the second order dynamic terms.

3 Applications

3.1 Benchmark example with recursive utility

In this section I illustrate the solution method on an important example featuring recursive utility. Kaltenbrunner and Lochstoer (2010) consider a standard real business cycle model augmented along two dimension: Epstein-Zin-Weil preferences and capital adjustment costs. The authors show that, in this framework, a high elasticity of inter-temporal substitution (calibrated separately from the coefficient of relative risk aversion) endogenously generates long-run risk in consumption growth and thereby results in the level of risk premia consistent with the data.

*Model.* The model is given as follows, I refer the reader to the original article for a detailed discussion of the assumptions of the model. Expectational equations can be

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9I would like to thank an anonymous referee for pointing this out.
with the rest of the equilibrium conditions (that can take an arbitrary functional form):

\[ V_t^{1 - \frac{1}{\psi}} - C_t^{1 - \frac{1}{\psi}} - \beta W_t^{1 - \frac{1}{\psi}} = 0, \]  

\[ C_t + I_t - A_t^{-\alpha} K_t^\alpha = 0, \]

\[ K_{t+1} - (1 - \delta) K_t - \left( \kappa_1 + \frac{\kappa_2}{1 - \frac{1}{\xi}} \left( \frac{I_{t+1}}{K_{t+1}} \right)^{1 - \frac{1}{\xi}} \right) K_t = 0, \]

\[ R_{t+1} - \kappa_2 \left( \frac{I_t}{K_t} \right)^{-\frac{1}{\xi}} \left( \alpha \left( \frac{A_{t+1}}{K_{t+1}} \right)^{1-\alpha} + \frac{1 - \delta + \kappa_1 + \frac{\kappa_2}{1 - \frac{1}{\xi}} \left( \frac{I_{t+1}}{K_{t+1}} \right)^{1 - \frac{1}{\xi}}}{\kappa_2 \left( \frac{I_{t+1}}{K_{t+1}} \right)^{-\frac{1}{\xi}}} - \frac{I_{t+1}}{K_{t+1}} \right) = 0, \]

\[ a_{t+1} - a_t - \mu - \sigma \epsilon_{t+1} = 0, \]

where \( X = e^x \) for all the variables.\(^{10}\)

Note that, unlike in the time-additive (constant relative risk aversion) case, the value function \( V_t \) enters the Euler equation (8) and as a result an additional equilibrium condition (10) has to define this additional variable. Furthermore, the certainty equivalent

\(^{10}\)When solving the model I induce stationarity by scaling all unit root variables by the level of productivity \( \tilde{X}_t = \frac{X_t}{x_{t-1}} \).
of the value function, \( W_t = \left( E_t \left( V_{t+1}^{1-\gamma} \right) \right)^{\frac{1}{1-\gamma}} \), also appears in the Euler equation, resulting in a second forward-looking equilibrium condition (9). These two features appeared somewhat problematic in the previous attempts to use linearization.

Backus, Routledge, and Zin (2007) consider a model similar to the above and chose to directly log-linearize the first-order and the envelope conditions of the dynamic programming problem:

\[
C_t^{-\frac{1}{\psi}} = -\beta E_t \left( V_t^{1-\gamma} \right)^{\frac{1}{1-\gamma}} E_t \left( V_{t+1}^{-\gamma} \frac{\partial V_{t+1}}{\partial K_{t+1}} \frac{\partial K_{t+1}}{\partial C_t} \right),
\]

\[
\frac{\partial V_t}{\partial K_t} = \frac{1}{\psi} V_t E_t \left( V_t^{1-\gamma} \right)^{\frac{1}{1-\gamma}} E_t \left( V_{t+1}^{-\gamma} \frac{\partial V_{t+1}}{\partial K_{t+1}} \frac{\partial K_{t+1}}{\partial K_t} \right).
\]

As discussed in detail in Caldara, Fernandez-Villaverde, Rubio-Ramirez, and Yao (2012), when first-order conditions are approximated, the approximation of the value function should be of one order higher than the desired approximation of the solution. This is very intuitive as first order derivatives of the value function enter the equations to be approximated. Therefore linearization cannot be implemented this way because the value function has to be approximated to the second order.\(^{11}\)

Uhlig (2010) uses a two-step approach to find an approximate solution to a model similar to the one considered in this section. First a log-linear solution is found without adjusting for risk. Then asset pricing implication are derived from the asset pricing equation using the properties of normal distribution:

\[
E_t \left( \exp \left( \ln \beta - \frac{1}{\psi} (c_{t+1} - c_t) + \left( \frac{1}{\psi} - \gamma \right) (v_{t+1} - w_t) + r_{i,t+1} \right) \right) = 1,
\]

\(^{11}\)For instance, it is easy to check that the Backus, Routledge, and Zin (2007) approach will not result in a value function approximation that is homogeneous of degree one in capital and labor.
where \( r_{i,t+1} \) is the log-return of asset \( i \). This approach had been used earlier by Lettau (2003) for the time-additive constant relative risk aversion case. I argue that in the case with recursive utility proceeding this way ignores an important risk adjustment term from equation (9) that corresponds to the certainty equivalent of future utility

\[
w_t - E_t v_{t+1} = \frac{1}{2} (1 - \gamma) \text{Var}_t (v_{t+1}).
\]

Consider for instance the log risk free rate \( r_{f,t} \). The two-step approximation gives

\[
r_{f,t} = -\ln \beta + \frac{1}{\psi} E_t (c_{t+1} - c_t) - \frac{1}{2} \text{Var}_t (m_{t+1}) + \frac{1}{2} (1/\psi - \gamma) (1 - \gamma) \text{Var}_t (v_{t+1}).
\]

The first two terms, related to time preference and expected consumption growth, are standard and appear in simple first order approximation. The last term depends on the volatility of the stochastic discount factor and captures the effect of risk on precautionary savings effect. My method adds another term to the risk free rate expression:

\[
r_{f,t} = -\ln \beta + \frac{1}{\psi} E_t (c_{t+1} - c_t) - \frac{1}{2} \text{Var}_t (m_{t+1}) + \frac{1}{2} (1/\psi - \gamma) (1 - \gamma) \text{Var}_t (v_{t+1}).
\]

This additional term captures the effect of preferences towards the timing of resolution of uncertainty\(^{12}\). It is non-zero when \( \frac{1}{\psi} \neq \gamma \). In other words, it is absent with constant relative risk aversion utility, but cannot be ignored with general Epstein-Zin-Weil utility.

Table 1 gives the numerical assessment of the bias if the risk adjustment in the certainty

\(^{12}\)The intuition for this extra effect in the risk free rate is related to the agents preference for the early resolution of uncertainty. If agents have a preference for early resolution of uncertainty (\( \gamma > \frac{1}{\psi} \)) and want to hedge against rather than bet on changing economic conditions (\( \gamma > 1 \)) they will prefer to bring consumption forward thus increasing the risk free rate.
equivalent is missing.

[TABLE 1 HERE]

Calibration and moments. I follow Kaltenbrunner and Lochstoer (2010) in calibrating the model. More precisely I set $\alpha = 0.36$, $\delta = 0.021$, $\xi = 18$, $\psi = 1.5$, $\gamma = 5$, $\beta = 0.998$, $\mu = 0.004$, $\sigma = 0.041$. Table 2 reports the moments of the model obtained by the authors using value function iteration and compares them to the ones obtained using risk-adjusted log-linearization, and standard first- and second-order perturbation in logs. I note that, with the exception of first-order perturbation, all solution methods produce similar results.

[TABLE 2 HERE]

The log-linear and risk adjusted optimal policies for consumption and value function can be written as

$$\hat{c}_t = -0.1737 + \begin{bmatrix} 0.6756 & 0.3244 \end{bmatrix} \begin{bmatrix} \hat{k}_t \\ \hat{a}_t \end{bmatrix},$$

$$v_t = -9.7631 + \begin{bmatrix} 0.0145 & 0.9855 \end{bmatrix} \begin{bmatrix} \hat{k}_t \\ \hat{a}_t \end{bmatrix}.$$  

(11)

(12)

Using second order perturbation I can write:

$$\hat{c}_t = -0.1716 + \begin{bmatrix} 0.6756 & 0.3244 \end{bmatrix} \begin{bmatrix} \hat{k}_t \\ \hat{a}_t \end{bmatrix} + \begin{bmatrix} 0.0019 & -0.0019 \\ -0.0019 & 0.0019 \end{bmatrix} \begin{bmatrix} \hat{k}_t \\ \hat{a}_t \end{bmatrix},$$

$$v_t = -9.7463 + \begin{bmatrix} 0.0145 & 0.9855 \end{bmatrix} \begin{bmatrix} \hat{k}_t \\ \hat{a}_t \end{bmatrix} + \begin{bmatrix} 0.0067 & -0.0067 \\ -0.0067 & 0.0067 \end{bmatrix} \begin{bmatrix} \hat{k}_t \\ \hat{a}_t \end{bmatrix}.$$  

(13)

(14)
The second order dynamic (quadratic) terms in equations (13) and (14) are small and, as a result, constant terms that account for risk adjustment are approximately equal for the two solution methods.

My method can also be applied to calculate the welfare cost of business cycles. The welfare cost can be measured by the constant terms in equations (12) equal to $-9.7631$ for my method and (14) equal to $-9.7463$ for second order perturbation. See Caldara, Fernandez-Villaverde, Rubio-Ramirez, and Yao (2012).

**Accuracy tests.** In order to look beyond the selected first and second moments and test the overall quality of the approximation I implement Den Haan and Marcet (1994) accuracy test. It allows me to evaluate the performance of the approximation without knowing the exact solution and non-locally. The latter point is of particular interest for an approximation around the non-stochastic steady state.

Den Haan and Marcet (1994) derive the distribution for cumulated residuals of expectational equations

\[ u_{t+1} = \exp (\Theta \hat{x}_t + \Theta_{+} \hat{x}_{t+1}) - 1, \]

possibly conditioning on $x_t$ instruments, under the null hypothesis that equilibrium conditions hold exactly i.e. the model is solved exactly:

\[ T B_{T}^{T} A_{T}^{-1} B_{T} \overset{D}{\to} \chi_{pq}^2 \text{ as } T \to \infty \]

where

\[ B_{T} = \frac{\sum_{t} u_{t+1} \otimes h(x_t)}{T}, \quad S = \frac{E \left[ (u_{t+1} \otimes h(x_t)) (u_{t+1} \otimes h(x_t))^\top \right]}{T} \quad \text{and} \quad A_{T} = \hat{S} \]
and $p$ and $q$ are the number of expectational equations and instruments respectively. I simulate 500 economies each with 3500 periods (first 500 discarded). I compute Den Haan and Marcet (1994) (DHM) statistics pertaining to the Euler equation (8) and compare their distribution to the one corresponding to the exact solution. Results are reported in Table 3 and on Figure 1. Risk-adjusted log-linearization method produces an accurate solution: the distribution of DHM statistic for the risk-adjusted log-linearized solution is virtually identical to the one implied by the exact solution. In particular, the proportion of DHM statistic realizations below $5\%$ and above $95\%$ critical values are close to the theoretical $5\%$. Moreover, the performance of risk-adjusted log-linearized solution and standard second-order perturbation solution in the test is very similar.

[TABLE 3 HERE]

[FIGURE 1 HERE]

*Extreme calibrations.* Finally, I test the robustness of the method outside the economically motivated range of parameters. I follow Caldara, Fernandez-Villaverde, Rubio-Ramirez, and Yao (2012) and perform accuracy tests for calibrations where the coefficient of relative risk aversion ($\gamma$) and the elasticity of inter-temporal substitution ($\psi$) are set to high values.\textsuperscript{13} Results are reported in Table 3 and on Figure 2. Increasing $\gamma$ from 5 to 30 or $\psi$ from 1.5 to 2, does not worsen the performance of the risk-adjusted log-linearized solution in the accuracy test: the proportion of DHM statistic realizations below $5\%$ and above $95\%$ critical values are close to the theoretical $5\%$. When I increase both parameters to 50 and 3 respectively the solution starts to become less accurate: 10.8\% of DHM statistic realizations lie above the 95\% critical value. I note that the values $\gamma = 30$ and

\textsuperscript{13}Tests for low values of these parameters are not reported: in line with Caldara, Fernandez-Villaverde, Rubio-Ramirez, and Yao (2012) they imply a good performance of all approximation methods.
\( \psi = 2 \) are outside the range typically considered in the long-run risk literature and are higher than empirical estimates; the values \( \gamma = 50 \) and \( \psi = 3 \) lie well outside that range.

TABLE 4 HERE

FIGURE 2 HERE

3.2 Habit formation and high capital adjustment costs

In this section I modify the model presented above by assuming preferences with habit formation (instead of Epstein-Zin-Weil utility):

\[
U_t = E_t \sum_{k=0}^{\infty} \beta^k \left( C_{t+k} - \psi C_{t+k-1} \right)^{1-\gamma}.
\]

The example is based on Jermann (1998) asset pricing model. It illustrates how the proposed method can handle additional non-linearity introduced by endogenously time-varying risk aversion together with high capital adjustment costs, and can be seen as an additional robustness check.

Model. The model is given as follows, I refer the reader to the original article for a detailed discussion of the assumptions of the model. The forward looking equations can be written in the special (1) form:

\[
E_t \exp (\ln \beta - \gamma (h_{t+1} + q_{t+1} - h_t - q_t) + r_{t+1}) = 1,
\]

(15)

\[
E_t \exp (-\gamma (c_{t+1} + s_{t+1}) - w_t) = 1
\]

with the rest of the equilibrium conditions (that can take an arbitrary functional form):

\[
H_{t+1} - C_{t+1} + \psi C_t = 0,
\]
\[ S_{t+1} - 1 + \frac{\psi C_t}{C_{t+1}} = 0, \]
\[ Q_t - 1 + \frac{\psi \beta W_t}{H_t} = 0, \]
\[ C_t + I_t - Z_t A_t^{1-\alpha} K_t^\alpha = 0, \]
\[ K_{t+1} - (1 - \delta) K_t - \left( \kappa_1 + \frac{\kappa_2}{1 - \frac{1}{\xi}} \left( \frac{I_{t+1}}{K_{t+1}} \right)^{1-\frac{1}{\xi}} \right) K_t = 0, \]
\[ R_{t+1} - \kappa_2 \left( \frac{I_t}{K_t} \right)^{-\frac{1}{\xi}} \left( \alpha Z_{t+1} \left( \frac{A_{t+1}}{K_{t+1}} \right)^{1-\alpha} + \frac{1 - \delta + \kappa_1 + \frac{\kappa_2}{1 - \frac{1}{\xi}} \left( \frac{I_{t+1}}{K_{t+1}} \right)^{1-\frac{1}{\xi}}}{\kappa_2 \left( \frac{I_{t+1}}{K_{t+1}} \right)^{-\frac{1}{\xi}}} - \frac{I_{t+1}}{K_{t+1}} \right) = 0, \]
\[ z_{t+1} = \rho z_t + \sigma \epsilon_{t+1}, \]
\[ a_{t+1} - a_t = \mu, \]

where \( X = e^x \) for all the variables.

**Calibration and moments.** I follow Jermann (1998) in calibrating the model. More precisely I set \( \alpha = 0.36, \delta = 0.025, \xi = 0.23, \beta \mu^{1-\gamma} = 0.99, \gamma = 5, \psi = 0.82, \mu = \log (1.005), \rho = 0.95, \sigma = 0.01. \) I note that in this section the parameter \( \xi \) (the elasticity of the investment to capital ratio to Tobin’s q) is calibrated to a lower value compared to Section 3.1, implying higher capital adjustment costs. The value \( \xi = 0.23 \) implies capital adjustment costs in the higher end of the range typically used in the asset-pricing literature.

Table 4 reports the moments of the model obtained by the author using projection methods and compares them to the ones obtained using risk-adjusted log-linearization, and standard first and second-order perturbation in logs. I note that, with the exception
of first-order perturbation, all solution methods produce similar results.

Accuracy tests. As in the previous section, I perform the DHM accuracy test by simulating 500 economies each with 3500 periods (first 500 discarded) and look at the distribution of cumulated residuals corresponding to Euler equation (15). Results are reported in Table 5 and on Figure 3. Risk-adjusted log-linearization method produces an accurate solution: the distribution of DHM statistic for the risk-adjusted log-linearized solution is close to the one implied by the exact solution. In particular, the proportion of DHM statistic realizations below 5% and above 95% critical values are close to the theoretical 5%. Moreover, the performance of risk-adjusted log-linearized solution and standard second-order perturbation solution in the test is very similar.

Extreme calibrations. Finally, I test the robustness of the method outside the benchmark set of parameters. In particular, I set the risk aversion ($\gamma$) and habit formation ($\psi$) to higher values. Note that these two parameters have a different interpretation in the context of this section and therefore their calibration cannot be compared to the one in Section 3.1. Results are reported in Table 5 and on Figure 4. Increasing $\gamma$ from 5 to 7.5 or $\psi$ from 0.82 to 0.85, does not significantly worsen the performance of the risk-adjusted log-linearized solution in the accuracy test: the proportion of DHM statistic realizations below 5% and above 95% critical values are close to the theoretical 5%. When I increase both parameters to 10 and 0.88 respectively the solution becomes less accurate: 15.4% of DHM statistic realizations lie above the 95% critical value.
3.3 Highly persistent stochastic volatility

This section illustrates how the proposed approximation technique can be used to study models with stochastic volatility. With standard perturbation algorithms, at least a third order approximation is typically required to capture the first-order dynamic effects of changes in volatility on the variables of the model. In other words, one has to compute higher order derivatives in order to compute what are in reality linear effects of volatility.\footnote{I will illustrate this point with Dynare++ software package. Benigno, Benigno, and Nistico (2010) show how to modify Schmitt-Grohe and Uribe (2004) framework in a way that allows to account for stochastic volatility by computing second order derivatives only. Because what the authors are after are in fact terms linear in volatility, my paper goes a step further and shows how stochastic volatility can be accommodated in log-linearized framework.} The same is true for risk-adjustments. I will show how these effects can be accurately accounted for in a log-linearized framework.

In addition, while various software packages can deal with third and higher order approximations in a reasonable amount of time, they remain effectively a ‘black box’. In this section I will illustrate how a closed form approximate solution can deliver interesting insights into the mechanisms of the model.

Model. I consider the real business cycle model with Epstein-Zin-Weil preferences from Section 3.1, but assume stochastic volatility and, for simplicity, no capital adjustment costs. Expectational equations can be written in the special (1) form:

\[
E_t \exp \left( \ln \beta - \frac{1}{\psi} (c_{t+1} - c_t) + \left( \frac{1}{\psi} - \gamma \right) (v_{t+1} - w_t) + r_{t+1} \right) = 1,
\]
\[ E_t \exp ((1 - \gamma) (v_{t+1} - w_t)) = 1, \]

with the rest of the equilibrium conditions:

\[ V_t^{1 - \frac{1}{\sigma}} - C_t^{1 - \frac{1}{\sigma}} - \beta W_t^{1 - \frac{1}{\sigma}} = 0, \]

\[ K_{t+1} - (1 - \delta)K_t - A_t^{1 - \alpha} K_t^\alpha + C_t = 0, \]

\[ R_{t+1} - 1 + \delta - \alpha A_t^{1 - \alpha} K_t^{\alpha - 1} = 0, \]

\[ a_{t+1} - a_t - \mu - \sigma_t \epsilon_{1,t+1} = 0, \]

\[ \sigma_t^2 - (1 - \varphi) \theta - \varphi \sigma_t^2 - \omega \epsilon_{2,t+1} = 0, \]

where \( X = e^x \) for all the variables. Note that the variance-covariance matrix of shocks is affine in the state variables:

\[
\begin{pmatrix}
\sigma_t & 0 \\
0 & \omega
\end{pmatrix}
\begin{pmatrix}
\sigma_t & 0 \\
0 & \omega
\end{pmatrix}^\top
= 
\begin{pmatrix}
\sigma_t^2 & 0 \\
0 & \omega^2
\end{pmatrix}
= 
\begin{pmatrix}
0 & 0 \\
0 & \omega^2
\end{pmatrix}
+ 
\begin{pmatrix}
1 & 0 \\
0 & 0
\end{pmatrix}
\sigma_t^2.
\]

An analytical solution of the model above can provide guidance for the design of the consumption based asset pricing models. Denoting by \( x_y \) the log-linearization coefficient of \( x \) with respect to \( y \), (endogenous) consumption growth can be written as:

\[ \Delta c_{t+1} = \text{const} + \Delta c_a a_t + \Delta c_k k_t + \Delta c_\sigma \sigma_t^2 + c_a \sigma_t \epsilon_{1,t+1} + c_\sigma \omega \epsilon_{2,t+1}, \] (16)

and the process for consumption growth variance as:
\[ \text{Var}_t(\Delta c_{t+1}) = (1 - \varphi) \left( c_a^2 \theta + c_a^2 \omega^2 \right) + \varphi \text{Var}_{t-1}(\Delta c_t) + c_a^2 \omega \varepsilon_t^2. \] \quad (17)

Using (17) I can re-write (16) as:

\[ \Delta c_{t+1} = \text{const} + \Delta c_a \hat{a}_t + \Delta c_k \hat{k}_t + \left( \frac{\varphi - 1 + c_k k_c}{c_a^2} \right) \text{Var}_{t+1}(\Delta c_{t+2}) + c_a \sigma_t \varepsilon_{t+1} + c_\omega \omega \varepsilon_{t+1}. \]

\[ \Delta c_{\sigma_c} \] \quad (18)

In the long-run risk literature consumption growth and consumption growth volatility are typically assumed to be independent. In a real business cycle framework volatility shocks have first order effects on real quantities such as consumption through precautionary savings mechanism: \( c_\sigma \neq 0 \). The negative initial response of consumption to volatility shock \( c_\sigma < 0 \) implies that 1) correlation between consumption volatility and realized consumption growth is negative

\[ \text{Cov}_t(\Delta c_{t+1}, \text{Var}_{t+1}(\Delta c_{t+2})) = c_\sigma c_a^2 \omega^2 < 0, \]

and 2) higher volatility implies higher expected future consumption growth

\[ \Delta c_{\sigma_c} = \left( \frac{\varphi - 1 + c_k k_c}{c_a^2} \right) c_\sigma > 0. \]

See Backus, Routledge, and Zin (2008) on the importance of the dependence between consumption growth and consumption growth volatility.

\textit{Calibration and moments.} I calibrate volatility shocks to (endogenously) reproduce
consumption volatility dynamics that are assumed as given in Bansal, Kiku, and Yaron (2009). Other parameters are standard to the business cycle and long-run risk literatures. More precisely I set \( \alpha = 0.34, \delta = 1 - 0.9^{1/2}, \beta = 0.98^{1/2}, \gamma = 5, \psi = 1.5, \mu = 0.0015, \theta = 3.81 \times 10^{-4}, \varphi = 0.999, \omega = 3.39 \times 10^{-5} \). This calibration exercise is itself made possible by the approximate analytical solution: given the Bansal, Kiku, and Yaron (2009) calibration of consumption growth variance process, namely \( \theta^c = 5.18 \times 10^{-5}, \varphi^c = 0.999, \omega^c = 2.80 \times 10^{-6} \), I use (17) to invert for the parameters of productivity growth variance. Note that the value \( \varphi = \varphi^c = 0.999 \) implies that variance shocks are extremely persistent and is at the high end of the range used in the long-run risk literature, see Beeler and Campbell (2012).

Figure 5 reports the impulse response of consumption to volatility shock. I note that first- and second-order perturbation solution fail to capture this response. At least a third-order expansion is required to account for a fall in consumption driven by an increase in precautionary savings. Risk-adjusted log-linearization produces a consumption response that is effectively identical to the one implied by the third- and fourth-order perturbation.

[FIGURE 5 HERE]

Table 6 reports the moments of the calibrated model obtained using different orders of approximation. I note that risk-adjusted log-linearization is able to capture the effect of stochastic volatility on the risk-free rate and equity risk premium that are not accounted for by the second- and fourth-order perturbation, and appear only in higher order expansions.

[TABLE 8 HERE]

*Accuracy tests and extreme calibrations.* I perform the DHM accuracy test by simulating 500 economies each with 3500 periods (first 500 discarded) and look at the dis-
tribution of cumulated residuals corresponding to Euler equation (15). Results for the benchmark and more extreme calibrations are reported in Table 7 and on Figure 6. I note that risk-adjusted log-linearization method produces an accurate solution for a range of calibrations: the distribution of DHM statistic is close to the one implied by the exact solution. In particular, the proportion of DHM statistic realizations below 5% and above 95% critical values are close to the theoretical 5%. However, as pointed out by Beeler and Campbell (2012), highly persistent stochastic volatility combined with recursive preferences significantly increases the role of risk adjustments. As a result, compared to the model without stochastic volatility in Section 3.1, a smaller increase in preference parameters leads to a less accurate solution.

[TABLE 9 HERE]

[FIGURE 6 HERE]

4 Conclusion

I describe a tractable way to find accurate analytical risk adjustments in a log-linearized framework, i.e. without the need to compute second or higher order derivatives. The method applies to a large class of models including real business cycle models with stochastic volatility and recursive preferences, two features that had appeared somewhat problematic in the context of approximation methods. The solution is well suited to investigate economic mechanisms, describe the time series properties, estimate the model and deal with stochastic volatility. I illustrate the performance of the method on several examples of interest. Finally, I show the theoretical relationship between my solution and higher-order perturbation methods.
References


Econometrica, 57(4), 937–969.


Table 1: Potential Risk-free Rate Bias in the Model with Recursive Preferences

This table illustrates the potential bias resulting from computing risk adjustments for only a subset of forward looking equations of the model. It reports the unconditional mean of the risk-free rate in the Kaltenbrunner and Lochstoer (2010) model for the fully risk-adjusted solution and the case where the certainty equivalent term \( w_t - E_t v_{t+1} = \frac{1}{2} (1 - \gamma) Var_t (v_{t+1}) \) is ignored.

<table>
<thead>
<tr>
<th>Methods</th>
<th>(fully) Risk-adjusted log-linearization</th>
<th>Risk adjusting only asset pricing equations</th>
</tr>
</thead>
<tbody>
<tr>
<td>( E(r_f) ) %</td>
<td>0.76%</td>
<td>-4.90%</td>
</tr>
</tbody>
</table>
Table 2: Key Moments for the Model with Recursive Preferences

This table reports the volatilities of consumption, output, and investment growth, the unconditional mean of the risk-free rate, and the equity risk premium in the model with recursive preferences. Fully numerical results are as reported in Kaltenbrunner and Lochstoer (2010) and are produced by the authors using value function iteration.

<table>
<thead>
<tr>
<th></th>
<th>Risk-adjusted log-linearization</th>
<th>First-order perturbation</th>
<th>Second-order perturbation</th>
<th>Fully numerical</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma (\Delta c) %$</td>
<td>2.74</td>
<td>2.74</td>
<td>2.74</td>
<td>2.72</td>
</tr>
<tr>
<td>$\sigma (\Delta c) / \sigma (\Delta y)$</td>
<td>0.52</td>
<td>0.52</td>
<td>0.52</td>
<td>0.52</td>
</tr>
<tr>
<td>$\sigma (\Delta i) / \sigma (\Delta y)$</td>
<td>1.90</td>
<td>1.90</td>
<td>1.90</td>
<td>1.83</td>
</tr>
<tr>
<td>$E (r_f) %$</td>
<td>0.76</td>
<td>1.87</td>
<td>0.76</td>
<td>0.82</td>
</tr>
<tr>
<td>$E (r_e - r_f) %$</td>
<td>1.64</td>
<td>0.00</td>
<td>1.64</td>
<td>1.59</td>
</tr>
</tbody>
</table>
Table 3: DHM Statistic Quantiles in the Model with Recursive Preferences

This table reports the proportion of the Den Haan and Marcet (1994) statistic realizations below (above) the lower (upper) 5% threshold under the null of the exact solution for a range of calibrations of the model with recursive preferences. Results are based on 500 simulated economies each with 3500 periods (first 500 discarded).

<table>
<thead>
<tr>
<th>Calibration</th>
<th>Lower 5%</th>
<th>Upper 5%</th>
</tr>
</thead>
<tbody>
<tr>
<td>Baseline calibration</td>
<td>6.80%</td>
<td>5.20%</td>
</tr>
<tr>
<td>Calibration with $\gamma = 30$</td>
<td>5.00%</td>
<td>5.40%</td>
</tr>
<tr>
<td>Calibration with $\psi = 2$</td>
<td>6.60%</td>
<td>5.20%</td>
</tr>
<tr>
<td>Calibration with $\gamma = 50$ and $\psi = 3$</td>
<td>6.20%</td>
<td>10.8%</td>
</tr>
</tbody>
</table>
Table 4: Key Moments for the Model with Habit Formation

This table reports the volatilities of consumption, output, and investment growth, the unconditional mean of the risk-free rate, and the equity risk premium in the model with habit formation. Fully numerical results are as reported in Jermann (1998) and are produced by the author using projection methods.

<table>
<thead>
<tr>
<th></th>
<th>Risk-adjusted log-linearization</th>
<th>First-order perturbation</th>
<th>Second-order perturbation</th>
<th>Fully numerical</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma (\Delta c)%$</td>
<td>0.38</td>
<td>0.38</td>
<td>0.38</td>
<td>0.38</td>
</tr>
<tr>
<td>$\sigma (\Delta c) / \sigma (\Delta y)$</td>
<td>0.38</td>
<td>0.38</td>
<td>0.38</td>
<td>0.38</td>
</tr>
<tr>
<td>$\sigma (\Delta i) / \sigma (\Delta y)$</td>
<td>2.76</td>
<td>2.76</td>
<td>2.76</td>
<td>2.77</td>
</tr>
<tr>
<td>$E(r_f)%$</td>
<td>1.10</td>
<td>4.43</td>
<td>1.10</td>
<td>1.55</td>
</tr>
<tr>
<td>$E(r_e - r_f)%$</td>
<td>5.92</td>
<td>0.00</td>
<td>5.92</td>
<td>5.90</td>
</tr>
</tbody>
</table>
Table 5: DHM Statistic Quantiles in the Model with Habit Formation

This table reports the proportion of the Den Haan and Marcet (1994) statistic realizations below (above) the lower (upper) 5% threshold under the null of the exact solution for a range of calibrations of the model with habit formation and high capital adjustment costs. Results are based on 500 simulated economies each with 3500 periods (first 500 discarded).

<table>
<thead>
<tr>
<th>Calibration</th>
<th>Lower 5%</th>
<th>Upper 5%</th>
</tr>
</thead>
<tbody>
<tr>
<td>Baseline calibration</td>
<td>6.00%</td>
<td>3.40%</td>
</tr>
<tr>
<td>Calibration with $\gamma = 7.5$</td>
<td>6.60%</td>
<td>5.80%</td>
</tr>
<tr>
<td>Calibration with $\psi = 0.85$</td>
<td>4.40%</td>
<td>6.00%</td>
</tr>
<tr>
<td>Calibration with $\gamma = 10$ and $\psi = 0.88$</td>
<td>3.80%</td>
<td>15.4%</td>
</tr>
</tbody>
</table>
Table 6: Key Moments for the Model with Stochastic Volatility

This table reports the volatilities of consumption, output, and investment growth, the unconditional mean of the risk-free rate, and the equity risk premium in the model with stochastic volatility. Second-, fourth-, and sixth-order perturbation solution results are produced using Dynare++ software.

<table>
<thead>
<tr>
<th></th>
<th>Risk-adjusted log-linearization</th>
<th>Second-order perturbation</th>
<th>Fourth-order perturbation</th>
<th>Sixth-order perturbation</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma (\Delta c) %$</td>
<td>2.55</td>
<td>1.96</td>
<td>2.54</td>
<td>2.40</td>
</tr>
<tr>
<td>$\sigma (\Delta c) / \sigma (\Delta y)$</td>
<td>0.53</td>
<td>0.43</td>
<td>0.52</td>
<td>0.53</td>
</tr>
<tr>
<td>$\sigma (\Delta i) / \sigma (\Delta y)$</td>
<td>2.39</td>
<td>2.40</td>
<td>2.35</td>
<td>2.52</td>
</tr>
<tr>
<td>$E (r_f) %$</td>
<td>1.49</td>
<td>2.68</td>
<td>2.71</td>
<td>1.64</td>
</tr>
<tr>
<td>$E (r_e - r_f) %$</td>
<td>2.63</td>
<td>0.94</td>
<td>0.98</td>
<td>2.51</td>
</tr>
</tbody>
</table>
Table 7: DHM Statistic Quantiles in the Model with Stochastic Volatility

This table reports the proportion of the Den Haan and Marcet (1994) statistic realizations below (above) the lower (upper) 5% threshold under the null of the exact solution for a range of calibrations of the model with highly persistent stochastic volatility. Results are based on 500 simulated economies each with 3500 periods (first 500 discarded).

<table>
<thead>
<tr>
<th>Calibration</th>
<th>Lower 5%</th>
<th>Upper 5%</th>
</tr>
</thead>
<tbody>
<tr>
<td>Baseline calibration</td>
<td>6.00%</td>
<td>4.80%</td>
</tr>
<tr>
<td>Calibration with $\gamma = 10$</td>
<td>5.80%</td>
<td>6.00%</td>
</tr>
<tr>
<td>Calibration with $\psi = 1.75$</td>
<td>5.40%</td>
<td>3.40%</td>
</tr>
<tr>
<td>Calibration with $\gamma = 15$ and $\psi = 2$</td>
<td>3.20%</td>
<td>10.0%</td>
</tr>
</tbody>
</table>
This figure shows the distribution of the Den Haan and Marcet (1994) statistic under the null of the exact solution in the model with recursive preferences. Results are based on 500 simulated economies each with 3500 periods (first 500 discarded). The figure reports the same result for the second-order perturbation solution. First-order perturbation solution implies very large realizations of the DHM statistic and corresponding results are not reported for presentational reasons.
Figure 2: Extreme Calibrations of Recursive Preferences

This figure shows the distribution of the Den Haan and Marcet (1994) statistic under the null of the exact solution in the model with recursive preferences for a range of extreme calibrations. Results are based on 500 simulated economies each with 3500 periods (first 500 discarded).
Figure 3: DHM Statistic Distribution in the Model with Habit Formation

This figure shows the distribution of the Den Haan and Marcet (1994) statistic under the null of the exact solution in the model with habit formation and high capital adjustment costs. Results are based on 500 simulated economies each with 3500 periods (first 500 discarded). The figure reports the same result for the second-order perturbation solution. First-order perturbation solution implies very large realizations of the DHM statistic and corresponding results are not reported for presentational reasons.
Figure 4: Extreme Calibrations of Habit Formation

This figure shows the distribution of the Den Haan and Marcet (1994) statistic under the null of the exact solution in the model with habit formation and high capital adjustment costs for a range of extreme calibrations. Results are based on 500 simulated economies each with 3500 periods (first 500 discarded).
Figure 5: Consumption Impulse Responses to Volatility Shocks

This figure reports the responses of consumption to a one standard deviation shock to the variance of productivity growth computed using first-, second-, third-, and fourth-order perturbation and risk-adjusted log-linearization. Perturbation results are obtained using Dynare ++ software.
Figure 6: DHM Statistic Distribution in the Model with Stochastic Volatility

This figure shows the distribution of the Den Haan and Marcet (1994) statistic under the null of the exact solution in the model with highly persistent stochastic volatility for the benchmark and a range of extreme calibrations. Results are based on 500 simulated economies each with 3500 periods (first 500 discarded).