Asset Prices in Affine Real Business Cycle Models

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Abstract

We develop a tractable way to study jointly macroeconomic quantities and asset prices in a class of real business cycle models. The main ingredients are affine structure of shocks and recursive preferences. The framework is compatible with endowment economy models commonly used in asset pricing literature. We show theoretically the close link of the method to standard perturbation techniques and find that the quantitative difference between the two is insignificant for several models of interest.

1 Introduction

In this note we show a way to solve real business cycle models with general affine structure of shocks and recursive preferences. We look for an analytical solution by transforming the model into a system of approximate log-linear difference equations and exploit the normality of shocks as it is common in finance literature. The aim is to have a simple, tractable framework allowing us to investigate the implications of rich information structure (in conjunction with Epstein-Zin-Weil preferences) for the co-movement of macroeconomic quantities and asset prices. We compare the technique to perturbation methods which are the standard in macroeconomic literature and show that the two are closely related. Moreover, in a realistic calibration of several interesting models the difference is not quantitatively important.

The idea is essentially based on Backus, Routledge and Zin (2008). We differ from these authors in the set of equilibrium conditions we choose to approximate. Namely, the Euler equation and the definition of the value function instead of the first-order and envelope conditions. We argue that this is more consistent and we explicitly relate this

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approach to higher order perturbation methods. Importantly (as will be explained below) we also allow for both stationary and non-stationary shocks in productivity.

Real business cycle literature typically assumes one un-anticipated shock to productivity and relies on various real frictions and special preference structure to generate desired dynamics. The interest of a richer information structure was pointed out by two strands of articles. Croce (2008) studies the long-run productivity risks with asset pricing implications as the main focus. Schmitt-Grohe and Uribe (2008) and Jaimovich and Rebelo (2008) look at the macro-economic co-movements driven by anticipated shocks, or news about productivity several periods ahead in the future. To our knowledge the impact of time varying uncertainty about productivity shocks on macroeconomic variables and asset prices has not been studied very extensively. Justiniano and Primiceri (2006) estimate the importance of stochastic volatility in total factor productivity shocks. Bloom, Floetotto, and Jaimovich (2009) set the task to study the implications of volatility regimes on macroeconomic variables.

Eraker (2008) and Eraker and Shaliastovich (2008) generalise the Bansal and Yaron (2004) setting and provide guidance on how to solve generalised affine endowment economy models. Their approach is closely related to the way we approximate the solution to the generic real business cycle model.

We compare our approximation to perturbation methods presented in Judd (1998). In particular, we choose a very tractable framework described in Schmitt-Grohe and Uribe (2004).

2 Setup

We extend the standard real business cycle framework with a representative agent, one good and physical capital by assuming Epstein-Zin-Weil (Epstein and Zin, 1989, 1991, Weil, 1989) preferences and allowing for a very general affine structure for shocks to productivity as well as stochastic volatility.

The representative consumer maximises the utility function defined recursively

$$\max_{C_t} U_t$$

where

$$U_t = \left(C_t^{1-1/\psi} + \beta (E_t(U_{t+1}^{1-\gamma}))^{1-1/\psi}\right)^{-1/1-\psi}$$

Unlike CRRA utility function, Epstein-Zin recursive preferences allow us to separate the elasticity of intertemporal substitution from the coefficient of relative risk aversion (see Epstein and Zin, 1989). The parameter $\gamma$ controls agents relative risk aversion and $\psi$ his elasticity of intertemporal substitution. The standard power utility can be obtained as
a special case by setting $\gamma = 1/\psi$. This separation has an important implication for the agents preferences towards the early resolution of uncertainty. In the power utility case investor is indifferent towards the timing of resolution of uncertainty, if $\gamma > 1/\psi$ ($\gamma < 1/\psi$) investor prefers early (late) resolution of uncertainty. Intuitively, with $\gamma > 1/\psi$ agents propensity to smooth consumption across states is greater than propensity to smooth consumption across time.

The consumption good is produced according to a Cobb-Douglas production function

$$Y_t = Z_t A_t^{1-\alpha} K_t^\alpha$$

The law of motion of capital is given by

$$K_{t+1} = (1 - \delta)K_t + Y_t - C_t$$

In addition lets define the marginal product of capital

$$R_t = \alpha \left( \frac{A_t}{K_t} \right)^{1-\alpha} + (1 - \delta)$$

$A_t$ and $Z_t$ are respectively the non-stationary and the stationary components of productivity.

$$\ln A_{t+1} - \ln A_t = x^1_{t+1}$$
$$\ln Z_{t+1} = x^2_{t+1}$$

The main ingredient of our paper is the specification for the exogenous state variables driving the technology shocks. We assume general affine dynamics for exogenous state variables

$$x_{t+1} = \Lambda_t + \Sigma_t \epsilon_{t+1}$$

with

$$\Lambda_t = H_0 + H_1 x_t$$
$$\left( \Sigma_t \Sigma_t^T \right)_{ij} = (G_0)_{ij} + (G_1)_{ij} x_t$$

where $H_0$ is $(n \times 1)$, $H_1$ is $(n \times n)$, $G_0$ is $(n \times n)$, $G_1$ is $(n \times n \times n)$ and $\epsilon_t \sim N(0, I_n)$.Lets also define $(1 \times n)$ vectors $\iota_j$ such that $x^j_t = \iota_j x_t$. 

3
3 Solution

3.1 Stationarised version and equilibrium conditions

The value function

\[ V(K_t, A_t) = \max_{C_t} (U_t) \]

Define the scaled variables \( \tilde{X}_t = \frac{X_t}{A_{t-1}} \) and the scaled value function \( \tilde{V}_t = V(\tilde{K}_t, \tilde{A}_t).U_t \), \( Y_t, K_{t+1} \) are homogeneous of degree 1 in \( A_t \) and \( K_t \), therefore the value function is hom1 in \( A_t \) and \( K_t \) as well. We can scale the problem by \( A_{t-1} \).

As shown in the Appendix A the optimal policy is determined by the Euler equation in scaled variables

\[
E_t \left[ \beta \tilde{A}_t^{-1/\psi} \left( \frac{\tilde{V}_{t+1}}{E_t(\tilde{V}_{t+1}^{-\gamma})^{1/\gamma}} \right)^{1/\psi-\gamma} \left( \frac{\tilde{C}_{t+1}}{\tilde{C}_t} \right)^{-1/\psi} R_{t+1} \right] = 1
\]  

(1)

with state variables evolution

\[
\tilde{K}_{t+1} = (1-\delta) \tilde{K}_t \tilde{A}_t^{-1} + Z_t \tilde{A}_t^{-\alpha} \tilde{K}_t^\alpha - \tilde{C}_t \tilde{A}_t^{-1} \\
R_t = (1-\delta) + \alpha Z_t \tilde{A}_t^{1-\alpha} \tilde{K}_t^{-1} \\
\ln \tilde{A}_t = x_t^1 \\
\ln Z_t = x_t^2
\]

As the value function enters the Euler equation we will need to introduce another equation, namely the definition of the value function

\[
\tilde{V}_t^{1-1/\psi} = \max_{\tilde{C}_t} \left( \tilde{C}_t^{1-1/\psi} + \tilde{A}_t^{-1-1/\psi} \beta (E_t(\tilde{V}_{t+1}^{-\gamma}))^{(1-1/\psi)} \right)
\]  

(2)

In the case with no uncertainty we can solve the model in closed form, see Appendix B. We are going to approximate these equations defining the equilibrium around the non-stochastic steady state\(^1\).

\(^1\)Backus Routledge and Zin (2007) log-linearise a different set of conditions. Namely they choose the first order and the envelope conditions of the dynamic programming problem. We argue that the choice of the Euler equation and the definition of the value function is more consistent. As discussed in Caldara et al. (2009) when we work with first order conditions (the so-called Value Function Perturbation approach) we need to take the derivatives of the value function one order higher than the order of approximation of decision rules. Equivalent to this is expanding all equilibrium conditions to the same order (the so-called Equilibrium Conditions Perturbation). Log-linearising the envelope condition as in Backus Routledge and Zin (2007) amounts to finding the second order derivative in the value function expansion. This is insufficient and inconsistent with the first order log-linearisation. For instance, Backus Routledge and Zin (2007) approach will not result in a value function approximation which is homogeneous of degree
3.2 Log-linearisation of the state variables

The strategy is to seek an approximate analytical solution by transforming the model into a system of approximate log-linear difference equations. Let’s first start with log-linearisation of the state variables dynamics. Define \( y = \ln Y \) and \( \hat{y} \) the deviation from the non-stochastic steady state values for scaled and stationary variables. Exogenous state variables are log-linear by assumption. \( \hat{k}_{t+1} \) and \( r_t \) can be log-linearised as follows (see Appendix C for details):

\[
\hat{k}_{t+1} \approx k_k \hat{k}_t + k_a \hat{a}_t + k_z \hat{z}_t + k_c \hat{c}_t \\
\hat{r}_t \approx r_{ak} (\hat{a}_t - \hat{k}_t) + r_z \hat{z}_t
\]

3.3 Log-linearisation of decision rules

We assume that the optimal consumption rule and the value function are log-linear in the state variables. We are going to solve for the coefficients using the equilibrium conditions.

\[
\hat{v}_t = v_0 + v_k \hat{k}_t + v_x \hat{x}_t \\
\hat{c}_t = c_0 + c_k \hat{k}_t + c_x \hat{x}_t
\]

The Euler equation (1) can be written as

\[
E_t (e^{m_{t+1} + r_{t+1}}) = 1
\]

which implies the following condition

\[
E_t (m_{t+1} + r_{t+1}) + \frac{1}{2} Var_t (m_{t+1} + r_{t+1}) = 0
\]

where

\[
m_{t+1} = \ln \beta + (1/\psi - \gamma) \left( \hat{v}_{t+1} - E_t \hat{v}_{t+1} + \frac{1}{2} (1 - \gamma) Var_t (\hat{v}_{t+1}) \right) - 1/\psi \hat{a}_t - 1/\psi (\hat{c}_{t+1} - \hat{c}_t)
\]

As the value function enters the stochastic discount factor we also have to approximate it by log-linearising its definition (2)

\[
\zeta_1 \hat{v}_t = \zeta_2 \hat{c}_t + \zeta_3 \left( \hat{a}_t + E_t \hat{v}_{t+1} + \frac{1}{2} (1 - \gamma) Var_t (\hat{v}_{t+1}) \right)
\]

Note that at each step we exploit the log-linear form of all the expressions of interest and the normality of the shocks to the exogenous state variables. After regrouping the terms in the state variables we obtain a system of equations for the coefficients of the optimal consumption rule and the value function. In the most general form this is a one in capital and productivity. We therefore suggest using equilibrium conditions for log-linearisation.
system of quadratic equations which has to be solved numerically. In the next section we present two examples in which closed form solution can be found. Namely the cases where \( x_{t+1} \) is \( VAR(1) \) and a simple growth model with stochastic volatility. The derivations are provided in Appendix D.

### 3.4 Quantities and Prices

The log-linear structure of the model is convenient to derive and study the time series properties of the variables of interest. For illustrative purpose we will focus on just some of them.

The log consumption growth is

\[
g_{t+1}^c = \hat{c}_{t+1} - \hat{c}_t + \tilde{a}_t
\]

The one-period risk-free rate is defined by

\[
r_f^t = - \ln E_t M_{t+1}
\]

and has the following expression

\[
r_f^t = - \ln \beta - \frac{1}{2} (1/\psi - \gamma) (1 - \gamma) \text{Var}_t (\tilde{v}_{t+1}) - \frac{1}{2} \text{Var}_t (m_{t+1}) + 1/\psi E_t (g_{t+1}^c)
\]

Let's define the entropy of the stochastic discount factor as \( \ln E_t M_{t+1} - E_t \ln M_{t+1} \). Alvarez and Jermann (2005) show that mean excess returns, defined as differences of logs of gross returns, place a lower bound on the entropy. We can interpret it as the Hansen-Jagannathan-type (Hansen and Jagannathan, 1991) measure of risk premia. Because of conditional normality the entropy depends only on the second moment and has the following simple expression

\[
\ln E_t M_{t+1} - E_t \ln M_{t+1} = \frac{1}{2} \text{Var}_t (m_{t+1})
\]

Returns on any asset \( i \) satisfies \( E_t \left( e^{m_{t+1} + r_{t+1}^i} \right) = 1 \). To the first order all expected returns are the same \( E_t (r_{t+1}^i) = -E_t (m_{t+1}) \). Using log-normality to adjust for risk we can show that the risk premium of any asset \( i \) is

\[
E_t \left( r_{t+1}^i - r_f^t \right) + \frac{1}{2} \text{Var}_t (r_{t+1}^i) = -\text{Cov}_t \left( r_{t+1}^i, m_{t+1} \right)
\]

where \( \frac{1}{2} \text{Var}_t (r_{t+1}^i) \) is a Jensen’s inequality correction term.
For any dividend stream defined as

\[ d_t^i = d_0^i + d_k^i \hat{k} + d_x^i x_t \]

the price of the corresponding asset can be computed using the standard Campbell-Shiller approximation of returns and assuming that the log price-dividend ratio is affine in state variables as it is common in consumption-based asset pricing models (see for example Eraker, 2008).

4 Relation to higher-order perturbation methods

Our model can be specified in Schmitt-Grohe and Uribe (2004) form. The solution to the recursively defined model is given by equilibrium conditions written as

\[ E_t f (y_{t+1}, y_t, x_{t+1}, x_t) = 0 \]  \( (3) \)

where the policy function for control variable vector \( y_t \) and law of motion for state variables \( x_t \) are

\[
\begin{align*}
    y_t &= g(x_t, \sigma) \\
    x_{t+1} &= h(x_t, \sigma) + \sigma \epsilon_{t+1}
\end{align*}
\]

The idea of perturbation methods is to approximate the functions \( g \) and \( h \) around the point \( \sigma = 0 \) and \( x_t = \bar{x} \), where \( \bar{x} \) is defined by the solution to \( f (\bar{y}, \bar{y}, \bar{x}, \bar{x}) = 0 \). The derivatives \( g \) and \( h \) with respect to \( x \) and \( \sigma \) are found by differentiating (3) and evaluating the derivatives at the steady state which gives us a system of equations. Note that perturbation methods can be applied to any transformation of the variables in the model, for instance we can specify (3) in terms of the logs of the variables.

With the assumption \( E_t (\epsilon_{t+1}) = 0 \) Schmitt-Grohe and Uribe (2004) show that \( g_\sigma = h_\sigma = g_{\sigma x} = h_{\sigma x} = 0 \). This implies that the first order approximation is not affected by volatility of the shocks, the unconditional means of the variables are equal to the non-stochastic steady state values. Therefore, to the first order all returns are the same in equilibrium, risk premia are eliminated. Second-order approximation produces constant risk premia which depend on \( g_{\sigma \sigma} \) terms.
The resulting optimal rules when we approximate up to the second order are

\[
[g(x_t, \sigma)]^i = [g(\bar{x}, 0)]^i + [g_x(\bar{x}, 0)]^i (x - \bar{x}) + \frac{1}{2} (x - \bar{x})^T [g_{xx}(\bar{x}, 0)]^i (x - \bar{x}) + \frac{1}{2} [g_{\sigma\sigma}(\bar{x}, 0)]^i [\sigma] [\sigma]
\]

(4)

\[
[h(x_t, \sigma)]^i = [h(\bar{x}, 0)]^i + [h_x(\bar{x}, 0)]^i (x - \bar{x}) + \frac{1}{2} (x - \bar{x})^T [h_{xx}(\bar{x}, 0)]^i (x - \bar{x}) + \frac{1}{2} [h_{\sigma\sigma}(\bar{x}, 0)]^i [\sigma] [\sigma]
\]

We can show, see Appendix I, that the standard perturbation solution and the log-linearisation approach are closely related. For models without stochastic volatility the log-linear approximation can be written as follows

\[
[g(x_t, \sigma)]^i = [g(\bar{x}, 0)]^i + [g_x(\bar{x}, 0)]^i (x - \bar{x}) + \frac{1}{2} [g_{\sigma\sigma}(\bar{x}, 0)]^i [\sigma] [\sigma]
\]

\[
[h(x_t, \sigma)]^i = [h(\bar{x}, 0)]^i + [h_x(\bar{x}, 0)]^i (x - \bar{x}) + \frac{1}{2} [h_{\sigma\sigma}(\bar{x}, 0)]^i [\sigma] [\sigma]
\]

Compared to (4) there are two differences. First, we drop the quadratic terms \(g_{xx}\) and \(h_{xx}\). Second, \(g_{\sigma\sigma}, h_{\sigma\sigma}\) and \(g_{\sigma\sigma}^*, h_{\sigma\sigma}^*\) are not exactly the same. Schmitt-Grohe and Uribe (2004) show how to compute \(g_{\sigma\sigma}\) and \(h_{\sigma\sigma}\) from other first and second order terms. \(g_{\sigma\sigma}^*\) and \(h_{\sigma\sigma}^*\) can be computed in exactly the same way except for, again, ignoring a term in \(g_{xx}\). In the next section we will present numerical results showing that the difference between the two methods is not significant for a class of models.

The main drawback of the first order perturbation is that risk adjustments embedded in \(g_{\sigma\sigma}\) and \(h_{\sigma\sigma}\) terms are omitted. If second order terms \(g_{xx}\) and \(h_{xx}\) are not important for a particular model, then the log-linearisation presented in this paper allows to capture both the dynamics of the variables\(^2\) and the adjustments for risk with a reasonable degree of accuracy.

We conjecture that for models with stochastic volatility the method results in adding selectively some higher order terms to the first order perturbation solution. Stochastic volatility models are difficult to deal with using perturbation techniques as higher order terms have to be computed. Our method allows to obtain a satisfactory approximation.

5 Specific models

In this section we consider two important examples in which closed-form expression for coefficients of optimal rules is available.

\(^2\)For instance the second moments of the series generated by the first order approximation only are correct to the second order accuracy (Schmitt-Grohe and Uribe, 2004).
5.1 VAR(1) state variables

Assume the vector of exogenous state variables is first order vector auto-regressive

\[ x_{t+1} = H_0 + H_1 x_t + H_2 \varepsilon_{t+1} \]

where \( H_0 \) is \((n \times 1)\), \( H_1 \) is \((n \times n)\), \( H_2 \) is \((n \times n_c)\). In other terms we assume an affine model with constant volatility. Appendix E provides the closed-form solution.

An interesting application is Schmitt-Grohe and Uribe (2008) anticipated shocks model. We assume that the non-stationary (leaving out the stationary shocks for parsimony reasons) TFP shocks are autoregressive and subject to anticipated \((\varepsilon^0, \varepsilon^1, \varepsilon^2)\) and unanticipated \((\varepsilon)\) innovations.

\[ x^1_t = (1 - \rho) \lambda + \rho x^1_{t-1} + \varepsilon_t + \varepsilon^0_{t-1} + \varepsilon^1_{t-2} + \varepsilon^2_{t-3} \]

The agent learns about innovation \( \varepsilon_t \) at date \( t \) and it affects the productivity at the same date. Innovations \( \varepsilon^0_{t-1}, \varepsilon^1_{t-2}, \varepsilon^2_{t-3} \) are anticipated one, two and three periods ahead - they affect date-\( t \) productivity, but are period \( t - 1, t - 2, t - 3 \) information respectively. Therefore, at date \( t \) the agent learns about 4 shocks \( \varepsilon_t, \varepsilon^0_t, \varepsilon^1_t \) and \( \varepsilon^2_t \), affecting productivity immediately and in one, two and three periods ahead. We assume all shocks are independent. Additional lagged innovations can easily be incorporated in to the recursive formulation of the problem by increasing the number of state variables. The matrices \( H_0, H_1, \) and \( H_2 \) are

\[
H_0 = \begin{pmatrix}
(1 - \rho) \lambda \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0
\end{pmatrix},
H_1 = \begin{pmatrix}
\rho & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix},
H_2 = \begin{pmatrix}
\sigma & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & \sigma^0 & 0 & 0 \\
0 & 0 & \sigma^1 & 0 \\
0 & 0 & 0 & \sigma^2 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

Asset pricing implication of this model are studied in detail in Malkhozov and Shamloo (2009). Here we would like to compare the optimal policy functions obtained with log-linearisation and second order perturbation technique. We consider the following calibration

<table>
<thead>
<tr>
<th>( \alpha )</th>
<th>( \delta )</th>
<th>( \lambda )</th>
<th>( \rho )</th>
<th>( \beta )</th>
<th>( \psi )</th>
<th>( \gamma )</th>
<th>( \sigma )</th>
<th>( \sigma^0 )</th>
<th>( \sigma^1 )</th>
<th>( \sigma^2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.34</td>
<td>0.025</td>
<td>0.006</td>
<td>0.14</td>
<td>0.995</td>
<td>1.5</td>
<td>10</td>
<td>0.0059</td>
<td>0.023</td>
<td>0.013</td>
<td>0.011</td>
</tr>
</tbody>
</table>
The consumption rule and the value function produced by the two respective methods
are given by\(^3\)

\[
\hat{c}_t = -\frac{1}{2} \cdot 0.0910 \\
+ \begin{pmatrix}
0.7025 & 0.3380 & 0.1098 & 0.2891 & 0.2750 & 0.2616 & 0.2891 & 0.2750 & 0.2891 \\
\end{pmatrix} \begin{pmatrix}
\dot{k}_t \\
\dot{x}_t \\
\end{pmatrix}
\]

\[
\hat{c}_t = -\frac{1}{2} \cdot 0.0906 \\
+ \begin{pmatrix}
0.7025 & 0.3380 & 0.1098 & 0.2891 & 0.2750 & 0.2616 & 0.2891 & 0.2750 & 0.2891 \\
\end{pmatrix} \begin{pmatrix}
\dot{k}_t \\
\dot{x}_t \\
\end{pmatrix}
+ \text{quadratic terms}
\]

\[
\hat{v}_t = -\frac{1}{2} \cdot 1.5726 \\
+ \begin{pmatrix}
0.0387 & 1.1174 & 0.0039 & 1.1144 & 1.1144 & 1.1084 & 1.1144 & 1.1144 & 1.1144 \\
\end{pmatrix} \begin{pmatrix}
\dot{k}_t \\
\dot{x}_t \\
\end{pmatrix}
\]

\[
\hat{v}_t = -\frac{1}{2} \cdot 1.5725 \\
+ \begin{pmatrix}
0.0387 & 1.1174 & 0.0039 & 1.1144 & 1.1144 & 1.1084 & 1.1144 & 1.1144 & 1.1144 \\
\end{pmatrix} \begin{pmatrix}
\dot{k}_t \\
\dot{x}_t \\
\end{pmatrix}
+ \text{quadratic terms}
\]

As we can see the first order dynamics are exactly the same. The difference between risk
adjustments is not significant for both consumption and the value function. Finally (see
Appendix H for numerical values) we can verify that additional quadratic terms do not
contribute significantly to the dynamics of the variables.

### 5.2 Stochastic volatility in a simple growth model

Lets introduce stochastic volatility in the simple version of the growth model. The shock
structure can be described as follows

\[
x_{t+1}^1 = \mu + \sigma_t \varepsilon_{t+1} \\
x_{t+1}^2 = 0 \\
\sigma_{t+1}^2 = (1 - \varphi) \theta + \varphi \sigma_{t}^2 + \omega \varepsilon_{t+1}
\]

where \( E_t (\varepsilon_{t+1} \varepsilon_{t+1}) = 0 \). See Appendix F for the solution details.

\(^3\)If in the perturbation algorithm the \( g_{xx} \) is forced to be 0 we obtain exactly the same answer as with
log-linearisation.
The conditional variance of consumption growth \( \text{Var}_t(\delta c_{t+1}) = (\sigma_e^c)^2 = c_e^2 \sigma_e^2 + c_e^2 \omega^2 \). It inherits the form of the productivity variance process

\[
(\sigma_e^c)^2 = (1 - \varphi) \left( c_e^2 \theta + c_e^2 \omega^2 \right) + \varphi (\sigma_e^c)^2 + c_e^2 \omega \epsilon_{t+1}
\]

This is also the form assumed by several recent consumption-based asset pricing models. See for example Bansal and Yaron (2004), Backus, Routledge and Zin (2008) and Beeler and Campbell (2009). Given a particular calibration for the exogenous consumption process we can easily back-engineer productivity shock variance process needed to obtain consumption variance dynamics estimated or assumed in the literature. We will assume the following monthly parameters for the growth model

<table>
<thead>
<tr>
<th>( \alpha )</th>
<th>( \delta )</th>
<th>( \mu )</th>
<th>( \beta )</th>
<th>( \gamma )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.33</td>
<td>1 - 0.9^{\frac{1}{12}}</td>
<td>0.02^{\frac{1}{12}}</td>
<td>0.98^{\frac{1}{12}}</td>
<td>5</td>
</tr>
</tbody>
</table>

Some popular calibration of the variance of the consumption growth summarised in Beeler and Campbell (2009) include

<table>
<thead>
<tr>
<th>( \theta^c )</th>
<th>( \varphi^c )</th>
<th>( \omega^c )</th>
</tr>
</thead>
<tbody>
<tr>
<td>BKY1</td>
<td>0.00002916</td>
<td>0.980</td>
</tr>
<tr>
<td>BY</td>
<td>0.00006084</td>
<td>0.987</td>
</tr>
<tr>
<td>BKY2</td>
<td>0.00005184</td>
<td>0.999</td>
</tr>
</tbody>
</table>

To obtain these dynamics in equilibrium of an simple real business cycle model we will need to make the following assumptions about the variance of the productivity shock
Interestingly, even when innovations to productivity growth and productivity growth variance are uncorrelated, innovation to consumption growth and consumption growth variance are.

\[
\text{Cov}(g_{t+1}, \sigma_{t+1}^c)^2 = c_o c^2 \omega^2
\]

This covariance is constant, which implies that correlation varies in time with the level of volatility. Consumption based asset pricing models assume ad hoc a zero correlation. The real business cycle analysis suggests that this is unlikely to emerge in equilibrium. The importance of the correlation between consumption growth and consumption growth variance for asset pricing has been highlighted by Backus, Routledge and Zin (2008). Figure 1 shows correlations for various values of parameters assuming the level of variance is equal to the long-term mean \( \theta \). Higher elasticity of intertemporal substitution implies a stronger negative correlation.
Figure 1. Correlation between consumption growth and variance of consumption growth for different calibrations and values of $\psi$.

Asset pricing implications of stochastic volatility of productivity are studied in detail in Malkhizov and Shamloo (2009). Here (Figures 2 and 3) we present the responses macro variables to productivity and variance shocks.

Figure 2. Impulse responses of capital and consumption (logs) to a one standard deviation shock to productivity (above) and variance (below), BY calibration, $\psi = 1.5$. 
Figure 3. Impulse responses of capital and consumption (logs) to a one standard deviation shock to productivity (above) and variance (below), BY calibration, $\psi = 0.2$.

6 Conclusion

We suggest a way to solve real business cycle models using approximation techniques common in asset pricing literature. Even if eventually we prefer perturbation methods for their generality, we argue that log-linearisation as presented in this paper is a very convenient tool that allows to capture correctly not only quantities dynamics, but also asset pricing and welfare implications. We show precisely how the method is related to standard higher order perturbation approach.
References


A Equilibrium conditions

The value function

\[ V(K_t, A_t) = \max_{C(K_t, Z_t)} (U_t) \]

\[ V_t = \max_{C(K_t, Z_t)} \left( (1 - \beta)C_t^{1-1/\psi} + \beta(E_t(V_{t+1})^{1-1/\gamma})^{1-1/\psi} \right) \]

Define the scaled variables \( \tilde{X}_t = \frac{X_t}{A_{t-1}} \) and the scaled value function \( \tilde{V}_t = V(\tilde{K}_t, \tilde{A}_t).U_t \), \( Y_t, K_{t+1} \) are h.o.d in \( A_t \) and \( K_t \), therefore the value function is h.o.d in \( A_t \) and \( K_t \) as well. In particular we can scale the problem by \( A_{t-1} \).

\[ V(K_t, A_t) = A_{t-1}V\left( \frac{K_t}{A_{t-1}}, \frac{A_t}{A_{t-1}} \right) \]

In scaled variables the equilibrium conditions can be written as

\[ \frac{V_t}{A_{t-1}} = \max_{C(K_t, A_t)} \left( \left( \frac{C_t}{A_t} \right)^{1-1/\psi} + \frac{1}{A_{t-1}^{1-1/\psi}} \beta(E_t(V_t^{1-1/\gamma})^{1-1/\gamma})^{1-1/\psi} \right) \]

\[ \frac{V_t}{A_{t-1}} = \max_{C(K_t, A_t)} \left( \left( \frac{C_t}{A_t} \right)^{1-1/\psi} + \frac{1}{A_{t-1}^{1-1/\psi}} \beta(E_t(V_t^{1-1/\gamma})^{1-1/\gamma})^{1-1/\psi} \right) \]

\[ \tilde{V}_t = \max_{C(K_t, Z_t)} \left( \tilde{C}_t^{1-1/\psi} + \tilde{A}_t^{1-1/\psi} \beta(E_t(V_t^{1-1/\gamma})^{1-1/\gamma})^{1-1/\psi} \right) \]

The first order condition with respect to \( \tilde{C}_t \) and the envelope condition are as follows

\[ \tilde{C}_t^{1-1/\psi} = \beta \tilde{A}_t^{1-1/\psi} E_t \left[ \tilde{V}_{t+1}^{1-1/\gamma} \right] \]

\[ \tilde{V}_{\tilde{K}_t} = \beta \tilde{A}_t^{1-1/\psi} V_t^{1-1/\psi} E_t \left[ \tilde{V}_{t+1}^{1-1/\gamma} \right] \]

Iterating the envelope condition one period forward and combining it with the first order condition we obtain the Euler equation for consumption and an expression for \( \tilde{V}_{\tilde{K}_t} \)

\[ \beta E_t \left[ \tilde{A}_t^{1-1/\psi} V_t^{1-1/\gamma} \left( \tilde{V}_{t+1}^{1-1/\gamma} \right)^{1-\gamma} \tilde{V}_{t+1}^{1-1/\gamma} \right] = 1 \]

Notice that \( R_t \) can be expressed in term of scaled variables
\[ R_t = (1 - \delta) + \alpha Z_t \tilde{A}_t^{1-\alpha} \tilde{K}_t^{\alpha-1} \]

Lets sum up the relations defining equilibrium

\[
E_t \left[ \beta \tilde{A}_t^{-1/\psi} E_t \left[ \tilde{V}_{t+1}^{1-\gamma} \right] \right]^{\frac{1-1/\psi}{1-\gamma}} \tilde{V}_{t+1}^{1-\gamma+1/\psi} \frac{\tilde{C}_t^{1-1/\psi}}{C_t^{-1/\psi}} R_{t+1} = 1
\]

\[
\tilde{V}_t^{1-\psi} - \max_{C(\tilde{K}_t, \tilde{Z}_t)} \left( \tilde{C}_t^{1-1/\psi} + \tilde{A}_t^{1-1/\psi} \beta (E_t(\tilde{V}_{t+1}^{1-\gamma}))^{\frac{1-1/\psi}{1-\gamma}} \right) = 0
\]

\[
\tilde{K}_{t+1} - (1 - \delta) \tilde{K}_t \tilde{A}_t^{-1} - Z_t \tilde{A}_t^{-\alpha} \tilde{K}_t^{\alpha} + \tilde{C}_t \tilde{A}_t^{-1} = 0
\]

B Non-stochastic steady state

The non-stochastic steady state

\[
x = (I - H)\alpha^{-1} H_0
\]

\[
\tilde{A} = \exp(x^1)
\]

\[
Z = \exp(x^2)
\]

\[
R = \beta^{-1} \tilde{A}^{1/\psi}
\]

\[
\tilde{K} = \tilde{A} \left[ \frac{\beta^{-1} \tilde{A}^{1/\psi} - (1 - \delta)}{\alpha Z} \right]^{\frac{1}{\alpha - 1}}
\]

\[
\tilde{Y} = Z \tilde{A}^{1-\alpha} \tilde{K}^{\alpha}
\]

\[
\tilde{C} = (1 - \delta) \tilde{K} + \tilde{Y} - \tilde{K} \tilde{A}
\]

\[
\tilde{V} = \tilde{C} \left( \frac{1}{1 - \beta \tilde{A}^{1-\frac{1}{\psi}}} \right)^{\frac{1}{1-\psi}}
\]

C Log-linearisation of state variables

\( \hat{k}_{t+1} \) can be log-linearised as follows

\[
\hat{k}_{t+1} \approx k_k \hat{k}_t + k_a \hat{a}_t + k_z \hat{z}_t + k_c \hat{c}_t
\]

where
\[ k_k = (1 - \delta)\hat{A}^{-1} + \alpha Z\hat{A}^{-\alpha}\hat{K}^{\alpha - 1} \]
\[ k_a = -(1 - \delta)\hat{A}^{-1} - \alpha Z\hat{A}^{-\alpha}\hat{K}^{\alpha - 1} + \hat{C}\hat{K}^{-1}\hat{A}^{-1} \]
\[ k_z = -\hat{C}\hat{K}^{-1}\hat{A}^{-1} = -k_k - k_a \]
\[ k_c = Z\hat{A}^{-\alpha}\hat{K}^{\alpha - 1} \]

Additionally, let's log-linearise \( r_t \)

\[ \dot{r}_t \approx r_{ak}\left(\dot{a}_t - \dot{k}_t\right) + r_z\dot{z}_t \]

where

\[ r_{ak} = \alpha(1 - \alpha)Z\hat{A}^{1 - \alpha}\hat{K}^{\alpha - 1}R^{-1} \]
\[ r_z = \alpha Z\hat{A}^{1 - \alpha}\hat{K}^{\alpha - 1}R^{-1} \]

## D General affine case

The log of the stochastic discount factor is

\[ m_{t+1} = \ln \beta + (1/\psi - \gamma)\left(\tilde{v}_{t+1} - \frac{1}{(1 - \gamma)}\ln E_t(e^{(1-\gamma)\tilde{v}_{t+1}})\right) - 1/\psi\tilde{a}_t - 1/\psi(\tilde{c}_{t+1} - \tilde{c}_t) \]

Because of the assumed log-linearity \( \tilde{v}_t \) of and the normality of shocks it can be written\(^4\)

\[ m_{t+1} = \ln \beta + (1/\psi - \gamma)\left(\tilde{v}_{t+1} - E_t\tilde{v}_{t+1} + \frac{1}{2}(1 - \gamma)Var_t(\tilde{v}_{t+1})\right) - 1/\psi\tilde{a}_t - 1/\psi(\tilde{c}_{t+1} - \tilde{c}_t) \]

The Euler equation (1) can be written as \( E_t(e^{m_{t+1} + r_{t+1}}) = 1 \) which again using the log-normal structure of the model implies the following condition

\[ E_t(m_{t+1} + r_{t+1}) + \frac{1}{2}Var_t(m_{t+1} + r_{t+1}) = 0 \]

Some preliminary expressions. First,

\[ -1/\psi\tilde{a}_t - 1/\psi(\tilde{c}_{t+1} - \tilde{c}_t) + r_{t+1} = \]

\(^4\)Useful result about certainty equivalents under log-normality. If \( \ln x \sim N(\mu, \nu) \) then \( \frac{1}{\alpha}\ln E(x^\alpha) - E(\ln x) = \frac{1}{2}\alpha\nu^2 \).
\[-1/\psi \left[ k_c c_k^2 + (k_k - 1 + \psi r_{ak} k_c) c_k + \psi r_{ak} k_k \right] \hat{k}_t \]

\[-1/\psi \left[ \xi_1 (I + (c_k + \psi r_{ak}) k_a I - \psi r_{ak} H_1) + \nu_2 ((c_k + \psi r_{ak}) k_2 I - \psi \tau_2 H_1) \right] \hat{x}_t \]

\[-1/\psi [c_x \Sigma_t - \psi (r_{ak} l_1 + r_{\varepsilon} l_2) \Sigma_t] \varepsilon_{t+1} \]

\[-1/\psi [c_0 (c_k + \psi r_{ak}) k_e] \]

Next\(^5\),

\[
Var_t (\hat{\varepsilon}_{t+1}) = (v_x \Sigma_t) (v_x \Sigma_t)^T = v_x \Sigma_t \Sigma_t^T v_x^T = v_x G_0 v_x + v_x G_1 v_x \hat{x}_t
\]

Finally,

\[
Var_t (m_{t+1} + \tau_{t+1}) = (l \Sigma_t) (l \Sigma_t)^T = l \Sigma_t \Sigma_t^T l^T = l G_0 l^T + l G_1 l^T \hat{x}_t
\]

where

\[
l = (-\gamma + 1/\psi) v_x - 1/\psi (c_x - \psi (r_{ak} l_1 + r_{\varepsilon} l_2))
\]

As the value function enters the Euler equation we need to approximate it by log-linearising its definition (2) to complete the solution. Again using log-normality we can rewrite (2)

\[
e^{(1-1/\psi) \hat{\varepsilon}_t} = e^{(1-1/\psi) \hat{\varepsilon}_t} + \beta e^{(1-1/\psi) (\hat{a}_t + E_t \hat{\varepsilon}_{t+1} + \frac{1}{2} (1-\gamma) Var_t (\hat{\varepsilon}_{t+1}))}
\]

and linearise it

\[
\zeta_1 \hat{\varepsilon}_t = \zeta_2 \hat{\varepsilon}_t + \zeta_3 \left( \hat{a}_t + E_t \hat{\varepsilon}_{t+1} + \frac{1}{2} (1-\gamma) Var_t (\hat{\varepsilon}_{t+1}) \right)
\]

where

\[
\zeta_1 = \hat{\varepsilon}^{1-\frac{1}{\psi}}
\]

\[
\zeta_2 = \hat{\varepsilon}^{1-\frac{1}{\psi}}
\]

\[
\zeta_3 = \beta A^{1-\frac{1}{\psi}} \hat{\varepsilon}^{1-\frac{1}{\psi}}
\]

Regrouping of terms in the conditions implied by the Euler equation and the linearisation of the value function definition give us the following system of equations for

\(^5\kappa_2 G_1 \kappa_2^T\) is a \((1 \times n)\) vector with \(k^{th}\) element equal to \(\sum_{i,j} \kappa_{2,i} G_{1,ijk} \kappa_{2,j} \).

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\[(c_0, c_k, c_x, v_0, v_k, v_x)\]

\[
k_k c_k + (k_k - 1 + \psi r_{ak} k_c) c_k + \psi r_{ak} k_k = 0
\]

\[
\begin{align*}
l_1 (I + (c_k + \psi r_{ak}) k_a I - \psi r_{ak} H_1) &+ l_2 ((c_k + \psi r_{ak}) k_z I - \psi \tau_2 H_1) \\
+ c_x ((-1 + (c_k + \psi r_{ak}) k_c) I + H_1) - \frac{1}{2} \psi ((\gamma - 1/\psi) (1 - \gamma) v_x G_1 v_x + I G_1^T I^T) & = 0
\end{align*}
\]

\[
c_0 (c_k + \psi r_{ak}) k_c - \frac{1}{2} \psi ((\gamma - 1/\psi) (1 - \gamma) (v_x G_0 v_x + v_x G_1 v_x) + I G_0^T + I G_1^T x) = 0
\]

\[
\zeta_1 v_k - \zeta_2 c_k - \zeta_3 v_k (k_k + k_c c_k) = 0
\]

\[
\begin{align*}
\zeta_1 v_x - \zeta_2 c_x - \zeta_3 \left( l_1 + v_k (k_a t_1 + k_z t_2 + k_c c_x) + v_x H_1 + \frac{1}{2} (1 - \gamma) v_x G_1 v_x \right) & = 0 \\
\zeta_1 v_0 - \zeta_2 c_0 - \zeta_3 \left( v_0 + v_k k_c c_0 + \frac{1}{2} (1 - \gamma) (v_x G_0 v_x + v_x G_1 v_x) \right) & = 0
\end{align*}
\]

In the most general case this is a system of quadratic equations we have to solve numerically.

### E Solution with VAR(1) state variables

\[
c_k = \frac{-(k_k - 1 + \psi r_{ak} k_c) \pm \sqrt{(k_k - 1 + \psi r_{ak} k_c)^2 - 4 k_k k_c \psi r_{ak}}}{2 k_c}
\]

\[
c_x = \left( l_1 (I + (c_k + \psi r_{ak}) k_a I - \psi r_{ak} H_1) \right) \left( ((-1 + (c_k + \psi r_{ak}) k_c) I + H_1)^{-1} \right) \left( \frac{1}{2} \psi ((\gamma - 1/\psi) (1 - \gamma) L_1 L_1^T + ((-\gamma + 1/\psi) L_1 - 1/\psi L_2) ((-\gamma + 1/\psi) L_1 - 1/\psi L_2)^T) \right)
\]

\[
c_0 = \frac{(c_k + \psi r_{ak}) k_c}{(c_k + \psi r_{ak}) k_c}
\]

\[
v_k = \frac{\zeta_2 c_k}{\zeta_1 - \zeta_3 (k_k + k_c c_k)}
\]

\[
v_x = \frac{(\zeta_2 c_x + \zeta_3 (l_1 + v_k (k_a t_1 + k_z t_2 + k_c c_x))) (\zeta_1 I - \zeta_3 H_1)^{-1}}{\zeta_1 - \zeta_3}
\]

\[
v_0 = \frac{\zeta_2 c_0 + \zeta_3 (v_k k_c c_0 + \frac{1}{2} (1 - \gamma) L_1 L_1^T)}{\zeta_1 - \zeta_3}
\]

where

\[
L_1 = v_x H_2
\]

\[
L_2 = (c_x - \psi (r_{ak} t_1 + r_z t_2)) H_2
\]
**F  Stochastic volatility model solution**

\[
\begin{align*}
    c_k &= \frac{- (k_k - 1 + \psi r_{ak} k_c) \pm \sqrt{(k_k - 1 + \psi r_{ak} k_c)^2 - 4 k_c k_k \psi r_{ak}}}{2 k_c} \\
    c_x &= \frac{1 + (c_k + \psi r_{ak}) k_a}{1 - (c_k + \psi r_{ak}) k_c} \\
    c_\sigma &= \frac{1/2 \psi \left( (\gamma - 1/\psi) (1 - \gamma) v_x^2 + (-1/\psi (c_x - \psi r_{ak}) + (-\gamma + 1/\psi) v_x)^2 \right)}{-1 + \varphi + (c_k + \psi r_{ak}) k_c} \\
    c_0 &= \frac{1/2 \psi \left( (\gamma - 1/\psi) (1 - \gamma) (v_x^2 \omega^2 + \sigma^2 \theta) + (-1/\psi c_\sigma \omega + (-\gamma + 1/\psi) \sigma \omega)^2 \right)}{(c_k + \psi r_{ak}) k_c} \\
    v_k &= \frac{\zeta_2 c_k}{\zeta_1 - \zeta_3 (k_k + k_c k_c)} \\
    v_x &= \frac{\zeta_2 c_x + \zeta_3 (1 + v_k (k_a + k_c x))}{\zeta_1} \\
    v_\sigma &= \frac{\zeta_2 v_0 + \zeta_3 (v_k c_0 + \zeta_1 (1 - \gamma) v_x^2)}{\zeta_1 - \zeta_3 \varphi} \\
    v_0 &= \frac{\zeta_2 c_0 + \zeta_3 (v_k k_c c_0 + \frac{1}{2} (1 - \gamma) (v_x^2 \omega^2 + \sigma^2 \theta))}{\zeta_1 - \zeta_3} \\
\end{align*}
\]

**G  Log-linearised model dynamics with VAR(1) state variables**

\[
\begin{align*}
    \begin{pmatrix}
        \hat{c}_t \\
        \hat{v}_t
    \end{pmatrix}
    &= \begin{pmatrix}
        c_0 & c_k & c_x \\
        v_0 & v_c & v_x
    \end{pmatrix} \begin{pmatrix}
        \hat{k}_t \\
        \hat{x}_t
    \end{pmatrix} \\
    \begin{pmatrix}
        \hat{k}_{t+1} \\
        \hat{x}_{t+1}
    \end{pmatrix}
    &= \begin{pmatrix}
        k_k c_0 \\
        0_{n \times 1}
    \end{pmatrix} + \begin{pmatrix}
        k_k + k_x c_k & k_a t_1 + k_c t_2 & k_c x_x \\
        0_{n \times 1} & H_1
    \end{pmatrix} \begin{pmatrix}
        \hat{k}_t \\
        \hat{x}_t
    \end{pmatrix} + \begin{pmatrix}
        0_{1 \times n} \\
        H_2
    \end{pmatrix} \epsilon_{t+1} \\
\end{align*}
\]

\[
\begin{align*}
    \begin{pmatrix}
        \hat{c}_t \\
        \hat{v}_t
    \end{pmatrix}
    &= \begin{pmatrix}
        c_0 & c_k & c_x & c_\sigma \\
        v_0 & v_c & v_x & v_\sigma
    \end{pmatrix} \begin{pmatrix}
        \hat{k}_t \\
        \hat{x}_t
    \end{pmatrix} \\
    \begin{pmatrix}
        \hat{k}_{t+1} \\
        \hat{a}_{t+1} \\
        \hat{\sigma}^2_{t+1}
    \end{pmatrix}
    &= \begin{pmatrix}
        k_k c_0 \\
        0 \\
        0
    \end{pmatrix} + \begin{pmatrix}
        k_k + k_x c_k & k_x + k_c c_x & k_c c_\sigma & \kappa_{t+1} \\
        0 & 0 & 0 & 0 \\
        0 & 0 & \varphi & \kappa_{t+1}
    \end{pmatrix} \begin{pmatrix}
        \hat{k}_{t+1} \\
        \hat{a}_{t+1} \\
        \hat{\sigma}^2_{t+1}
    \end{pmatrix} + \begin{pmatrix}
        0 & 0 \\
        \sigma_t & 0 \\
        0 & \omega
    \end{pmatrix} \epsilon_{t+1}
\end{align*}
\]

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H Second order terms for VAR(1) model

Quadratic terms of respectively consumption and value function optimal rules for the anticipated shocks model calibration

\[ 0.0472 -0.0505 -0.0526 -0.0230 -0.0180 -0.0135 -0.0230 -0.0180 -0.0230 -0.0174 -0.0152 -0.0132 -0.0174 -0.0152 -0.0174 \]
\[ -0.0505 0.0542 0.0543 0.0269 0.0211 0.0159 0.0269 0.0211 0.0269 0.0202 0.0178 0.0156 0.0202 0.0178 0.0202 \]
\[ -0.0526 0.0543 0.0727 0.0117 0.0057 0.0022 0.0117 0.0057 0.0117 0.0135 0.0074 0.0036 0.0135 0.0074 0.0135 \]
\[ -0.0230 0.0269 0.0117 0.0277 0.0224 0.0170 0.0277 0.0224 0.0277 0.0227 0.0188 0.0166 0.0200 0.0188 0.0200 \]
\[ -0.0180 0.0211 0.0057 0.0224 0.0233 0.0183 0.0224 0.0233 0.0224 0.0108 0.0187 0.0175 0.0108 0.0187 0.0108 \]
\[ -0.0135 0.0159 0.0022 0.0170 0.0013 0.0192 0.0170 0.0013 0.0192 0.0170 0.0051 0.0099 0.0174 -0.0051 0.0099 0.0051 \]
\[ -0.0230 0.0269 0.0117 0.0277 0.0224 0.0170 0.0277 0.0224 0.0277 0.0227 0.0188 0.0166 0.0200 0.0188 0.0200 \]
\[ -0.0180 0.0211 0.0057 0.0224 0.0233 0.0183 0.0224 0.0233 0.0224 0.0108 0.0187 0.0175 0.0108 0.0187 0.0108 \]
\[ -0.0230 0.0269 0.0117 0.0277 0.0224 0.0170 0.0277 0.0224 0.0277 0.0227 0.0188 0.0166 0.0200 0.0188 0.0200 \]
\[ -0.0174 0.0202 0.0135 0.0200 0.0108 0.0051 0.0200 0.0108 0.0200 0.0200 0.0218 0.0128 0.0071 0.0218 0.0128 \]
\[ -0.0152 0.0178 0.0074 0.0188 0.0187 0.0099 0.0188 0.0187 0.0188 0.0128 0.0207 0.0122 0.0071 0.0207 0.0122 \]
\[ -0.0132 0.0156 0.0036 0.0166 0.0175 0.0174 0.0166 0.0175 0.0166 0.0071 0.0122 0.0071 0.0122 0.0071 0.0122 \]
\[ -0.0174 0.0202 0.0135 0.0200 0.0108 0.0051 0.0200 0.0108 0.0200 0.0200 0.0218 0.0128 0.0071 0.0218 0.0128 \]
\[ -0.0152 0.0178 0.0074 0.0188 0.0187 0.0099 0.0188 0.0187 0.0188 0.0128 0.0207 0.0122 0.0071 0.0207 0.0122 \]
\[ -0.0174 0.0202 0.0135 0.0200 0.0108 0.0051 0.0200 0.0108 0.0200 0.0200 0.0218 0.0128 0.0071 0.0218 0.0128 \]

I Relation to perturbation methods

To illustrate the meaning of section consider a simple model which nevertheless captures all the elements of the result. Control \( y_t \) is a function of state variables \( x_t \) and a parameter \( \sigma \) scaling uncertainty. State variables evolution is assumed linear and so doesn’t need to be approximated.

\[ y_t = g(x_t, \sigma) \]
\[ x_{t+1} = h_x x_t + \sigma \eta_{t+1} \]

The equilibrium condition allowing us to approximate \( g \) is exponential in \( y_{t+1} \) and \( x_{t+1} \)

\[ E_t \left( e^{\alpha y_{t+1} + \beta x_{t+1} + \gamma} \right) = 1 \]

At the steady state (\( \sigma = 0 \))

\[ e^{\alpha y + \gamma} = \frac{1}{\alpha} \]

\[ y = -\frac{\gamma}{\alpha} \]
Taking the first derivative with respect to \( x_t \) and evaluating it at the steady state we can obtain \( g_x \)

\[
E_t \left( e^{\alpha y_{t+1} + \beta x_{t+1}} \left( \alpha g_x h_x + \beta h_x \right) \right) = 0
\]

\[
g_x = \frac{\beta}{\alpha}
\]

Similarly taking the second derivative

\[
E_t \left( e^{\alpha y_{t+1} + \beta x_{t+1}} \left( (\alpha g_x h_x + \beta h_x)^2 + \alpha g_{xx} h_x \right) \right) = 0
\]

\[
g_{xx} = 0
\]

Next, we take the first derivative with respect to \( \sigma \) and are able to verify the general result that \( g_\sigma \) is equal to zero

\[
E_t \left( e^{\alpha y_{t+1} + \beta x_{t+1}} \left( \alpha g_\sigma + \alpha g_x \eta_{t+1} + \beta \eta \epsilon_{t+1} \right) \right) = 0
\]

Finally, we take the second derivative with respect to \( \sigma \)

\[
E_t \left( e^{\alpha y_{t+1} + \beta x_{t+1}} \left( (\alpha g_\sigma + \alpha g_x \eta_{t+1} + \beta \eta \epsilon_{t+1})^2 + \alpha g_{\sigma \sigma} + \alpha g_{xx} \eta^2 \epsilon_{t+1}^2 \right) \right) = 0
\]

\[
g_\sigma = 0 \text{ and in our particular example } g_{xx} = 0 \text{ therefore }
\]

\[
g_{ss} = -\frac{(\alpha g_x + \beta)^2 \eta^2}{\alpha}
\]

Now consider solving the model using lognormality. We assume

\[
y_t = g_x x_t + g_0
\]

\[
x_{t+1} = h_x x_t + \sigma \eta \epsilon_{t+1}
\]

The equilibrium condition implies

\[
E_t \left( \alpha y_{t+1} + \beta x_{t+1} + \gamma \right) + \frac{1}{2} \text{Var}_t \left( \alpha y_{t+1} + \beta x_{t+1} + \gamma \right) = 0
\]

\[
(\alpha g_x h_x + \beta h_x) x_t + \alpha g_0 + \gamma + \frac{1}{2} (\alpha g_x + \beta)^2 \eta^2 = 0
\]

Regrouping the terms

\[
g_x = \frac{\beta}{\alpha}
\]

\[
g_0 = -\gamma + \frac{1}{2} \frac{(\alpha g_x + \beta)^2 \eta^2}{\alpha}
\]
We verify exactly that
\[ g_0 = y + \frac{1}{2} g_{ss} \]
This will not hold exactly if \( g_{xx} \neq 0 \), which in our setup would have been the case if \( h_{xx} \neq 0 \). In other terms we compute \( g_{ss}^* = 2(g_0 - y) \) in the same way as \( g_{ss} \) is computed using standard perturbation methods except for ignoring second order term in \( g_{xx} \).