Some Remarks on Measures of Risk Aversion and on Their Uses

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In this essay, measures of risk aversion are looked upon from the point of view of the "states of nature" approach to decision making under uncertainty. This point of view turns out to be quite useful in the formulation of axioms concerning the change in risk aversion as certain environmental parameters (e.g., wealth) vary. A simple portfolio selection problem (with one risky security) is used as a setting for the discussion.

I

Consider a random event \( E \), and let \( \sim E \) denote its complement. (The term "random event" is taken here in the colloquial sense of "something that could happen" rather than in the formal sense of a measurable subset of a sample space on which there is defined a probability measure.) We may define a gamble on the event \( E \) and its complement as a real valued function on the set \( \{E, \sim E\} \). In other words, a gamble on \( E \) and its complement is an ordered pair of real numbers, say \( \langle x_1, x_2 \rangle \), where \( x_1 \) is the number of dollars to be received if \( E \) occurs, and \( x_2 \) is the number of dollars to be received if \( E \) fails to occur. Thus, for example, the pair \( \langle -2, 1 \rangle \) is the gamble whose holder pays two dollars if \( E \) occurs and receives one dollar if \( E \) fails to occur. Let \( \geq \) be a decision maker's preference ordering of the set of all such money gambles on \( E \) and its complement. Assume that the relation \( \geq \) satisfies the following standard axioms:

I. \( \geq \) is a complete weak ordering.

II. \( \geq \) is monotone, in the sense that if \( x \geq y \), then \( x > y \).\(^1\)

III. \( \geq \) is continuous. In other words, for all gambles \( y \), the sets \( \{x | x \geq y\} \) and \( \{x | y \geq x\} \) are closed.

IV. \( \geq \) is convex. That is, the set \( \{x | x \geq y\} \) is convex for every gamble \( y \).

In the present context, Axiom IV goes by the name of risk aversion.

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\(^1\) Let \( x = \langle x_1, x_2 \rangle \) and \( y = \langle y_1, y_2 \rangle \). Then, \( x \gg y \) means \( x_1 > y_1 \) and \( x_2 > y_2 \). \( x \geq y \) means \( x \geq y \) but not \( y \geq x \).
In standard consumer theory, we require all commodity bundles to be nonnegative. Here, commodity bundles are gambles, and the latter must be allowed to have negative components. However, nonnegativity is replaced by boundedness from below. Specifically, if we let $W$ be the decision maker’s current wealth, then the only gambles which he can consider are ones where possible losses do not exceed $W$. In other words, the set of all admissible gambles is given by

$$\{(x_1, x_2) | x_1 \geq -W \text{ and } x_2 \geq -W\}.$$ 

The preference ordering $\succeq$ is assumed to be defined on this set.

Let the symbol $\hat{0}$ be used to denote the origin, $\langle 0, 0 \rangle$. The decision maker will accept a gamble $x$ if, and only if, $x \succeq \hat{0}$. Therefore, let us refer to the set $\{x | x \succeq \hat{0}\}$ as the decision maker’s acceptance set. Our axioms tell us that the acceptance set is convex, and that it is bounded below by the graph of a continuous, nonincreasing function, passing through the origin. We shall refer to this function as the decision maker’s acceptance frontier.

Now consider two decision makers, Mr. A and Mr. B, and let their acceptance sets be denoted $S_A$ and $S_B$, respectively. Mr. A and Mr. B could, of course, be the same individual, observed at different wealth levels, at different levels of information, etc. It seems quite natural to say that Mr. A is more risk averse than Mr. B if $S_A$ is a subset of $S_B$, i.e., if every gamble which is acceptable to $A$ is also acceptable to $B$.

The relation “is more risk averse than . . .” is a partial ordering of the set of decision makers. It is a global concept. The corresponding local concept may be defined as follows: We say that Mr. A is locally more risk averse than Mr. B if there exists an open disc $D$, centered at the origin, such that $D \cap S_A$ is a subset of $D \cap S_B$. In other words, A is locally more risk averse than B if all sufficiently small gambles which are acceptable to A are also acceptable to B. The relation “is locally more risk averse than . . .” is also a partial ordering of decision makers, and it obviously contains the relation “is more risk averse than . . .”. Given that we are dealing with a partial ordering, it might be of interest to see under what conditions two decision makers are comparable in this ordering.

Let $f$ be a decision maker’s acceptance frontier. For convenience, assume that $f$ is differentiable at the origin. Then, the quantity $p$, defined by

$$p = \frac{f'(0)}{f'(0) - 1}$$

would be more accurate, but also more cumbersome, to say in this case that Mr. A is at least as risk averse as Mr. B.

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is referred to as the decision maker’s *subjective probability* for the event $E$. This definition follows the Ramsey–Savage notion of deriving the subjective probability of the event $E$ from the odds at which the decision maker is willing to make small bets on $E$. It is, of course, possible to define subjective probability in an analogous fashion also in the case where the acceptance frontier is not differentiable at the origin. Note, however, that in the absence of differentiability, subjective probability is no longer unique.

**Remark 1.** In order for Mr. $A$ and Mr. $B$ to be comparable in the partial ordering “is locally more risk averse than . . .” it is necessary that the set of subjective probabilities for $E$ of one of them be contained in the set of subjective probabilities for $E$ of the other. In particular, if acceptance frontiers are differentiable at the origin, then a necessary condition for Mr. $A$ and Mr. $B$ to be comparable in the ordering is that their subjective probabilities coincide. In other words, $A$ and $B$ can only be comparable in the ordering if their acceptance frontiers are tangent (to each other) at the origin.

The proof of this remark is straightforward.

Suppose that Mr. $A$ and Mr. $B$ are, in fact, comparable in the ordering “is locally more risk averse than . . .”. Then, clearly, the second derivative of the acceptance frontier (if it exists) will provide us with criteria for determining who, among these two men, is higher on the ordering. Specifically, we have:

**Remark 2.** Let all acceptance frontiers be twice differentiable at 0, and assume that Mr. $A$ and Mr. $B$ are comparable in the ordering “is locally more risk averse than . . .”. Let the acceptance frontiers of Mr. $A$ and Mr. $B$ be denoted $f_A$ and $f_B$, respectively. Then, a necessary condition for Mr. $A$ to be locally more risk averse than Mr. $B$ is given by the inequality

$$f''_A(0) \geq f''_B(0),$$

and a sufficient condition for Mr. $A$ to be locally more risk averse than Mr. $B$ is given by the inequality

$$f''_A(0) > f''_B(0).$$

The proof, once again, is straightforward.

This last remark clearly suggests the use of the second derivative of the acceptance frontier, evaluated at the origin, as a measure of local

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3 The decision maker's *subjective odds* for the event $E$ are given by the quantity $-f'(0)$. 

risk aversion. Note that, by the term "measure of local risk aversion," we do not mean to imply that the second derivative of the acceptance frontier at the origin is a numerical representation of the ordering "is locally more risk averse than . . .". In fact, Mr. A could be strictly more risk averse (locally as well as globally) than Mr. B, while \( f'_A(0) = f'_B(0) \). This is simply a re-statement of the fact that it is not possible, in general, to deduce the behavior of a function in an interval from information about its derivatives at a point in the interval.

The decision maker need not be a maximizer of expected utility for our measure of local risk aversion to be applicable. However, if it happens that the decision maker is a maximizer of expected utility, then this measure leads immediately to the Arrow–Pratt \([1, 2]\) measure of risk aversion. For suppose that a decision maker, faced with the possibility of betting on an event \( E \), decides which bets to take on the basis of expected utility. Let his utility function be denoted \( u \), and let his current wealth be denoted \( W \). Finally, let his acceptance frontier be denoted \( f_W \), to indicate that at different wealth levels the decision maker will, in general, have different acceptance frontiers. Then, the acceptance frontier \( f_W \) is given by the equation

\[
p u(W + t) + (1 - p) u(W + f_W(t)) = u(W),
\]

where \( p \) is the probability which the decision maker assigns to the event \( E \). We shall assume that \( u \) is twice differentiable, which implies that \( f_W \) is also twice differentiable. Upon differentiating once, we find, as indeed we should, that

\[
p = \frac{f'_W(0)}{f'_W(0) - 1},
\]

and upon differentiating again, we obtain

\[
f''_W(0) = \frac{p}{(1-p)^2} \left[ - \frac{u''(W)}{u'(W)} \right].
\]

The quantity in brackets is precisely the Arrow–Pratt index of risk aversion. In other words, the risk aversion measure \( f''_W(0) \) is proportional to the Arrow–Pratt measure, since the probability \( p \) cannot depend upon \( W \). (Indeed, we know that the probability \( p \) is common to all the decision makers who are comparable in the ordering "is locally more averse than . . ."."

We see from this last equation that the Arrow–Pratt index of local risk aversion has the same shortcoming that \( f''_W(0) \) has: It could happen that the Arrow–Pratt index is equal at two wealth levels, and yet the

\footnote{To get this result, one needs to have \( u'(W) > 0 \). But this follows from our strong monotonicity axiom.}
decision maker could be strictly more risk averse (in terms of accepted gambles) at one of these wealth levels than at the other.

Now, what about the Arrow–Pratt measure of relative risk aversion? Can it be constructed in a similar fashion? The answer is in the affirmative. Let the pair of real numbers \( \langle r_1, r_2 \rangle \) stand for the following gamble: “You will receive an amount equal to 100\(r_1\) percent of your wealth if \(E\) occurs, and you will receive an amount equal to 100\(r_2\) percent of your wealth if \(E\) fails to occur.” Thus, for example, the pair \(<-1/2, 1/2>\) will now stand for the contract under which the decision maker undertakes to pay one half of his wealth if \(E\) occurs, in return for which he is to receive one half again on his wealth if \(E\) fails to occur. Let us refer to such contracts as relative gambles. The decision maker’s preference ordering over regular gambles induces a preference ordering over relative gambles, in the following obvious manner: A relative gamble \(\langle r_1, r_2 \rangle\) is preferred or equivalent to a relative gamble \(\langle r'_1, r'_2 \rangle\) if, and only if, the regular gamble \(\langle Wr_1, Wr_2 \rangle\) is preferred or equivalent to the regular gamble \(\langle Wr'_1, Wr'_2 \rangle\), where \(W\) denotes current wealth. The set of all relative gambles preferred or equivalent to the origin is, as before, the decision maker’s acceptance set (for relative gambles) and it is bounded below by the graph of a function, say \(g_w\), where, once again, we use \(W\) to denote current wealth. If \(f_w\) is the decision maker’s acceptance frontier for regular gambles, then the relationship between \(g_w\) and \(f_w\) is given by

\[
g_w(t) = \frac{1}{W} f_w(Wt),
\]

for all \(t\). This follows immediately from the relationship between the decision maker’s preference ordering for regular gambles and his preference ordering for relative gambles.

By differentiating this last equation twice, we find that

\[
g''_w(0) = Wf''_w(0).
\]

Now, the quantity \(g''_w(0)\) may be regarded as a natural measure of the decision maker’s aversion to relative risk. The argument which leads to this observation is completely analogous to our previous argument, where \(f''_w(0)\) was proposed as a natural measure of aversion to absolute (i.e., non-relative) risk. And if the decision maker happens to be a maximizer of expected utility, then we can substitute for \(f''_w(0)\) in the foregoing equation, and obtain

\[
g''_w(0) = \frac{p}{(1-p)^3} \left[ -\frac{Wu''(W)}{u'(W)} \right].
\]

The expression in brackets is, of course, the Arrow–Pratt index of relative risk aversion.
Measures of risk aversion have one major use: They facilitate the formulation of certain axioms on decision making under uncertainty, and these axioms may, in turn, lead to empirically meaningful conclusions. Here, for example, are three fairly reasonable axioms which make use of the notion of "measuring" risk aversion: (1) Greater wealth can never lead to greater risk aversion. (2) Greater information (about the physical nature of the various random events) can never lead to greater risk aversion. (3) Greater family size can never lead to lower risk aversion. These axioms (and others like them) have obvious uses in predicting the behavior of decision makers. However, a great deal depends on exactly how each axiom is stated. Indeed, each of the three statements above is not really an axiom, but a heading for a whole class of possible axioms which, while similar in spirit, may differ greatly. In order to explore this issue a little bit further, let us turn to the following "classical" query in the economics of uncertainty: How will variations in wealth affect a decision maker's portfolio of risky and non-risky securities? The answer, as everyone knows, depends on the way in which variations in wealth affect risk aversion. Consider the simplest possible portfolio selection problem: A decision maker is presented with a security which bears a net rate of return of $\alpha$ per dollar if the event $E$ occurs, and $\beta$ per dollar if $E$ fails to occur. To get a nontrivial case, we shall assume that of the two numbers, $\alpha$ and $\beta$, one is positive and the other is negative; say $\alpha > 0$, $\beta < 0$. Assume also that the decision maker can buy this security or sell short in any amount, provided he has adequate resources to cover his losses. Let $W$ be the decision maker's wealth, and let $A$ be the amount of the security purchased. Then, $A$ is restricted by the inequalities $-W/\alpha \leq A \leq -W/\beta$, where a negative value of $A$ indicates selling short. (If selling short is not permitted, then $A$ is restricted by the inequalities $0 \leq A \leq -W/\beta$.) Now, the purchase of an amount $A$ of securities is clearly equivalent to engaging in the gamble $<\alpha A, \beta A>$. Thus, the decision maker's opportunity locus is given by the set

$$\left\{\left.\frac{\alpha A}{\alpha} - \frac{W}{\alpha} \leq A \leq -\frac{W}{\beta}\right\}\right.$$

This is a downward sloping line segment, which passes through the origin. Its slope is given by the ratio $\beta/\alpha$. The negative of this ratio, $-\beta/\alpha$, will be referred to as the market odds on the event $E$. We may also refer to the quantities $-\beta/(\alpha - \beta)$ and $\alpha/(\alpha - \beta)$ as the market probabilities of $E$ and $\sim E$, respectively. The decision maker will invest in the risky security in such a way as to maximize his preferences. In other words, he will choose $A^*$ in such a way that, among all admissible points of the form
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Fig. 1.

\[ \langle zA, \beta A \rangle, \text{ the point } \langle zA^*, \beta A^* \rangle \text{ will be on the highest indifference curve. (See Fig. 1.)} \]

It is obvious, from Fig. 1, that the decision maker will buy, sell short, or stay put according to whether or not there is a divergence between his subjective probability and the market probability for the event \( E \). Specifically, we have:

**Remark 3.** Refraining from both purchasing and selling short is optimal if, and only if, the market probability for \( E \) coincides with a subjective probability for \( E \).\(^5\) If all the subjective probabilities for \( E \) exceed the market probability for \( E \), then purchasing is optimal. Finally, if all the subjective probabilities for \( E \) fall short of the market probability for \( E \), then selling short is optimal. In the case where the subjective probability for \( E \) is unique and where the decision maker's preferences are strictly convex (so that the optimal action is also unique) the foregoing assertion

\(^5\) Recall that, in the non-differentiable case, subjective probability is not unique.
reduces to the following: Purchasing is optimal if, and only if, the decision maker's subjective probability for $E$ exceeds the market probability, and selling short is optimal if, and only if, the market probability for $E$ exceeds the decision maker's subjective probability.

It is possible also to state the assertions in Remark 3 in terms of the expected rate of return of the risky security. For example, in the case of a unique subjective probability and strictly convex preferences, we have the following: The decision maker will buy or sell short if, and only if, the expected rate of return from the risky security, reckoned with respect to his subjective probability distribution, is different from 0. He will buy if (and only if) the expected rate of return is positive, and he will sell short if (and only if) the expected rate of return is negative. And if selling short is not permitted, then it is still true that the decision maker will buy if, and only if, the expected return from the risky security is positive. This observation should be compared with Arrow's result in [1], where he gets the same condition for a decision maker who maximizes expected utility.

Let us return, however, to our attempt at determining how variations in wealth affect investment in the risky security, whether in purchases or in short sales. We are looking for a proposition to the effect that if risk aversion is nonincreasing with wealth, then investment in the risky security (whether via purchases or via short sales) is nondecreasing with wealth. To this end, we must state exactly in what sense we mean risk aversion to be nonincreasing with wealth. The following axiom seems reasonable:

**Axiom V.** For every pair of wealth levels, $W$ and $W'$, satisfying $W' \geq W$, and for every gamble $x$, if $x$ is accepted at the wealth level $W$, then it is also accepted at the wealth level $W'$.

If Axiom V holds, then we shall say that the decision maker exhibits *nonincreasing global risk aversion*.

It seems natural at this point to examine the following assertion: If the decision maker satisfies Axioms I–V, then his investment in the risky security (whether in purchases or in short sales) is nondecreasing in wealth.

Unfortunately, this assertion is false. A counterexample is contained in Fig. 2. The preference ordering depicted in Fig. 2 satisfies Axioms I–V, and yet, with market odds as depicted in the figure, we find investment in the risky security decreasing as wealth increases. Indeed, at low wealth levels, investment in the risky security decreases (as wealth increases) by more than the increase in wealth. In other words, Axiom V is not even
strong enough to rule out inferiority of the conditional commodities on which the preference ordering is based.\textsuperscript{6}

Axiom V does lead to the following results:

\textit{Remark 4}. If the decision maker's preferences satisfy nonincreasing global risk aversion, then there exists a real number $p$, such that $p$ is a subjective probability for $E$ at all wealth levels.

\textit{Proof}. By Remark 1, the set of subjective probabilities shrinks as wealth increases. In other words, if $W$ and $W'$ are two wealth levels satisfying $W \geq W'$, and if $p$ is a subjective probability at $W'$, then $p$ is also a subjective probability at $W$. Now, it is easy to establish that, at

\textsuperscript{6}Fig. 2 shows also that there is no hope of getting a \textit{local} theorem to the effect that, say, if local risk aversion is decreasing locally, then small changes in wealth will always lead to changes in the same direction in the decision maker's investment in the risky security.
each wealth level, the set of subjective probabilities is compact. Therefore, there exists a subjective probability common to all wealth levels.

**Remark 5.** If the decision maker's preferences satisfy nonincreasing global risk aversion, and if purchasing is optimal at one wealth level, while selling short is optimal at another (or the same) wealth level, then staying put (i.e., neither purchasing nor selling short) is optimal at both wealth levels.

**Proof.** This follows from Remarks 3 and 4.

This last remark says that, under nonincreasing global risk aversion, if the decision maker engages in purchases of the risky security at one wealth level, then there is no reason for him to switch to selling short at another wealth level, and vice versa. Let us now proceed to motivate the axiom from which it follows that the amount of the risky security purchased (or sold short) is nondecreasing with wealth.

Consider a decision maker, and assume that this decision maker has already contracted a gamble $x = \langle x_1, x_2 \rangle$, either because he had chosen to do so, or because he had been made to do so. In other words, the decision maker is already under an obligation which will bring him $x_1$ dollars if $E$ occurs, and $x_2$ dollars if $E$ fails to occur. Now suppose that a further gamble, say $y = \langle y_1, y_2 \rangle$, is offered to him. Let us examine whether it is reasonable to postulate the following: If the decision maker accepts the gamble $y$ (on top of $x$) at his present wealth level, then he will also accept $y$ (on top of $x$) at all higher wealth levels. At first blush, this postulate may not seem unreasonable. However, on second thought, we see that it is, in fact, much too strong. The decision maker's willingness to engage in the gamble $y$ may stem from a desire to hedge against his existing bet, $x$. Now, the desire to hedge against a given gamble may very well be greater at low wealth levels than at high ones. The above-mentioned postulate precludes this possibility. To get around this problem, it is necessary to introduce an explicit distinction between bets on $E$ and bets on $\sim E$: Let $z = \langle z_1, z_2 \rangle$ be a gamble. If $z_1 \geq z_2$, then we shall say that $z$ is a bet on $E$; and if $z_2 \geq z_1$, then we shall say that $z$ is a bet on $\sim E$. Assume, once again, that the decision maker has already contracted a gamble $x$, and that he is now being offered a gamble $y$, on top of $x$. Clearly, the gamble $y$ can be considered a hedge against the gamble $x$ only if $x$ and $y$ are bets on opposite events, that is, only if $x$ is a bet on $E$ and $y$ is a bet on $\sim E$, or if $x$ is a bet on $\sim E$ and $y$ is a bet on $E$. The axiom which we shall now state refers only to the case in which hedging considerations do not arise, namely to the case where $x$ and $y$ are bets on the same event. If $x$ is, say, a bet on $E$ and $y$ is also a bet on $E$, then
it seems entirely reasonable to postulate that if the decision maker accepts $y$ (on top of $x$) at a low wealth level, then he also accepts $y$ (on top of $x$) at higher wealth levels. In other words, if the decision maker is willing to go farther out on a limb at a given wealth level, then there is no reason why he should not be willing to do the same thing at higher wealth levels. This is, in fact, the axiom which we are looking for.

**Axiom V'**. Let $x$ and $y$ be gambles, and let $w$ be a nonnegative real number. If $x$ and $y$ are bets on the same event and $x + y \succeq x$, then $x + y + \hat{w} \succeq x + \hat{w}$, where $\hat{w}$ is defined by $\hat{w} = \langle w, w \rangle$.

We shall refer to Axiom V', for want of a better name, as the axiom of nonincreasing total risk aversion. Nonincreasing total risk aversion implies nonincreasing global risk aversion, but not conversely.

**Remark 6.** Assume that Axioms I–IV and V' are satisfied, and let $W$ and $W'$ be two levels of wealth, satisfying $w' \geq W$. Suppose that, at the level $W$, investing an amount $A$ in the risky security is optimal. If $A \geq 0$, then there exists an $A'$, satisfying $A' \geq A$, such that investing $A'$ is optimal at the level $W'$. And if $A \leq 0$, then there exists an $A'$, satisfying $A' \leq A$, such that investing $A'$ is optimal at $W'$.

**Proof.** Assume, without loss of generality, that the ordering $\succeq$ describes the decision maker's preferences when his wealth level is $W$. Consider the case where $A \geq 0$, and define $x = \langle \alpha A, \beta A \rangle$. $x$ is a bet on the event $E$. Now let $A^*$ be an optimal amount of investment at the wealth level $W'$. If $A^* \geq A$, there is nothing more to prove. Assume, therefore, that $A^* < A$. By Remark 5, we may assume also that $A^* \geq 0$. Define a real number $w$ and a gamble $x'$ by

$$w = W' - W,$$

$$x' = \langle w + \alpha A^*, w + \beta A^* \rangle.$$

The $x'$ is a gamble which represents optimal investment at the wealth level $W'$, when viewed from the wealth level $W$. Since $A^* \geq 0$, $x'$ is a bet on the event $E$. Writing $\hat{w} = \langle w, w \rangle$, we obtain

$$x = x' - \hat{w} + y,$$

where $y = \langle \alpha(A - A^*), \beta(A - A^*) \rangle$. Since $A^* < A$, we have that $y$, like $x$ and $x'$, is a bet on the event $E$. And since $x$ is optimal at the wealth level $W$, and $x' - \hat{w}$ is feasible at the same wealth level, we have

$$x' - \hat{w} + y \succeq x' - \hat{w}.$$
From Axiom V', it now follows, since $x'-\hat{\omega}$ and $y$ are bets on the same event, that

$$x' + y \succeq x'.$$

Now, $x' + y = x + \hat{\omega}$. Hence, we have

$$x + \hat{\omega} \succeq x'.$$

But $x'$ is optimal at the wealth level $W'$, and $x + \hat{\omega}$ is feasible at the same wealth level. Therefore, the foregoing assertion tells us that $x + \hat{\omega}$ must also be optimal at $W'$. And since $x + \hat{\omega}$ is the gamble which represents an investment of $A$ at the wealth level $W'$, we have the desired result. The proof for the case where $A \leq 0$ is analogous.

If the decision maker's preferences are strictly convex, then the optimal amount of investment in the risky security is unique, and then Remark 6 reduces to the assertion that investment in the risky security is non-decreasing in absolute value with wealth.

Axiom V' is, in fact, the weakest assumption which implies that investment in any arbitrary risky security is nondecreasing with wealth. Specifically, we have:

**Remark 7.** Given Axioms I-IV, if the conclusion in Remark 6 holds for all possible market odds, then Axiom V' is satisfied.

**Proof.** Assume that $x$ and $y$ are bets on $E$, such that $x+y \succeq x$, while $x + \hat{\omega} \succeq x + y + \hat{\omega}$, where $\hat{\omega} = \langle w, w \rangle$ and $w > 0$. Let the components of $y$ be denoted $y_1$ and $y_2$. We know, since $y$ is a bet on $E$, that $y_1 \geq y_2$. However, since $x+y \succeq x$ and $x + \hat{\omega} \succeq x + y + \hat{\omega}$, we have, by monotonicity of preferences, that $y_1 > 0$ and $y_2 < 0$. Hence, the straight line, call it $L$, which passes through both $x$ and $x+y$, is negatively sloped. We may look upon an appropriate segment\(^7\) of the line $L$ as the opportunity locus for investment in a risky security at some wealth level. Let this wealth level be denoted $W$. Similarly, let the straight line which passes through both $x + \hat{\omega}$ and $x + y + \hat{\omega}$ be denoted $L'$. Then, an appropriate segment of $L'$ can be looked upon as the opportunity locus for investments in the same risky security, but at a higher wealth level, namely at the level $W' = W + w$. By continuity of preferences, both of these opportunity loci possess optimal points. Furthermore, the assertion $x+y \succeq x$, in conjunction with convexity of preferences, implies that the set of optimal points on $L$ either has the point $x$ at its extreme or lies entirely to the southeast of $x$. On the other hand, the assertion that $x + \hat{\omega} \succ x + y + \hat{\omega}$,

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\(^7\) Namely, the segment of gambles which do not involve losses that are greater than the decision maker's wealth.
again in conjunction with the convexity of preferences, implies that the set of optimal points on $L'$ lies strictly to the northwest of the point $x + \hat{w}$. This contradicts the conclusion of Remark 6.

We may summarize Remarks 6 and 7, for the case where the decision maker's preference ordering is strictly convex, as follows: The assertion that investment in a risky security is nondecreasing with wealth is equivalent to nonincreasing total risk aversion.

The response of investment in a risky security to variations in wealth was investigated, for the case where the decision maker is a maximizer of expected utility, by Arrow [1]. In order to obtain that investment in a risky security is nondecreasing with wealth, Arrow assumes that the Arrow-Pratt measure of local risk aversion is nonincreasing everywhere, as wealth increases. Our Remark 7 above tells us that, in the presence of the dominance axiom, i.e., the axiom needed to obtain the expected utility hypothesis, Arrow's assumption is actually strong enough to imply nonincreasing total risk aversion. It is simple to verify that the converse of this assertion is also true: In the presence of the dominance axiom, nonincreasing total risk aversion implies that the Arrow-Pratt measure of local risk aversion is globally nonincreasing. Thus, for an expected utility maximizer, investment in a risky security increases with wealth, for all securities and at all wealth levels, if and only if the Arrow-Pratt measure of risk aversion is decreasing with wealth everywhere. Note that the corresponding local assertion (locally decreasing risk aversion implies locally increasing investment in a risky security) is not true.

III

A brief comment may be in order, concerning what happens if, instead of just two events, $E$ and $\sim E$, we have a family $\{E_1, E_2, \ldots, E_n\}$ of $n$ mutually exclusive and exhaustive events—or states of nature, as they are often called. In this case, a gamble is given by an $n$-tuple of real numbers, say $\langle x_1, x_2, \ldots, x_n \rangle$, where $x_i$ is the amount of money to be received if the event $E_i$ occurs. Axioms I–IV do not have to be changed at all to accommodate this new situation. Given that Axioms I–IV hold, one can define the decision maker's subjective probability distribution (at a given wealth level) as an $n$-tuple $\langle p_1, p_2, \ldots, p_n \rangle$ which determines a supporting hyperplane to the set of accepted gambles at the origin, appropriately normalized. (The monotonicity axiom, Axiom II, guarantees that the numbers $p_1, p_2, \ldots, p_n$ will all be nonnegative.)

The axiom of nonincreasing global risk aversion is the same in the case of $n$ events as in the case of two. However, in order to write down the
The axiom of nonincreasing total risk aversion, we must first give a definition of what we mean by two gambles being bets on the same event. Let \( x = \langle x_1, \ldots, x_n \rangle \) and \( y = \langle y_1, \ldots, y_n \rangle \) be gambles. We say that \( x \) and \( y \) are bets on the same (composite) event if the inequality \((x_i - x_j)(y_i - y_j) \geq 0\) holds for all \( i \) and \( j \). With this definition, we may interpret the axiom of nonincreasing total risk aversion as an axiom for the \( n \)-event case.

A risky security in the \( n \)-event case is characterized by an \( n \)-tuple of real numbers, \( \langle \alpha_1, \ldots, \alpha_n \rangle \), where \( \alpha_i \) is the rate of return per dollar invested, if the event \( E_i \) occurs. Define \( \alpha^* = \max \alpha_i \) and \( \alpha_* = \min \alpha_i \). We know that \( \alpha^* > 0 \) and \( \alpha_* < 0 \), or else we get a trivial portfolio problem. Let \( A \) be the amount invested in the risky security, and let \( W \) be the decision maker's current wealth. Then, the opportunity locus for investments is given, if selling short is permitted, by

\[
\left\{ \langle \alpha_1 A, \ldots, \alpha_n A \rangle \left| -\frac{W}{\alpha^*} \leq A \leq -\frac{W}{\alpha_*} \right. \right\},
\]

and, if selling short is not permitted, by

\[
\left\{ \langle \alpha_1 A, \ldots, \alpha_n A \rangle \left| 0 \leq A \leq -\frac{W}{\alpha_*} \right. \right\}.
\]

Note that these opportunity loci are line segments in \( n \)-space (and not sections of hyperplanes, as is the case in standard consumer theory).

Remarks 1 and 3–7 are all valid in this new setting, and Remark 2 is valid after an appropriate modification. All the proofs remain virtually unchanged.

IV

It is, perhaps, worthwhile to point out, in conclusion, that the intuitive appeal of the axioms which were discussed above depends rather crucially on the nature of the random events being considered. Specifically, if the occurrence or non-occurrence of a certain random event is, in and of itself, a matter relevant to the decision maker's well-being, then Axioms V and V' lose much of their appeal. For example, if we let \( E \) and \( \sim E \) stand for the decision maker's health and ill-health, respectively, then it is no longer reasonable to assert that all the gambles which are accepted at a given wealth level will also be accepted at higher wealth levels. Furthermore, the very notion of defining subjective probabilities from the odds at which one is willing to bet loses its appeal when random events are, in and of themselves, relevant to the decision maker's preferences. This is why Ramsey [3] chose to work with what he called ethically neutral events (or ethically neutral propositions), i.e., with events that are not directly relevant to the decision maker's preferences.
Only in one case is the lack of "ethical neutrality" easily overcome. If the occurrence of a random event is equivalent, from the decision maker's point of view, to a monetary gain (or loss) of fixed magnitude, then all that is required to make Axioms V and V' sound perfectly reasonable is a suitable prior translation of the coordinate system. The theorems which follow from these transcribed axioms are themselves transcribed versions of the theorems which hold under ethical neutrality. If the occurrence of a random event is directly relevant to the decision maker's preferences but is not equivalent to a fixed monetary gain (or loss), then there no longer exists a straightforward way to alter the axioms so as to restore their appeal.

REFERENCES