Risk in the recursive business cycle model

Or: Volatility and business cycles

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November 29, 2011

Abstract

We explore the impact of risk on business cycles. We make two changes to the standard model: risk is stochastic and preferences are recursive. We use loglinear approximations to show that the sign of the impact of risk on consumption depends on the intertemporal elasticity of substitution and the magnitude depends on risk aversion. This two-factor model (productivity and risk) accounts for traditional business cycle facts, including the volatility and comovement of expenditure components, and some new ones, including the cyclical behavior of risk premiums and so-called “wedges” in labor supply and investment.

JEL Classification Codes: E32, E44, G12.

Keywords: recursive preferences, stochastic volatility, risk premiums.

*Preliminary and incomplete: no guarantees of accuracy or sense. We welcome comments, including references to related papers we inadvertently overlooked.

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1 Introduction

Hall: last three recessions were financial...

Dating back to Kydland and Prescott (1982), and probably before, macroeconomists have ignored risk, in the sense that decision rules in business cycle models have typically been computed from approximations of deterministic versions in the neighborhood their stationary points. This is justified, if at all, by reference to linear-quadratic models, in which decision rules are invariant to risk. This is a reasonable approach for many purposes, but it seems to us to miss one of the key elements of the interaction of real decisions and financial markets, where the dynamics of risk arguably play an important role. We find it hard to think about the recent past without reference to risk and its impact on both real decisions and asset prices.

Mark Holmes (1995), in his popular book on approximation methods, says (more or less, this is a paraphrase):

Computer methods are capable of solving many problems, but they often do not provide much insight. The principle objective of approximation methods, as far as I am concerned, is to provide a reasonably accurate solution that leads to an understanding of the problem.

Notation: Variables: \( x_t \), \( x \) is mean or steady state. Functions and derivatives: \( J(x_t) \) is a function, \( J_x(x_t) \) its derivative; shorthand is \( J_t, J_{kt} \).

2 Preliminaries

We define recursive preferences and show how one of the ingredients, the certainty equivalent function, can be computed in simple cases.

Recursive preferences

Overview: preferences across time and states

\[
U_t = V[u_t, \mu_t(U_{t+1})] \\
V = \text{“time aggregator”} \text{ (time preference)} \\
\mu = \text{“certainty equivalent”} \text{ (risk preference)}
\]

Comments:
• Can’t really separate time and risk preference, an example of an old problem (Kihlstrom-Mirman).

• Typically we’ll set \( u_t = c_t \) (the “growth model”), but in the “business cycle model” we include leisure: \( u_t = c_t (1 - n_t) \).

• For future reference, note that increasing transformations of utility reflect the same preference ordering. For example, we can use \( U' = h(U) \), with \( h \) increasing, as long as we adjust the certainty equivalent appropriately:

\[
U'_t = h(U_t) = h \left( V[u_t, \mu_t(U_{t+1})] \right) = h \left( V \left[ u_t, \mu_t \left\{ h^{-1} [h(U_{t+1})] \right\} \right] \right) = V' \left[ u_t, \mu'_t(U'_{t+1}) \right]
\]

with \( V'(x, y) = h[V(x, y)] \) and \( \mu'(x) = \mu[h^{-1}(x)] \).

Flavors include:

• Kreps-Porteus: \( \mu \) is expected utility

• Epstein-Zin: now-standard functional forms, Chew-Dekel risk preference

• Ambiguity aversion: Epstein-Schneider, Hansen-Sargent, many others

All described in Backus, Routledge, and Zin (2004, Sections 3-5).

Our focus is the constant elasticity version of Kreps-Porteus/Epstein-Zin/Weil:

\[
U_t = \left[ (1 - \beta) c_t^\rho + \beta \mu_t(U_{t+1})^\rho \right]^{1/\rho}
\]

\[
\mu_t(U_{t+1}) = (E_t U_{t+1}^\alpha)^{1/\alpha}
\]

with \( \alpha, \rho \leq 1 \). From BCZ: Additive power utility is a special case with \( \alpha \cdot \rho = 0 \). In standard terminology, \( \rho < 1 \) captures time preference (with intertemporal elasticity of substitution \( 1/(1 - \rho) \)) and \( \alpha < 1 \) captures risk aversion (with coefficient of relative risk aversion over future utility of \( 1 - \alpha \)). The terminology is a useful shortcut, but it’s somewhat misleading: \( \alpha \) describes risk aversion over future utility, which depends on (among other things) \( \rho \). As in other multigood environments, there is no clear separation between preference across goods and preference across states.

Show how we can convert to additive case:

\[
U_t^\rho / \rho = (1 - \beta) c_t^\rho / \rho + \beta \mu_t(U_{t+1})^\rho / \rho
\]

Subtract one and take limit as \( \rho \to 0 \) to get log version.

**Loglinear certainty equivalents**

Example 1: let \( \log x_{t+1} \sim N(a_t, b_t) \).
Expectations and certainty equivalents:
\[
E_t(x_{t+1}) = \exp(a_t + b_t/2)
\]
\[
E_t(x_{t+1}^\alpha) = \exp(aa_t + \alpha^2 b_t/2)
\]
\[
\mu_t(x_{t+1}) = \left[E_t(x_{t+1}^\alpha)\right]^{1/\alpha} = \exp[a_t + (\alpha/2)b_t]
\]
\[
\log \mu_t(x_{t+1}) = a_t + \frac{\alpha}{2}b_t
\]

This example based on normal innovations, but the idea does not. Jump model with time-varying intensity has similar structure, but the second term has a different form. ...

Example 2: let \(x\) be combination of normal and Poisson mixture...

The idea we use later: make \(\log U_{t+1}\) linear so certainty equivalents have simple form.

## 3 The recursive growth model

We use the term “growth model” to refer to the real business cycle model with fixed labor supply. The model consists of preferences over current and future consumption, a technology for transforming current output into physical capital, and shocks to productivity. A specific version follows.

### 3.1 The model

Preferences are characterized by the time aggregator (1) and certainty equivalent (2) described earlier with \(u_t = c_t\).

The technology is summarized by the law of motion for capital:
\[
k_{t+1} = f(k_t, a_t n_t) - c_t \\
= [\omega k_t^\nu + (1 - \omega)(a_t n_t)^\nu]^{1/\nu} + (1 - \delta)k_t
\]  

Elasticity of substitution between capital and labor is \(1/(1 - \nu)\). In the growth model, we set \(n_t = n = 1\) (constant). For the most part, we use the Cobb-Douglas version, \(f(k_t, a_t n_t) = k_t^\omega(a_t n_t)^{1-\omega} + (1 - \delta)k_t\), which is (roughly) the limit as \(\nu \to 0\).

Productivity growth \(g_t = a_t/a_{t-1}\) follows
\[
\log g_t = \log g + e^T x_t
\]
\[
x_{t+1} = Ax_t + v_t^{1/2} B w_{1t+1}
\]
\[
v_{t+1} = (1 - \varphi_v) v + \varphi_v v_t + \tau w_{2t+1}
\]
\[\{w_{1t}, w_{2t}\} \text{ independent standard normals}\]

It’s helpful here to specify the growth rate rather than the level, but both fit into this framework. The level is
\[
\log a_{t+1} = \log a_t + g_{t+1} = \log a_t + \log g + e^T x_{t+1}.
\]

If productivity is stationary, then \(\log a_t\) is “overdifferenced” but still of the form described above. The unit root is useful in generating realistic asset returns (Alvarez and Jermann, 2005).
3.2 Scaling the Bellman equation

The social planner’s problem for this economy is summarized by the Bellman equation,

$$J(k_t, x_t, v_t) = \max_{\tilde{c}_t} \left\{ \tilde{c}_t, \mu_t[gt+1J(k_{t+1}, x_{t+1}, v_{t+1})] \right\}$$

subject to the laws of motion (3.2.1)-(3.2.4) and initial conditions.

Since everything here is $h^1$, we can scale the Bellman equation by $a_t$. If we define the scaled variables $\tilde{k}_t = k_t/a_t$, $\tilde{c}_t = c_t/a_t$, etc, the scaled Bellman equation is

$$J(\tilde{k}_t, x_t, v_t) = \max_{\tilde{c}_t} \left\{ \tilde{c}_t, \mu_t[gt+1J(\tilde{k}_{t+1}, x_{t+1}, v_{t+1})] \right\}$$

subject to the scaled law of motion for capital,

$$\tilde{k}_{t+1} = \left[ f(\tilde{k}_t, n) - \tilde{c}_t \right]/gt+1,$$

and the other laws of motion. We’ll use the scaled problem in what follows.

3.3 Alternative representations of the value function

[Cumbersome, could be shorter...]

The conventions for representing the value function differ between the additive and recursive cases. With recursive preferences, it’s convenient to use the $h^1$ forms. With additive preferences, it’s convenient to use a power transformation to make the rhs of the Bellman equation additive. This changes the first-order and envelope conditions.

We start with the $h^1$ recursive form, as in the scaled Bellman equation

$$J(\tilde{k}_t, x_t, v_t) = \max_{\tilde{c}_t} \left\{ (1 - \beta)\tilde{c}_t^\rho + \beta \mu_t[gt+1J(\tilde{k}_{t+1}, x_{t+1}, v_{t+1})]^\rho \right\}]^{1/\rho},$$

subject to the appropriate laws of motion and initial conditions.

One useful transformation makes the Bellman equation additive over time (but not states):

$$J(\tilde{k}_t, x_t, v_t)^\rho/\rho = \max_{\tilde{c}_t} (1 - \beta)\tilde{c}_t^\rho/\rho + \beta \mu_t[gt+1J(\tilde{k}_{t+1}, x_{t+1}, v_{t+1})]^\rho/\rho.$$

This works when $\rho \neq 0$. Otherwise, subtract one from each of the terms and take the limit as $\rho \to 0$:

$$\log J(\tilde{k}_t, x_t, v_t) = \max_{\tilde{c}_t} (1 - \beta)\tilde{c}_t^\rho/\rho + \beta \log \mu_t[gt+1J(\tilde{k}_{t+1}, x_{t+1}, v_{t+1})],$$

corresponding to an IES of $\sigma = 1$. We can redefine the value functions in the two cases as $J'(\tilde{k}_t, x_t, v_t) = J(\tilde{k}_t, x_t, v_t)^\rho/\rho$ and $J'(\tilde{k}_t, x_t, v_t) = \log J(\tilde{k}_t, x_t, v_t)$.

The transformation leaves the problem unchanged, but derivatives have a different form. For example, if we use $J'_t = J_t^\rho/\rho$, the derivative is $J'_{kt} = J_{kt}^{\rho-1}J_{kt}$. We’ll see examples of this below.
4 The additive growth model

What follows is a close relative of Campbell’s (1994) loglinear solution. The difference lies in using dynamic programming — what we might call recursive methods if that weren’t confusing in this context.

We’re looking specifically for a loglinear decision rule — something like

$$\log \tilde{c}_t = h_{c0} + h_{ck} \log \tilde{k}_t + h_{cx} x_t + h_{cv} v_t$$

(11)

with coefficients \((h_{c0}, h_{ck}, h_{cx}, h_{cv})\) to be determined.

We’ll see:

- Growth model depends only on the derivative of the value function, not the value function itself. Similar to other applications of dynamic programming in this respect.
- A loglinear approximation requires a loglinear value function.
- The loglinear approximation has a feature common to LQ problems: the coefficient \(h_{ck}\) of the controlled state variable is independent of the laws of motion of the shocks. See Anderson, Hansen, McGrattan, and Sargent (1996) and Hansen and Sargent (1980).

4.1 The additive model

In the additive case \(\alpha = \rho\), the Bellman equation becomes

$$J(\tilde{k}_t, x_t, v_t)^{\rho} / \rho = \max_{\tilde{c}_t} (1 - \beta) \tilde{c}_t^{\rho} / \rho + \beta E_t [g_{t+1}^{\rho} J(\tilde{k}_{t+1}, x_{t+1}, v_{t+1})^{\rho} / \rho]$$

or

$$J'(\tilde{k}_t, x_t, v_t) = \max_{\tilde{c}_t} (1 - \beta) \log \tilde{c}_t + \beta E_t [g_{t+1} \log J'(\tilde{k}_{t+1}, x_{t+1}, v_{t+1})]$$

where \(J'_t = J_t^{\rho} / \rho\). The log case is the limit as \(\rho \to 0\):

$$\log J(\tilde{k}_t, x_t, v_t) = \max_{\tilde{c}_t} (1 - \beta) \log \tilde{c}_t + \beta E_t [\log g_{t+1} + \log J(\tilde{k}_{t+1}, x_{t+1}, v_{t+1})]$$

or

$$J'(\tilde{k}_t, x_t, v_t) = \max_{\tilde{c}_t} (1 - \beta) \log \tilde{c}_t + \beta E_t [\log g_{t+1} + J'(\tilde{k}_{t+1}, x_{t+1}, v_{t+1})],$$

(12)

where \(J'_t = \log J_t\).
4.2 The Brock-Mirman example

This is a common example in graduate courses, because the value function has a simple loglinear form. See Sargent (1987, pp 24-27) and many other places. The problem is a special case in which ρ = 0, δ = 1, and the law of motion for capital is

\[
\tilde{k}_{t+1} = \frac{f(\tilde{k}_t, n) - \tilde{c}_t}{g_{t+1}} = \frac{(\tilde{k}_t^\omega - \tilde{c}_t)}{g_{t+1}}.
\]

For future reference, note that

\[
\sigma = \frac{1}{1 - \rho} = 1
\]

and steady state values of various functions and derivatives include

\[
f_k = \omega \tilde{k}_t^\omega - 1, \quad f_kk = \omega - 1, \quad \text{and} \quad \tilde{k} = (\tilde{k}_t^\omega - c)/g \text{ or } 1 - f_k/g = -(c/kg).
\]

The Bellman equation has the form \((12)\). We guess a value function of the form

\[
J_t = p_1 + p_k \log \tilde{k}_t + p_x^T x_t + p_v v_t.
\]

Brute force substitution:

\[
E_t (\log g_{t+1} + J_{t+1}) = E_t \left[ p_1 + p_k \log \tilde{k}_{t+1} + (p_x^T - p_k e^T)x_{t+1} + p_v v_{t+1} \right]
= p_1 + (1 - \phi_v) v + p_k \log(\tilde{k}_t^\omega - \tilde{c}_t) + (p_x^T - p_k e^T)Ax_t + p_v \phi_v v_t.
\]

The first-order and envelope conditions are

\[
(1 - \beta)/\tilde{c}_t = \beta p_k/(\tilde{k}_t^\omega - \tilde{c}_t)
\]

\[
p_k/\tilde{k}_t = \beta p_k \omega \tilde{k}_t^{\omega - 1}/(\tilde{k}_t^\omega - \tilde{c}_t)
\]

The latter gives us the decision rule(s)

\[
\tilde{k}_t^\omega - \tilde{c}_t = \omega \beta \tilde{k}_t^\omega
\]

\[
\tilde{c}_t = (1 - \omega \beta) \tilde{k}_t^\omega.
\]

In terms of our loglinear decision rule \((13)\), we have \(h_{c0} = \log(1 - \omega \beta), h_{ck} = \omega, \) and \(h_{cx} = h_{cv} = 0\). Note that risk does not affect consumption decision.

We don’t need this, but for future reference we derive the coefficients of the value function. The envelope condition implies

\[
p_k = (1 - \beta)\omega/(1 - \omega \beta).
\]

The others follows from of the Bellman equation:

\[
x_t: \quad p_x^T = (1 - \beta)^{-1} p_k e^T A
\]

\[
v_t: \quad p_v = 0
\]

constant: \(p_1 = \log(1 - \omega \beta) + [\beta/(1 - \beta)] [\log(\omega \beta) + (1 - \phi_v)v].\)

The last one, of course, has no bearing on the decision rule.

4.3 General loglinear approximation

The general additive problem starts with the scaled Bellman equation

\[
J(\tilde{k}_t, x_t, v_t) = \max_{\tilde{c}_t} \frac{(1 - \beta)\tilde{c}_t^\omega}{\rho} + \beta E_t [g_{t+1}^v J(\tilde{k}_{t+1}, x_{t+1}, v_{t+1})],
\]

(13)
subject to laws of motion and initial conditions. Here $J$ stands for the transformed value function.

The first-order and envelope conditions:

\[
(1 - \beta)c_t^{\beta-1} = \beta E_t(g_{t+1}^{\beta-1}J_{kt+1}) \tag{14}
\]

\[
J_{kt} = \beta E_t(g_{t+1}^{\beta-1}J_{kt+1})f_{kt}, \tag{15}
\]

which imply

\[
(1 - \beta)c_t^{\beta-1} = J_{kt}/f_{kt}. \tag{16}
\]

Evidently all we need here is a loglinear approximation of $J_{kt}$, the derivative of the value function. The value function itself doesn’t appear.

We follow a variant of the traditional guess and verify.

**Step 1: find deterministic steady state.** This curious phrase means to set shocks equal to zero and find the stationary point.

If we take the foc, the ec, and the law of motion for capital, and express them in terms of this deterministic steady state, we have the three equations,

\[
(1 - \beta)c_t^{\beta-1} = \beta g^{\beta-1}J_k
\]

\[
\beta g^{\beta-1}f_k = 1
\]

\[
(g + \delta - 1)\tilde{k} = k^\omega - \tilde{c}
\]

which we solve for the unknowns $(\tilde{c}, \tilde{k}, J_k)$. [?? Define $\tilde{\beta} = \beta g^{\beta-1}$?][?? CES version of last eq.]

**Step 2: approximate and solve.** The tradition is to use linear or loglinear approximations, or equivalent quadratic approximations of value function. We use loglinear approximations of the relevant conditions, including a guess of the derivative of the value function. This guess then leads to a Riccati-like equation, which is solved for the value function parameters.

(a) Guess. We guess the derivative of the value function has the form

\[
\log J_{kt} = p_1 + (p_k - 1) \log \tilde{k}_t + p_x^T x_t + p_v v_t
\]

with $(p_1, p_k, p_x, p_v)$ to be determined. The reason for the $(p_k - 1)$ is to maintain a connection to the previous example, where we specified $J$.

(b) Other loglinearizations. Ignore intercepts. Law of motion:

\[
\log \tilde{k}_{t+1} = (f_k/k) \log \tilde{k}_t - (c/kg) \log \tilde{c}_t - c^T x_{t+1}.
\]

The marginal product of capital:

\[
f_{kt} = \omega \tilde{k}_t^{\omega - 1} + (1 - \delta)
\]

\[
\log f_{kt} = (f_{kk} \tilde{k}/f_k) \log \tilde{k}_t = (f_{kk} \tilde{k}/f_k) \log \tilde{k}_t = (\omega - 1) \log \tilde{k}_t.
\]

(c) Decision rule. The combined foc/ec gives us

\[
\log c_t = \sigma(\log f_{kt} - \log J_{kt})
\]

\[
= \sigma[1 - p_k + (f_{kk} \tilde{k}/f_k)] \log k_t - \sigma p_x^T x_t - \sigma p_v v_t.
\]
Note that $h_{ck}$ depends only on the $k$ terms, the $x$ terms are irrelevant.

(d) Verify. Now substitute into the cc and solve for the $p$s. In the spirit of our “approximate the deterministic problem” approach, we solve the modified cc in which we apply the expectation to the log of the equation:

$$\log J_{kt} = E_t[(\rho - 1) \log g_{t+1} + \log J_{k_{t+1}}] + \log f_{kt}.$$ 

It’s clear here we’re not taking risk seriously, something we’ll come back to. The lhs is our guess. The rhs is

$$\text{rhs} = (p_k - 1) E_t \log \tilde{k}_{t+1} + (p_\tau + (\rho - 1) e^\top) E_t x_{t+1} + p_v E_t v_{t+1} + (f_{kk} \tilde{k}/f_k) \log \tilde{k}_t$$

If we link up terms from lhs and rhs, we get

$$\log \tilde{k}_t : 0 = \sigma(c/\tilde{kg})(p_k - 1)^2 - [1 - (f_k/g) + \sigma(c/\tilde{kg})(f_{kk} \tilde{k}/f_k)](p_k - 1) + (f_{kk} \tilde{k}/f_k)$$

$$x_t : 0 = [(\rho - p_k) e^\top + (p_k - 1)(c/\tilde{kg})\sigma(I - A)^{-1}] x_t + \{p_v \varphi_v + (p_k - 1)(c/\tilde{kg})\sigma\} v_t.$$

The first equation is central, describes problem and characterizes the response of consumption to capital. The last one shows that risk is irrelevant here.

(d) Clean up. We can find $p_1$ from the steady state value of $J_k$ and the other coefficients.

Comments:

- We’re missing risk. Even the law of motion changes if we add the innovation: $E(a) = \exp[v/2(1 - \varphi^2)] > 1$. This isn’t a huge change numerically, but it’s indicative.

- The dependence of the approximate decision rule on the capital stock — namely, $h_{ck}$ does not depend on the dynamics of productivity $a_t$. Standard result in LQ literature. See, for example, Anderson, Hansen, McGrattan, and Sargent (1996) and Hansen and Sargent (1980).

- Limiting results based on IES, $f_{kk}$.

4.4 Two examples

Example 1: verify Brock-Mirman. The special case $\rho = 1$, $\sigma = 1/(1 - \rho) = 1$, $\delta = 1$. Derivative term $f_{kk} \tilde{k}/f_k = \omega - 1$. The lom gives us $1 = f_k/g - c/\tilde{kg}$.

[Need to work out the rest. There’s an issue here about how to map the ec solution to one based on the be. Should be the same, but we need to show it.]

Example 2: a numerical approximation. In practice, we do this together with calibration. It’s helpful here to introduce the additional variable $y_t = k_t^\omega (a_t n_t)^{1-\omega}$ or (in our case) $y = k^\omega$. We use the macro ratios: $k/y = 10$ (meaning $\log(k/y) = \log 10$), $c/y = 0.75$ (ditto). And we use these values
for basic parameters: \( \omega = 1/3 \) (capital’s share), \( \delta = 0.025 \) (quarterly depreciation). We’ll leave the productivity process alone for now other than to set \( a = 1 \).

This automatically satisfies the law of motion for capital (the second one above). The first depends on

\[
f_k = \omega (y/k) + (1 - \delta) = 1.0083
\]

and implies \( \beta = 1/f_k = 0.9917 \). Since \( k/y = k/k^\omega = k^{1-\omega} = 10 \), \( k = 10^{1/(1-\omega)} = 31.6228 \) and \( y = k/10 = 3.1623 \). That gives us \( c = 2.3717 \). Finally, we have

\[
J_k = c^{\rho-1}(1-\beta)/\beta.
\]

If \( \rho = -1 \), we have \( J_k = 0.003513 \). We don’t need this yet, but it will come up later on.

5 Solving the recursive growth model

Here we need the value function as well as its derivative — its certainty equivalent shows up in the Bellman equation.

It’s convenient to solve the model in its original hdl1 form, although this gives us some extra terms. The first-order and envelope conditions are

\[
(1 - \beta)\bar{c}_t^{\rho-1} = \beta \mu_t(g_{t+1}J_{t+1})^{\rho-\alpha}E_t[(g_{t+1}J_{t+1})^{\alpha-1}J_{kt+1}]
\]

\[
J_{kt} = J_t^{\rho-1}J_{kt}/f_{kt}
\]

They imply the more useful condition for consumption:

\[
(1 - \beta)\bar{c}_t^{\rho-1} = J_t^{\rho-1}J_{kt}/f_{kt}
\]  \hspace{1cm} (17)

We use this condition below to generate the decision rule for consumption.

Digression. This looks different from the additive case even when we set \( \alpha = \rho \). The reason is that it’s based on a different version of preferences. If we used \( J'_{kt} = J_t^\rho/\rho \), then \( J'_{kt} = J_t^{\rho-1}J_{kt} \) or \( J_{kt} = J_t^{\rho-1}J_{kt}/J_t^{\rho-1} \). In terms of \( J' \), the consumption condition and ec are

\[
(1 - \beta)\bar{c}_t^{\rho-1} = J_t^{\rho-1}J_{kt}/f_{kt}
\]

\[
J'_{kt} = \beta \mu_t(g_{t+1}J_{t+1})^{\rho-\alpha}E_t[(g_{t+1}J_{t+1})^{\alpha-1}\rho^{-1}g_{t+1}^{\rho-1}J_{kt+1}]/f_{kt}
\]

Note that in the additive case (when \( \alpha = \rho \)), only the derivative of the value function matters. Otherwise, the value function itself shows up. From Stan: I’ve now checked this lots of different ways (eg, start with the unscaled model then scale the focs, or just assert what the Euler equation has to be then scale), and the equations hold up. That \( g_{t+1}^{\rho-1} \) plays an important role: the Euler equation depends on consumption growth raised to the \( \rho - 1 \), not scaled consumption growth. The extra \( g_{t+1} \) term makes that work.
5.1 The recursive Brock-Mirman example

We know in the additive case that if we set $\rho = 0$, $\delta = 1$, and $f(k, n) = k^\omega$, the value function is loglinear. The same holds with recursive preferences. For future reference, note: $\rho = 0, \sigma = 1/(1-\rho)$, $f_k = \omega k^{\omega - 1}$, $f_k k/f_k = \omega - 1$, and $1 = f_k/g - c/kg$. This is a recursive version of the problem described in Sargent’s red book (Dynamic Macro Theory, pp 24-27). We take the same approach.

1. The scaled Bellman equation is
\[
\log J(\tilde{k}_t, x_t, v_t) = \max_{\tilde{c}_t} (1 - \beta) \log \tilde{c}_t + \beta \log \mu_t[g_{t+1}J(\tilde{k}_{t+1}, x_{t+1}, v_{t+1})]
\]
subject to
\[
\tilde{k}_{t+1} = [\tilde{k}_t^\omega - \tilde{c}_t]/g_{t+1} \quad x_{t+1} = Ax_t + v_t^{1/2}Bw_{t+1} \quad v_{t+1} = (1 - \varphi_v)v + \varphi_v v_t + \tau w_{2t+1}
\]
plus initial conditions

Here $\log J_t$ is the value function, but this notation is closer to our treatment. We guess the value function
\[
\log J_t = p_0 + p_k \log \tilde{k}_t + p_x^T x_t + p_v v_t.
\]

2. We find the certainty equivalent in steps:
\[
\log(g_{t+1}J_{t+1}) = p_0 + p_k \log(\tilde{k}_t^\omega - \tilde{c}_t) + (p_x^T - p_k e^T)x_{t+1} + p_v v_{t+1}
\]
\[
= [p_0 + (1 - \varphi_v)v] + p_k \log(\tilde{k}_t^\omega - \tilde{c}_t) + (p_x^T - p_k e^T)(Ax_t + Bv_t^{1/2}w_{t+1})
\]
\[
+ p_v (\varphi_v v_t + \tau w_{2t+1}).
\]

Therefore
\[
\log \mu_t(g_{t+1}J_{t+1}) = [p_0 + (1 - \varphi_v)v] + p_k \log(\tilde{k}_t^\omega - \tilde{c}_t) + (p_x^T - p_k e^T)Ax_t + p_v \varphi_v v_t
\]
\[
+ (\alpha/2)V_x v_t + (\alpha/2)v^2.
\]

where
\[
V_x = (p_x - p_k e)^T B B^T (p_x - p_k e).
\]

3. The first-order and envelope conditions are
\[
(1 - \beta)/\tilde{c}_t = \beta p_k/(\tilde{k}_t^\omega - \tilde{c}_t)
\]
\[
p_k/\tilde{k}_t = \beta p_k \omega \tilde{k}_t^{\omega - 1}/(\tilde{k}_t^\omega - \tilde{c}_t)
\]
The latter gives us the decision rule(s)
\[
\tilde{k}_t^\omega - \tilde{c}_t = \omega \beta \tilde{k}_t^\omega
\]
\[
\tilde{c}_t = (1 - \omega/\beta)\tilde{k}_t^\omega.
\]
The former then implies
\[
p_k = (1 - \beta)\omega/(1 - \omega/\beta).
\]
Note again that risk does not affect consumption decision.

4. The other coefficients of the value function follow from the Bellman equation:
\[
x_t: \quad p_x^T = \beta(p_x^T - p_k e^T)A = p_k e^T A(I - \beta A)^{-1}
\]
\[
v_t: \quad p_v = \beta \varphi_v p_v + (\alpha/2)V_x = (\alpha/2)V_x(1 - \beta \varphi_v)^{-1}.
\]
5.2 General loglinear approximation

Work through the solution, ignoring most of the intercepts...

1. Preliminary steps.

- Guess value function (and its derivative):
  \[ J_t = p_0 + p_1 k^p_t \exp(p^\top x_t + p_v v_t) \]
  \[ \Rightarrow J_{kt} = p_k p_1 k^{p_k-1} \exp(p^\top x_t + p_v v_t) \]

- Various loglinearizations (ignoring intercepts):
  \[ \log J_t = d(p_k \log k_t + p^\top x_t + p_v v_t) \quad [d = (J - p_0)/J] \]
  \[ \log J_{kt} = (p_k - 1) \log k_t + p^\top x_t + p_v v_t \]
  \[ \log(J^{p-1} J_{kt}) = [(1 + (\rho - 1)d) p_k - 1] \log \tilde{k}_t + [1 + (\rho - 1)d] (p^\top x_t + p_v v_t) \]
  \[ \log f_{kt} = (f_{kk} \tilde{k}/f_k) \log k_t \]
  \[ \log f_{kt} - \log J_{kt} = [1 - p_k + \{f_{kk} \tilde{k}/f_k\}] \log \tilde{k}_t - p^\top x_t - p_v v_t \]
  \[ \log \tilde{k}_{t+1} = (f_k/g) \log \tilde{k}_t - (\varphi/\tilde{k}g) \log \tilde{c}_t - \log g_{t+1}. \]

- Approximate Bellman equation:
  \[ \log J_t = \rho^{-1} \log \left[ (1 - \beta) e^{\rho \log \tilde{c}_t} + \beta e^{\rho \log \mu_t(g_{t+1} J_{t+1})} \right] \]
  \[ \approx b_0 + (1 - b_1) \log \tilde{c}_t + b_1 \log \mu_t(g_{t+1} J_{t+1}). \]
  This is exact if \( \rho = 0 \), in which case \( b_0 = 0 \) and \( b_1 = \beta \).

2. Decision rule for consumption. Start with (11), loglinear version is

\[ \log \tilde{c}_t = \log J_t + \sigma (\log f_{kt} - \log J_{kt}) \]
\[ = \left[ \sigma + (d - \sigma) p_k + \sigma (f_{kk} \tilde{k}/f_k) \right] \log \tilde{k}_t + (d - \sigma) (p^\top x_t + p_v v_t) \]

where \( \sigma = 1/(1 - \rho) \) is the IES.

3. Evaluate certainty equivalent \( \mu_t(g_{t+1} J_{t+1}) \).

\[ \log(g_{t+1} J_{t+1}) = \log g_{t+1} + d(p_k \log k_{t+1} + p^\top x_{t+1} + p_v v_{t+1}) \]
\[ = e^\top x_{t+1} + dp_k[(f_{kk}/g) \log \tilde{k}_t - (c/\tilde{k}g) \log \tilde{c}_t - e^\top x_{t+1}] + dp^\top x_{t+1} + dp_v v_{t+1} \]
\[ = dp_k \left\{ (f_{kk}/g) - (\varphi/\tilde{k}g) [\sigma + (d - \sigma) p_k + \sigma (f_{kk} \tilde{k}/f_k)] \right\} \log \tilde{k}_t \]
\[ + [(1 - dp_k) e^\top + dp^\top x] A - (c/\tilde{k}g) dp_k (d - \sigma) p^\top x \]
\[ + dp_v [\varphi_v - (c/\tilde{k}g) dp_k (d - \sigma)] v_t \]
\[ + [(1 - dp_k) e + dp^\top x v_t^{1/2}] B w_{t+1} + dp_v \tau w_{2t+1} \]
That gives us the usual mean plus variance over two term:

\[
\log \mu_t(g_{t+1}J_{t+1}) = dp_k \left\{ (f_k/g) - (c/kg)(\sigma + (d - \sigma)p_k + \sigma(f_kk/f_k)) \right\} \log \tilde{k}_t \\
+ \left\{ (1 - dp_k)e^T + dp_x^T A - (c/kg)dp_k(d - \sigma)p_k^T \right\} x_t \\
+ \{ dp_x[\varphi_v - (c/kg)dp_k(d - \sigma)] + (\alpha/2)V_x \},
\]

where

\[
V_x = [dp_x + (1 - dp_k)e]^T BB^T [dp_x + (1 - dp_k)e] > 0
\]
is the contribution of \(x_{t+1}\) to the conditional variance of \(\log(g_{t+1}J_{t+1})\).

4. Use Bellman equation to determine coefficients of value function. Riccati equation for \(p_k\):

\[
0 = b_1d(\sigma - d)(c/kg)p_k^2 + \left[ d + (1 - b_1)(\sigma - d) - b_1d(f_k/g) + b_1d(c/kg)(1 + fkkk/f_k) \right] p_k \\
- (1 - b_1)\sigma(1 + fkkk/f_k)
\]

Others...

Algorithm: start with steady state, compute \((b_1, d)\), the solve problem. Recompute as needed.

### 5.3 Two examples

**Example 1: verify Brock-Mirman.** The idea is to take the general loglinear approximation and specialize it to this case.

1. Decision rule. With \(\rho = 1\) and \(d = 1\) (a guess), consumption satisfies (ignoring constants)

\[
\log \hat{\epsilon}_t = \log \hat{J}_{kt} - \log \hat{J}_t - \log f_{kt} = 1 + (fkkk/f_k) \log \tilde{k}_t = \omega \log \tilde{k}_t.
\]

Note that \(p_k\) drops out, as it did above. Also that consumption doesn’t depend on \(x_t\) or \(v_t\).

2. Law of motion. The capital equation is

\[
\tilde{k}_{t+1} = \tilde{k}_t^\omega - \hat{\epsilon}_t \\
\Rightarrow \log \tilde{k}_{t+1} = (f_k/g) \log \tilde{k}_t - (c/kg) \log \tilde{\epsilon}_t - \log g_{t+1} \\
= (\omega/g) \log \tilde{k}_t - \log g_{t+1}.
\]

The last step follows from the steady state condition.

3. Find the certainty equivalent again:

\[
\log(g_{t+1}J_{t+1}) = p_0 + p_k \log \tilde{k}_{t+1} + (p_z^T - p_k e^T) x_{t+1} + p_v v_{t+1} \\
= [p_0 + (1 - \varphi_v)e] + p_k \omega \log \tilde{k}_t + (p_z^T - p_k e^T)(Ax_t + Bu_t^{1/2}w_{2t+1}) \\
+ p_v(\varphi_v v_t + \omega w_{2t+1}) \\
\log \mu_t(g_{t+1}J_{t+1}) = [p_0 + (1 - \varphi_v)e] + p_k \omega \log \tilde{k}_t + (p_z^T - p_k e^T)Ax_t + p_v \varphi_v v_t \\
+(\alpha/2)V_x v_t + (\alpha/2)\tau^2.
\]
4. Substitute into Bellman equation:

$$\log J_t = (1 - \beta) \log \tilde{c}_t + \beta \log \mu_t (g_{t+1} J_{t+1})$$

implies

$$\log \tilde{k}_t : p_k = (1 - \beta) \omega + \beta \omega p_k \Rightarrow p_k = (1 - \beta) \omega / (1 - \beta \omega).$$

Etc. I.e, we get the same answer.

Example 2: a numerical approximation.

5.4 Discussion

- Impact of vol on consumption depends on IES.
- Contrast with BY: consumption depends on volatility. And impact of $x$ depends on IES. What’s the mrs?
- Show how interest rate responds to $v$
- Capital part of decision rule independent of risk. Extension of Tallarini...
- Capital coefficient $p_k$. Show positive and less than one?
- Dynamics of the capital stock.

6 Applications

1. Wedges... Which ones do we have?
2. News... Jaimovich-Rebelo, Schmitt-Grohe-Uribe,... Show why news isn’t enough...
3. Pricing kernel — compare to BY... Interest rate and vol a la Atkeson-Kehoe...
A Appendix

A.1 Certainty equivalents for Poisson risks

A.2 Expressions for miscellaneous derivatives

\( f_k, f_{kk}, \ldots \)

A.3 Campbell’s approximation

Solve for \( h_{kk} \).

Solve for \( p_k \). Derive by setting \( \alpha = \rho \) in recursive model...

Relation to our solution.
References


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