Econometric Asset Pricing Modelling

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ABSTRACT
The purpose of this paper is to propose a general econometric approach to no-arbitrage asset pricing modelling based on three main ingredients: (i) the historical discrete-time dynamics of the factor representing the information, (ii) the stochastic discount factor (SDF), and (iii) the discrete-time risk-neutral (RN) factor dynamics. Retaining an exponential-affine specification of the SDF, its modelling is equivalent to the specification of the risk-sensitivity vector and of the short rate, if the latter is neither exogenous nor a known function of the factor. In this general framework, we distinguish three modelling strategies: the direct modelling, the RN constrained direct modelling, and the back modelling. In all the approaches, we study the internal consistency conditions (ICCs), implied by the absence of arbitrage opportunity assumption, and the identification problem. The general modelling strategies are applied to two important domains: security market models and term structure of interest rates models. In these contexts, we stress the usefulness (and we suggest the use)
of the RN constrained direct modelling and of the back modelling approaches, both allowing us to conciliate a flexible (non-Car) historical dynamics and a Car (compound autoregressive) RN dynamics leading to explicit or quasi-explicit pricing formulas for various derivative products. Moreover, we highlight the possibility to specify asset pricing models able to accommodate non-Car historical and non-Car RN factor dynamics with tractable pricing formulas. This result is based on the notion of (RN) extended Car process that we introduce in the paper, and which allows us to deal with sophisticated models such as Gaussian and inverse Gaussian GARCH-type models with regime-switching, or Wishart quadratic term structure models. (JEL C1, C5, G12)

KEYWORDS: back modelling, Car and extended Car processes, direct modelling, identification problem, internal consistency conditions, Laplace transform, risk-neutral constrained direct modelling

Financial econometrics and no-arbitrage asset pricing remain rather disconnected fields mainly because the former is essentially based on discrete-time processes (like, for instance, VAR, GARCH and stochastic volatility models or switching regime models) and the latter is in general based on continuous-time diffusion processes, jump-diffusion processes, and Lévy processes. Recently, a few papers have tried to build a bridge between these two literatures [see Heston and Nandi (2000), Garcia, Ghysels, and Renault (2003), and Christoffersen, Heston, and Jacobs (2006) for the econometrics of option pricing; Gourieroux, Monfort, and Polimenis (2003), Dai, Le, and Singleton (2006), Dai, Singleton, and Yang (2007), and Monfort and Pegoraro (2007) for interest rates models; Gourieroux, Monfort and Sufana (2005) for exchange rates models; Gourieroux, Monfort and Polimenis (2006) for credit risk models], and the aim of the present work is in the same spirit. More precisely, the general objective of our paper is organized in the following four steps.

First, we propose a general and flexible pricing framework based on three main ingredients: (i) the discrete-time historical ($\mathbb{P}$) dynamics of the factor ($w_t$, say) representing the information (in the economy) used by the investor to price assets; (ii) the (one-period) stochastic discount factor (SDF) $M_{t,t+1}$, defining the change of probability measure between the historical and risk-neutral world; (iii) the discrete-time risk-neutral (RN or $\mathbb{Q}$) factor dynamics. The central mathematical tool used in the description of the historical and RN dynamics of the factor is the conditional log-Laplace transform (or cumulant generating function). The SDF is assumed to be exponential-affine [see Gourieroux and Monfort (2007)], and its specification is equivalent to the specification of a risk-sensitivity vector ($\alpha_t$, say) and of the short rate $r_t$, if the latter is neither exogenous nor a known function of the factor. Moreover, the notion of risk sensitivity is linked to the usual notion of market price of risk in a way that depends on the financial context (security markets or interest rates).
Second, we focus on the tractability of this general framework, in terms of explicit or quasi-explicit derivative pricing formulas, by defining the notion of extended Car (ECar) process, based on the fundamental concept of Car (compound autoregressive, or discrete-time affine) process introduced by Darolles, Gourieroux, and Jasiak (2006). More precisely, we first recall that the discrete-time Car approach is much more flexible than the corresponding continuous-time affine one, since, although every discretized continuous-time affine model is Car, the converse is not true. In other words, the Car family of processes is much wider than the discretized affine family, mainly because of the time consistency constraints (embedding condition) applying to the latter [see Darolles, Gourieroux, and Jasiak (2006), Gourieroux, Monfort, and Polimenis (2003, 2006), Monfort and Pegoraro (2006a, 2006b, 2007)]. Then, thanks to the concept of ECar process we define, we show that, even if the starting factor in our pricing model \( (w_{1,t}, \text{say}) \) is not Car in the RN world (implying, in principle, pricing difficulties), there is the possibility to find a second factor \( (w_{2,t}, \text{say}) \), possibly function of the first one, such that the extended process \( w_t = (w'_{1,t}, w'_{2,t})' \) turns out to be RN Car. The process \( \{w_{1,t}\} \) is called (risk-neutral) extended Car. If the RN dynamics is extended Car, the whole machinery of multihorizon complex Laplace transform, truncated real Laplace transform, and inverse Fourier transform of Car-based pricing procedures [see Bakshi and Madan (2000), and Duffie, Pan and, Singleton (2000)] becomes available.

Third, in this general asset pricing setting, we formalize three modelling strategies: the direct modelling strategy, the RN constrained direct modelling strategy, and the back modelling strategy. Since the three elements of the general framework, namely the \( \mathbb{P} \)-dynamics, the SDF \( M_{t,t+1} \), and the \( \mathbb{Q} \)-dynamics, are linked together (through the SDF change of probability measure), each strategy proposes a parametric modelling of two elements, the third one being a by-product. In the direct modelling strategy, we specify the historical dynamics and the SDF, that is to say, the risk-sensitivity vector and the short rate and, thus, the RN dynamics is obtained as a by-product. In the second strategy, the RN constrained direct modelling strategy, we specify the \( \mathbb{P} \)-dynamics and constrain the RN dynamics to belong to a given family, typically the family of Car or ECar processes. In this case, the risk-sensitivity vector characterizing the SDF is obtained as a by-product. Finally, in the back modelling strategy (the third strategy), we specify the \( \mathbb{Q} \)-dynamics, the short rate process \( r_t \), as well as the risk-sensitivity vector \( \alpha_t \) and, consequently, the historical dynamics is obtained as by-product. Thus, we get three kinds of econometric asset pricing models (EAPMs). In these strategies we carefully take into account the following important points: (i) the status of the short rate; (ii) the internal consistency conditions (ICCs) ensuring the compatibility of the pricing model with the absence of arbitrage opportunity principle [the ICCs are conveniently (explicitly) imposed through the log-Laplace transform]; (iii) the identification problem; and (iv) the possibility to have a \( \mathbb{Q} \)-dynamics of Car or extended Car type. In this respect, two of the proposed strategies, the back modelling and the RN constrained direct modelling strategies, are particularly attractive since they control for the RN dynamics and they allow for a rich class of nonlinear historical dynamics.
Moreover, these two approaches may be very useful for the computation of the (exact) likelihood function. For instance, in the back modelling approach, the nonlinear historical conditional density function is easily deduced from the, generally tractable (known in closed form), probability density function (pdf) in the RN world and from the possibly complex, but explicitly specified, risk-sensitivity vector.

Fourth, we apply these strategies to two important domains: security market models and interest rate models. In the first domain, we show how the back modelling strategy provides quasi-explicit derivative pricing formulas even in sophisticated models such as the RN switching regimes GARCH models generalizing those proposed by Heston and Nandi (2000) and Christoffersen, Heston, and Jacobs (2006). In the second domain, we show how both the back modelling and the RN constrained direct modelling strategies provide models able to generate, at the same time, nonlinear historical dynamics and tractable pricing procedures. In particular, we show how the introduction of lags and switching regimes lead to a rich and tractable modelling of the term structure of interest rates [see Monfort and Pegoraro (2007)].

The strategies formalized in this paper have been already used, more or less explicitly, in the continuous-time literature. However, it is worth noting that, very often, rather specific direct modelling or back modelling strategies are used: the dynamics of the factor is assumed to be affine under the historical (the RN, respectively) probability, the risk-sensitivity vector (and the short rate) is specified as affine function of the factor, and the RN (historical, respectively) dynamics is found to be also affine once the Girsanov change of probability measure is applied. These strategies could be called “basic” direct and back modelling strategies.

If we consider the option pricing literature, the stochastic volatility (SV) diffusion models [based on Heston (1993)] with jumps [in the return and/or volatility dynamics] of Bates (2000), Pan (2002), and Eraker (2004) are derived following this basic direct modelling strategy. The pricing models proposed by Bakshi, Cao, and Chen (1997, 2000) can be seen as an application of the basic back modelling approach, given that they work directly under the RN (pricing) probability measure. Even if these (affine) parametric specifications are able to explain relevant empirical features of asset price dynamics, the introduction of nonlinearities in the \( \mathbb{P} \)-dynamics of the factor seems to be very important, as suggested by Chernov et al. (2003) and Garcia, Ghysels, and Renault (2003).

In the continuous-time term structure literature, for instance, Duffie and Kan (1996) and Cheridito, Filipovic, and Kimmel (2007) follow the basic direct

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modelling strategy, while Dai and Singleton (2000, 2002) and Duffee (2002) use the basic back modelling counterpart. In other words, the classes of completely and essentially affine term structure models are derived following a basic direct or back modelling strategy. A notable exception, however, is given by the semi-affine model of Duarte (2004). Following a back modelling approach, he proposes a square-root bond pricing model, which is affine under the RN probability, but not under the historical, given the nonaffine specification of the market price of risk. This nonlinearity improves the model’s ability to match the time variability of the term premium, but it is not able to solve the tension between the matching of the first and the second conditional moments of yields (Dai and Singleton 2002; Duffee 2002), and it makes the estimation more difficult (less precise) given that the likelihood function of yield data becomes intractable.

The last example highlights the kind of limits typically affecting the continuous-time setting: the affine specification is necessary (under both $\mathbb{P}$ and $\mathbb{Q}$ measures) to make the econometric analysis of the model tractable and, therefore, certain relevant nonlinearities are missed. As indicated above, we can overcome these limits in our discrete-time asset pricing setting if the right strategy is followed. For instance, Dai, Le, and Singleton (2006), following a well-chosen back modelling strategy, propose a nonlinear discrete-time term structure model, which nests (the discrete-time equivalent of) the specifications adopted in Duffee (2002), Duarte (2004), and Cheridito, Filipovic, and Kimmel (2007). In their work, the $\mathbb{Q}$-dynamics of the factor is Car, the market price of risk is assumed to be a nonlinear (polynomial) function of the factor, the $\mathbb{P}$-dynamics is not Car, and the likelihood function of the bond yield data is known in closed form. This nonlinearity is shown to significantly improve the statistical fit and the out-of-sample forecasting performance of the nested models.

The paper is organized as follows. In Section 1, we define the historical and RN dynamics of the factor, and the SDF. In Section 2, we briefly review Car processes and their main properties, introduce the important notion of (internally and externally) extended Car (ECar) process, provide several examples, and briefly describe the pricing of derivative products when the underlying asset is Car (or ECar) in the RN world. In Section 3, we discuss the status of the short rate, describe the various modelling strategies for the specification of an EAPM, and present the associated inference problem. Sections 4 and 5 consider, respectively, applications to econometric security market models and to econometric term structure models, while, in Section 6, we present an example of security market model with stochastic dividends and short rate. Section 7 concludes, and the proofs are gathered in the Appendices.

1 HISTORICAL AND RN DYNAMICS

1.1 Information and Historical Dynamics

We consider an economy between dates 0 and $T$. The new information in the economy at date $t$ is denoted by $w_t$, the overall information at date $t$ is
Let us denote by \( L_{2_t} \) the (Hilbert) space of square integrable functions \( g(w_t) \). Following Hansen and Richard (1987), we consider the following assumptions:

**A1** (Existence and uniqueness of a price): Any payoff \( g(w_s) \) of \( L_{2_s} \), delivered at \( s \), has a unique price at any \( t < s \), for any \( w_t \), denoted by \( p_t[g(w_s)] \), function of \( w_t \).

**A2** (Linearity and continuity):
- \( p_t[\lambda_1 g_1(w_s) + \lambda_2 g_2(w_s)] = \lambda_1 p_t[g_1(w_s)] + \lambda_2 p_t[g_2(w_s)] \) (law of one price)
- if \( g_n(w_s) \xrightarrow{n \to \infty} 0 \), \( p_t[g_n(w_s)] \xrightarrow{n \to \infty} 0 \).

**A3** (Absence of arbitrage opportunity): At any \( t \in \{0, \ldots, T\} \) it is impossible to constitute a portfolio (of future payoffs), possibly modified at subsequent dates, such that: (i) its price at \( t \) is nonpositive; (ii) its payoffs at subsequent dates are non-negative; and (iii) there exists at least one date \( s > t \) such that the net payoff, at \( s \), is strictly positive with a strictly positive conditional probability at \( t \).

Under **A1**, **A2**, and **A3**, a conditional version of the Riesz representation theorem implies, for each \( t \in \{0, \ldots, T - 1\} \), the existence and uniqueness of the SDF \( M_{t+1}(w_{t+1}) \), belonging to \( L_{2_{t+1}} \), such that the price at date \( t \) of the payoff \( g(w_s) \) delivered at \( s > t \) is given by (see Appendix A):

\[
p_t \left[ g(w_s) \right] = E_t \left[ M_{t+1} \cdots M_{s-1} g(w_s) \right].
\]
Moreover, under A3, $M_{t,t+1}$ is positive for each $t \in \{0, \ldots, T - 1\}$. The process $M_{lt} = \prod_{j=0}^{t-1} M_{j,j+1}$ is called the state price deflator over the period $\{0, \ldots, t\}$.

Since $L_{2,t+1}$ contains 1, the price at $t$ of a zero-coupon bond maturing at $t + 1$ is

$$B(t, 1) = \exp(-r_{t+1}) = E_t(M_{t,t+1}),$$

where $r_{t+1}$ is the predetermined (that is, known at $t$) geometric short rate between $t$ and $t + 1$.

### 1.3 Exponential–Affine SDF

We assume that $M_{t,t+1}(w_{t+1})$ has an exponential-affine form

$$M_{t,t+1} = \exp\left[\alpha_t(w_t) w_{t+1} + \beta_t(w_t)\right],$$

where $\alpha_t$ is the factor-loading or risk-sensitivity vector.\(^4\) Since $\exp(-r_{t+1}) = E_t(M_{t,t+1}) = \exp[\psi_t(\alpha_t|w_t) + \beta_t]$, the SDF can also be written as

$$M_{t,t+1} = \exp\left[-r_{t+1}(w_t) + \alpha_t'(w_t) w_{t+1} - \psi_t(\alpha_t|w_t)\right].$$

In the case where $w_{t+1}$ is a vector of geometric returns of basic assets or a vector of yields, the risk-sensitivity vector $\alpha_t(w_t)$ can be seen, respectively, as the opposite of a market price of risk vector, or as a market price of risk vector (see Appendix B for a complete proof). More precisely, if we consider the vector of arithmetic returns $\rho_{A,t+1}$ of the basic assets in the first case, and of zero-coupon bonds in the second case, the arithmetic risk premia $\pi_{At} = E_t(\rho_{A,t+1}) - r_{A,t+1}e$ (where $r_{A,t+1}$ is the arithmetic risk-free rate, and where $e$ denotes the unitary vector) is given by $\pi_{At} = -\exp(r_{t+1})\Sigma_t\alpha_t$ in the first case, and it is $\pi_{At} = \exp(r_{t+1})\Sigma_t\alpha_t$ in the second case ($\Sigma_t$ is the conditional variance–covariance matrix of $w_{t+1}$ given $w_t$).

### 1.4 RN Dynamics

The joint historical distribution of $w_T$, denoted by $\mathbb{P}$, is defined by the conditional distribution of $w_{t+1}$ given $w_t$, characterized either by the pdf $f_t(w_{t+1}|w_t)$ or the Laplace transform $\varphi_t(u|w_t)$, or the log-Laplace transform $\psi_t(u|w_t)$.

\(^4\)The justification of this exponential-affine specification is now well documented in the asset pricing literature. First, this form naturally appears in equilibrium models like CCAPM [see e.g. Cochrane (2005)], consumption-based asset pricing models either with habit formation or with Epstein–Zin preferences [see, among the others, Bansal and Yaron (2004), Campbell and Cochrane (1999), Eraker (2007), Garcia, Meddahi, and Tedongap (2006), Garcia, Renault, and Semenov (2006)]. Second, in general continuous-time security market models, the discretized version of the SDF is exponential-affine (Gourieroux and Monfort 2007). Third, the exponential-affine specification is particularly well adapted to Laplace transform, which is a central tool in discrete-time asset pricing theory [see e.g. Bertholon, Monfort, and Pegoraro (2006), Darolles, Gourieroux, and Jasiak (2006), Gourieroux, Jasiak, and Sufana (2004), Gourieroux, Monfort, and Polimenis (2003, 2006), Monfort and Pegoraro (2006a, 2006b, 2007), Pegoraro (2006), Polimenis (2001)].
The RN dynamics is another joint distribution of $w_{t+1}$, denoted by $Q$, defined by the conditional pdf, with respect to the corresponding conditional historical probability, given by

$$d^Q_t(w_{t+1}|w_t) = \frac{M_{t+1}(w_{t+1})}{E_t[M_{t+1}(w_{t+1})]} = \exp(r_{t+1})M_{t+1}(w_{t+1}).$$

So, the RN conditional pdf (with respect to the same measure as the corresponding conditional historical probability) is

$$f^Q_t(w_{t+1}|w_t) = f_t(w_{t+1}|w_t)d^Q_t(w_{t+1}|w_t),$$

and the conditional pdf of the conditional historical distribution with respect to the RN one is given by

$$d^P_t(w_{t+1}|w_t) = \frac{1}{d^Q_t(w_{t+1}|w_t)}.$$

When the SDF is exponential-affine, we have the convenient additional result

$$d^Q_t(w_{t+1}|w_t) = \frac{\exp(\alpha_t'w_{t+1} + \beta_t)}{E_t[\exp(\alpha_t'w_{t+1} + \beta_t)]} = \exp[\alpha_t'w_{t+1} - \psi_t(\alpha_t)],$$

so $d^Q_t$ is also exponential-affine. It is readily seen that the conditional RN Laplace transform of the factor $w_{t+1}$, given $w_t$, is [see Gourieroux and Monfort (2007)]

$$\varphi^Q_t(u|w_t) = \frac{\varphi_t(u + \alpha_t)}{\varphi_t(\alpha_t)}$$

and, consequently, the associated conditional RN log-Laplace transform is

$$\psi^Q_t(u) = \psi_t(u + \alpha_t) - \psi_t(\alpha_t). \quad (3)$$

Conversely, we get

$$d^P_t(w_{t+1}|w_t) = \exp[-\alpha_t'w_{t+1} + \psi_t(\alpha_t)]$$

and, taking $u = -\alpha_t$ in $\psi^Q_t(u)$, we can write

$$\psi^Q_t(-\alpha_t) = -\psi_t(\alpha_t) \quad (4)$$

and, replacing $u$ by $u - \alpha_t$, we obtain

$$\psi_t(u) = \psi^Q_t(u - \alpha_t) - \psi^Q_t(-\alpha_t). \quad (5)$$
We also have
\[
d_t^p(w_{t+1} | w_t) = \exp \left[ - \alpha_t' w_{t+1} - \psi_t^p(-\alpha_t) \right],
\]
\[
d_t^Q(w_{t+1} | w_t) = \exp \left[ \alpha_t' w_{t+1} + \psi_t^Q(-\alpha_t) \right].
\]

2 Car AND EXTENDED Car (ECar) PROCESSES

For the sake of completeness, we present in this section a brief review of the Car (or discrete-time affine) processes and of their main properties [for more details, see Darolles, Gourieroux, and Jasiak (2006), Gourieroux and Jasiak (2006), and Gourieroux, Monfort, and Polimenis (2006)]. We will also introduce the notion of extended Car process, which will be very useful in the rest of the paper. All the processes \{y_t\} considered will be such that \(y_t\) is a function of the information at time \(t: w_t\).

2.1 Car\((1)\) Processes

An \(n\)-dimensional process \(\{y_t\}\) is called Car\((1)\) if its conditional Laplace transform
\[
\varphi_t(u | y_t) = E[\exp(u' y_{t+1}) | y_t],
\]
is of the form
\[
\varphi_t(u | y_t) = \exp[a_t(u)' y_t + b_t(u)], \quad u \in \mathbb{R}^n,
\]
where \(a_t\) and \(b_t\) may depend on \(t\) in a deterministic way. The log-Laplace transform \(\psi_t(u | y_t) = \log \varphi_t(u | y_t)\) is therefore affine in \(y_t\), which implies that all the conditional cumulants and, in particular, the conditional mean and the conditional variance-covariance matrix, are affine in \(y_t\). Let us consider some examples of Car\((1)\) processes.

2.1.1 Gaussian AR\((1)\) processes. If \(y_{t+1}\) is a Gaussian AR\((1)\) process defined by:
\[
y_{t+1} = \mu + \rho y_t + \epsilon_{t+1}
\]
where \(\epsilon_{t+1}\) is a Gaussian white noise distributed as \(N(0, \sigma^2)\), then the process is Car\((1)\) with \(a(u) = up\) and \(b(u) = u\mu + \frac{\sigma^2}{2}u^2\).

2.1.2 Compound Poisson processes (or integer valued AR\((1)\) processes). If \(y_{t+1}\) is defined by
\[
y_{t+1} = \sum_{i=1}^{w} z_{it} + \epsilon_{t+1},
\]
where the \(z_{it}\)'s follow independently the Bernoulli distribution \(B(\rho)\) of parameter \(\rho \in ]0, 1[\), and the \(\epsilon_{t+1}\)'s follow independently (and independently from the \(z_{it}\)'s) a Poisson distribution \(P(\lambda)\) of parameter \(\lambda > 0\). It is easily seen that \(\{y_t\}\) is Car\((1)\) with \(a(u) = \log[\rho \exp(u) + 1 - \rho]\) and \(b(u) = -\lambda[1 - \exp(u)]\).
In particular, the correlation between $y_{t+1}$ and $y_t$ is given by $\rho$, and we can write $y_{t+1} = \lambda + \rho y_t + \eta_{t+1}$, where $\eta_{t+1}$ is a martingale difference and, therefore, \{y_t\} is an integer valued weak AR(1) process.

2.1.3 Autoregressive Gamma processes (ARG(1) or positive AR(1) processes). The ARG(1) process $y_{t+1}$ is the exact discrete-time equivalent of the square-root (Cox-Ingersoll-Ross) diffusion process, and it can be defined in the following way:

$$
\frac{y_{t+1}}{\mu} | z_{t+1} \sim \gamma(v + z_{t+1}), \quad v > 0,
$$

$$
z_{t+1} | y_t \sim \mathcal{P}(\rho y_t / \mu), \quad \rho > 0, \mu > 0,
$$

where $\gamma$ denotes a Gamma distribution, $\mu$ is the scale parameter, $v$ is the degree of freedom, $\rho$ is the correlation parameter, and $z_t$ is the mixing variable. The conditional probability density function $f(y_{t+1} | y_t; \mu, v, \rho)$ (say) of the ARG(1) process is a mixture of Gamma densities with Poisson weights. It is easy to verify that \{y_t\} is Car(1) with $a(u) = \frac{\rho u}{1 - u \mu}$ and $b(u) = -v \log(1 - u \mu)$. Moreover, we have

$$
y_{t+1} = v \mu + \rho y_t + \eta_{t+1},
$$

where $\eta_{t+1}$ is a martingale difference sequence, so \{y_t\} is a positive weak AR(1) process with $E[y_{t+1} | y_t] = v \mu + \rho y_t$ and $V[y_{t+1} | y_t] = v \mu^2 + 2 \rho \mu y_t$.

It is also possible, thanks to the recursive methodology followed by Monfort and Pegoraro (2006b), to build discrete-time multivariate autoregressive Gamma processes. A notable advantage of the vector ARG(1) process, with respect to the continuous-time analogue, is given by its conditional probability density (and likelihood) function known in closed-form even in the case of conditionally correlated scalar components. Indeed, the multivariate CIR process has a known discrete transition density only in the case of uncorrelated components and, therefore, in continuous-time this particular case, only, opens the possibility for an exact maximum likelihood estimation approach.

2.1.4 Wishart autoregressive processes (or positive definite matrix valued AR(1) processes). The Wishart autoregressive (WAR) process $y_{t+1}$ is a process valued in the space of $(n \times n$) symmetric positive definite matrices, such that its conditional historical log-Laplace transform is given by

$$
\psi_t(\Gamma) = \log\{E_t \exp(\text{Tr} \, \Gamma y_{t+1})\}
= \text{Tr}[\Gamma M (I_n - 2 \Sigma \Gamma)^{-1} M y_t] - K \log \det[(I_n - 2 \Sigma \Gamma)],
$$

where $\Gamma$ is a $(n \times n)$ matrix of coefficients, which can be chosen symmetric (since, with obvious notations, $\text{Tr}(\Gamma y_{t+1}) = \sum_{ij} \Gamma_{ij} y_{ij,t+1} = \sum_{i \leq j} (\Gamma_{ij} + \Gamma_{ji}) y_{ij,t+1}$). This
dynamics is Car(1) and, if $K$ is integer, it can be defined as

$$y_t = \sum_{k=1}^{K} x_{k,t} x_{k,t}^\prime \quad (K \geq n),$$

$$x_{k,t+1} = M x_{k,t} + \varepsilon_{k,t+1}, \quad k \in \{1, \ldots, K\},$$

$$\varepsilon_{k,t+1} \overset{\text{p}}{\sim} \text{IN}(0, \Sigma), \quad k \in \{1, \ldots, K\},$$

Moreover, we have

$$y_{t+1} = M y_t M^\prime + k \Omega + \eta_{t+1},$$

where $\eta_{t+1}$ is a matrix martingale difference. So, $\{y_t\}$ is a positive definite matrix valued AR(1) process. Note that, if $n = 1$, $\Gamma = u$, $M = m$, and $\Omega = \sigma^2$, relation (7) reduces to $\psi^\prime(u) = \left[ 1 + \frac{2}{\sigma^2} \log(1 - 2\sigma^2 u) \right]$, and $\{y_t\}$ is found to be an ARG(1) process with $\rho = m^2$, $v = k/2$, $\mu = 2\sigma^2$. This means that the WAR process is a multivariate (matrix) generalization of the ARG(1) process.

### 2.2.1 Internally extended Car(1) processes

**Definition.** A process $\{y_{1,t}\}$ is said to be ECar(1) if there exists a process $\{y_{2,t}\}$ such that $y_t = (y_{1,t}, y_{2,t})$ is Car(1). Moreover, if the $\sigma$-algebra $\sigma(y_{1,t})$ spanned by $y_{1,t}$ is equal to $\sigma(y_{2,t})$, $\{y_{1,t}\}$ will be called internally extended Car(1) process. Otherwise, if $\sigma(y_{1,t}) \subset \sigma(y_{2,t})$, $\{y_{1,t}\}$ will be called externally extended Car(1) process.

### 2.2.2 Internally extended Car(1) processes

**Car(p) processes.** The process $y_{1,t+1}$ is Car(p) if its conditional log-Laplace transform satisfies

$$\psi_l(u \mid y_t) = \sum_{i=1}^{p} a_{i,t}(u)^\prime y_{t+1-i} + b_{i}(u), \quad u \in \mathbb{R}^n. \quad (9)$$

It is easily seen that the process $y_t = (y_{1,t}, y_{2,t})$, with $y_{1,t} = (y_{1,t-1}, \ldots, y_{1,t-p+1})$, is Car(1) (Darolles, Gourieroux, and Jasiak 2006), and that $\sigma(y_{1,t}) = \sigma(y_{2,t})$.
starting from a Car(1) process, we can easily construct index-Car(p) processes such as ARG(p) and Gaussian AR(p) processes (Monfort and Pegoraro 2007).

**ARMA processes.** If we consider an ARMA(1, 1) process \( \{ y_{1,t} \} \) defined by

\[
y_{1,t+1} - \varphi y_{1,t} = \varepsilon_{t+1} - \theta \varepsilon_t,
\]

where \( \varepsilon_{t+1} \sim IIN(0, \sigma^2) \), it is well known that \( y_{t+1} \) is not Markovian and, consequently, it is not Car(1), or even Car(p). However, using the state-space representation of ARMA processes, we have that the process \( y_t = (y_{1,t}, \varepsilon_t) \)' satisfies

\[
y_{t+1} = \begin{bmatrix} \varphi & -\theta \\ 0 & 0 \end{bmatrix} y_t + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \varepsilon_{t+1}.
\]

This means that \( \{ y_t \} \) is Car(1) since it is a Gaussian bivariate AR(1) process, and that \( \{ y_{1,t} \} \) is a ECar(1) process. Clearly, \( \sigma(y_{1,t}) = \sigma(y_1) \), so \( \{ y_{1,t} \} \) is an internally ECar(1). It is important to observe that, in the bivariate AR(1) representation (10), one eigenvalue of the autoregressive matrix is equal to zero and, therefore, this process has no continuous-time bivariate Ornstein–Uhlenbeck analogue, since in this kind of process the autoregressive matrix \( \Phi \) (say) is of the form \( \Phi = \exp(A) \). This result is also a consequence of the fact that a discrete-time ARMA(\( p, q \)), with \( q \geq p \), cannot be embedded in a continuous-time ARMA (CARMA) process (Brockwell 1995; Huzii 2007). This example of extended Car process can obviously be generalized to ARMA(\( p, q \)) and VARMA(\( p, q \)) processes. The VARMA model belongs also to the class of generalized affine models proposed in finance by Feunou and Meddahi (2007) to provide tractable derivative prices.

**GARCH-type processes.** Let us consider the process \( \{ y_{1,t} \} \) defined by

\[
\begin{cases}
y_{1,t+1} = \mu + \varphi y_{1,t} + \sigma_{t+1} \varepsilon_{t+1}, \\
\sigma_{t+1}^2 = \omega + \alpha \varepsilon_t^2 + \beta \sigma_t^2,
\end{cases}
\]

where \( \varepsilon_{t+1} \sim IIN(0, 1) \). \( \{ y_{1,t} \} \) is not Car(1), but the (extended) process \( y_t = (y_{1,t}, \sigma_{t+1}^2) \)' is Car(1). Indeed, we have that

\[
E \left[ \exp \left( u y_{1,t+1} + v \sigma_{t+1}^2 \right) \mid y_{1,t}, \sigma_{t+1}^2 \right] = \exp \left[ \left( u \mu + v \omega - \frac{1}{2} \log(1 - 2v\omega) \right) + u \varphi y_{1,t} + \left( v\beta + \frac{u^2}{2(1 - 2v\omega)} \right) \sigma_{t+1}^2 \right].
\]

Therefore, \( \{ y_{1,t} \} \) is ECar(1) and \( \sigma(y_{1,t}) = \sigma(y_1) \). Section 4.6 shows that this result still applies when switching regimes are introduced. Observe that this model [also called Heston and Nandi (2000) model] is not a generalized affine one, and it belongs to the class of generalized nonaffine models mentioned in Feunou and Meddahi (2007).
2.2.2 Externally extended Car(1) processes

Quadratic transformation of Gaussian AR(1) processes. Let us consider the following Gaussian AR(1) process:

$$x_{t+1} = \mu + \rho x_t + \epsilon_{t+1}, \quad \epsilon_{t+1} \sim \text{IIN}(0, \sigma^2).$$

If $\mu = 0$ the process $y_{1,t} = x_t^2$ is Car(1). If $\mu \neq 0$, the process $y_{1,t} = x_t^2$ is not Car; however, it can be shown that $y_t = (y_{1,t}, x_t)'$ is Car(1) (Gourieroux and Sufana 2003) and, thus, $y_{1,t}$ is ECar(1) (see Section 5.4 for a proof in a multivariate context). Obviously, we have $\sigma(y_{1,t}) \subset \sigma(y_t)$.

Switching regimes Gaussian AR(1) processes. In the classical Gaussian AR(1) model defined in Section 2.1.1, the conditional distribution of $y_{1,t+1}$ given $y_t$, has a skewness $\tilde{\mu}_3 = 0$ and a kurtosis $\tilde{\mu}_4 = 3$. If we want to introduce a more flexible specification for $\tilde{\mu}_3$ and $\tilde{\mu}_4$, the first possibility is to assume that $\epsilon_{t+1}$ is still a zero mean, unit variance white noise, but with a distribution belonging to some parametric family [like, for instance, the truncated Gram–Charlier expansion used by Jondeau and Rockinger (2001) to price foreign exchange options, or the nonparametric (SNP) distribution employed by Léon, Mencia, and Sentana (2007) for European-type option pricing]. However, this approach has some drawbacks: the set of possible pair of conditional skewness–kurtosis of $y_{1,t+1}$ (i.e., the set of skewness and kurtosis generated by $\epsilon_{t+1}$) is not the maximal set $D = \{(\tilde{\mu}_3, \tilde{\mu}_4) \in \mathbb{R} \times \mathbb{R}_+^*: \tilde{\mu}_4 \geq \tilde{\mu}_3^2 + 1\}$ and, moreover, $\tilde{\mu}_3$ and $\tilde{\mu}_4$ do not depend on $y_t$. One way to solve these problems is to consider a two-state switching regimes Gaussian AR(1) process $\{y_{1,t}\}$ given by

$$y_{1,t+1} = \mu' y_{2,t+1} + \rho y_{1,t} + (\sigma' y_{2,t+1}) \epsilon_{t+1},$$

where $\epsilon_{t+1} \sim \text{IIN}(0,1), \mu' = (\mu_1, \mu_2), \sigma' = (\sigma_1, \sigma_2)$, and where $\{y_{2,t}\}$ is a two-state homogeneous Markov chain [as defined in Section 2.1.5] with $\pi(e_1, e_1) = p$ and $\pi(e_2, e_2) = q$, independent of $\{\epsilon_t\}$. The Laplace transform of $y_{1,t+1}$, conditionally to $y_{1,t}$, is not exponential-affine, but it is easy to verify that the bivariate process $y_t = (y_{1,t}, y_{2,t})'$ is Car(1) (Monfort and Pegoraro 2007). In other words, $y_{1,t}$ is an externally ECar(1) process, given the additional information introduced by the Markov chain.

Given that the probability density function of $y_{1,t+1}$, conditionally to $y_{2,t}$, is a mixture of the Gaussian densities $n(y_{1,t+1}; \mu_j + \rho y_{1,t}, \sigma_j^2)$, with $j \in \{1, 2\}$, this kind of Car process is able to generate (conditionally to $y_t$) stochastic skewness [$\tilde{\mu}_3(y_t')$, say] and kurtosis [$\tilde{\mu}_4(y_t')$, say] and, moreover, it is able to reach, for each time $t$, any possible pair of skewness–kurtosis in the domain of maximal size $D_t = \{((\tilde{\mu}_3(y_t'), \tilde{\mu}_4(y_t')) \in \mathbb{R} \times \mathbb{R}_+^*: \tilde{\mu}_4(y_t') \geq \tilde{\mu}_3(y_t')^2 + 1\}$ [see Bertholon, Monfort, and Pegoraro (2006) for a formal proof].

It is important to highlight that these features do not characterize just the distribution of $y_{1,t+1}$ conditionally to both its own past $(y_{1,t})$ and the past of the latent variable $(z_t)$. Indeed, the distribution of $y_{1,t+1}$, conditionally only to its own past $(y_{1,t})$, is still a mixture of Gaussian distributions with probability density function...
given by
\[
    f(y_{1,t+1} \mid y_{1,t}) = n(y_{1,t+1} \mid \mu_1 + \rho y_{1,t}, \sigma_1^2)[p P(y_{2,t} = e_1 \mid y_{1,t}) + (1 - q) P(y_{2,t} = e_2 \mid y_{1,t})] + n(y_{1,t+1} \mid \mu_2 + \rho y_{1,t}, \sigma_2^2)[(1 - p) P(y_{2,t} = e_1 \mid y_{1,t}) + q P(y_{2,t} = e_2 \mid y_{1,t})].
\]

In Section 4, we will see that, thanks to the exponential-affine specification (2) of the SDF, these statistical properties (used to describe the dynamics of geometric returns) are transferred from the historical to the RN distribution, with important pricing implications.

**Stochastic volatility in mean processes.** We can specify also an SV in mean AR(1) process defined by
\[
    y_{1,t+1} = \mu_1 + \mu_2 y_{2,t+1} + \rho y_{1,t} + y_{1,t+1}^{1/2},
\]
where \( \varepsilon_{t+1} \sim \text{IID}(0, 1) \), and where \( \{y_{2,t}\} \) is an ARG(1) process, as defined in Section 2.1.3, independent of \( \{\varepsilon_t\} \). The process \( \{y_{1,t}\} \) is an externally ECar(1) since \( y_t = (y_{1,t}, y_{2,t}) \) is Car(1) and \( \sigma(y_{1,t}) \subset \sigma(y_t) \). We can also consider a n-variate SV in mean AR(1) process defined by
\[
    y_{1,t+1} = \mu + R y_{1,t} + \left[ \begin{array}{c} \text{Tr} S_1 y_{2,t+1} \\ \vdots \\ \text{Tr} S_n y_{2,t+1} \end{array} \right] + y_{2,t+1}^{1/2} \varepsilon_{t+1},
\]
where \( \varepsilon_{t+1} \sim \text{IID}(0, I) \), of size \( n \), \( R \) is a \((n \times n)\) matrix, the \( S_i \)'s are \((n \times n)\) symmetric matrices, and \( \{y_{2,t}\} \) is an \( n \)-dimensional WAR process independent of \( \{\varepsilon_t\} \).

In this multivariate setting, \( \{y_{1,t}\} \) is an \( n \)-dimensional ECar(1) process, because \( y_t = (y_{1,t}', \text{vech}(y_{2,t}))' \) is Car(1) [see Gourieroux, Jasiak, and Sufana (2004) and, in continuous time, Burasci, Porchia, and Trojani (2007), Da Fonseca, Grasselli, and Tebaldi (2007a, 2007b), Da Fonseca, Grasselli, and Ielpo (2008)].

### 2.5 Pricing with RN Car(1) or ECar(1) Processes

It is well known that if \( \{y_t\} \) is Car(1) with conditional Laplace transform \( \varphi_t(u \mid y_t) = \exp[a(u)'y_t + b(u)] \), the multihorizon (conditional) Laplace transform takes the following exponential-affine form:
\[
    E_t[\exp(u_{t+1}'y_{t+1} + \cdots + u_T'y_T)] = \exp[A_T(t)'y_t + B_T(t)],
\]
where the functions \( A_T \) and \( B_T \) are easily computed recursively, for \( j \in \{T - t - 1, \ldots, 0\} \), by
\[
    A_T(t + j) = a_{t+j+1}[u_{t+j+1} + A_T(t + j + 1)], \quad B_T(t + j) = b_{t+j+1}[u_{t+j+1} + A_T(t + j + 1)] + B_T(t + j + 1),
\]
starting from the terminal conditions \( A_T(T) = 0, B_T(T) = 0 \).

If we want to determine the price at \( t \) of a payoff \( g(y_{T,t}) \) at \( T \), we have to compute a conditional expectation under the RN probability, namely \( E_t^Q[\exp(r_{t+1} + \cdots +}
If \( \{ y_t \} \) is \( \text{Car}(1) \) or \( \text{ECar}(1) \) in the RN world, this computation leads to explicit or quasi-explicit pricing formulas for several derivative products. For instance, if the one-period risk-free rate \( r_{t+1} \) is exogenous or affine in \( y_t \) and if 
\[
g(y_T) = \left[ \exp(\mu_1 y_T) - \exp(\mu_2 y_T) \right]^+, \]
where \( y_{t,T} = (y_t', \ldots, y_T')' \), the computation reduces to two truncated multihorizon Laplace transforms which, in turn, are obtained by simple integrals based on the untruncated complex Laplace transform easily deduced from the recursive equations given above (Bakshi and Madan 2000; Duffie, Pan, and Singleton 2000; Gourieroux, Monfort, and Polimenis 2003; Monfort and Pegoraro 2007).

3 ECONOMETRIC ASSET PRICING MODELS (EAPMs)

The true value of the various mathematical tools introduced in Section 1, for instance \( \psi_t, M_{t+1}, \text{or } \psi^Q_t \), are not known by the econometrician and, therefore, they have to be specified and parameterized. In other words, we have to specify an econometric asset pricing model (EAPM). What we really need, in order to derive explicit or quasi-explicit pricing formulas, is a factor \( w_{t+1} \), which is \( \text{Car} \) or \( \text{ECar} \) under the RN probability, while its historical dynamics does not necessarily belong to this family of processes [see also Duarte (2004) and Dai, Le, and Singleton (2006)]. In other words, the tractability of the asset pricing model is associated with a conditional log-Laplace transform \( \psi^Q_t \), which is affine in \( w_t \), while the specification and parameterization of \( \psi_t \) can be more general.

We are going to present three ways of specifying an EAPM: the direct modelling, the RN constrained direct modelling, and the back modelling. In all approaches, we first need to make more precise the status of the short rate \( r_{t+1} \).

3.1 The Status of the Short Rate

The short rate \( r_{t+1} \) is a function of \( w_t \). This function may or may not be known by the econometrician. It is known in two main cases:

1. \( r_{t+1} \) is exogenous, i.e., \( r_{t+1}(w_t) \) does not depend on \( w_t \), and, therefore, \( r_{t+1}(\cdot) \) is a known constant function of \( w_t \);

2. \( r_{t+1} \) is an endogenous factor, i.e., \( r_{t+1} \) is a component of \( w_t \).

If the function \( r_{t+1}(w_t) \) is unknown, it has to be specified parametrically. So we assume that the unknown function belongs to a family,

\[
\left\{ r_{t+1}(w_t, \tilde{\theta}), \tilde{\theta} \in \tilde{\Theta} \right\},
\]

where \( r_{t+1}(\cdot, \cdot) \) is a known function.

3.2 Direct Modelling

In the direct modelling approach, we first specify the historical dynamics, i.e., we choose a parametric family for the conditional log-Laplace transform \( \psi_t(u | w_t) \):

\[
\left\{ \psi_t(u | w_t, \theta_1), \theta_1 \in \Theta_1 \right\}. \tag{11}
\]
Then, we have to specify the SDF:

\[ M_{t,t+1} = \exp \left[ \alpha_t(w_t)w_{t+1} + \beta_t(w_t) \right] \]

\[ = \exp \left[ -r_{t+1}(w_t) + \alpha'_t(w_t)w_{t+1} - \psi_t(\alpha_t(w_t)) \right]. \]

Once \( r_{t+1} \) has been specified, according to its status described in Section 3.1, as well as \( \psi_t \), the remaining function to be specified is \( \alpha_t(w_t) \). We assume that \( \alpha_t(w_t) \) belongs to a parametric family:

\[ \{ \alpha_t(w_t, \theta_2), \theta_2 \in \Theta_2 \}. \]

Finally, \( M_{t,t+1} \) is specified as

\[ M_{t,t+1}(w_{t+1}, \theta) = \exp \left\{ -r_{t+1}(w_t, \tilde{\theta}) + \alpha'_t(w_t, \theta_2)w_{t+1} - \psi_t[\alpha_t(w_t, \theta_2)w_t, \theta_1] \right\}, \quad (12) \]

where \( \theta = (\tilde{\theta}', \theta_2') \in \tilde{\Theta} \times \Theta_1 \times \Theta_2 = \Theta \); note that \( \tilde{\Theta} \) may be reduced to one point.

This kind of modelling may have to satisfy some ICCs. Indeed, for any payoff \( g(w_s) \) delivered at \( s > t \), that has a price \( p(w_t) \) at \( t \), which is a known function of \( w_t \), we must have

\[ p(w_t) = E \left\{ M_{t,t+1}(\theta) \cdots M_{s-1,s}(\theta) g(w_s) \mid w_t, \theta_1 \right\} \quad \forall \ w_t, \theta. \quad (13) \]

These AAO pricing conditions may imply strong constraints on the parameter \( \theta \), for instance when components of \( w_t \) are returns of some assets or interest rates with various maturities (see Sections 4 and 5).

The specification of the historical dynamics (11) and of the SDF (12) obviously implies the specification of the RN dynamics

\[ \psi_t^Q(u|w_t, \theta_1, \theta_2) = \psi_t \left[ u + \alpha_t(w_t, \theta_2)w_t, \theta_1 \right] - \psi_t \left[ \alpha_t(w_t, \theta_2)w_t, \theta_1 \right]. \]

The particular case in which the historical dynamics is Car, \( \alpha_t(w_t, \theta_2) \) is an affine function of the factor, along with the short rate \( r_{t+1}(w_t, \tilde{\theta}) \), is the (discrete-time) counterpart of the basic direct modelling strategy frequently followed in continuous time.

### 3.3 RN Constrained Direct Modelling

In the previous kind of modelling, the family of RN dynamics \( \psi_t^Q(u|w_t) \) is obtained as a by-product and therefore is, in general, not controlled.

In some cases it may be important to control the family of RN dynamics and, possibly, the specification of the short rate, if we want to have explicit or quasi-explicit formulas for the price of some derivatives. For instance, it is often convenient to impose that the RN dynamics be described by a Car process. If we want, at the same time, to control the historical dynamics, for instance to have good fitting when \( w_t \) is observable, the by-product of the modelling becomes the factor-loading vector \( \alpha_t(w_t) \). More precisely, we may wish to choose a family \( \{ \psi_t(u|w_t, \theta_1), \theta_1 \in \Theta_1 \} \) and a family \( \{ \psi_t^Q(u|w_t, \theta^*), \theta^* \in \Theta^* \} \) such that, for any pair
(ψ_Q^t, ψ_t) belonging to these families, there exists a unique function α_t(w_t) denoted by α_t(w_t, θ_t, θ^*) satisfying

\[ ψ_t(Q|w_t) = ψ_t[u + α_t(w_t)|w_t] - ψ_t[α_t(w_t)|w_t]. \]

In fact, this condition may be satisfied only for a subset of pairs (θ_t, θ^*). In other words (θ_t, θ^*) belongs to Θ^*_1 strictly included in Θ_1 × Θ^*, but such that any θ_t ∈ Θ_1 and any θ^* ∈ Θ^* can be reached (see Section 4). Once the parameterization (θ, θ_t, θ^*) ∈ Θ × Θ^*_1 is defined, internal consistency conditions similar to (13) may be imposed.

### 3.4 Back Modelling

The final possibility is to parameterize first the RN dynamics ψ_Q^t(u|w_t, θ^*_t), and the short rate process r_{t+1}(w_t), taking into account, if relevant, internal consistency conditions of the form

\[ p(w_t) = E_t^Q \left[ \exp(-r_{t+1}(w_t, θ) - \cdots - r_s(w_s, θ))g(w_s)|w_s, θ^*_t^s \right], \quad ∀w_t, θ, θ^*_t. \quad (14) \]

Once this is done, the specification of α_t(w_t) is chosen, without any constraint, providing the family \( \{α_t(w_t, θ^*_2), θ^*_2 ∈ Θ^*_2\} \), and the historical dynamics is a by-product:

\[ ψ_t(u|w_t, θ^*_1, θ^*_2) = ψ_t(Q[u - α_t(w_t, θ^*_2)|w_t, θ^*_1] - ψ_t[−α_t(w_t, θ^*_2)|w_t, θ^*_1]). \]

The basic back modelling approach (frequently adopted in continuous time) is given by the particular case in which ψ_Q^t(u|w_t, θ^*_1), the short rate r_{t+1}(w_t, θ) and the risk-sensitivity vector α_t(w_t, θ^*_2) are assumed to be affine functions of the factor.

Also note that, if the RN conditional pdf \( f_t^Q(w_{t+1}|w_t, θ^*_1) \) is known in (quasi) closed form, the same is true for the historical conditional pdf:

\[ f_t(w_{t+1}|w_t, θ^*_1, θ^*_2) = f_t^Q(w_{t+1}|w_t, θ^*_1) \exp \left\{ -α'_t(w_t, θ^*_2)w_{t+1} - ψ_t[−α_t(w_t, θ^*_2)|w_t, θ^*_1] \right\}. \quad (15) \]

In particular, if \( w_t \) is observable, we can compute the likelihood function. However, the identification of the parameters (θ^*_1, θ^*_2), from the dynamics of the observable components of \( w_t \) must be carefully studied (see examples in Sections 4 and 5) and observations of derivative prices may be necessary to reach identifiability.

### 3.5 Inference in an Econometric Asset Pricing Model

In order to estimate an EAPM, we assume that the econometrician observes, at dates \( t ∈ \{0, \ldots, T\} \), a set of prices \( x_i \) corresponding to payoffs \( g_i(w_s), i ∈ \{1, \ldots, J_t\}, s > t \), given by (using the parameter notations of direct modelling):

\[ q_{ii}(w_t, θ) = E\left[ g_i(w_s)M_{t,s}(w_s, θ)|w_t, θ^*_1 \right], \quad i ∈ \{1, \ldots, J_t\}. \]
Therefore, we have two kinds of equations representing respectively the historical dynamics of the factors and the observations:

\[ w_t = \tilde{q}_t(w_{t-1}, \varepsilon_{1t}, \theta_1) \quad \text{(say)}, \]

\[ x_t = q_t(w_t, \theta), \]

where the first equation is a rewriting of the conditional historical distribution of \( w_t \) given \( w_{t-1}, \varepsilon_{1t} \), \( \varepsilon_{1t} \) is a white noise (which can be chosen Gaussian without loss of generality), \( x_t = (x_{t1}, \ldots, x_{tJ})' \) and \( \tilde{q}_t(w_t, \theta) = [q_{t1}(w_t, \theta), \ldots, q_{tJ}(w_t, \theta)]' \).

Note that, if \( r_{t+1} \) is not a known function of \( w_t \), we must have \( r_{t+1} = r_{t+1}(w_t, \tilde{\theta}) \) among Equations (17), and that if some components of \( w_t \) are observed, they should appear also in (17) without parameters.

System (16)–(17) is a nonlinear state space model and appropriate econometric methods may be used for inference in this system (in particular, maximum likelihood methods possibly based on Kalman filter, Kitagawa–Hamilton filter, simulations-based methods or indirect inference).

For given \( x_t \)'s, Equations (17) may have no solutions in \( w_t \)'s and, in this case, an additional white noise is often introduced leading to

\[ x_t = q_t(w_t, \theta) + \varepsilon_{2t}. \]

Moreover, when \( w_t \) is (partially) observable, \( \theta_1 \) may be identifiable from (16) and in this case a two step estimation method is available: (i) ML estimation of \( \theta_1 \) from (16); (ii) estimation of \( \theta_2 \), and possibly of \( \tilde{\theta} \), by nonlinear least square using (18) in which \( \theta_1 \) is replaced by its ML estimator (and, possibly, the unobserved components of \( w_t \) are replaced by their smoothed values).

4 APPLICATIONS TO ECONOMETRIC SECURITY MARKET MODELLING

4.1 General Setting

In an econometric security market model, we assume that the short rate \( r_{t+1} \) is exogenous and that the first \( K_1 \) components of \( w_t \), denoted by \( y_t \), are observable geometric returns of \( K_1 \) basic assets. The remaining \( K_2 = K - K_1 \) components of \( w_t \), denoted by \( z_t \), are factors not observed by the econometrician. Since the payoffs \( \exp(y_{jt+1}) \) delivered at \( t+1 \), for each \( j \in \{1, \ldots, K_1\} \), have a price at \( t \), which are known function of \( w_t \), namely 1, we have to guarantee internal consistency conditions. In the direct modelling approach, and in the RN constrained direct modelling one, these conditions are [using the notation of the (unconstrained) direct approach]:

\[ 1 = E_t \left\{ \exp(y_{jt+1} - r_{t+1} + \alpha_t(w_t, \theta_2)'w_{t+1} - \psi_t[\alpha_t(w_t, \theta_2)|w_t, \theta_1] \right\}, \quad j \in \{1, \ldots, K_1\} \]
or
\[ r_{t+1} = \psi_t \left[ \alpha_t(w_t, \theta_2) + e_j \mid w_t, \theta_1 \right] - \psi_t \left[ \alpha_t(w_t, \theta_2) \mid w_t, \theta_1 \right], \quad \forall w_t, \theta_1, \theta_2; \quad j \in \{1, \ldots, K_1\} . \]  
(19)

In the Back Modelling approach, these conditions are:
\[ r_{t+1} = \psi_t^Q(e_j \mid w_t, \theta_1^*), \quad \forall w_t, \theta_1^*; \quad j \in \{1, \ldots, K_1\} . \]  
(20)

If we consider the case where the factor \( w_{t+1} \) is a RN Car(1) process (the generalization to the case of a Car(\( p \)) process is straightforward), with conditional RN log-Laplace transform \( \psi_t^Q(u \mid w_t) = a^Q(u) w_t + b^Q(u) \), the internal consistency conditions (19) or (20) are given by (using the back modelling notation):
\[
\begin{cases}
    a^Q(e_j, \theta_1^*) = 0, \\
    b^Q(e_j, \theta_1^*) = r_{t+1}, \quad \forall \theta_1^*; \quad j \in \{1, \ldots, K_1\} .
\end{cases}
\]  
(21)

### 4.2 Back Modelling for Nonlinear Conditionally Gaussian Models

Let us consider a conditionally Gaussian setting, and let us assume that all the components of \( w_t \) are geometric returns (\( K_1 = K \)), that is, we consider \( w_t = y_t \). If we follow the back modelling approach, we specify, first, the RN Car(1) dynamics:
\[ y_{t+1} \mid y_t \sim Q \left[ m_t^Q(y_t, \theta_1^*), \Sigma_t^Q(\theta_1^*) \right], \]

or
\[ \psi_t^Q(u \mid y_t, \theta_1^*) = u^t m_t^Q(y_t, \theta_1^*) + \frac{1}{2} u^t \Sigma_t^Q(\theta_1^*) u. \]

Then, we impose the internal consistency conditions, which are (with obvious notations) given by
\[ r_{t+1} = m_{ji}^Q(y_j, \theta_1^*) + \frac{1}{2} \Sigma_{ji}^Q, \quad j \in \{1, \ldots, K\} , \]
(22)

and, consequently, the conditional RN distribution compatible with arbitrage restrictions is
\[ N \left[ r_{t+1} e - \frac{1}{2} v diag \Sigma_t^Q(\theta_1^*), \Sigma_t^Q(\theta_1^*) \right] , \]
i.e.,
\[ \psi_t^Q(u \mid y_j, \theta_1^*) = u^t r_{t+1} e - \frac{1}{2} u^t \Sigma_t^Q + \frac{1}{2} u^t \Sigma_t^Q u . \]

Finally, choosing any \( \alpha_t(y_j, \theta_2^*) \), we deduce the historical dynamics
\[ \psi_t(u \mid y_j, \theta_1^*, \theta_2^*) = u^t \left[ r_{t+1} e - \frac{1}{2} v diag \Sigma_t^Q(\theta_1^*) - \Sigma_t^Q(\theta_1^*) \alpha_t(y_j, \theta_2^*) \right] + \frac{1}{2} u^t \Sigma_t^Q(\theta_1^*) u, \]
which is not Car, in general, and therefore the process \( \{ y_t \} \) is not Gaussian. In other words, we have:

\[
y_{t+1} | y_t \sim N \left[ r_{t+1} e - \frac{1}{2} v \text{diag} \Sigma^Q(\theta^*_1) - \Sigma^Q(\theta^*_1) \alpha_t(y_t, \theta^*_2), \Sigma^Q(\theta^*_1) \right].
\]

Thus, for a given RN dynamics, we can reach any conditional historical mean of the factor, whereas the historical conditional variance–covariance matrix is the same as the RN one. Moreover, \( \theta^*_1 \) and \( \theta^*_2 \) can be identified from the dynamics of \( y_t \) only [see Gourieroux and Monfort (2007) for a derivation of conditionally Gaussian models using the direct modelling approach].

This modelling generalizes the basic Black–Scholes framework to the multivariate case, with arbitrary (nonlinear) historical conditional mean. Therefore, options with any maturity have standard Black–Scholes prices, but their future values are predicted using the joint non-Gaussian historical dynamics of the factor \( y_t \).

### 4.3 Back Modelling of Switching Regime Models

The class of conditionally mixed-normal models contains many static, dynamic, parametric, semi-parametric or nonparametric models (Bertholon, Monfort, and Pegoraro 2006; Garcia, Ghysels, and Renault 2003). Let us consider, for instance, the switching regime models. The factor \( w_t \) is equal to \( (y_t, z_t)' \), where \( y_t \) is an observable geometric return and \( z_t \) is a \( J \)-state homogeneous Markov chain, valued in \( (e_1, \ldots, e_J) \), and unobservable by the econometrician.

The direct modelling approach, described in Bertholon, Monfort, and Pegoraro (2006), has two main drawbacks. First, the ICC associated with the risky asset must be solved numerically for any \( t \). Second, the RN dynamics is not Car in general, and the pricing of derivatives needs simulations which, in turn, imply to solve the ICC for any \( t \) and any path.

Let us consider now the back modelling approach, starting from a Car RN dynamics defined by

\[
y_{t+1} = v_t + \rho y_t + v_1' z_t + v_2 z_{t+1} + (v_3' z_{t+1}) \xi_{t+1},
\]

where \( v_t \) is a deterministic function of \( t \) and where

\[
\xi_{t+1} | \xi_t, z_{t+1} \sim N(0, 1),
\]

\[
Q(z_{t+1} = e_j | y_t, z_{t-1}, z_t = e_i) = Q(z_{t+1} = e_j | z_t = e_i) = \pi_{ij}^*.
\]

In other words, \( z_t \) is an exogenous Markov chain in the risk-neutral world. The conditional RN Laplace transform is given by

\[
\varphi^Q_t(u, v) = E_t^Q \exp(uy_{t+1} + v' z_{t+1})
\]

\[
= \exp \left[ u(v_t + \rho y_t + v_1' z_t) \right] E_t^Q \exp \left[ (uv_2 + \frac{1}{2} u^2 v_3^2 + v)' z_{t+1} \right],
\]

(23)
\text{[}v_3^2\text{ is the vector containing the square of the components in } v_3\text{]} \text{ and we get }

\begin{equation}
\begin{aligned}
\psi_t^Q(u, v) &= \log \varphi_t^Q(u, v) \\
&= u(v_t + \rho y_t + v_1' z_t) + \Lambda'(u, v, v_2, v_3, \pi^*) z_t,
\end{aligned}
\end{equation}

where the $i$th component of $\Lambda(u, v, v_2, v_3, \pi^*)$ is

$$
\Lambda_i(u, v, v_2, v_3, \pi^*) = \log \sum_{j=1}^{I} \pi_{ij}^* \exp \left( u v_{2j} + \frac{1}{2} u^2 v_{3j}^2 + v_j \right).
$$

So, as announced, the joint RN dynamics of the process $(y_t, z_t')$ is Car since

$$
\psi_t^Q(u, v) = a_t^Q(u, v)' w_t + b_t^Q(u, v)
$$

with

$$
a_t^Q(u, v)' = [u \rho, u v_1' + \Lambda'(u, v, v_2, v_3, \pi^*)],
$$

$$
b_t^Q(u, v) = u v_t.
$$

The internal consistency condition is

$$
\psi_t^Q(1, 0) = r_{t+1}
$$

that is,

$$
-r_{t+1} + v_t + \rho y_t + v_1' z_t + \lambda'(v_2, v_3, \pi^*) z_t = 0 \quad \forall \ y_t, z_t,
$$

(24)

and where the $i$th component of $\lambda(v_2, v_3, \pi^*)$ is

$$
\lambda_i(v_2, v_3, \pi^*) = \log \sum_{j=1}^{I} \pi_{ij}^* \exp \left( v_{2j} + \frac{1}{2} v_{3j}^2 \right).
$$

Condition (24) implies, since $r_{t+1}$ and $v_t$ are deterministic functions of time,

$$
\left\{ \begin{array}{l}
\rho = 0, \\
v_1 = -\lambda(v_2, v_3, \pi^*), \\
v_t = r_{t+1}.
\end{array} \right.
$$

(25)

Finally, the RN dynamics compatible with the AAO conditions is

$$
y_{t+1} = r_{t+1} - \lambda'(v_2, v_3, \pi^*) z_t + v_2' z_{t+1} + (v_3' z_{t+1}) \xi_{t+1},
$$

(26)

where

$$
\xi_{t+1} \bigg| \xi_j, z_{t+1} \overset{\mathcal{Q}}{\sim} N(0, 1)
$$

(27)

$$
\mathcal{Q}(z_{t+1} = e_j \big| \xi_j, z_{t-1}, z_t = e_i) = \mathcal{Q}(z_{t+1} = e_j \big| z_t = e_i) = \pi_i^*.
$$
Note that, if \( v_2 \) is replaced by \( v_2 + c \), \( v'_2 z_{t+1} \) is replaced by \( v'_2 z_{t+1} + c \), and \(-\lambda' z_t \) by \(-\lambda' z_t - c \), so the RHS of (26) is unchanged and therefore we can impose, for instance, \( v_{2t} = 0 \).

The SDF is specified as

\[
\mathcal{M}_{t+1} = \exp \left[ -r_{t+1} + \gamma_t(w_t, \theta^*_2) y_{t+1} + \delta_t(w_t, \theta^*_2)' z_{t+1} - \psi_t(\gamma_t, \delta_t) \right],
\]

and the historical dynamics can then be deduced by specifying \( \gamma_t(w_t, \theta^*_2) \) and \( \delta_t(w_t, \theta^*_2) \) without any constraints (and assuming, for instance, \( \delta_{1t} = 0 \)). We get the log-Laplace transform

\[
\psi_t(u, v) = \psi^Q_t(u - \gamma_t, v - \delta_t) - \psi^Q_t(-\gamma_t, -\delta_t),
\]

where

\[
\psi^Q_t(u, v) = u(r_{t+1} - \lambda' z_t) + \Lambda'(u, v) z_t,
\]

and thus

\[
\psi_t(u, v) = u(r_{t+1} - \lambda' z_t) + [\Lambda(u - \gamma_t, v - \delta_t) - \Lambda(-\gamma_t, -\delta_t)]' z_t,
\]

with

\[
\Lambda_i(u - \gamma_t, v - \delta_t) - \Lambda_i(-\gamma_t, -\delta_t)
\]

\[
= \log \frac{\sum_{j=1}^l \pi^*_i \exp \left( -\gamma_t v_{2j} + \frac{1}{2} \gamma_t^2 v_{3j}^2 - \delta_{jt} \right) \exp \left[ u(v_{2j} - \gamma_t v_{3j}^2) + \frac{1}{2} u^2 v_{3j}^2 + v_j \right]}{\sum_{j=1}^l \pi^*_i \exp \left( -\gamma_t v_{2j} + \frac{1}{2} \gamma_t^2 v_{3j}^2 - \delta_{jt} \right) \exp \left[ u(v_{2j} - \gamma_t v_{3j}^2) + \frac{1}{2} u^2 v_{3j}^2 + v_j \right]}
\]

and

\[
\pi_{ij,t} = \frac{\pi^*_i \exp \left( -\gamma_t v_{2j} + \frac{1}{2} \gamma_t^2 v_{3j}^2 - \delta_{jt} \right)}{\sum_{j=1}^l \pi^*_i \exp \left( -\gamma_t v_{2j} + \frac{1}{2} \gamma_t^2 v_{3j}^2 - \delta_{jt} \right) \exp \left[ u(v_{2j} - \gamma_t v_{3j}^2) + \frac{1}{2} u^2 v_{3j}^2 + v_j \right]}
\]

Therefore, the historical dynamics is

\[
y_{t+1} = r_{t+1} - \lambda'(v_2, v_3, \pi^*) z_t + (v_2 - \gamma_t v_{3j}^2)' z_{t+1} + (v_3 z_{t+1}) \varepsilon_{t+1},
\]

where

\[
\varepsilon_{t+1} \mid \varepsilon_t, z_{t+1} \overset{p}{\sim} N(0, 1)
\]

\[
P(z_{t+1} = e_j \mid y_t, z_{t-1}, z_t = e_t) = \pi_{ij,t}
\]

\[
\lambda_i(v_2, v_3, \pi^*) = \log \sum_{j=1}^l \pi^*_i \exp \left( v_{2j} + \frac{1}{2} v_{3j}^2 \right),
\]
and

$$
\varepsilon_{t+1} = \xi_{t+1} + \gamma_t (v_3 z_{t+1}).
$$

(30)

Conditionally to $w_t$, the historical distribution of $y_{t+1}$ is a mixture of $J$ Gaussian distributions with means $(r_{t+1} - \lambda' z_t + v_{2j} - \gamma_t v_{3j}^2)$ and variances $v_{3j}^2$, and with weights given by $\pi_{ij,t}$, $j \in \{1, \ldots, J\}$, when $z_t = e_i$.

Since $\gamma_t$ and $\delta_t$ are arbitrary functions of $w_t$ (assuming, for instance, $\delta_J = 0$), we obtain a large class of historical (non-Car) switching regime dynamics, which can be matched with a Car switching regime RN dynamics. These features give the possibility to specify a tractable option pricing model able, at the same time, to provide historical and RN stochastic skewness and kurtosis, which are determinant to fit stock return and implied volatility surface dynamics [see the survey on econometrics of option pricing proposed by Garcia, Ghysels, and Renault (2003), where mixture models are studied, and the works of Bakshi, Carr, and Wu (2008) and Carr and Wu (2007), where the important role of stochastic skewness in currency options is analyzed].

As mentioned in Section 3.4, the identification problem must be discussed. Let us consider the case where $\gamma$ and $\delta$ are constant. In this case, the parameters $\pi_{ij}$ are constant and the identifiable parameters are $\pi_{ij}$, $v_3$, the vector of the $J$ coefficients of $z_{t+1}$ in (29), and $(J - 1)$ coefficients of $z_t$ [assuming, for instance, $\lambda_J = 0$], i.e., $J (J - 1) + 3J - 1 = J (J + 2) - 1$ parameters, whereas the parameters to be estimated are the $\pi_{ij}^*$, $v_2$ (with $v_{2J} = 0$), $v_3$, $\gamma$, $\delta$ (with $\delta_J = 0$), i.e., $J (J + 2) - 1$ parameters also. So all the parameters might be estimated from the observations of the $y_t$'s.

### 4.4 Back Modelling of Stochastic Volatility Models

We focus on the back modelling, starting from a Car representation of the RN dynamics of the factor $w_t = (y_t, \sigma_t^2)$, where $y_t$ is an observable geometric return, whereas $\sigma_t^2$ is an unobservable stochastic variance. More precisely, the RN dynamics is assumed to satisfy

$$
y_{t+1} = \lambda_t + \lambda_1 y_t + \lambda_2 \sigma_t^2 + (\lambda_3 \sigma_t^2) \xi_{t+1},
$$

(31)

where $\lambda_t$ is a deterministic function of $t$ and

$$
\xi_{t+1} \sim \mathcal{N}(0, 1)
$$

(32)

and where the conditional ARG(1, $\nu$, $\rho$) distribution [characterizing an autoregressive Gamma process of order one (ARG(1)) with unit scale parameter$^5$] is defined

\[\text{ARG}(1, \nu, \rho)\]

\[^5\text{See Darolles, Gourieroux, and Jasiak (2006), Gourieroux and Jasiak (2006), and Monfort and Pegoraro (2006b) for a presentation of single regime and regime-switching (scalar and vector) autoregressive Gamma processes.}\]
by the affine conditional RN log-Laplace transform

$$\psi^Q_t(v) = a^Q(v)\sigma_t^2 + b^Q(v),$$

where $a^Q(v) = \frac{\mu}{1 - \nu}, b^Q(v) = -v\log(1 - v), v < 1, \rho > 0, \nu > 0$. The conditional RN log-Laplace transform of $(y_{t+1}, \sigma_{t+1}^2)$ is

$$\psi^Q_t(u, v) = (\lambda_t + \lambda_1 y_t + \lambda_2 \sigma_t^2)u + \frac{1}{2}\lambda_3^2 \sigma_t^2 u^2 + a^Q(v)\sigma_t^2 + b^Q(v). \quad (33)$$

The internal consistency condition is

$$\psi^Q_t(1, 0) = r_{t+1}$$

or

$$r_{t+1} = \lambda_t + \lambda_1 y_t + \lambda_2 \sigma_t^2 + \frac{1}{2}\lambda_3^2 \sigma_t^2,$$

which implies

$$\lambda_t = r_{t+1}, \quad \lambda_1 = 0, \quad \lambda_2 = -\frac{1}{2}\lambda_3^2. \quad (34)$$

So, the RN dynamics compatible with the AAO restriction is given by (32) and

$$y_{t+1} = r_{t+1} - \frac{1}{2}\lambda_3^2 \sigma_t^2 + \lambda_3 \sigma_t \xi_{t+1},$$

that is,

$$\psi_t(u, v) = (r_{t+1} - \frac{1}{2}\lambda_3^2 \sigma_t^2)u + \frac{1}{2}\lambda_3^2 \sigma_t^2 u^2 + a^Q(v)\sigma_t^2 + b^Q(v). \quad (35)$$

The historical dynamics is defined by specifying $\gamma_t(w, \theta^*_2)$ and $\delta_t(w, \theta^*_2)$, and we get

$$\psi_t(u, v) = \psi_t^Q(u - \gamma_t, v - \delta_t) - \psi_t^Q(-\gamma_t, -\delta_t)$$

$$= (r_{t+1} - \frac{1}{2}\lambda_3^2 \sigma_t^2)u - \lambda_3 \sigma_t^2 \gamma_t u + \frac{1}{2}\lambda_3^2 \sigma_t^2 u^2$$

$$+ \left[ a^Q(v - \delta_t) - a^Q(-\delta_t) \right] \sigma_t^2 + b^Q(v - \delta_t) - b^Q(-\delta_t)$$

$$= (r_{t+1} - \frac{1}{2}\lambda_3^2 \sigma_t^2 - \lambda_3 \sigma_t^2 \gamma_t)u + \frac{1}{2}\lambda_3^2 \sigma_t^2 u^2 + a_t(v)\sigma_t^2 + b_t(v),$$

with

$$a_t(v) = \frac{\rho_t v}{1 - v\mu_t}, \quad b_t(v) = -v\log(1 - v\mu_t),$$

$$\rho_t = \frac{\rho}{(1 + \delta_t)^2}, \quad \mu_t = \frac{1}{1 + \delta_t}.$$

So, the only conditions, when we define the historical dynamics, are $\mu_t > 0$, i.e., $\delta_t > -1$, and $v < 1/\mu_t$. The historical dynamics can be written:

$$y_{t+1} = r_{t+1} - \frac{1}{2}\lambda_3^2 \sigma_t^2 - \lambda_3 \sigma_t^2 \gamma_t + \lambda_3 \sigma_t \xi_{t+1}, \quad (36)$$
where

\[ \epsilon_{t+1} | \epsilon_{t}, \sigma_{t+1}^2 \overset{\mathbb{P}}{\sim} N(0, 1), \]
\[ \sigma_{t+1}^2 | \epsilon_{t}, \sigma_{t}^2 \overset{\mathbb{P}}{\sim} \text{ARG}(\mu_t, \nu, \rho_t). \]  

(37)

Note that, the conditional historical distribution of \( \sigma_{t+1}^2 \), given \((\epsilon_t, \sigma_t^2)\), is given by the log-Laplace transform

\[ \psi_t(v) = \frac{\rho_t v}{1 - v \mu_t} \sigma_t^2 - v \log(1 - v \mu_t), \]

which is not affine in \( \sigma_t^2 \), except in the case where \( \delta_t \) is constant (or a deterministic function of \( t \)). Moreover, we have

\[ \epsilon_{t+1} = \xi_{t+1} + (\lambda_3 \sigma_t) \gamma_t. \]  

(38)

If \( \gamma_t \) and \( \delta_t \) are constant, the identifiable parameters are the coefficients of \( \sigma_t^2 \) and \( \sigma_t \epsilon_{t+1} \) in (36) as well as the two parameters of the ARG dynamics (with unit scale). So, we have four identifiable parameters. The parameters to be estimated are \( \lambda_3, \nu, \gamma, \delta \), i.e., five parameters. So these parameters are not identifiable from the dynamics of the \( y_t \)’s. Observations of derivative prices must be added.

In this example we have assumed \( \sigma_{t+1}^2 \sim \text{ARG}(1) \) and absence of instantaneous causality between \( y_{t+1} \) and \( \sigma_{t+1}^2 \) just for ease of exposition. It is possible to specify an SV model in which \( \sigma_{t+1}^2 \sim \text{ARG}(p) \), with an instantaneous correlation between the stock return and the SV. For instance, we can consider

\[
\begin{align*}
    y_{t+1} &= \lambda_t + \lambda_1 y_t + \lambda_2 \sigma_{t+1}^2 + \lambda_3 \sigma_t^2 + (\lambda_4 \sigma_{t+1}) \xi_{t+1}, \\
    \sigma_{t+1}^2 &= v + \varphi_1 \sigma_t^2 + \cdots + \varphi_p \sigma_{t-p+1}^2 + \eta_{t+1},
\end{align*}
\]

where \( \eta_{t+1} \) is an heteroscedastic martingale difference sequence. This specification generalizes the exact discrete-time equivalent of the SV diffusion model typically used in continuous time (and based on the CIR process). It has the potential features to explain not only the volatility smile in option data, but also to improve the fitting of the observed time varying persistence in stock return volatility [see Garcia, Ghysels, and Renault (2003), and the references therein]. Indeed, the conditional mean and variance of \( \sigma_{t+1}^2 \) show the following specifications: 
\[ E[\sigma_{t+1}^2 | \sigma_t^2] = v + \varphi_1 \sigma_t^2 + \cdots + \varphi_p \sigma_{t-p+1}^2 \] and 
\[ V[\sigma_{t+1}^2 | \sigma_t^2] = v + 2(\varphi_1 \sigma_t^2 + \cdots + \varphi_p \sigma_{t-p+1}^2). \]

### 4.5 Back Modelling of Switching GARCH Models with Leverage Effect: A First Application of Extended Car Processes

In this section, following a back modelling approach, we consider specifications generalizing those proposed by Heston and Nandi (2000) to the case where switching regimes are introduced in the conditional mean and conditional (GARCH-type) variance of the geometric return [see also Elliot, Siu, and Chan (2006)].

Like in Section 4.4, we assume \( w_t = (y_t, z_t) \), where \( y_t \) is an observable geometric return and \( z_t \) an unobservable \( J \)-state homogeneous Markov chain valued in
The new feature is the introduction of a GARCH effect (with leverage). More precisely, the RN dynamics is assumed to be of the following type:

\[ y_{t+1} = \nu_t + v_1 y_t + v'_2 z_t + v'_3 z_{t+1} + v_4 \sigma_{t+1}^2 + \sigma_{t+1} \xi_{t+1}, \quad (39) \]

where \( \nu_t \) is a deterministic function of \( t \) and

\[
\xi_{t+1} | \xi_j, z_{t+1} \overset{\sim}{\sim} N(0, 1), \\
\sigma_{t+1}^2 = \omega' z_t + \alpha_1 (\xi_t - \alpha_2 \sigma_t)^2 + \alpha_3 \sigma_t^2,
\]

and

\[
Q(z_{t+1} = e_j | y_t, z_{t-1}, z_t = e_i) = Q(z_{t+1} = e_j | z_t = e_i) = \pi^*_i j.
\]

Note that \( \sigma_{t+1}^2 \) is a deterministic function of \( (\xi_j, z_j) \), and therefore of \( w_t = (y_t, z_t) \). Also note that, following Heston and Nandi (2000), in this switching GARCH(1,1) model, \( \xi_t \) replaces the usual term \( \sigma_t \xi_t \) in the RHS of the equation giving \( \sigma_{t+1}^2 \) and the term \( \alpha_2 \sigma_t \) captures an asymmetric or leverage effect.

It is easily seen that the RN conditional log-Laplace transform of \( (y_{t+1}, z_{t+1}) \) is

\[
\psi^{Q}_t (u, v) = \log E^{Q}_t \exp(\nu y_{t+1} + v' z_{t+1}) = (v_t + v_1 y_t + v'_2 z_t + v_4 \sigma_{t+1}^2) u + \frac{1}{2} \sigma_{t+1}^2 u^2 + \Lambda'(u, v, v_3, \pi^*) z_t,
\]

where the \( i \)th component of \( \Lambda(u, v, v_3, \pi^*) \) is

\[
\Lambda_i(u, v, v_3, \pi^*) = \log \sum_{j=1}^{l} \pi^*_i j \exp(u v_3 j + v_j). \quad (41)
\]

The internal consistency condition, or AAO constraint, is

\[
\psi^{Q}_i (1, 0) = r_{t+1} \quad \forall w_t,
\]

implying

\[
r_{t+1} = \nu_t + v_1 y_t + v'_2 z_t + v_4 \sigma_{t+1}^2 + \frac{1}{2} \sigma_{t+1}^2 + \lambda'(v_3, \pi^*) z_t,
\]

where the \( i \)th component of \( \lambda(v_3, \pi^*) \) is given by

\[
\lambda_i(v_3, \pi^*) = \log \sum_{j=1}^{l} \pi^*_i j \exp(v_3 j) \quad (42)
\]

and, therefore, the arbitrage restriction implies:

\[
\begin{align*}
\nu_1 &= 0, \\
\nu_2 &= -\lambda(v_3, \pi^*), \\
\nu_4 &= -\frac{1}{2}, \\
\nu_t &= r_{t+1}.
\end{align*}
\]
Thus, Equation (39) becomes

$$y_{t+1} = r_{t+1} - \lambda(v_3, \pi^*)'z_t - \frac{1}{2}\sigma_{t+1}^2 + \psi'z_{t+1} + \sigma_{t+1}\xi_{t+1}$$  \hspace{1cm} (43)

with

$$\sigma_{t+1}^2 = \omega'z_t + \alpha_1(\xi_t - \alpha_2\gamma_t)^2 + \alpha_3\sigma_t^2,$$

$$\xi_{t+1} | \xi_t, z_{t+1} \sim N(0, 1),$$

$$\mathbb{Q}(z_{t+1} = e_j | y_{i,t}, z_{t-1}, z_t = e_i) = \mathbb{Q}(z_{t+1} = e_j | z_t = e_i) = \pi_{ij}^*,$$

(again, we can take $v_{3j} = 0$) which gives the RN dynamics compatible with the AAO restriction. The corresponding log-Laplace transform is

$$\psi_t^Q(u, v) = (r_{t+1} - \lambda'z_t - \frac{1}{2}\sigma_{t+1}^2) u + \frac{1}{2}\sigma_{t+1}^2 u^2 + \Lambda'(u, v, v_3, \pi^*)'z_t.$$  \hspace{1cm} (44)

The historical dynamics is obtained by specifying $\gamma_i(w_i, \theta_2^i)$ and $\delta_i(w_i, \theta_2^i)$, with, for instance $\delta_{f+1} = 0$, and in particular we have

$$\psi_t(u, v) = \psi_t^Q(u - \gamma_t, v - \delta_t) - \psi_t^Q(-\gamma_t, -\delta_t).$$

We obtain

$$\psi_t(u, v) = (r_{t+1} - \lambda'z_t - \frac{1}{2}\sigma_{t+1}^2) u + \frac{1}{2}\sigma_{t+1}^2 u^2 + [\Lambda(u - \gamma_t, v - \delta_t, v_3, \pi^*) - \Lambda(-\gamma_t, -\delta_t, v_3, \pi^*)]'z_t,$$

where

$$\Lambda_i(u - \gamma_t, v - \delta_t, v_3, \pi^*) - \Lambda_i(-\gamma_t, -\delta_t, v_3, \pi^*) = \log \sum_{j=1}^f \pi_{ij,t} \exp(uv_{3j} + v_j)$$

with

$$\pi_{ij,t} = \frac{\pi_{ij}^* \exp(-\gamma_t v_{3j} - \delta_{ij})}{\sum_{j=1}^f \pi_{ij}^* \exp(-\gamma_t v_{3j} - \delta_{ij})}. \hspace{1cm} (45)$$

So the nonaffine historical dynamics is given by

$$y_{t+1} = r_{t+1} - \lambda(v_3, \pi^*)'z_t - \frac{1}{2}\sigma_{t+1}^2 + \gamma_i(w_i, \theta_2^i)\sigma_{t+1}^2 + v_3z_{t+1} + \sigma_{t+1}\epsilon_{t+1}$$

$$\epsilon_{t+1} | \epsilon_{i,t}, z_{t+1} \sim N(0, 1),$$  \hspace{1cm} (46)

with

$$\sigma_{t+1}^2 = \omega'z_t + \alpha_1(\xi_t - \alpha_2\gamma_t)^2 + \alpha_3\sigma_t^2,$$

$$\mathbb{P}(z_{t+1} = e_j | y_{i,t}, z_{t-1}, z_t = e_i) = \pi_{ij,t}.$$
and, therefore, the equation giving $\sigma_{t+1}^2$ can be rewritten as

$$\sigma_{t+1}^2 = \omega z_t + \alpha_1 [\epsilon_t - (\alpha_2 + \gamma) \sigma_t] + \alpha_3 \sigma_t^2.$$ 

One may observe, from (44), that $w_{t+1} = (y_{t+1}, z'_{t+1})$ does not have a Car RN dynamics. So, the pricing seems a priori difficult. Fortunately, it can be shown (see Appendix C) that the (extended) factor $w_{t+1}^e := (y_{t+1}, z_{t+1}^2)$ is RN Car, that is, $w_{t+1}$ is an internally extended Car(1), and therefore the pricing methods based on Car dynamics apply. In particular, the RN conditional log-Laplace transform of $w_{t+1}^e$, given $w_e$, is

$$\psi_t(u, v, \bar{v}) = a_1^Q(u, v, \bar{v}) \bar{z}_t + a_2^Q(u, \bar{v}) \sigma_{t+1}^2 + b_1^Q(u, \bar{v}), \tag{47}$$

where

$$a_1^Q(u, v, \bar{v}) = \bar{\Lambda}(u, v, \bar{v}, \nu, \pi^*) - \lambda(\nu_3, \pi^*) u$$

with

$$\bar{\Lambda}(u, v, \bar{v}, \nu, \pi^*) = \log \sum_{j=1}^{J} \pi_j^* \exp(u \nu_j + \bar{v} \omega_j), \quad i \in \{1, \ldots, J\},$$

$$a_2^Q(u, \bar{v}) = -\frac{1}{2} u + \bar{v} (\alpha_1 \alpha_2^2 + \alpha_3) + \frac{(u - 2\alpha_1 \alpha_2 \bar{v})^2}{2(1 - 2\alpha_1 \bar{v})},$$

$$b_1^Q(u, \bar{v}) = u \rho_{t+1} - \frac{1}{2} \log(1 - 2\alpha_1 \bar{v}),$$

which is affine in $(z_{t+1}, \sigma_{t+1}^2)'$, with an intercept deterministic function of time.

Finally, let us consider the identification problem from the historical dynamics when functions $\gamma$ and $\delta$ are constant. In this case, we can identify from (46) $J$ coefficients of $z_{t+1}$, $(J - 1)$ coefficients of $z_t$, the coefficient of $\sigma_{t+1}^2$, $\omega$, $\alpha_1$, $(\alpha_2 + \gamma)$, $\alpha_3$, and $\pi_{ij}$, i.e., $3J + 3 + J(J - 1) = J(J + 2) + 3$ parameters. The parameters to be estimated are $\nu_3$ (with $\nu_{3j} = 0$), $\omega$, $\alpha_1$, $\alpha_2$, $\alpha_3$, $\pi_{ij}$, $\gamma$, $\delta$ (with $\delta_j = 0$), that is, $2(J - 1) + J + 4 + J(J - 1) = J(J + 2) + 2$ parameters. Therefore, the historical model is over identified.

### 4.6 Back Modelling of Switching IG GARCH Models: A Second Application of Extended Car Processes

The purpose of this section is to introduce, following the back modelling approach, several generalizations of the inverse Gaussian6 (IG) GARCH model proposed by Christoffersen, Heston, and Jacobs (2006). First, we consider switching regimes in the (historical and RN) dynamics of the geometric return $y_t$ and in the GARCH variance $\sigma_{t+1}^2$. Second, we price not only the factor risk but also the regime-shift

---

6The strictly positive random variable $y$ has an inverse Gaussian distribution with parameter $\delta > 0$ [denoted IG($\delta$)] if and only if its distribution function is given by $F(y; \delta) = \int_0^y \frac{1}{\sqrt{2\pi y^3}} \exp(-\frac{(t^2 - 2\delta^2)(1 - 2\delta)}{(1 - 2\delta)(1 - 2\delta)}) dt$. The generalized Laplace transform is $E[\exp(\eta y + \theta/y)] = \int_0^\delta \exp(\delta - \sqrt{(\delta^2 - 2\eta)(1 - 2\delta)}) d\delta$ and $E(y) = V(y) = \delta$ [see Christoffersen, Heston, and Jacobs (2006) for further details].
risk and, third, risk correction coefficients are in general time varying. The factor is given by \( w_t = (y_t, z_t)' \), where \( z_t \) is the unobservable \( J \)-state homogeneous Markov chain valued in \( \{e_1, \ldots, e_J\} \). The RN dynamics is given by

\[
y_{t+1} = v_t + v_1 y_t + v_2 z_t + v_3 z_{t+1} + v_4 \sigma_{t+1}^2 + \eta \xi_{t+1},
\]

where \( v_t \) is a deterministic function of \( t \) and

\[
\begin{align*}
\xi_{t+1} | \xi_t, z_{t+1} & \overset{\mathcal{Q}}{\sim} IG \left( \frac{\sigma_{t+1}^2}{\eta^2} \right), \\
\sigma_{t+1}^2 = \omega' z_t + \alpha_1 \sigma_t^2 + \alpha_2 \xi_t + \alpha_3 \frac{\sigma_t^4}{\xi_t},
\end{align*}
\]

with

\[
\mathcal{Q}(z_{t+1} = e_j | y_t', z_{t-1}, z_t = e_i) = \mathcal{Q}(z_{t+1} = e_j | z_t = e_i) = \pi_{ij}^*.
\]

The RN conditional log-Laplace transform of \((y_{t+1}, z_{t+1})\) is

\[
\psi_t^Q(u, v) = \log E_t^Q \exp(u y_{t+1} + v' z_{t+1}) = (v_t + v_1 y_t + v_2 z_t + v_3 \sigma_{t+1}^2) u + \Lambda'(u, v, v_3, \pi^*) z_t + \frac{\sigma_{t+1}^2}{\eta^2} \left[ 1 - (1 - 2\eta)^{1/2} \right],
\]

where the \( i \)th component of \( \Lambda(u, v, v_3, \pi^*) \) is given by (41). The absence of arbitrage constraint is \( \psi_t^Q(1, 0) = r_{t+1}, \forall w_t \), implying

\[
r_{t+1} = v_t + v_1 y_t + v_2 z_t + \lambda'(v_3, \pi^*) z_t + \sigma_{t+1}^2 \left( v_4 + \frac{1}{\eta^2} \left[ 1 - (1 - 2\eta)^{1/2} \right] \right),
\]

with the \( i \)th component of \( \lambda(v_3, \pi^*) \) given by (42). Therefore, the arbitrage restriction implies:

\[
\begin{align*}
v_1 &= 0, \\
v_2 &= -\lambda(v_3, \pi^*), \\
v_4 &= -\frac{1}{\eta^2} \left[ 1 - (1 - 2\eta)^{1/2} \right], \\
v_t &= r_{t+1}.
\end{align*}
\]

Thus, Equation (48) becomes

\[
y_{t+1} = r_{t+1} - \lambda(v_3, \pi^*) z_t - \frac{1}{\eta^2} \left[ 1 - (1 - 2\eta)^{1/2} \right] \sigma_{t+1}^2 + v_3 z_{t+1} + \eta \xi_{t+1}
\]

with

\[
\begin{align*}
\sigma_{t+1}^2 &= \omega' z_t + \alpha_1 \sigma_t^2 + \alpha_2 \xi_t + \alpha_3 \frac{\sigma_t^4}{\xi_t}, \\
\xi_{t+1} | \xi_t, z_{t+1} & \overset{\mathcal{Q}}{\sim} IG \left( \frac{\sigma_{t+1}^2}{\eta^2} \right), \\
\mathcal{Q}(z_{t+1} = e_j | y_t', z_{t-1}, z_t = e_i) &= \mathcal{Q}(z_{t+1} = e_j | z_t = e_i) = \pi_{ij}^*.
\end{align*}
\]
(again, we can take $\nu_3 = 0$) which gives the RN dynamics compatible with the AAO restriction. The corresponding log-Laplace transform is

$$
\psi_t^Q(u, v) = \left( r_{t+1} - \lambda' z_t - \frac{1}{\eta^2} \left[ 1 - (1 - 2\eta)^{1/2} \right] \sigma_t^2 \right)^2 u
+ \Lambda'(u, v, \nu_3, \pi^*) z_t + \frac{\sigma_{t+1}^2}{\eta^2} \left [ 1 - (1 - 2u\eta)^{1/2} \right ]. \tag{50}
$$

Given the specification of $\gamma(w, \theta_2^t)$ and $\delta_t(w, \theta_2^t)$ (with, for instance, $\delta_{t+1} = 0$), the conditional historical log-Laplace transform of the factor is given by

$$
\psi_t(u, v) = \left( r_{t+1} - \lambda' z_t - \frac{1}{\eta^2} \left[ 1 - (1 - 2\eta)^{1/2} \right] \sigma_t^2 \right)^2 u
+ \Lambda(u - \gamma_t, v - \delta_t, \nu_3, \pi^*) z_t
+ \frac{\sigma_{t+1}^2}{\eta^2} \left [ (1 + 2\gamma_t \eta)^{1/2} - [1 - 2(u - \gamma_t)\eta]^{1/2} \right ]
= \left( r_{t+1} - \lambda' z_t - \eta_t^{-3/2} \eta^{-1/2} \left[ 1 - (1 - 2\eta)^{1/2} \right] \sigma_{t+1}^2 \right) u
+ \Lambda(u - \gamma_t, v - \delta_t, \nu_3, \pi^*) z_t
+ \frac{\sigma_{t+1}^2}{\eta_t^2} \left [ 1 - (1 - 2u\eta_t)^{1/2} \right ],
$$

with $\Lambda_t(u - \gamma_t, v - \delta_t) - \Lambda_t(-\gamma_t, -\delta_t)$ specified by (45), and where $\eta_t = \frac{\eta}{\nu_3 + 2\eta}$ and $\sigma_{t+1}^2 = \sigma_{t+1}^2(\frac{\eta_t}{\nu_3})^{3/2}$. So, the nonaffine historical dynamics is given by

$$
y_{t+1} = r_{t+1} - \lambda'(\nu_3, \pi^*) z_t + \nu_3' z_{t+1} - \eta_t^{-3/2} \eta^{-1/2} \left[ 1 - (1 - 2\eta_t)^{1/2} \right] \sigma_{t+1}^2 + \eta_t \epsilon_{t+1}
$$

$$
\epsilon_{t+1} | \epsilon_t, z_{t+1} \sim IG \left( \frac{\sigma_{t+1}^2}{\eta_t^2} \right), \tag{51}
$$

with, using (49) and (51), $\eta \xi_{t+1} = \eta_t \epsilon_{t+1}$ and

$$
\sigma_{t+1}^2 = \sigma_{t+1}^2(\frac{\eta_t}{\nu_3})^{3/2}, \quad \sigma_{t+1}^2 = \sigma_{t+1}^2(\frac{\eta_t}{\nu_3})^{3/2}, \quad \sigma_{t+1}^2 = \sigma_{t+1}^2(\frac{\eta_t}{\nu_3})^{3/2}, \quad \sigma_{t+1}^2 = \sigma_{t+1}^2(\frac{\eta_t}{\nu_3})^{3/2}, \quad \sigma_{t+1}^2 = \sigma_{t+1}^2(\frac{\eta_t}{\nu_3})^{3/2}, \quad \sigma_{t+1}^2 = \sigma_{t+1}^2(\frac{\eta_t}{\nu_3})^{3/2}, \quad \sigma_{t+1}^2 = \sigma_{t+1}^2(\frac{\eta_t}{\nu_3})^{3/2},
$$

where

$$
\sigma_t^2 = \sigma_t^2(\frac{\eta_t}{\nu_3})^{3/2}, \quad \sigma_t^2 = \sigma_t^2(\frac{\eta_t}{\nu_3})^{3/2}, \quad \sigma_t^2 = \sigma_t^2(\frac{\eta_t}{\nu_3})^{3/2}, \quad \sigma_t^2 = \sigma_t^2(\frac{\eta_t}{\nu_3})^{3/2}, \quad \sigma_t^2 = \sigma_t^2(\frac{\eta_t}{\nu_3})^{3/2}, \quad \sigma_t^2 = \sigma_t^2(\frac{\eta_t}{\nu_3})^{3/2},
$$

As in the previous section, the factor $w_{t+1} = (y_{t+1}, z_{t+1}')$ is not a RN Car process, but it can be verified that the factor $w_{t+1}^Q = (y_{t+1}, z_{t+1}', \sigma_{t+1}^2)'$ is RN Car (see Appendix D), and that $w_{t+1}$ is an internally extended Car(1) process. Indeed, the RN conditional log-Laplace transform of $w_{t+1}^Q$, given $w_{t}^Q$, is

$$
\psi_t^Q(u, v, \tilde{v}) = a_1^Q(u, v, \tilde{v}) z_t + a_2^Q(u, \tilde{v}) \sigma_{t+1}^2 + b_1^Q(u, \tilde{v}), \tag{52}
$$
where

\[ a_1^Q(u, v, \tilde{v}) = \bar{A}(u, v, \tilde{v}, v_3, \omega, \pi^*) - \lambda(v_3, \pi^*)u \]

with \( \bar{A}_i(u, v, \tilde{v}, v_3, \omega, \pi^*) = \log \sum_{j=1}^{J} \pi^*_{ij} \exp(u v_3 j + v j + \tilde{v} \omega j), \quad i \in \{1, \ldots, J\} \),

\[ a_2^Q(u, \tilde{v}) = \tilde{v} \alpha_1 - \frac{1}{\eta^2} \left( u \left[ 1 - (1 - 2\eta)^{1/2} \right] + 1 - \sqrt{(1 - 2\tilde{v} \alpha_3 \eta^4)} \left( 1 - 2(u \eta + \tilde{v} \alpha_2) \right) \right), \]

\[ b_i^Q(u, \tilde{v}) = ur_{t+1} - \frac{1}{2} \log(1 - 2\tilde{v} \alpha_3 \eta^4), \]

which is affine in \((z', \sigma_{t+1}^2)'\), with an intercept deterministic function of time.

As far as the identification problem is concerned, with functions \( \gamma \) and \( \delta \) constant, we can identify, from the historical dynamics (51), \( 3J + J(J - 1) + 4 \) coefficients, while the parameters to be estimated are \( v_3 \) (with \( v_{3j} = 0 \)), \( \omega, \alpha_1, \alpha_2, \alpha_3, \pi^*_{ij}, \gamma, \delta \) (with \( \delta_j = 0 \)), and \( \eta \), that is, \( 2J - 1 \) + \( J + 5 + J(J - 1) = 3J + J(J - 1) + 3 \) parameters. Thus, as in the previous section, the historical model is over identified.

5 APPLICATIONS TO ECONOMETRIC TERM STRUCTURE MODELLING

It is well known that, if the RN dynamics of \( w_t \) is Car and if \( r_{t+1} \) is an affine function of \( w_t \), the term structure of interest rates \( \{r(t, h), h \in \{1, \ldots, H\}\} \) is easily determined recursively and is affine in \( w_t \) [see Gourieroux, Monfort, and Polimenis (2003), or Monfort and Pegoraro (2007)]. Indeed, if

\[ \psi^Q_t(u|w_t; \theta^*_1) = a^Q(u, \theta^*_1)'w_t + b^Q(u, \theta^*_1) \]

and \( r_{t+1} = \tilde{\theta}_1 + \tilde{\theta}_2'w_t \), then

\[ r(t, h) = -\frac{c'_h}{h} w_t - \frac{d_h}{h}, \quad (53) \]

where

\[
\begin{align*}
  c_h &= -\tilde{\theta}_2 + a^Q(c_{h-1}), \\
  d_h &= d_{h-1} - \tilde{\theta}_1 + b^Q(c_{h-1}), \\
  c_0 &= 0, \quad d_0 = 0.
\end{align*}
\]

Moreover, applying the transform analysis, various interest rates derivatives have quasi-explicit pricing formulas. Note that if the \( i \) th component of \( w_t \) is a rate \( r(t, h_i) \), \( i \in \{1, \ldots, K_1\} \), we must satisfy the internal consistency conditions

\[ c_{h_i} = -h_i \epsilon_i, \quad d_{h_i} = 0, \quad i \in \{1, \ldots, K_1\}. \]
Therefore, it is highly desirable to have a Car RN dynamics and this specification is obtained by one of the three modelling strategies described in Section 3. Let us consider some examples.

5.1 Direct Modelling of VAR($p$) Factor-Based Term Structure Models

For the sake of notational simplicity, we consider the one factor case, but the results can be extended to the multivariate case (Monfort and Pegoraro 2006a). We assume, for instance, that the factor $w_t$ is unobservable, and has a historical dynamics given by a Gaussian AR($p$) model,

$$w_{t+1} = v + \varphi_1 w_t + \ldots + \varphi_p w_{t+1-p} + \sigma \varepsilon_{t+1},$$

(55)

where $\varepsilon_{t+1} \sim \text{IN}(0, 1)$, $\varphi = (\varphi_1, \ldots, \varphi_p)'$ and $W_t = (w_t, \ldots, w_{t+1-p})'$. This dynamics can also be written as

$$W_{t+1} = \tilde{v} + \Phi W_t + \sigma \tilde{\varepsilon}_{t+1},$$

where $\tilde{v} = ve_1, \tilde{\varepsilon}_{t+1} = \varepsilon_{t+1}e_1$ [$e_1$ denotes the first column of the identity matrix $I_p$] and

$$\Phi = \begin{bmatrix} \varphi_1 & \ldots & \ldots & \varphi_{p-1} & \varphi_p \\ 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \ldots & \ldots & 1 & 0 \end{bmatrix}$$

is a $(p \times p)$ matrix.

The SDF takes the following exponential-affine form:

$$M_{t+1} = \exp[-r_t + \alpha_t w_{t+1} - \psi_t(\alpha_t)],$$

(56)

with

$$\psi_t(u) = (v + \varphi' W_t)u + \frac{1}{2} \sigma^2 u^2,$$

$$\alpha_t = \alpha_0 + \alpha' W_t$$

and the short rate is given by

$$r_{t+1} = \tilde{\theta}_1 + \tilde{\theta}_2' W_t.$$ 

If $r_{t+1} = w_t$, we have $\tilde{\theta}_2 = e_1$ and $\tilde{\theta}_1 = 0$.

The conditional RN log-Laplace transform is given by

$$\psi^0_t(u) = \psi_t(u + \alpha_t) - \psi_t(\alpha_t)$$

$$= (v + \varphi' W_t)u + \sigma^2 \alpha_t u + \frac{1}{2} \sigma^2 u^2$$

$$= [v + \sigma^2 \alpha_0 + (\varphi + \sigma^2 \alpha)' W_t]u + \frac{1}{2} \sigma^2 u^2.$$
Therefore, the RN dynamics of the factor is given by

\[ w_{t+1} = (v + \sigma^2 \alpha_0) + (\varphi + \sigma^2 \alpha) W_t + \sigma \xi_{t+1}, \]  

(57)

where \( \xi_{t+1} \sim IN(0, 1) \). Moreover, we have \( \varepsilon_{t+1} = \xi_{t+1} + (\alpha_0 + \alpha' W_t) \).

The yield-to-maturity formula at date \( t \) is given by [see Monfort and Pegoraro (2006a) for the proof]

\[ r(t, h) = -\frac{c'}{h} W_t - \frac{c_1}{h} \]  

(58)

with

\[
\begin{align*}
&c_h = -\tilde{\theta}_2 + \Phi' c_{h-1} + c_{1,h} \sigma^2 \alpha, \\
d_h = -\tilde{\theta}_1 + c_{1,h-1}(v + \sigma^2 \alpha_0) + \frac{1}{2} c_{1,h-1}^2 \sigma^2 + d_{h-1}, \\
c_0 = 0 \quad d_0 = 0.
\end{align*}
\]

The pricing model presented in this section is derived following the discrete-time equivalent of the basic direct modelling strategy typically used in continuous time (Duffie and Kan 1996; Cheridito, Filipovic, and Kimmel 2007). If we specify \( \alpha_t \) as a nonlinear function of \( W_t \), \( \psi^Q(u) \) turns out to be nonaffine (in \( W_t \)) and, therefore, we lose the explicit representation of the yield formula. We will see in Section 5.3 that we can go beyond this limit following the back modelling approach.

### 5.2 RN Constrained Direct Modelling of Switching VAR(\( p \))

#### Factor-Based Term Structure Models

Again for the sake of simplicity, we consider the univariate case [see Monfort and Pegoraro (2007) for extensions] where the factor is given by \( w_t = (x_t, z_t')' \), with \( z_t \) a \( J \)-state nonhomogeneous Markov chain valued in \( \{e_1, \ldots, e_J\} \). The first component \( x_t \) is observable or unobservable, \( z_t \) is unobservable, and the historical dynamics is given by

\[ x_{t+1} = v(Z_t) + \varphi_1(Z_t)x_t + \cdots + \varphi_p(Z_t)x_{t+1-p} + \sigma(Z_t)\varepsilon_{t+1}, \]  

(59)

where

\[
\varepsilon_{t+1} \mid \xi_{t}, z_{t+1} \overset{\mathbb{P}}{\sim} N(0, 1),
\]

\[
\mathbb{P}(z_{t+1} = e_j \mid x_t, z_{t-1}, z_t = e_i) = \pi(e_i, e_j; X_t),
\]

\[
Z_t = (z_t', \ldots, z_{t-p}'),
\]

\[
X_t = (x_t, \ldots, x_{t+1-p})'.
\]

Observe that the joint historical dynamics of \( (x_t, z_t')' \) is not Car. Functions \( v, \varphi_1, \ldots, \varphi_p, \sigma, \) and \( \pi \) are parameterized using a parameter \( \theta_1 \).

We specify the SDF in the following way:

\[
M_{t,t+1} = \exp \left[ -r_{t+1} + \Gamma(Z_t, X_t)\varepsilon_{t+1} - \frac{1}{2} \Gamma(Z_t, X_t)^2 - \delta(Z_t, X_t)z_{t+1} \right],
\]  

(60)
with \( \Gamma(Z_t, X_t) = \gamma(Z_t) + \tilde{\gamma}(Z_t)X_t \) and, in order to ensure that \( E_t M_{t+1} = \exp(-r_{t+1}) \), we add the condition

\[
\sum_{j=1}^{l} \pi(e_i, e_j, X_t) \exp[-\delta(Z_t, X_t)' e_j] = 1, \quad \forall Z_t, X_t.
\]

The short rate is given by

\[
r_{t+1} = \tilde{\theta}_1' X_t + \tilde{\theta}_2' Z_t,
\]

and, in the observable factor case \((x_t = r_{t+1})\), we have \( \tilde{\theta}_1 = e_1 \) and \( \tilde{\theta}_2 = 0 \).

It is easily seen that the RN dynamics is given by

\[
x_{t+1} = v(Z_t) + \gamma(Z_t) \sigma(Z_t) + \left[ \varphi(Z_t) + \tilde{\gamma}(Z_t) \sigma(Z_t) \right]' X_t + \sigma(Z_t) \xi_{t+1},
\]

\[
\xi_{t+1} | \xi_t, Z_{t+1} \sim N(0, 1),
\]

\[
Q(z_{t+1} = e_j | x_t, z_{t-1}, z_t = e_i) = \pi(e_i, e_j, X_t) \exp[-\delta(Z_t, X_t)' e_j].
\]

So, if we want the RN dynamics of \( w_t \) to be Car, we have to impose:

(i) \( \sigma(Z_t) = \sigma^* Z_t \) (linearity in \( z_t, \ldots, z_{t-p} \)),

(ii) \( \gamma(Z_t) = \frac{\nu^* Z_t - v(Z_t)}{\sigma^* Z_t} \),

(iii) \( \tilde{\gamma}(Z_t) = \frac{\varphi^* - \varphi(Z_t)}{\sigma^* Z_t} \),

(iv) \( \delta_j(Z_t, X_t) = \log \left[ \frac{\pi(z_t, e_j, X_t)}{\pi^*(z_t, e_j)} \right] \),

where \( \sigma^*, \nu^*, \varphi^* \) are free parameters, \( \pi^*(e_i, e_j) \) are the entries of an homogeneous transition matrix. All of these parameters constitute the parameter \( \theta^* \in \Theta^* \) introduced in Section 3.3. Also note that, because of constraints (62(i)) above, \( \theta \) and \( \theta^* \) do not vary independently.

So the RN dynamics is

\[
X_{t+1} = \Phi^* X_t + [\nu^* Z_t + (\sigma^* Z_t) \xi_{t+1}] e_1,
\]

\[
\Phi^* = \begin{bmatrix}
\varphi^*_1 & \cdots & \varphi^*_{p-1} & \varphi^*_p \\
1 & 0 & \cdots & 0 & 0 \\
0 & 1 & \cdots & 0 & 0 \\
\vdots & \ddots & \ddots & \vdots & \vdots \\
0 & \cdots & \cdots & 1 & 0
\end{bmatrix}
\]

is a \((p \times p)\) matrix,

\[
\xi_{t+1} | \xi_t, Z_{t+1} \sim N(0, 1),
\]

\[
Q(z_{t+1} = e_j | x_t, z_{t-1}, z_t = e_i) = \pi(z_{t+1} = e_j | z_t = e_i) = \pi^*_{ij}
\]

and the affine (in \( X_t \) and \( Z_t \)) term structure of interest rates is easily derived [see Monfort and Pegoraro (2007) for the proof, and Dai, Singleton, and Yang (2006)
for the case $p = 1$. The empirical study proposed in Monfort and Pegoraro (2007), shows that the introduction of multiple lags and switching regimes, in the historical and RN dynamics of the observable factor (short rate and spread between the long and the short rate), leads to term structure models, which are able to fit the yield curve and to explain the violation of the Expectation Hypothesis Theory, over both the short and long horizon, as well as or better than competing models like 2-Factor CIR, 3-Factor CIR, 3-Factor $\hat{A}_1(3)$ [using the Dai and Singleton (2000) notation] and the 2-Factor regime-switching CIR term structure model proposed by Bansal and Zhou (2002). Dai, Singleton, and Yang (2007) show the determinant role of priced, state-dependent regime-shift risks in capturing the dynamics of expected excess bond returns. Moreover, they show that the well-known hump-shaped term structure of volatility of bond yield changes is a low-volatility phenomenon.

5.3 Back Modelling of VAR$(p)$ Factor-Based Term Structure Models

Let us consider the (bivariate) case where $w_t$ is given by $[r(t, 1), r(t, 2)]'$. We want to impose the following Gaussian VAR(1) RN dynamics:

$$w_{t+1} = v + \Phi w_t + \xi_{t+1}, \quad (64)$$

where $\xi_{t+1} \overset{Q}{\sim} \mathcal{N}(0, \Sigma)$. In this case, the internal consistency conditions are satisfied if we impose, in (53) and (54), $\hat{\theta}_1 = 0, \hat{\theta}_2' = (1, 0), c_2 = -2\varepsilon_2,$ and $d_2 = 0$, or

$$\left\{ \begin{array}{l} -2\varepsilon_2 = a^Q \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \\ 0 = b^Q \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \end{array} \right. \quad (65)$$

where $a^Q(u) = \Phi' u$ and $b^Q(u) = u' v + \frac{1}{2} u' \Sigma u$. So, relation (65) becomes, with obvious notations:

$$\begin{bmatrix} \varphi_{11} \\ \varphi_{12} \end{bmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}, \quad \nu_1 = \frac{1}{2} \sigma_1^2,$$

and (64) must be written as:

$$\left\{ \begin{array}{l} r(t + 1, 1) = \frac{1}{2} \sigma_1^2 - r(t, 1) + 2r(t, 2) + \xi_{1,t+1}, \\ r(t + 1, 2) = v_2 + \varphi_{21} r(t, 1) + \varphi_{22} r(t, 2) + \xi_{2,t+1}, \end{array} \right. \quad (66)$$

with $\xi_t \overset{Q}{\sim} \mathcal{N}(0, \Sigma)$. Consequently, the RN conditional log-Laplace transform of $w_{t+1}$, compatible with the AAO restrictions is

$$\psi^Q_t(u) = u' \left[ \begin{pmatrix} \frac{1}{2} \sigma_1^2 \\ v_2 \end{pmatrix} + \begin{pmatrix} -1 & 2 \\ \varphi_{21} & \varphi_{22} \end{pmatrix} w_t \right] + \frac{1}{2} u' \Sigma u.$$
the yield-to-maturity formula will be affine in \( w_t \), as indicated by (53), and, moreover, independent of the specification of the factor-loading \( \alpha_t \). Now, if we move back to the historical conditional log-Laplace transform, we get

\[
\psi_t(u) = \psi^Q_t(u - \alpha_t) - \psi^Q_t(-\alpha_t) = u' \left[ \left( \frac{1}{2} \sigma_1^2 \right) + \left( \begin{array}{c} \varphi_2 \\varphi_2 \end{array} \right) w_t \right] - u' \Sigma \alpha_t + \frac{1}{2} u' \Sigma u.
\]

If we assume \( \alpha_t = \gamma + \Gamma w_t \), we get

\[
\psi_t(u) = u' \left\{ \left( \frac{1}{2} \sigma_1^2 \right) - \Sigma \gamma + \left[ \left( \begin{array}{c} \varphi_2 \\varphi_2 \end{array} \right) - \Sigma \Gamma \right] w_t \right\} + \frac{1}{2} u' \Sigma u,
\]

or, equivalently, we have the following Car \( \mathbb{P} \)-dynamics:

\[
w_{t+1} = \left( \frac{1}{2} \sigma_1^2 \right) - \Sigma \gamma + \left[ \left( \begin{array}{c} \varphi_2 \\varphi_2 \end{array} \right) - \Sigma \Gamma \right] w_t + \varepsilon_{t+1},
\]

where \( \varepsilon_{t+1} \sim \mathcal{N}(0, \Sigma) \) and \( \varepsilon_t = \xi_t + \Sigma (\gamma + \Gamma w_t) \). If \( \Gamma = 0 \), the historical dynamics of \( w_t \) is constrained, the parameters \( \Sigma, \varphi_{12}, \) and \( \varphi_{22} \) are identifiable from the observations on \( w_t \), whereas \( \gamma \) and \( \nu_2 \) are not. If \( \Gamma \neq 0 \), the historical dynamics of \( w_t \) is not constrained and only \( \Sigma \) is identifiable from the observations on \( w_t \).

Observe that, even if we assume \( \alpha_t \) to be nonlinear in \( w_t \), the interest rate formula is still affine (contrary to the direct modelling case of Section 5.1), and the historical conditional pdf of non-Car factor \( w_t \) remains known in closed form [see relation (15)]. This means that, at the same time, we have a tractable pricing model, we can introduce (non-Car) nonlinearities in the interest rate historical dynamics [as suggested, for instance, by Ait-Sahalia (1996)], and we maintain the possibility to estimate the parameters by exact maximum likelihood. Following the back modelling strategy, Dai, Le, and Singleton (2006) develop a family of discrete-time nonlinear term structure models [exact discrete-time counterpart of the models in Dai and Singleton (2000)] characterized by these three important features.

### 5.4 Direct Modelling of Wishart Term Structure Models and Quadratic Term Structure Models: A Third Application of Extended Car Processes

The Wishart Quadratic Term Structure model, proposed by Gourieroux and Sufana (2003), is characterized by an unobservable factor \( W_t \), which follows (under the historical probability) the WAR process introduced in Section 2.1.4. The SDF is defined by

\[
M_{t,t+1} = \exp \left[ \text{Tr}(C W_{t+1}) + d \right],
\]
where $C$ is a $(n \times n)$ symmetric matrix and $d$ is a scalar. The associated RN dynamics is defined by

$$
\psi_t^{(2)}(\Gamma) = \text{Tr} \left[ M' \left\{ (C + \Gamma)[I_n - 2(C + \Gamma)]^{-1} - C(I_n - 2C)^{-1} \right\} MW_t \right]
- \frac{K}{2} \log \det[(I_n - 2(I_n - 2C)^{-1} \Gamma)],
$$

which is also Car(1). The term structure of interest rates at date $t$ is affine in $W_t$ and given by

$$
r(t, h) = -\frac{1}{h} \text{Tr}[A(h)W_t] - \frac{1}{h} b(h), \quad h \geq 1,
$$

$$
A(h) = M'[C + A(h - 1)] [I_n - 2(C + A(h - 1))]^{-1} M,
$$

$$
b(h) = d + b(h - 1) - \frac{K}{2} \log \det[I_n - 2(C + A(h - 1))],
$$

$$
A(0) = 0, \quad b(0) = 0.
$$

In particular, if $K$ is integer, we get

$$
r(t, h) = -\frac{1}{h} \text{Tr} \left[ \sum_{k=1}^{K} A(h)x_{k,t}x_{k,t}' \right] - \frac{1}{h} b(h),
$$

$$
= -\frac{1}{h} \sum_{k=1}^{K} x_{k,t}'A(h)x_{k,t} - \frac{1}{h} b(h), \quad h \geq 1,
$$

which is a sum of quadratic forms in $x_{k,t}$. If $K = 1$, we get the standard Quadratic Term Structure Model, which is, therefore, a special affine model [see Beaglehole and Tenney (1991), Ahn, Dittmar, and Gallant (2002), Leippold and Wu (2002), Cheng and Scaillet (2007), and Buraschi, Cieslak, and Trojani (2008) for a generalization in the continuous-time general equilibrium setting].

We can also define a quadratic term structure model with a linear term, if the historical dynamics of $x_{t+1}$ is given by the following Gaussian VAR(1) process:

$$
x_{t+1} = m + Mx_t + \varepsilon_{t+1},
$$

$$
\varepsilon_{t+1} \overset{p}{\sim} IN(0, \Sigma).
$$

Indeed (as suggested by example c) in Section 2.3), the factor $w_t = [x_t', \text{vech}(x_t x_t')]'$ is Car(1), that is, $w_t$ is an extended Car process in the historical world (see Appendix E for the proof). Moreover, choosing

$$
M_{t,t+1} = \exp[C'x_{t+1} + \text{Tr}(Cx_{t+1}x_{t+1}') + d]
$$

$$
= \exp(C'x_{t+1} + x_{t+1}'C x_{t+1} + d), \quad (C \text{ is a symmetric } (n \times n) \text{ matrix}),
$$

(71)
the process \( w_t \) is also extended Car in the risk-neutral world. The term structure at date \( t \) is affine in \( w_t \), that is, of the form
\[
    r(t, h) = x_t' \Lambda(h)x_t + \mu(h)'x_t + \nu(h), \quad h \geq 1,
\]
where \( \Lambda(h) \), \( \mu(h) \), and \( \nu(h) \) follow recursive equations [see also Gourieroux and Sufana (2003), Cheng and Scaillet (2007), and Jiang and Yan (2006)].

6 An Example of Back Modelling for a Security Market Model with Stochastic Dividends and Short Rate

The purpose of this section is to consider an econometric security market model where the risky assets are dividend-paying assets and the short rate is endogenous. More precisely, the factor is given by \( w_t = (y_t, \delta_t, r_{t+1})' \), where

- \( y_t = (y_{1,t}, \ldots, y_{K_1,t})' \) denotes, for each date \( t \), the \( K_1 \)-dimensional vector of geometric returns associated with cum dividend prices \( S_{j,t} \), \( j \in \{1, \ldots, K_1\} \);
- \( \delta_t = (\delta_{1,t}, \ldots, \delta_{K_1,t}) \) is the associated \( K_1 \)-dimensional vector of (geometric) dividend yields, and denoting \( \tilde{S}_{j,t} \) as the ex dividend price of the \( j \)th risky asset, we have \( S_{j,t} = \tilde{S}_{j,t} \exp(\delta_{j,t}) \);
- \( r_{t+1} \) denotes the (predetermined) stochastic short rate for the period \([t, t+1]\).

Observe that, compared to the setting of Section 4.1 (where \( r_{t+1} \) was exogenous), this model proposes a more general \( K \)-dimensional factor \( w_t \) (with \( K = 2K_1 + 1 \)), where we jointly specify \( y_t, \delta_t \) (which is considered as an observable factor), and the short rate \( r_{t+1} \). It would be straightforward to add an unobservable factor \( z_t \).

Following the back modelling approach, we propose an RN Gaussian VAR(1) dynamics for the factor and the conditional distribution of \( w_{t+1}, \) given \( w_t \), is assumed to be Gaussian with mean vector \((A_0 + A_1 w_t)\) and variance–covariance matrix \( \Sigma \). The process \( w_{t+1} \) is, therefore, a Car(1) process with a conditional RN Laplace transform given by
\[
    \varphi_Q(u | w_t) = E_Q[\exp(u' w_{t+1})] = \exp\left[a_Q(u)' w_t + b_Q(u)\right],
\]
where the functions \( a_Q \) and \( b_Q \) are the following:
\[
    \begin{cases}
        a_Q(u) = A_1' u, \\
        b_Q(u) = A_0' u + \frac{1}{2} u' \Sigma u.
    \end{cases}
\]

The RN dynamics can also be written as
\[
    w_{t+1} \overset{Q}{=} A_0 + A_1 w_t + \xi_{t+1},
\]
\[
    \xi_{t+1} \overset{Q}{\sim} II N(0, \Sigma).
\]
The AAO restrictions, applied to the $K_1$-dimensional vector $y_{t+1}$, are given by

$$E_t^Q \left[ \exp[\log \left( \frac{S_{j,t+1}}{\tilde{S}_{j,t}} \right) \right] = \exp(r_{t+1}) , \quad j \in \{1, \ldots, K_1\},$$

$$\iff E_t^Q \left[ \exp(y_{j,t+1}) \right] = \exp(r_{t+1} - \delta_{j,t}), \quad j \in \{1, \ldots, K_1\},$$

$$\iff \begin{cases} a^Q(e_j) = A_1' e_j = e_K - e_{j+K_1}, \quad j \in \{1, \ldots, K_1\}, \\ b^Q(e_j) = A_0' e_j + \frac{1}{2} e_j' \Sigma e_j = 0, \quad j \in \{1, \ldots, K_1\}. \end{cases}$$

This means that the first $K_1$ rows of $A_1$ and the first $K_1$ components of $A_0$ are, for $j \in \{1, \ldots, K_1\}$, respectively given by $(e_K - e_{j+K_1})'$ and $-\frac{1}{2} \sigma^2_j$ [where $e_K$ and $e_{j+K_1}$ denote, respectively, the $K$th and the $(j + K_1)$th column of the identity matrix $I_K$], while $\sigma^2_j$ is the $(j, j)$-term of $\Sigma$. In other words, the $K_1$ first equations of (73) are

$$y_{j,t+1} = -\frac{1}{2} \sigma^2_j + r_{t+1} - \delta_{j,t} + \xi_{j,t+1}, \quad j \in \{1, \ldots, K_1\}.$$

Then, coming back to the historical dynamics of $w_t$, we get:

$$\psi_t(u) = \psi_t^Q(u - \alpha_t) - \psi_t^Q(-\alpha_t)$$

$$= (a^Q(u - \alpha_t) - a^Q(-\alpha_t))' w_t + b^Q(u - \alpha_t) - b^Q(-\alpha_t)$$

$$= u' A_1 w_t + u' A_0 + \frac{1}{2} (u - \alpha_t)' \Sigma (u - \alpha_t) - \frac{1}{2} \alpha_t' \Sigma \alpha_t$$

$$= u' (A_0 + A_1 w_t - \Sigma \alpha_t) + \frac{1}{2} u' \Sigma u.$$  \hspace{1cm} (74)

So, if we impose $\alpha_t = (\alpha_0 + \alpha w_t)$, the historical dynamics of the factor is also Gaussian VAR(1) with a modified conditional mean vector equal to $[A_0 - \Sigma \alpha_0 + (A_1 - \Sigma \alpha) w_t]$ and the same variance–covariance matrix $\Sigma$, that is,

$$w_{t+1} = A_0 - \Sigma \alpha_0 + (A_1 - \Sigma \alpha) w_t + \varepsilon_{t+1},$$

$$\varepsilon_{t+1} \overset{\mathbb{P}}{\sim} \text{IN}(0, \Sigma), \quad \text{and} \quad \xi_{t+1} = \xi_{t+1} + \Sigma (\alpha_0 + \alpha w_t).$$

We notice that, under the historical probability, any VAR(1) distribution can be reached, but only $\Sigma$ is identifiable. If we add the constraint $\alpha = 0$, then the historical dynamics of $w_t$ is constrained, and $A_0$ and $\alpha_0$ are not identifiable.

### 7 Conclusions

In this paper, we have proposed a general econometric approach to no-arbitrage asset pricing modelling based on three main elements: (i) the historical discrete-time dynamics of the factor representing the information, (ii) SDF, and (iii) the RN discrete-time factor dynamics. We have presented three modelling strategies: the direct modelling, the RN constrained direct modelling, and the back modelling. In all the approaches, we have considered the internal consistency conditions, induced by the AAO restrictions, and the identification problem. These three approaches have been explained explicitly for several discrete-time security market models and affine term structure models. In all cases, we have indicated the important role played by the RN constrained direct modelling and the back modelling.
strategies in determining, at the same time, flexible historical dynamics and CAR RN dynamics leading to explicit or quasi-explicit pricing formulas for various contingent claims. Moreover, we have shown the possibility to derive asset pricing models able to accommodate non-CAR historical and RN factor dynamics with tractable pricing formulas. This result is achieved when the starting RN non-CAR factor turns out to be a RN extended CAR process. These strategies, already implicitly adopted in several papers, clearly could be the basis for the specification of new asset pricing models leading to promising empirical analysis.

APPENDIX A: PROOF OF THE EXISTENCE AND UNIQUENESS OF $M_{t,t+1}$ AND OF THE PRICING FORMULA (1)

Using A1 and A2, the Riesz representation theorem implies

$$\forall s > t, \forall w_t, \exists M_{t,s}(w_s), \text{unique, such that } \forall g(w_s) \in L_{2s},$$

$$p_t[g(w_s)] = E[M_{t,s}(w_s)g(w_s) | w_t].$$

In particular, the price at $t$ of a zero-coupon bond with maturity $s$ is $E[M_{t,s}(w_s) | w_t]$. A3 implies that $\mathbb{P}[M_{t,s} > 0 | w_t] = 1, \forall t, s \in \{0, \ldots , T\}$, since otherwise the payoff $1(M_{t,s} \leq 0)$ at $s$, would be such that $\mathbb{P}[1(M_{t,s} \leq 0) > 0 | w_t] > 0$ and $p_t[1(M_{t,s} \leq 0)] = E_t[M_{t,s}1(M_{t,s} \leq 0)] \leq 0$, contradicting A3.

Relation (1) will be shown if we prove that, $\forall t < r < s$, $w_t, g(w_s) \in L_{2s}$ we have

$$p_t[g(w_s)] = p_t[p_r[g(w_s)]]$$

Let us show, for instance, that if (with obvious notations) $p_t(g_s) > p_t[p_r(g_s)]$, we can construct over the time interval $[t, s]$ a sequence of portfolios with strictly positive payoff at $s$, with zero payoffs at any date $r \in [t, s]$, and with price zero at $t$, contradicting A3. The sequence of portfolios is defined by the following trading strategy:

- at $t$: buy $p_t(g_s)$, (short) sell $g_s$, buy $\frac{p_t(g_s) - p_t[p_r(g_s)]}{E[M_{t,s} | w_t]}$ zero-coupon bonds with maturity $s$, generating a zero payoff;
- at $r$: buy $g_s$ and sell $p_r(g_s)$, generating a zero payoff;
- at $s$: the net payoff is $g_s - g_s + \frac{p_t(g_s) - p_t[p_r(g_s)]}{E[M_{t,s} | w_t]} > 0$.

A similar argument shows that $p_t(g_s) < p_t[p_r(g_s)]$ contradicts A3 and, therefore, relation (1) is proved.

APPENDIX B: RISK PREMIA AND MARKET PRICE OF RISK

A.1 Notation

In this appendix $[f_i(e_i)]$ will denote, for given scalar or row $K$-vectors $f_i(e_i), i \in \{1, \ldots , K\}$, the $K$-vector or the $K \times K$ matrix $(f_i(e_1)', \ldots , f_i(e_K)')'$ with rows $f_i(e_i), i \in \{1, \ldots , K\}; e$ will denote the $K$-dimensional unitary vector.
A.2 Geometric and Arithmetic Risk Premia

Let \( p_t \) be the price at \( t \) of any given asset. The geometric return between \( t \) and \( t + 1 \) is
\[
\rho_{G,t+1} = \log \left( \frac{p_{t+1}}{p_t} \right),
\]
whereas the arithmetic return is
\[
\rho_{A,t+1} = \frac{p_{t+1}}{p_t} - 1 = \exp(\rho_{G,t+1}) - 1.
\]
In particular, for the risk-free asset we have
\[
\rho_{G,t+1}^f = r_{t+1},
\]
\[
\rho_{A,t+1}^f = \exp(r_{t+1}) - 1 = r_{A,t+1}.
\]
So, we can define two risk premia of the given asset as:
\[
\pi_{Gt} = E_t(\rho_{G,t+1}) - r_{t+1},
\]
\[
\pi_{At} = E_t(\rho_{A,t+1}) - r_{A,t+1} = E_t[\exp(\rho_{G,t+1})] - \exp(r_{t+1}).
\]
Note that the arithmetic risk premia have the advantage to satisfy \( \pi_{At}(\lambda) = \sum_{j=1}^J \lambda_j \pi_{A_j} \), if \( \pi_{At}(\lambda) \) is the risk premium of the portfolio defined by the shares in value \( \lambda_j \) for the asset \( j \). Let us now consider two important particular cases in order to have more explicit forms of these risk premia and to obtain intuitive interpretations of the factor-loading vector \( \alpha_t \) [see also Dai, Le, and Singleton (2006) for a similar analysis].

A.3 The Factor is a Vector of Geometric Returns

If \( w_{t+1} \) is a \( K \)-vector of geometric returns, the vectors of risk premia \( \pi_{Gt} \) and \( \pi_{At} \) whose entries are
\[
\pi_{Gi,j} = e_i^\prime \psi_t^{(1)}(0) - r_{t+1}, \quad i \in \{1, \ldots, K\},
\]
(where \( \psi_t^{(1)} \) is the gradient of \( \psi_t \) and \( e_i \) is the \( i \)th column of the identity matrix \( I_K \)),
\[
\pi_{Ai,j} = \varphi_t(e_i) - \exp(r_{t+1}), \quad i \in \{1, \ldots, K\}.
\]
Moreover, we have the pricing identities:
\[
1 = E_t[\exp[e_i^\prime w_{t+1} + \alpha_i^\prime w_{t+1} - r_{t+1} - \psi_t(\alpha_t)]], \quad i \in \{1, \ldots, K\},
\]
that is
\[
\exp(r_{t+1}) = \frac{\varphi_t(\alpha_t + e_i)}{\varphi_t(\alpha_t)} = \varphi_t^{\ominus}(e_i).
or

\[ r_{t+1} = \psi_t(a_t + e_t) - \psi_t(a_t) = \psi_t^Q(e_t). \]

So, for each \( i \in \{1, \ldots, K\} \), the risk premia can be written as

\[
\pi_{Gt,i} = e_i^T \psi_t^{(1)}(0) - \psi_t(a_t + e_t) + \psi_t(a_t), \\
\pi_{At,i} = \varphi_t(e_t) - \varphi_t(a_t) + \varphi_t(a_t).
\]

Note that, for \( \alpha_t = 0 \), i.e., when the historical and the RN dynamics are identical, we have

\[
\pi_{Gt,i} = m_{it} - \psi_t(e_t) \neq 0, \quad i \in \{1, \ldots, K\},
\]

where \( m_{it} \) denotes the conditional mean of \( w_{i,t+1} \) given \( w_t \) and

\[
\pi_{At,i} = 0, \quad i \in \{1, \ldots, K\}.
\]

So the arithmetic risk premia seem to have more natural properties. Moreover, considering first-order expansions around \( \alpha_t = 0 \) and neglecting conditional cumulants of order strictly larger than 2 (which are zero in the conditionally gaussian case), we get:

\[
\pi_{Gt} \simeq -\frac{1}{2} v \text{diag}(\Sigma_t) - \Sigma_t \alpha_t, \quad (A3) \\
\pi_{At} \simeq -\exp(r_{t+1}) \Sigma_t \alpha_t, \quad (A4)
\]

where \( v \text{diag}(\Sigma_t) \) is the vector whose entries are the diagonal terms of \( \Sigma_t \), and \( \Sigma_t \) is the conditional variance–covariance matrix of \( w_{t+1} \) given \( w_t \). So, \( \alpha_t \) can be viewed as the opposite of a market price of risk vector. We will see in the proof below that the expression of \( \pi_{Gt} \) is exact in the conditionally Gaussian case.

### A.4 Proof of Relations (A3) and (A4)

We have seen above that the geometric risk premium can be written as:

\[
\pi_{Gt} = \psi_t^{(1)}(0) - [\psi_t(a_t + e_t)] + \psi_t(a_t)e.
\]

Using a first-order expansion of \( \pi_{Gt} = \pi_{Gt}(a_t) \) around \( \alpha_t = 0 \), we obtain

\[
\pi_{Gt} \simeq \psi_t^{(1)}(0) - [\psi_t(e_t)] - [\psi_t^{(1)}(e_t)']\alpha_t + (\psi_t^{(1)}(0)\alpha_t)e,
\]

and neglecting conditional cumulants of order \( \geq 3 \), we can write:

\[
\pi_{Gt} \simeq m_t - m_t - \frac{1}{2} v \text{diag} \Sigma_t - (m'\alpha_t)e - \Sigma_t \alpha_t + (m'\alpha_t)e \\
\simeq -\frac{1}{2} v \text{diag} \Sigma_t - \Sigma_t \alpha_t.
\]
If we consider now the arithmetic risk premium, and apply the same procedure, we get:

\[
\pi_{At} = [\phi_t(e_t)] - \left[ \frac{\phi_t(\alpha_t + e_t)}{\phi_t(\alpha_t)} \right]
\]

\[
\simeq \left[ \phi_t(e_t) - \phi_t(e_t) \left( 1 + \frac{\phi_t^{(1)}(e_t')\alpha_t}{\phi_t(e_t)} - \phi_t^{(1)}(0)\alpha_t \right) \right]
\]

\[
\simeq \left[ - \phi_t(e_t)(\psi_t^{(1)}(e_t')\alpha_t - \psi_t^{(1)}(0)\alpha_t) \right]
\]

\[
\simeq - \text{diag}[\phi_t(e_t)]((m_t'\alpha_t)e + \Sigma_t\alpha_t - (m_t'\alpha_t)e)
\]

\[
\simeq - \text{diag}[\phi_t(e_t)]\Sigma_t\alpha_t
\]

\[
\simeq - \exp(r_{t+1})\Sigma_t\alpha_t,
\]

since \( \phi_t(e_t) = E_t \exp(w_{i,t+1}) \simeq E_t^Q \exp(w_{i,t+1}) = \exp(r_{t+1}) \).

In the conditionally Gaussian case, where

\[
\phi_t(u) = \exp \left( m_t' u + \frac{1}{2} u' \Sigma_t u \right), \quad \psi_t(u) = m_t' u + \frac{1}{2} u' \Sigma_t u,
\]

the geometric risk premium becomes

\[
\pi_{Gt} = \psi_t^{(1)}(0) - [\phi_t(\alpha_t + e_t)] + \phi_t(\alpha_t)e
\]

\[
= m_t - \left[ m_t' (\alpha_t + e_t) + \frac{1}{2} (\alpha_t + e_t)' \Sigma_t (\alpha_t + e_t) - m_t' \alpha_t - \frac{1}{2} \alpha_t' \Sigma_t \alpha_t \right]
\]

\[
= - \frac{1}{2} v \text{diag} \Sigma_t - \Sigma_t \alpha_t,
\]

while the arithmetic risk premium is

\[
\pi_{At} = [\phi_t(e_t)] - \left[ \frac{\phi_t(\alpha + e_t)}{\phi_t(\alpha_t)} \right]
\]

\[
= \left[ \exp \left( m_{it} + \frac{1}{2} \Sigma_{ii,t} \right) \right] - \left[ \exp \left( m_{it} + \frac{1}{2} \Sigma_{ii,t} + e_t' \Sigma_t \alpha_t \right) \right]
\]

\[
= \left[ \exp \left( m_{it} + \frac{1}{2} \Sigma_{ii,t} \right) \right] (1 - \exp(e_t' \Sigma_t \alpha_t))
\]

\[
\simeq - \text{diag} \left[ \exp \left( m_{it} + \frac{1}{2} \Sigma_{ii,t} \right) \right] \Sigma_t \alpha_t = - \exp(r_{t+1}) \Sigma_t \alpha_t,
\]

since \( \phi_t(e_t) = \exp(m_{it} + \frac{1}{2} \Sigma_{ii,t}) \).
A.5 The Factor is a Vector of Yields

Let us denote by \( r(t, h) \) the yield at \( t \) with residual maturity \( h \); if \( B(t, h) \) denotes the price at \( t \) of the zero coupon bond with time to maturity \( h \), we have

\[
r(t, h) = -\frac{1}{h} \log [B(t, h)].
\]

We assume that the components of \( w_{t+1} \) are

\[
w_{t+1,i} = h_i r(t + 1, h_i), \quad i \in \{1, \ldots, K\},
\]

where \( h_i \) are various integer residual maturities; this definition of \( w_{t+1,i} \) leads to simpler notations than the equivalent definition \( w_{t+1,i} = r(t + 1, h_i) \). The payoffs \( B(t + 1, h_i) = \exp(-w_{t+1,i}) \) have price at \( t \) equal to

\[
B(t, h_i + 1) = \exp [-(h_i + 1)r(t, h_i + 1)].
\]

So, we have

\[
1 = E_t \{\exp[-w_{t+1,i} + (h_i + 1)r(t, h_i + 1) + \alpha_i r_{t+1} - \psi_t(\alpha_i)]\}, \quad i \in \{1, \ldots, K\}, \tag{A5}
\]

that is

\[
r_{t+1} = \psi_t(\alpha_t - e_i) - \psi_t(\alpha_i) + (h_i + 1)r(t, h_i + 1),
\]

or

\[
\exp(r_{t+1}) = \frac{\varphi_t(\alpha_t - e_i)}{\varphi_t(\alpha_i)} \exp [(h_i + 1)r(t, h_i + 1)].
\]

The risk premia associated with the geometric returns,

\[
\log \left[ \frac{B(t + 1, h_i)}{B(t, h_i + 1)} \right] = -w_{t+1,i} + (h_i + 1)r(t, h_i + 1),
\]

are the vectors with components

\[
\pi_{Gt,i} = -E_t(w_{t+1,i}) + (h_i + 1)r(t, h_i + 1) - r_{t+1}
\]

\[= -e_i \psi_t^{(1)}(0) - \psi_t(\alpha_t - e_i) + \psi_t(\alpha_i), \tag{A6}
\]

and

\[
\pi_{At,i} = \exp [(h_i + 1)r(t, h_i + 1)] \varphi_t(-e_i) - \exp(r_{t+1})
\]

\[= \exp [(h_i + 1)r(t, h_i + 1)] \left[ \varphi_t(-e_i) - \frac{\varphi_t(\alpha_t - e_i)}{\varphi_t(\alpha_i)} \right]. \tag{A7}
\]
Expanding relations (A6) and (A7) around \( \alpha_t = 0 \), and neglecting conditional cumulants of order strictly larger than 2, we get

\[
\pi_{Gi} \simeq -\frac{1}{2} \text{diag}(\Sigma_i) + \Sigma_i \alpha_t, \quad (A8)
\]
\[
\pi_{Ai} \simeq \exp(r_{t+1}) \Sigma_i \alpha_t, \quad (A9)
\]

where \( \Sigma_i \) is the conditional variance–covariance matrix of \( w_{i+1} \) given \( w_i \). So, \( \alpha_t \) can be viewed as a market price of risk vector. Moreover, the formula for \( \pi_{Gi} \) is exact in the conditionally Gaussian case.

### A.6 Proof of Relations (A8) and (A9)

Following the same procedure presented above, the geometric risk premium associated with \( w_{i+1} = (h_1 r(t+1, h_1), \ldots, h_K r(t+1, h_K)) \) can be written as

\[
\pi_{Gi} = -\psi_i^{(1)}(0) - \left[ \psi_i(\alpha_t - e_i) \right] + \psi_i(\alpha_t)e \\
\simeq -\psi_i^{(1)}(0) - \left[ \psi_i(-e_i - \psi_i^{(1)}(-e_i)\alpha_t) + (\psi_i^{(1)}(0)\alpha_t) \right]e \\
\simeq -m_i - \left( -m_i + \frac{1}{2} \text{diag} \Sigma_i + (m_i'\alpha_t)e - \Sigma_i \alpha_t + (m_i'\alpha_t)e \right) \\
\simeq -\frac{1}{2} \text{diag} \Sigma_i + \Sigma_i \alpha_t,
\]

while, the arithmetic risk premium is

\[
\pi_{Ai} = \left[ \exp((h_i + 1)r(t, h_i + 1)) \left( \varphi_i(-e_i) - \frac{\varphi_i(\alpha_t - e_i)}{\varphi_i(\alpha_t)} \right) \right] \\
\simeq \left[ \exp((h_i + 1)r(t, h_i + 1))(\varphi_i(-e_i) - \varphi_i(-e_i)(1 + \psi^{(1)}(-e_i)\alpha_t - \psi^{(1)}(0)\alpha_t)) \right] \\
\simeq -\text{diag}[\varphi_i(-e_i) \exp((h_i + 1)r(t, h_i + 1))][\psi^{(1)}(-e_i)\alpha_t - \psi^{(1)}(0)\alpha_t] \\
\simeq \text{diag}[\varphi_i(-e_i) \exp((h_i + 1)r(t, h_i + 1))] \Sigma_i \alpha_t \\
\simeq \exp(r_{t+1}) \Sigma_i \alpha_t,
\]

since \( \varphi_i(-e_i) = E_i \exp[-h_i r(t + 1, h_i)] = E_i B(t + 1, h_i) \simeq E_i^Q B(t + 1, h_i) = \exp(r_{t+1}) \times B(t, h_i + 1) \).

In the conditionally Gaussian case, we have

\[
\pi_{Gi} = -\frac{1}{2} \text{diag} \Sigma_i + \Sigma_i \alpha_t \quad \text{and} \quad \\
\pi_{Ai} \simeq \text{diag} \left[ \exp \left( -m_{ii} + \frac{1}{2} \Sigma_{ii,i} \right) \exp((h_i + 1)r(t, h_i + 1)) \right] \Sigma_i \alpha_t = \exp(r_{t+1}) \Sigma_i \alpha_t,
\]

given that \( \varphi_i(-e_i) = \exp(-m_{ii} + \frac{1}{2} \Sigma_{ii,i}) \).
APPENDIX C: SWITCHING GARCH MODELS AND EXTENDED Car PROCESSES

The purpose of this appendix is to show, in the context of Section 4.6, that under the RN probability, even if \( w_{t+1} = (y_{t+1}, z'_{t+1})' \) is not a Car process, the extended factor \( w^e_{t+1} = (y_{t+1}, z'_{t+1}, \sigma^2_{t+1})' \) is Car. The proof of this result is based on the following two lemmas.

**Lemma 1.** For any vector \( \mu \in \mathbb{R}^n \) and any symmetric positive definite \((n \times n)\) matrix \( Q \), the following relation holds:

\[
\int_{\mathbb{R}^n} \exp(-u'Qu + \mu'u)du = \frac{\pi^{n/2}}{(\text{det } Q)^{1/2}} \exp\left(\frac{1}{4} \mu'Q^{-1}\mu\right).
\]

**Proof.** The LHS of the previous relation can be written as

\[
\int_{\mathbb{R}^n} \exp\left[-\left(u - \frac{1}{2}Q^{-1}\mu\right)'Q\left(u - \frac{1}{2}Q^{-1}\mu\right)\right] \exp\left(\frac{1}{4} \mu'Q^{-1}\mu\right)du
\]

\[
= \frac{\pi^{n/2}}{(\text{det } Q)^{1/2}} \exp\left(\frac{1}{4} \mu'Q^{-1}\mu\right),
\]

given that the \( n \)-dimensional Gaussian distribution \( N(\frac{1}{2}Q^{-1}\mu, (2Q)^{-1}) \) admits unit mass.

**Lemma 2.** If \( \varepsilon_{t+1} \sim N(0, I_n) \), we have:

\[
E_t\{\exp[\lambda'\varepsilon_{t+1} + \varepsilon'_{t+1}V\varepsilon_{t+1}]\}
\]

\[
= \frac{1}{[\text{det}(I - 2V)]^{1/2}} \exp\left[\frac{1}{2} \lambda'(I - 2V)^{-1}\lambda\right].
\]

**Proof.** From Lemma 1, we have:

\[
E_t\{\exp[\lambda'\varepsilon_{t+1} + \varepsilon'_{t+1}V\varepsilon_{t+1}]\}
\]

\[
= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \exp\left[-u'\left(\frac{1}{2}I - V\right)u + \lambda'u\right] du
\]

\[
= \frac{1}{2^{n/2}[\text{det}(\frac{1}{2}I - V)]^{1/2}} \exp\left[\frac{1}{4} \lambda'\left(\frac{1}{2}I - V\right)^{-1}\lambda\right]
\]

\[
= \frac{1}{[\text{det}(I - 2V)]^{1/2}} \exp\left[\frac{1}{2} \lambda'(I - 2V)^{-1}\lambda\right].
\]

**Proposition.** In the context of Section 4.6, the process \( w^e_{t+1} = (y_{t+1}, z'_{t+1}, \sigma^2_{t+1})' \) is Car(1) under the RN probability.
Proof. We have:

\[ y_{t+1} = r_{t+1} - \lambda' z_t - \frac{1}{2} \sigma_{t+1}^2 + \nu'_3 z_{t+1} + \sigma_{t+1} \xi_{t+1}, \]

\[ \xi_{t+1} \big| \xi_t, z_{t+1} \sim N(0, 1), \]

\[ \sigma_{t+1}^2 = \omega' z_t + \alpha_1(\xi_t - \alpha_2 \sigma_t)^2 + \alpha_3 \sigma_t^2, \]

\[ \mathbb{Q}(z_{t+1} = e_j | \mathbf{y}_t, z_{t-1}, z_t = e_i) = \pi_{ij}^*, \]

So, the conditional RN Laplace transform of \((y_{t+1}, z'_{t+1}, \sigma_{t+2}^2)'\) is

\[ \varphi_t^\mathbb{Q}(u, v, \tilde{v}) = E_t^\mathbb{Q} \exp \left( u y_{t+1} + v' z_{t+1} + \tilde{v} \sigma_{t+2}^2 \right) \]

\[ = E_t^\mathbb{Q} \exp \left\{ u \left( r_{t+1} - \lambda' z_t - \frac{1}{2} \sigma_{t+1}^2 + \nu'_3 z_{t+1} + \sigma_{t+1} \xi_{t+1} \right) \right. \]

\[ + v' z_{t+1} + \tilde{v} \left[ \omega' z_{t+1} + \alpha_1(\xi_t - \alpha_2 \sigma_t)^2 + \alpha_3 \sigma_t^2 \right] \} \]

\[ = \exp \left\{ u \left( r_{t+1} - \lambda' z_t - \frac{1}{2} \sigma_{t+1}^2 \right) + \tilde{v} \alpha_1 \sigma_{t+1}^2 + \tilde{v} \alpha_3 \sigma_{t+1}^2 \right\} \]

\[ E_t^\mathbb{Q} \exp[\xi_{t+1} \sigma_{t+1}(u - 2 \alpha_1 \alpha_2 \tilde{v}) + \tilde{v} \alpha_1 \xi_{t+1}^2 + (uv_3 + v + \tilde{v} \omega)' z_{t+1}]. \]

Using Lemma 2:

\[ \varphi_t^\mathbb{Q}(u, v, \tilde{v}) = \exp \left[ u \left( r_{t+1} - \lambda' z_t - \frac{1}{2} \sigma_{t+1}^2 \right) + \tilde{v} \alpha_1 \sigma_{t+1}^2 + \tilde{v} \alpha_3 \sigma_{t+1}^2 \right] \]

\[ \times \exp \left[ -\frac{1}{2} \log(1 - 2\alpha_1 \tilde{v}) + \frac{(u - 2 \alpha_1 \alpha_2 \tilde{v})^2}{2(1 - 2\alpha_1 \tilde{v})} \sigma_{t+1}^2 + \bar{\Lambda}'(u, v, \tilde{v}, \omega, \nu_3, \pi^*) z_t \right], \]

where the \(i\)th component of \(\bar{\Lambda}(u, v, \tilde{v}, \omega, \nu_3, \pi^*)\) is given by

\[ \bar{\Lambda}_i(u, v, \tilde{v}, \omega, \nu_3, \pi^*) = \log \sum_{j=1}^{l} \pi_{ij}^* \exp(u \nu_{3j} + v_j + \tilde{v} \omega_j), \]

and relation (47) is proved.

**APPENDIX D: SWITCHING IG GARCH MODELS AND EXTENDED CAR PROCESSES**

In this appendix we show, in the context of Section 4.7, that under the RN probability, even if \(w_{t+1} = (y_{t+1}, z'_{t+1})'\) is not a Car process, the extended factor \(w_{t+1}^e = (y_{t+1}, z'_{t+1}, \sigma_{t+2}^2)'\) is Car.
Proposition. In the context of Section 4.7, the process \(w_t = (y_t, z_t, \sigma_t^2)\) is \(Car(1)\) under the RN probability.

Proof. Let us recall Equation (49):

\[
y_{t+1} = r_{t+1} - \lambda'z_t - \frac{1}{\eta^2} \left[1 - (1 - 2\eta)^{1/2}\right] \sigma_{t+1}^2 + v'z_{t+1} + \eta \xi_{t+1}
\]

with

\[
\sigma_{t+1}^2 = \omega'z_t + \alpha_1 \sigma_t^2 + \alpha_2 \xi_t + \alpha_3 \frac{\sigma_t^4}{\xi_t}
\]

\[
\xi_{t+1} | \xi_j, z_{t+1} \sim IG \left(\frac{\sigma_{t+1}^2}{\eta^2}\right),
\]

\[
Q(z_t = e_j | y_j, z_{t-1}, z_t = e_i) = Q(z_t = e_j | z_t = e_i) = \pi_{i,j}^n,
\]

So, the conditional RN Laplace transform of \(w_t = (y_t, z_t, \sigma_t^2)\) is

\[
\varphi_t^Q(u, v, \bar{v}) = E_t^Q \exp \left( u y_{t+1} + v'z_{t+1} + \bar{v} \sigma_{t+1}^2 \right)
\]

\[
= E_t^Q \exp \left\{ u \left( r_{t+1} - \lambda'z_t - \frac{1}{\eta^2} \left[1 - (1 - 2\eta)^{1/2}\right] \sigma_{t+1}^2 + v'z_{t+1} + \eta \xi_{t+1} \right) \right\}
\]

\[
+ v'z_{t+1} + \bar{v} \left[ \omega'z_{t+1} + \alpha_1 \sigma_{t+1}^2 + \alpha_2 \xi_{t+1} + \alpha_3 \frac{\sigma_{t+1}^4}{\xi_{t+1}} \right] \}
\]

\[
= \exp \left\{ u \left( r_{t+1} - \lambda'z_t - \frac{1}{\eta^2} \left[1 - (1 - 2\eta)^{1/2}\right] \sigma_{t+1}^2 \right) + \bar{v} \alpha_1 \sigma_{t+1}^2 \right\}
\]

\[
E_t^Q \exp \left[ (\mu \eta + \bar{v} \alpha_2) \xi_{t+1} + \frac{\bar{v} \alpha_3 \sigma_{t+1}^4}{\xi_{t+1}^3} + (\mu v_3 + v + \bar{v} \omega)'z_{t+1} \right].
\]

Using the formula of the generalized Laplace transform of an inverse Gaussian distribution given in footnote 4 (Section 4.7):

\[
\varphi_t^Q(u, v, \bar{v}) = \exp \left\{ u \left( r_{t+1} - \lambda'z_t - \frac{1}{\eta^2} \left[1 - (1 - 2\eta)^{1/2}\right] \sigma_{t+1}^2 \right) + \bar{v} \alpha_1 \sigma_{t+1}^2 \right\}
\]

\[
\times \exp \left[ -\frac{1}{2} \log(1 - 2\bar{v} \alpha_3 \eta^4) + \frac{1}{\eta^2} \left( 1 - \sqrt{1 - 2\bar{v} \alpha_3 \eta^4} \right) \left( 1 - 2(\mu \eta + \bar{v} \alpha_2) \right) \right] \sigma_{t+1}^2
\]

\[
+ \tilde{\Lambda}'(u, v, \bar{v}, v_3, \omega, \pi^*) \pi_i^* z_i \right],
\]

where the \(i\)th component of \(\tilde{\Lambda}(u, v, \bar{v}, v_3, \omega, \pi^*)\) is given by

\[
\tilde{\Lambda}_i(u, v, \bar{v}, v_3, \omega, \pi^*) = \log \sum_{j=1}^{I} \pi_{ij}^* \exp(\mu v_{3j} + v_j + \bar{v} \omega_j),
\]

and relation (52) is proved.
APPENDIX E: QUADRATIC TERM STRUCTURE MODELS AND EXTENDED CAR PROCESSES

Given the Gaussian VAR(1) process defined by relation (70), we have that, for any real symmetric matrix $V$, the conditional historical Laplace transform of $(x_{t+1}, x'_{t+1})$ is given by

$$E_t \exp[u'x_{t+1} + \text{Tr}(Vx_{t+1}x'_{t+1})] = \exp\{u'm + u'Mx_t + \text{Tr}[m'm' + Mx_tM' + Mx_tM']\}$$

$$E_t \exp[u'\varepsilon_{t+1} + \text{Tr}[\varepsilon_{t+1}\varepsilon'_{t+1} + m\varepsilon'_{t+1} + \varepsilon_{t+1}m' + Mx_t\varepsilon'_{t+1} + \varepsilon_{t+1}x'M']$$

$$E_t \exp\{[u' + 2(m + Mx_t)'V]\varepsilon_{t+1} + \varepsilon'_{t+1}V\varepsilon_{t+1}\}$$

and, using Lemma 2 in Appendix C, we can write:

$$E_t \exp[u'x_{t+1} + \text{Tr}(Vx_{t+1}x'_{t+1})] = \exp\{u'm + m'Vm + (M'u + 2M'Vm)'x_t + x'M'VMx_t$$

$$+ \frac{1}{2}[u' + 2(m + Mx_t)'V](I - 2V)[u' + 2V(m + Mx_t)] - \frac{1}{2} \log \det(I - 2V),$$

which is exponential-affine in $[x'_t, \text{vech}(x_t)]$.

REFERENCES


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