Asset Pricing with Unforeseen Contingencies

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Abstract

Data suggests that macro asset pricing models based solely on aggregate consumption perform poorly. Yet, there is no consensus over what alternative primitive pricing factors should accompany or take the place of aggregate consumption. We assume an economy where agents are like economists: they know that factors other than aggregate consumption have welfare implication, but they can’t specifically and explicitly state what these factors are. Instead, agents attribute the additional source of uncertainty to ‘unforeseen contingencies’ whose behavioral impact is observationally equivalent to individual taste shocks.

We investigate an economy of heterogeneous agents that cannot specify all exogenous welfare-relevant events, and characterize the appropriate equilibrium concept when securities can only trade on demand and price contingent events. We establish the existence of an equilibrium for a class of parametric models in which aggregating unforeseen contingencies across agents can lead to non-consumption pricing factors. To fit the stylized facts, (i) non-consumption factors must dominate the pricing kernel and contribute to the variation of the wealth-consumption ratio, (ii) markets must be incomplete, and the set of claims that are traded endogenously determined, (iii) agents’ preferences with respect to unforeseen contingencies must be non-expected utility, and (iv) although non-consumption pricing factors can be conditionally uncorrelated with aggregate consumption shocks, they must be correlated with shocks to expected consumption growth.

JEL Classification: D51, D52, D58, G12, G13

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1 Introduction

The macroeconomic asset-pricing literature is motivated by the failure of neo-classical models to describe empirical moments of macroeconomic variables, namely: the equity premium puzzle of Mehra and Prescott (1985), the volatility puzzle (Campbell (2000)), and the risk-free rate puzzle of Weil (1989).\(^1\) Dissonance between theory and data is partly due to the necessary compromises inherent in model selection (in the simplest case, a representative agent who is indifferent to the timing of resolution of uncertainty), and partly due to the fact that consumption is much smoother than stock prices and performs poorly as a cross-sectional pricing factor. Recent work on habit formation (Campbell and Cochrane (1999, 2000)) seems helpful in establishing a more consistent story based on aggregate consumption only. In the end however, it seems that all theoretical models based solely on aggregate consumption have to somehow ratchet the aggregate absolute risk aversion to levels that are not ‘esthetically’ pleasing while they continue to perform poorly in cross-sectional tests.\(^2\) On the other hand, there is recent empirical work by Lettau and Ludvigson (2001ab, 2002) that indicates that the wealth-consumption ratio plays an important role in the pricing of risk. Overall, there seems to be a consensus that an asset pricing model where the fundamental shocks are those corresponding to aggregate consumption cannot fully capture the litany of stylized macroeconomic facts.

The aim of this paper is to motivate the addition of non-consumption based factors to the standard consumption based general equilibrium asset pricing model. The main difficulty is simple enough to state: if agents care about fundamental systematic economic factors other than consumption or derivatives of consumption,\(^3\) then what are these factors and how can one identify them? The answer is not obvious and we are no better than our colleagues in identifying additional primitive or non-derived factors directly;\(^4\) instead,

\(^1\)A list of references includes, but is not limited to Kocherlakota (1996), Abel(1990), Constantinides (1990), Epstein and Zin (1989), and Sundaresan (1989). A frequently missed early paper on consumption based asset pricing is Rubinstein (1976).

\(^2\)Although the aggregate of agents need not please anyone esthetically, one should also keep in mind that it is risk tolerance and not risk aversion that is aggregated in equilibrium (Wilson (1968)). In other words, one theoretically expects the economy’s risk attitudes to be dominated by those who are least risk averse. For example, in a frictionless economy, the presence of a single classically risk-neutral agent is enough to drive the aggregate risk aversion of the entire economy to zero.

\(^3\)In referring to fundamental factors, we mean primitives of choice. This includes consumption but excludes derived quantities such as wealth, interest rates and prices. Derivatives of consumption include factors that depend on consumption history (as in habit formation) or any other functions of past and future (e.g., forecasts of) consumption.

\(^4\)Candidates may include higher moments of income distribution (status), the changes in the constituents of the
we attempt to shed light on the issue indirectly by assuming that agents in the economy themselves realize that their future welfare depends on things other than consumption, but they cannot fully characterize the additional variables in terms of fundamentals or non-derived quantities. Such agents know that something can and will affect them in the future but cannot specify exactly what it will be. In the decision theory literature, these events are said to be ‘unforeseen’ since they cannot be exogenously specified within a standard information filtration.

To model our agents’ preferences we make use of recent work in axiomatic decision theory with unforeseen contingencies (see Kreps (1979, 1992), Dekel, Lipman and Rustichini (2001), and Kraus and Sagi (2002)). A general conclusion of this literature is that an agent making rational decisions in the face of unforeseen contingencies will act as if she experiences private taste shocks. In other words, since agents cannot point to specific external events that affect their welfare, they instead postulate ‘utility states’ that quantify changes in private welfare, but do not specify the cause of the change. Moreover, these agents will not generally obey the standard expected utility axioms with respect to their private states. To risk oversimplification for the sake of intuition, suppose an agent’s future welfare is a function, \( v(c, x) \), that depends on both consumption, \( c \), and some other factor, \( x \), but the agent cannot identify \( x \). From the agent’s perspective, the situation is equivalent to random future taste shocks (corresponding to future realizations of the unidentified ‘missing variable’, \( x \)).

We consider the aggregate behavior of agents along the lines suggested by this literature and analyze an economy that is populated by heterogeneous agents who experience private taste shocks. A particularly attractive feature of our approach, consistent with the spirit of unforeseen contingencies, is the fact that we do not need to make specific modeling assumptions regarding the evolution of the private taste shocks. All that is required is that agents’ portfolio choices reflect the possibility that private taste shocks can take place. Moreover, to steer clear of artificial or exogenous constraints on the economy we allow agents to trade any Arrow-Debreu security on what we term publicly observable states. These we take to be aggregate consumption (i.e., demand) or price related events. If, in aggregate, taste shocks influence prices – as might be the case if a single unforeseen event similarly affects all agents’ demand for securities – then some component of taste shocks can be hedged by trading. Generally, however, not all components of an agent’s taste shocks have publicly observable price impact. Thus securities in our economy trade only on a portion of welfare relevant events commodity bundle comprising ‘aggregate consumption’, non-tradeable endowment shocks, etc. Moreover, it may very well be the case that it is not any particular mix of such candidate variables that comprises the ‘missing factor’, but that different variables contribute to asset pricing at different times.
and markets are incomplete. This leads to a complication in analyzing an equilibrium: deriving prices requires a knowledge of the asset mix, and since all price contingent claims can be traded, the asset mix requires knowledge of prices. One way out of this conundrum is to complete the market by including contracts on all private taste shocks – an approach that seems unrealistic if private actions cannot be fully observed by all. Thus an important and non-trivial part of our approach is to provide a framework in which an \textit{equilibrium endogenously determines both the events that are publicly observable and the consequent set of tradeable assets in an incomplete market.}

We investigate a class of parametric models for which such an equilibrium generally exists. After aggregating over agents’ consumption/investment decisions, we derive a three-factor equilibrium nesting the Lucas (1978) model.\textsuperscript{5} The three factors whose dynamics govern asset prices are, (i) per capita consumption growth, (ii) proportional changes in a measure of average relative risk aversion, and (iii) a variable that measures the departure of preferences from expected utility theory. The second factor is potentially a non-consumption source of priced volatility since average relative risk aversion aggregates ‘utility states’ across agents. The contribution of this factor to the equity premium depends on the \textit{relative volatility} of aggregate risk aversion. Allowing this volatility to be high results in a high market price of risk even when the level of aggregate relative risk aversion is low. Variations in the third factor correspond to an average agent over (resp. under) weighting, relative to expected utility, high future risk-tolerance states when assessing current welfare.\textsuperscript{6} Over (resp. under) weighting leads to state prices that are high (resp. low) relative to what would be obtained under the assumption of expected utility. Thus the third factor can be said to capture swings between ‘bullish’ and ‘bearish’ attitudes towards securities in the market. This too is a source of volatility in state prices.

At first blush, the two ‘new factors’ are something of an ‘expected’ disappointment since they are empirically unobserved. The news is not entirely bad, however, as we can show that, under various conditions, changes in these variables are linked to changes in the wealth-consumption ratio; the latter, in principle, is observable and allows one to ‘back’ out the behavioral parameters corresponding to the non-consumption factors. To account for

\textsuperscript{5}Recently, several researchers have begun to investigate the aggregation of agents with heterogeneous preferences. Some references include Dumas (1989), Wang (1996) and Chan and Kogan (2002). Our method of aggregation and choice of preferences is closest in spirit to Rubinstein (1974).

\textsuperscript{6}The deviation from expected utility described above does not require false or irrational ‘average beliefs’ about future risk tolerance. Such deviations may occur when beliefs are correct if the manner in which they enter the utility function is not linear.
both the high equity risk premium and the poor cross sectional power of aggregate consumption alone as a pricing factor, one needs to attribute a great deal of the variation in state prices to the presence of shocks uncorrelated with aggregate consumption growth. We do this by setting the non-consumption variables to be conditionally uncorrelated with aggregate consumption shocks to the degree that the market price of risk for consumption is negligible. However, to calibrate to the remaining stylized facts in a simple framework, the non-consumption component cannot be unconditionally independent of aggregate consumption; otherwise, shocks to the wealth-consumption ratio would depend only on shocks to consumption and/or the term structure of interest rates – neither being consistent with facts. Another property of the model, is that deviations from expected utility are necessary to match key empirical moments. Thus, although some states cannot be hedged (since they cannot be publicly observed), their existence has an important impact on macro-economic variables.

Section 2 discusses the intuition and literature behind the economic model. Section 3 of the paper outlines the general model and presents our aggregation theorem. Section 4 provides further discussion and results on how our model may be calibrated to stylized facts. Section 5 concludes.

2 Discussion of setting and literature

We now informally describe the economic setting. In addition to facing the usual uncertainty, each agent in our single good exchange economy has to contend with the fact that there are other contingencies (i.e., unrelated to aggregate consumption) that might affect her. However, like us, the agent cannot explicitly list all future contingencies that can affect her. Recent literature on unforeseen contingencies gives normative insight into agents’ choice behavior under such circumstances: the impact of unforeseen contingencies can be consistently taken into account by incorporating additional subjective utility states into the already existing objective exogenous filtration.7

Since this is likely to be unfamiliar to most financial economists, we provide a simple illustration: consider an agent facing an exogenously given filtration of states as in Figure

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7See Kreps (1979, 1992), Dekel, Lipman and Rustichini (2001) and Kraus and Sagi (2002). The term unforeseen contingencies is now a technical one that defines exogenous events that are not adapted to the current exogenous filtration as seen by the agent.
Figure 1: A consumption-decision tree. Squares (circles) denote decision (chance) nodes and $z_{i,j}$ is the bundle consumed at date $i$ and branch $j$.

1. Squares (circles) denote decision (chance) nodes and $z_{i,j}$ is the bundle consumed at date $i$ and branch $j$. In the conventional approach to dynamically consistent choice, the agent can determine what her choice will be at $x_{1,4}$ even though she is presently at date 0. For instance, if she knows that she will prefer the upper branch of $x_{1,4}$, then in considering her date 0 choices she can simplify the tree and ‘erase’ the lower branch at $x_{1,4}$ - this is essentially a pre-commitment to the upper branch. If there are unforeseen contingencies, however, the agent may be reluctant to ‘erase’ a future choice, or in other words, to pre-commit to a future choice branch. How does one correctly represent such behavior? The answer is illustrated in Figure 2. There the agent is willing to pre-commit to the upper branch (and thus erase the lower branch) of $x_{1,4}$, but is reluctant to do so at any of the branches of $x_{1,1}$. She behaves as if there are at least two unlisted ‘states’: one in which she will prefer the upper branch,
Figure 2: Assume that in Figure 1 the agent knows at date 0 that she will prefer the upper branch of $x_{1,4}$, but is reluctant to pre-commit to either branch in $x_{1,1}$. The induced tree, shown above, ‘erases’ the lower branch at $x_{1,4}$, but transforms the choice node at $x_{1,1}$ into a chance node corresponding to preference states.
and one in which she will prefer the lower branch. The unlisted ‘states’ are not exogenous states, but subjective preference states that can only be inferred by the agent’s aversion to commitment, and they can be represented by two different utility states at date 1 (these are illustrated by a bold chance node in Figure 2). As suggested in Figure 2, a backward induction procedure generally results in a tree without multi-branch choice nodes, and where there are two types of chance nodes: those that can be associated with exogenous events, and those that are private to the agent. Thus, unforeseen contingencies are synonymous with changing tastes or private taste shocks.

To provide additional motivation for the relevance of such endogenous preference states, consider the following exercise in introspection: would you (would anyone) commit to a contingent stream of consumption for all future dates based only on what you know today about future consumption/production opportunities? In particular, consider that a contingent consumption stream delivers a basket or index of goods at each point in time that depends on exogenous events that the agent can prespecify. By committing to a particular stream of commodity baskets today, one is foregoing the possibility of enjoying alternative baskets containing goods of which one cannot currently conceive. Although a rational agent may not conceive of all future available goods, she does recognize her own limitations and refuses to commit to a contingent consumption stream. Instead, she values flexibility, reasoning that something can happen that will cause her to deviate from a contingent plan. Although neither she nor outside observers can name what event might do so, such deviation from a plan can be viewed as a ‘taste shock.’

Uncertainty with respect to future preferences is neither new nor unreasonable, and there is a literature on ‘changing tastes’ and associated utility for flexibility dating to the 60’s.\(^8\) Alternatively, one can view the preference structure of each of our agents as that of reduced-form utility. In this context, the private preference states of Figure 2 can be interpreted as proxying for residual utility possibilities that remain after imperfectly aggregating a consumer’s bundle of goods into a single index that is homogeneous across agents. In other words, the agent feels differently about her allotment of the aggregate good in node \(x_{1,1}\) because she is actually consuming a different amount of some unlisted commodities in each of these branches. Ultimately, though, we defend our use of unorthodox preferences by claiming that it is unreasonable to suggest that agents can tell now how they will feel later about some future consumption contingency.\(^9\)


\(^9\)We also note that although our agents have non-standard preferences, they are inter-temporally consistent – the
The figures illustrate the decision problem of a single agent. In a multi-agent economy such as ours, it is sensible to assume that certain contingencies relevant to all agents are also largely unforeseen by all of them; these can be viewed as private taste shocks that share a common component. We are particularly interested in the possibility that common components of taste shocks are not correlated with aggregate consumption: it is mainly in this way that the model departs from the conventional paradigm of state dependent preferences (e.g., habit formation models) and ‘non-consumption’ factors are introduced into the pricing kernel.

The preference states (i.e., the branches at $x_{1,1}$ in Figure 2) of individual agents are known only to them. However, in a general equilibrium, related components of agents’ taste shocks may have an impact on prices.¹⁰ Such price impact allows agents to infer the common component of private taste shocks (although the identities of agents experiencing the different shocks cannot generally be determined). We allow for a security market where every aggregate consumption or price-related event can be hedged.¹¹ Since prices reveal the ‘correlated’ parts of private taste shocks, the set of observable states must be endogenously determined by trading, and thus the asset mix in the economy is determined by the equilibrium. It is not hard to see that if every unforeseen contingency is common to all agents, no truly private state exists and the market is complete - a classic exchange economy follows with the possibility of state dependent aggregate risk aversion. On the other hand, market incompleteness arises from the failure of some or any private shock to have macroeconomic significance. The last few statements clarify that an economy with unforeseen contingencies ought to explicitly aggregate heterogeneous agents’ preferences to arrive at a pricing kernel. This is especially true if the market is incomplete, since a representative agent does not, in general, exist.

¹⁰E.g., if the same unforeseen contingency increases the appetites of all agents for mangoes, then such an event will have an impact on the price of mangoes.

¹¹Note that it is possible that the source of the event (e.g., development of the internet) is not foreseen, yet the impact on welfare is anticipated by prices (e.g., something might improve everyone’s ability to communicate and pool information). By betting on technology stocks, one can hedge breakthroughs in technology even though one cannot possibly characterize what these breakthroughs might actually be.
3 General Equilibrium With Unforeseen Contingencies

The assumed economy is one of pure exchange, consisting of \( N \) agents with state dependent preferences.\(^{12} \) There are \( T \) periods and one perishable consumption good that is produced according to some exogenous process (all of the good produced at date \( t \) must be consumed at date \( t \)). There are two basic types of states: nature states that determine the amount of aggregate good available each period, and individual preference states corresponding to private taste shocks. Nature and preference states give rise to macro and micro events, depending on whether or not the event under consideration is observable by all agents.

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<th>OBSERVABILITY OF EVENTS - ‘macro’ vs. ‘micro’</th>
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Before we define precisely what it is that we mean by ‘macro’ or observable states, we introduce some technical definitions and notation. Let \( \Omega \) be a finite set and \( \mathcal{F} = (\mathcal{F}_1, \ldots, \mathcal{F}_T) \) be a sequence of \( \sigma \)-algebras such that \( \mathcal{F}_1 = \{\Omega, \emptyset\} \), \( \mathcal{F}_{t+1} \subseteq \mathcal{F}_t \), and \( \mathcal{F}_T \) contains all subsets of \( \Omega \). \( \mathcal{F} \) is an information filtration and represents knowledge that becomes progressively finer as \( t \) increases. We refer to elements of \( \mathcal{F}_t \) as date-\( t \) ‘events’. Each \( \mathcal{F}_t \) is generated by a unique partition of \( \Omega \), which we denote via \( \mathcal{A}_t \). An arbitrary atom of the date-\( t \) partition, \( a_t \in \mathcal{A}_t \), is referred to as a ‘date-\( t \) state’. Note that \( a_{t+1} \in \mathcal{A}_{t+1} \Rightarrow a_{t+1} \subseteq a_t \) for one and only one \( a_t \in \mathcal{A}_t \). Diagrammatically, \( a_t \) is a possible path (or history) in a decision tree with an ending point at date \( t \).

**Definition 1.** A state-price system adapted to \( \mathcal{F} \) is a strictly positive real-valued mapping over states, \( \phi: \bigcup_{t=1}^T \mathcal{A}_t \to \mathbb{R}_{++} \), and an associated set at each date \( t \in \{1, \ldots, T\} \) and state \( a_t \in \mathcal{A}_t \): \( \{\phi(a_s|a_t)\}_{a_s \subseteq a_t} \), where

\[
\phi(a_s|a_t) = \begin{cases} 
\phi(a_s) & t < s \leq T \text{ and } a_s \subseteq a_t \\
0 & \text{otherwise}
\end{cases}
\]  

Intuitively, a state-price system represents a set of Arrow-Debreu prices for \( \mathcal{F} \) at each date \( t \) and each state \( a_t \). The \( \phi(a_s) \)'s represent the date-1 Arrow-Debreu prices, and \( \phi(a_s|a_t) \)

\(^{12} N \) can be taken to infinity.
represents the price of an Arrow-Debreu claim that pays in state $a_s$ given that the current state is $a_t$. Note that for now this is simply an interpretation; we have said nothing about the contents of $\Omega$, nor established that an equilibrium set of prices exists.

Assume that associated with each $\omega \in \Omega$ is an unconditional probability, $\pi(\omega) > 0$. The unconditional probability of a state at date $t$, $a_t \in A_t$, is simply $\pi(a_t) = \sum_{\omega \in a_t} \pi(\omega)$. For $t < u$ and any two states, $a_t \in A_t$ and $a_u \in A_u$ such that $a_u \subseteq a_t$, Bayes’ Rule gives the probability of the state $a_u$ conditional on the occurrence of $a_t$ as $\pi(a_u|a_t) = \pi(a_u)/\pi(a_t)$. If $\mathcal{F}$ corresponds to a filtration of macro states, then we assume that the associated probabilities are ‘objective’ and that all agents agree on their values. In other words, the $\pi$’s are part of the tree structure characterizing the macro states.

To form a filtration of ‘macro states’, events in $\mathcal{F}$ have to be observable by all. Specifically, macro states ought to be related only to realizations of aggregate consumption and prices. How can one be sure that two states in $A_t$ are distinguishable in this way? First, note that if the event tree originating from $a_t$ does not have the same basic structure as the one originating from $a'_t$, then the two states are distinguishable. If the two trees do happen to have the same structure but have a different history, then they are still distinguishable. Finally, if two states share the same history and are at the root of separate but structurally identical trees then there is only one way to distinguish them: the two trees must disagree at some corresponding future node on the realization of aggregate consumption (demand) or on the state price associated with that node.

To state this formally, two states with the same history, (i.e., $a_t \cup a'_t \subseteq a_{t-1}$ for some $a_{t-1} \in A_{t-1}$), share the same ‘probability tree’ structure whenever there exists an isomorphism, $I : \{a_s\}_{a_s \subseteq a_t} \to \{a_s\}_{a_s \subseteq a'_t}$ such that $I(a_t) = a'_t$, and for any $s > t$, $\pi(a_s|a_t) = \pi(I(a_s)|a'_t)$ and $a_{s+1} \subseteq a_s \Leftrightarrow I(a_{s+1}) \subseteq I(a_s)$. Diagrammatically, this represents a situation in which the state-probability tree with root at $a_t$ is identical to the one with root at $a'_t$; $I$ is the isomorphism that establishes that the structures are identical.

**Definition 2 (Observable/Macro States).** Given a state-price system generated by $\phi(\cdot)$ and probabilities generated by $\pi(\cdot)$, $\mathcal{F}$ is a filtration of observable/macro states whenever aggregate consumption, $C(\cdot)$, is adapted to $\mathcal{F}$ and the following two properties hold:

i) for any two distinct states, $a_t, a'_t \in A_t$, with the same history and who share a tree isomorphism, $I$, there is some $a_s \subseteq a_t$ such that either $C(a_s) \neq C(I(a_s))$, or there is some $a_{s+1} \subseteq a_t$ such that $\phi(a_{s+1}|a_s) \neq \phi(I(a_{s+1})|I(a_s))$. 

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ii) \(F\) is adapted to the information filtration generated by the intersection of all agents’ private information sets

The definition is recursive: at date \(T\), the price system for future state claims is degenerate, thus observable states simply index the possible realizations of aggregate consumption; at date \(T-1\), the possible states index both different realizations of aggregate consumption and different sets of state prices for hedging date \(T\) aggregate consumption. By backward induction one can show that the first part of the definition also rules out ‘coin toss’ or ‘sunspot’ states that are not related to fundamentals (e.g., discount rates or aggregate consumption). The second requirement excludes situations in which the ‘observable’ filtration contains more information than can be achieved by ‘pooling’ all agents’ information sets.\(^{13}\) Note that \(F\) is at least as fine as the coarsest filtration to which aggregate consumption is adapted; i.e., \(F\) contains at least as many states as can be distinguished via aggregate consumption history.

The set of ‘macro’ states, those that can be observed by everyone, includes all of the nature (i.e., aggregate consumption) states, but it also includes anything else that affects prices; in particular, any aggregated taste shocks that are not correlated with aggregate consumption and yet impact prices will induce additional states. Some taste shocks have no impact on prices because they are not correlated across individuals - those are assumed to be unobservable and are labeled as ‘micro’ states.

Note that the definition of observable states is circular: to identify a filtration of observable states, one requires a set of prices for Arrow-Debreu claims on \(F\)-events; the set of Arrow-Debreu claims over \(F\), in turn, depends on which states are identified as belonging to \(F\). In other words, \(F\) must be determined *endogenously* as part of a general equilibrium. In large measure, it is this property that differentiates our approach from that of the standard complete market exchange economy (where the filtration of states is exogenously specified).

We now turn to a description of preferences. For ‘micro’ states to be meaningful, one must consider an economy with heterogeneous agents. We assume a preference structure that is amenable to closed form solutions. To motivate our assumptions, it is first helpful to consider the case in which there are no micro states (i.e., all states are observable to all agents). Our benchmark is time-separable expected utility preferences under objective risk. Assume a filtration of observable states, \(F\), exists. Agents’ preferences are constructed as follows: at date \(t\) and macro state \(a_t \in A_t\), agent \(i\) must think ahead and consider the indirect utility generated from future wealth, \(w_{t+1}^{i}\), in state \(a_{t+1}\), where \(a_{t+1} \subseteq a_t\) refers to a

\(^{13}\)See Kreps (1977) page 34.
date $t + 1$ macro state. For a particular pair of date $t$ consumption and date $t + 1$ realization of wealth, $(c_t, w^{i}_{t+1})$, the contribution to the agent’s utility in state $a_{t+1}$ is,

$$U_t(c_t, w^{i}_{t+1}, a_{t+1}) = u(c_t) + \beta V^*_t(w^{i}_{t+1}, a_{t+1})$$

where $0 < \beta < 1$, $V^*_t$ is the indirect utility function for date $t + 1$, and $u(\cdot)$ is the agent’s utility of date $t$ consumption, taken to be time independent. Note that we have assumed time separability and that the indirect utility function is state dependent (as is usually the case when the investment opportunity set is time varying).

Assuming expected utility with respect to macro-economic states (i.e., the $a_{t+1}$’s), the agent’s utility over the pair of current consumption and random future wealth, $(c_t, \tilde{w}^{i}_{t+1})$, is

$$\sum_{a_{t+1} \subseteq a_t} \pi(a_{t+1}|a_t) U_t(c_t, \tilde{w}^{i}_{t+1}, a_{t+1}) = u(c_t) + \beta \sum_{a_{t+1} \subseteq a_t} \pi(a_{t+1}|a_t) V^*_t(\tilde{w}^{i}_{t+1}, a_{t+1})$$

where the $\pi(a_{t+1}|a_t)$’s are the conditional probabilities for the realization of macro states considered earlier and known to all agents. Finally, the indirect utility is,

$$V^*_t(w^{i}_{t}, a_t) \equiv \max_{(c_t, \tilde{w}^{i}_{t+1}) \in B(w^{i}_{t})} \sum_{a_{t+1} \subseteq a_t} \pi(a_{t+1}|a_t) U_t(c_t, \tilde{w}^{i}_{t+1}, a_{t+1})$$

where $B(w^{i}_{t})$ is the agent’s budget set.

We now introduce micro states. Specifically, suppose agent $i$ has different possible preference realizations in a given macro state. A normative theory of unforeseen contingencies (e.g., Kraus and Sagi (2002)) can give an indication of how time consistent preferences incorporating subjective (i.e., micro) preference states can be constructed: let $s^i_t$ denote agent $i$’s current preference state, and $V^*_t(w^{i}_{t+1}, a_{t+1})$ denote the date $t + 1$ indirect utility function associated with the objective macro state, $a_{t+1}$, and the future possible subjective micro state $s^i_j$, $j \in 1, \ldots, n$; to consistently account for possible micro states, one can positively aggregate all the date $t + 1$ indirect utility functions given $a_{t+1}$ and indexed by subjective micro states. That is, instead of Eqn. (2) one writes:

$$U_t^{s^i_t}(c_t, w^{i}_{t+1}, a_{t+1}) = u_{s^i_t}(c_t) + \beta \varphi^{s^i_t}_t \left(V^{s^i_1}_t, \ldots, V^{s^i_n}_t\right)$$

where $s^i_t$ denotes the current (i.e., date $t$) preference state and $s^i_1, \ldots, s^i_n$ denote the possible date $t + 1$ preference states conditional on macro state $a_{t+1}$.\footnote{We could, and perhaps should, index the date $t + 1$ preference states with a time index (i.e., $t + 1$) and the agent’s identity (i.e., $j$). Since, however, we later assume that agents have the same possible set of preference states, we feel that the additional detail does not justify the resulting confounding notation. By contrast, referring to the current or realized preference state of the agent (i.e., $s^i_t$) is necessary, since different agents realize different taste shocks.} $\varphi^{s^i_t}_t$ is increasing in all its
arguments. Note that the subjective preference state affects attitudes towards consumption, inter-temporal substitution, and risk.

In what follows, we assume a particular form for \( \varphi^s_{it} \) that reduces to Eqn. (2) when there is only a single micro state. One advantage to our recursive form is parsimony, while the other is the obvious similarity with other forms in the literature (see, especially, Epstein and Zin (1989)); specifically, consider

\[
U^s_{it}(c_t, w^t_{t+1}, a^t_{t+1}) = u^s_{it}(c_t) + \beta v^s_{it}\left( \mathbb{E}^i\left[ v^s_{i't}(V^s_{i't+1}) \mid a_{t+1} \right] \right)
\]

The operator, \( \mathbb{E}^i\left[ \cdot \mid a_{t+1} \right] \) is a positive weighting function for the date \( t + 1 \) micro states (indexed by \( s' \in \{s'_1, s'_2, \ldots\} \)). It is reminiscent of an ‘expectation’ over micro states in that it is assumed to have the following properties:

\[
\mathbb{E}^i\left[ 1 + f(s') \mid a_{t+1} \right] = 1 + \mathbb{E}^i\left[ f(s') \mid a_{t+1} \right]
\]
\[
\mathbb{E}^i\left[ af(s') \mid a_{t+1} \right] = a\mathbb{E}^i\left[ f(s') \mid a_{t+1} \right]
\]

for any real-valued function, \( f(s') \), over the micro states, and

\[
\mathbb{E}^i\left[ f(s') \mid a_{t+1} \right] > \mathbb{E}^i\left[ g(s') \mid a_{t+1} \right]
\]

whenever \( f(s') \geq g(s') \) for each \( s' \) with strict inequality over a measurable set of micro states. In Eqn. (6) the indirect utility function for state \( s'_j \) is first converted to a consumption ‘certainty equivalent’ via the transformation, \( v^s_{i't}^{-1}(V^s_{i't+1}) \). The certainty equivalents are then aggregated by the weighting function, \( \mathbb{E}^i \). The result is then transformed back to a consumption utility equivalent via \( v^s_{it}(\cdot) \). Note that if there is only a single micro state, \( s' \equiv s^*_i \), Eqn. (6) reduces to Eqn. (2). There are several possibilities to consider:

1. \( \mathbb{E}^i \) is linear and the weights correspond to objective probabilities.
2. \( \mathbb{E}^i \) is linear but the weights are not ‘objective’ probabilities.
3. \( \mathbb{E}^i \) is non-linear (e.g., a Choquet integral) and may utilize ‘objective’ probabilities or otherwise.

The first case is valid when the agent’s tastes change, but the distribution of tastes is stationary; the second case is valid whenever tastes are not stationary - realizations of past tastes give little or no information about the future distribution of tastes (e.g., the agent’s tastes
change in different ways as she ages). In the last case the agent is simply not an expected utility maximizer when considering micro states - she responds differently to different sources of uncertainty (i.e., micro versus macro states). These distinctions are far from inconsequential. Later we show that there are two necessary conditions for calibrating the model to empirical stylized facts: (i) the market is incomplete, and (ii) either agents’ preferences with respect to unforeseen contingencies are non-expected utility, or they cannot deduce objective probabilities for the evolution of their own preferences (due perhaps to non-stationarity).

The analogous expression to Eqn. (3) is
\[
\sum_{a_{t+1} \subseteq a_t} \pi(a_{t+1}|a_t)U_t^q(c_t, w_{t+1}^i, a_{t+1}) = u_s^i(c_t) + \beta \sum_{a_{t+1} \subseteq a_t} \pi(a_{t+1}|a_t)\left( \mathbb{E}^i\left[ v_{s'}^{-1}\left( V_{s'^*}^t \right) \right| a_{t+1} \right) \right) \ (7)
\]

This state-dependent preference structure is consistent with the general recursive utility form,\(^{15}\) as well as the utility for flexibility discussed in the ‘unforeseen contingencies’ and ‘changing tastes’ literature.\(^{16}\) In particular, since the preferences are not necessarily ‘expected utility’ with respect to preference-states, this formulation belongs to the class of non-Expected Utility general equilibrium models.

3.1 Equilibrium

Although an agent, by introspection, can observe her own current preferences, we assume that her actions (e.g., investment and consumption) and wealth are unobservable to others. Under these assumptions, moral hazard prevents anyone from insuring herself against a micro state - the intuition is that no agent can hedge against a mood change that is not related to aggregate sentiments because her private actions are not observable. Markets can be said, however, to be pseudo-complete in a practical sense: contracts on all macro states are available and marketed. All publicly observed risk but not all private uncertainty can be hedged. In particular, agents whose tastes do not change can completely hedge their future consumption by purchasing a set of contingent claims at the first session of trade.

Just prior to the beginning of the \(t^{th}\) period, the date-\(t\) macro and micro states are revealed. Following the realization of preference and nature states, each agent begins period \(t\) with a claim to current consumption and a portfolio of claims to future (macro state


\(^{16}\)See, for example, Kreps (1979, 1991), Dekel, Lipman and Rustichini (2001) and Kraus and Sagi (2002).
contingent) amounts of the consumption good. The total value of each agent’s endowment of claims (in terms of current consumption) forms that agent’s budget. Following this but still during period \( t \), a market opens for trading current and future claims. If the markets are incomplete, in that there are unobserved micro states, agents will typically continue to trade after the initial round (date 1). By trading, agents can revise their holdings subject to the budget constraint by choosing a desired amount of consumption for period \( t \) and a set of state contingent claims for consumption in the remaining periods. After trade, agents consume their share of current production of the consumption good.

Before trading begins at date \( t \) and state \( a_t \), agent \( i \) has \( c_{t-1}^i(a_u) \) units of the state contingent claim that pays out in the event \( a_u \in \mathcal{A}_u \) (\( u \geq t \)). These are traded between the agents when markets subsequently open to achieve the new equilibrium allocations of state claims, \( c_t^i(a_u) \) for \( u > t \), plus a current consumption allotment, \( c_t^i \). It is important to note that, in the presence of non-degenerate micro states, the portfolio holdings and personal consumption for each agent will not be adapted to \( \mathcal{F} \) (hence the additional time-subscript). Equilibrium is achieved when at each date every agent trades to maximize her current period utility function from Eqn. (7) and the following market clearing conditions are met:

\[
C(a_u) = \begin{cases} 
\frac{1}{N} \sum_{i=1}^{N} c_t^i(a_u) & \forall \ u > t \\
\frac{1}{N} \sum_{i=1}^{N} c_t^i & \ u = t
\end{cases}
\]  

(8)

where \( C(a_t) \) is the economy’s per-capita aggregate consumption of the good produced in the event \( a_t \). Note that although \( c_t^i \) need not be adapted to the filtration of macro states, this is clearly required of the aggregate of all date \( t \) allocations.

**Definition 3.** An equilibrium is a quintuplet \( (\mathcal{F}, \pi(\cdot), \{\phi(\cdot)\}, \{C(\cdot)\}, \{e_t^i\}_{i=1}^{N}) \) where,

i) \( \{\phi(\cdot)\} \) is a complete set of state prices adapted to \( \mathcal{F} \).

ii) \( \mathcal{F} \) is a filtration of observable states given \( \phi(\cdot) \) and \( \pi(\cdot) \).

iii) \( \{C(\cdot)\} \) is an aggregate consumption process adapted to \( \mathcal{F} \).

iv) \( \{e_t^i\}_{i=1}^{N} \) is an initial endowment of state claims satisfying the market clearing condition (Eqn. (8)).

v) at each date every agent maximizes a recursive utility function of the form in Eqn. (7) by trading in \( \mathcal{F} \)-Arrow-Debreu state claims with state prices given by \( \{\phi(\cdot)\} \) and subject to her budget constraint.
vi) at each date agents’ consumption and portfolio holdings satisfy the market clearing equation, Eqn. (8).

There are two main differences between our equilibrium concept and the standard one. First, \( \mathcal{F} \) is determined endogenously and consistently with the set of state prices adapted to it; second, the evolution of any individual agent’s optimal consumption or portfolio holdings need not be adapted to \( \mathcal{F} \) (since an agent’s individual consumption may depend on both the macro event and her micro state).

At this stage, we do not have general results on the existence or efficiency of an equilibrium. We expect that a competitive equilibrium with macro-complete markets is not generally Pareto efficient relative to all possible macro-complete market equilibria. If agents are forced to trade in a different pattern than implied by a competitive equilibrium, it may be possible to reveal more private states and thereby introduce more macro states; the new hedging opportunities may make everyone better off. For now, we restrict ourselves to investigating the existence of an equilibrium in a parametric model.

### 3.2 Equilibrium with HARA-like Preferences

We now demonstrate the existence of an equilibrium under specific parametric assumptions. Our choice of parameterization is governed by economic considerations as well as those necessitated by the requirement of analytic tractability. To facilitate the transparency of the model we discuss the rationale behind the assumptions immediately following their statement. Assume that agent \( i \) has preferences given by Eqn. (7) with:

\[
\begin{align*}
    u_{s_{it}}(x) &= \frac{(s_{it}^x + 1)^{1-\gamma}}{s_{it}^{\gamma}(1-\gamma)} \Delta, \quad \gamma > 0, \quad \gamma \neq 1 \\
    v_{s_{it}}^{-1}(x) &= u_{s_{it}}^{-1}(x) + \frac{1}{s_{it}^{\gamma}}
\end{align*}
\]

(9)-(10)

All agents have utility of current consumption given by (9)-(10) with identical power, \( \gamma \), but possibly different realizations of \( s_{it} > 0 \).\(^{17}\) The final assumption needed concerns the nature of the weighting operator, \( \mathbb{E}^i \). We assume little about it save for the following:

\[
\mathbb{E}^i \left[ \frac{1}{s_{t+1}^{a_{t+1}}} \mid a_{t+1} \right] = \frac{1}{F(a_{t+1})s_{t}^{\gamma}}
\]

(11)

\(^{17}\)The case \( \gamma = 1 \) (i.e., log utility) can be treated with minor modification. The same is true of constant absolute risk aversion (exponential) utility.
where \( F(a_{t+1}) > 0 \) depends only on the macro state \( a_{t+1} \) and is identical across individuals.

Consider first the utility of consumption in (9). We note that its implied relative risk aversion for consumption at date \( t \) is

\[
\gamma \frac{s_i^t c}{1 + s_i^t c}
\]

where \( c \) is the amount consumed. Since \( \gamma \) is identical across agents and \( s_i^t \) is specific to agent \( i \), the changing tastes correspond to private shocks to relative risk aversion. Moreover, since \( s_i^t \) is assumed positive, \( \gamma \) acts as universal upper bound on the relative risk aversion of any individual. Another reason for choosing a quasi-homothetic utility for consumption is the well known simple aggregation properties that such a function affords when \( \gamma \) is identical across agents. Fortunately, this can hold even when inter-temporal utility is not of the standard separable form.\(^{18}\) It is possible to consider other quasi-homothetic forms; for instance, \( u_{s_i^t}(x) \sim (s_i^t x - 1)^{1-\gamma} \). The advantage to the latter is the familiar ‘threshold’ suggestive of linear habit formation (as used by Campbell and Cochrane (1999)). Moreover, in such a case one trades the upper bound on relative risk aversion for a lower bound. A distinct disadvantage is that, once aggregated, one has to impose additional exogenous assumptions on the consumption process so as to explicitly avoid crossing an aggregated ‘threshold’. Another reason we avoid the threshold form is to distinguish our work from linear habit formation in which some of the explanatory power depends on unrealistically large levels of relative risk aversion.

The constant \( \Delta \) in (9) represents the time interval between decisions; one can arrive at a continuous time limit by taking \( \Delta \to 0 \). The normalization of utility by \( \frac{1}{s_i^t} \) for our purposes is merely a convention; the optimal demand for consumption and securities would not change if we were to instead write \( u_{s_i^t}(x) = \frac{(s_i^t x + 1)^{1-\gamma}}{(1-\gamma)\Delta} \). The assumption over \( u_{s_i^t}^{-1}(x) \) is dictated by both parsimony and tractability - note that it introduces no additional parameters other than those already incorporated into \( u_{s_i^t}(x) \). Specifically, Eqn. (6) becomes

\[
U_{s_i^t}(c_t^i, w_{t+1}^i, a_{t+1}) = \frac{(s_i^t c_t^i + 1)^{1-\gamma}}{s_i^t(1-\gamma)} \Delta + \frac{1}{1-\gamma} \beta \mathbb{E}^i\left[ \left( \frac{s_i^t}{s_t^i} \right)^{\frac{1}{1-\gamma}} \left( (1-\gamma) V_{t+1}^{s_t^i + 1} \right)^{\frac{1}{1-\gamma}} \Big| a_{t+1} \right]^{1-\gamma}
\]

The above form is somewhat reminiscent of Epstein-Zin (1989) recursive utility. Note that in their formalism, the operator \( \mathbb{E}^i \) also need not correspond to expected utility. There are important differences, however. Our indirect utility is additionally weighted by the state-dependent term, \( \left( \frac{s_i^t}{s_t^i} \right)^{\frac{2}{1-\gamma}} \), and the aggregation is only over micro (i.e., preference) states

\(^{18}\)It is worth noting that by assuming all agents have exponential utility for consumption, private shocks to the personal discount factor, \( \beta \), can also be aggregated. Such shocks would enhance the explanatory power of the model at the cost of introducing additional degrees of freedom.
and not over consumption states. The additional weighting factor ensures a closed form solution.

\( E^i \) is an aggregation operator that depends on the agent’s preferences. It is consistent with a variety of hypotheses: for instance, agents can be probabilistically sophisticated (Machina and Schmeidler (1992)) or ambiguity averse (Gilboa and Schmeidler (1989)) about future risk tolerance. To help with interpretation, note that the absolute risk tolerance of consumption implied by (9) is \( \frac{1}{\gamma}(c + \frac{1}{s^i_{t+1}}) \). The change to risk tolerance brought about by a taste shock is \( \frac{1}{\gamma} \left( \frac{1}{s^i_{t+1}} - \frac{1}{s^i_t} \right) \). If \( E^i \) is a linear weighting scheme that coincides with objective probabilities for the evolution of tastes, then \( \frac{1}{\gamma s^i_t} \left( \frac{1}{F(a_{t+1})} - 1 \right) \) should be viewed as the forecast or objective expected value of the change in the agent’s risk tolerance due to taste shocks, while \( \frac{1}{\gamma} \left( \frac{1}{F(a_{t+1})} - 1 \right) \) is the forecast of the percentage change in a measure of risk tolerance (the \( \frac{1}{s^i_t} \)’s). Note that the latter is the same across all agents and therefore represents the average percentage change in the \( \frac{1}{s^i_t} \)'s. If, on the other hand, \( E^i \) is non-linear or non-probabilistic – an interpretation that is particularly valid if past shifts in taste are not useful for predicting future shifts, or if agents’ attitudes towards micro states are not ‘separable’ across states – then \( F(a_{t+1}) \) corresponds to a ‘preference distorted forecast’ of the agent’s relative risk tolerance in the next period. In particular, if \( \left( \frac{1}{F(a_{t+1})} - 1 \right) \) is larger than the average percentage change in the \( \frac{1}{s^i_t} \)'s, then the typical agent is ‘optimistic’ or ‘bullish’ in the sense that, on average, high risk tolerance micro states receive more weight in the typical agent’s preferences. A similar remark applies to ‘pessimistic’ or ‘bearish’ agents. In summary, the deviations of \( E^i \) from a linear and objective weighting scheme (which may not be available to agents) can be interpreted in terms of aggregate tendencies towards optimism or pessimism regarding future risk tolerance. We will shortly return to these interpretations.

On the practical side, (11) is a necessary assumption to guarantee that the indirect utility function has a stationary ‘power form’ mimicking that of the utility for consumption. Introducing heterogeneity to the drift \( F(a_{t+1}) \) entails two disadvantages: the aggregator \( v^{-1}_{s^i_t}(x) \) will have to be adjusted to ensure closed form portfolio choices, and the degrees of freedom introduced into the model will generally increase.

At date \( T \) each agent is assumed to have the following utility for consumption:

\[
U^{s^i_T}(c^i_T) = \frac{(s^i_T c^i_T + 1)^{1-\gamma}}{s^i_T (1 - \gamma)} \Delta
\]  

\( 19 \)This is only possible if agents have had enough time to ‘learn’ the evolution process of their own tastes, not to mention that the process can be ‘learned’ in the first place (i.e., a stationarity assumption).
The preference assumptions can be used to calculate agents’ optimal portfolios given a filtration of macro states, \( F \), and a complete set of state claims over the macro states.

**Proposition 1.** Under assumptions (9)-(11), an agent with wealth \( w^i_t \) at date \( t \) optimally consumes \( c^i_t \):

\[
c^i_t = \rho(a_t)^{-\frac{1}{\gamma}} w^i_t + \frac{\rho(a_t)^{-\frac{1}{\gamma}} - 1}{s^i_t}
\]

leading to a date \( t \) indirect utility function given by:

\[
V^*_t s^i_t(w^i_t, a_t) = \rho(a_t) z(a_t) \frac{(s^i_t \frac{w^i_t}{z(a_t)} + 1)^{1-\gamma}}{s^i_t(1-\gamma)}
\]

where \( z(a_T) = \Delta, \rho(a_T) = 1 \), while \( z(a_t) \) and \( \rho(a_t) \) are defined recursively via

\[
z(a_t) \equiv \Delta + \sum_{a_{t+1} \subseteq a_t} \frac{\phi(a_{t+1}|a_t)}{F(a_{t+1})} z(a_{t+1})
\]

\[
\rho(a_t)^{\frac{1}{\gamma}} z(a_t) \equiv \Delta + \sum_{a_{t+1} \subseteq a_t} \phi(a_{t+1}|a_t) \left( \frac{\phi(a_{t+1}|a_t)}{\beta \pi(a_{t+1}|a_t)} \right)^{-1/\gamma} \rho(a_{t+1})^{\frac{1}{\gamma}} z(a_{t+1})
\]

**Proof:** See appendix.

Agent \( i \)'s wealth is calculated as the present value of the state claims she owns at the beginning of period \( t \). Each of \( z(a_t) \) and \( \rho(a_t) \equiv \rho(a_t)^{\frac{1}{\gamma}} z(a_t) \) can be interpreted as the value of a unit coupon consol bond under an adjusted set of state prices. In particular, if \( F(a_{t+1}) \) is always identically equal to one, then \( z(a_t) \) is the price of a consol bond. Both \( z(a_t) \) and \( \rho(a_t) \) depend only on the macro states (i.e., the investment opportunity set). The micro state-dependence of optimal consumption and the indirect utility function is due to the explicit dependence on \( s^i_t \).
One can aggregate the marginal rates of substitution across agents to derive state prices:

**Proposition 2.** Under assumptions (9)-(11), if an equilibrium exists then state prices satisfy Eqn. (1) and

\[
\phi(a_{t+1}|a_t) = \begin{cases} 
\beta\pi(a_{t+1}|a_t)\rho(a_{t+1}) \left( \frac{G(t) w(a_{t+1})}{\pi(a_{t+1})} - \frac{1}{G(t) G(a_t + 1)} \right)^{-\gamma} & t < T \\
0 & t = T 
\end{cases}
\]  

(17)

where,

\[
\frac{1}{G(t)} = \frac{1}{N} \sum_{i=1}^{N} \frac{1}{s^i_t}
\]

(18)

and

\[
w(a_t) \equiv C(a_t)\Delta + \sum_{a_{t+1} \subseteq a_t} \phi(a_{t+1}|a_t)w(a_{t+1})
\]

(19)

**Proof:** See appendix.

\[ \square \]

\(w(a_t)\) is aggregate wealth: the present value of current and all future per-capita consumption. \(G(t)\) is a measure of aggregate risk aversion across investors and is the only quantity in Eqn. (17) that directly depends on the realization of investors’ utility functions (i.e., the \(s^i_t\)'s). Note also that prices do not depend on the distribution of initial endowments across agents.

**Theorem 1.** Assume (9)-(11) and that \(G(t)\) has a finite number of possible realizations at each date \(t\). Let \(\{e^i_0\}_{i=1}^{N}\) be the set of initial endowments, \(\mathcal{F}\) be the filtration of states generated by per-capita consumption at all dates and by realizations of \(G(t)\) for dates \(t \leq T\), and let \{\(\phi(\cdot)\}\} be the state price system in (17). Then for any set of market clearing initial endowments and almost every parameterization of \(F(a_{t+1})\), \((\mathcal{F}, \pi(\cdot), \{\phi(\cdot)\}, \{C(\cdot)\}, \{e^i_t\}_{i=1}^{N})\) is an equilibrium.

**Proof:** By definition, aggregate consumption and the realizations of \(G(t)\) are adapted to \(\mathcal{F}\). Defining \(\phi(a_s) \equiv \phi(a_s|a_1)\), a simple induction argument shows that \(\phi(\cdot)\) is adapted to \(\mathcal{F}\). Since the state prices result from aggregating over the optimal consumption-investment decisions of all agents and then applying the market clearing conditions, to prove \((\mathcal{F}, \pi(\cdot), \{\phi(\cdot)\}, \{C(\cdot)\}, \{e^i_t\}_{i=1}^{N})\) is an equilibrium we need only establish that for almost every parameterization of \(F(a_{t+1})\), every event in \(\mathcal{F}\) is observable.
To do this, first note that $F$ satisfies the second requirement of Definition 2. Now consider $a_t \neq a'_t \in \mathcal{A}_t$ such that $a_t \cup a'_t \subseteq a_{t-1}$ for some $a_{t-1} \in \mathcal{A}_{t-1}$. If the probability trees originating at $a_t$ and $a'_t$ are identical in both structure and assignment of aggregate consumption and state prices, then they also coincide on the evolution of $z(\cdot), \rho(\cdot)$ and $w(\cdot)$. In particular, Eqn. (17) implies that either $G(a_t) = G(a'_t)$ or $\phi(a_{t+1} | a_t)$ is independent of $G(a_t)$ for every $a_{t+1} \subseteq a_t$. If $G(a_t) = G(a'_t)$ then by the assumption that $F$ is generated by $C_t$ and $G_t$, it must be that $a_t = a'_t$ - a contradiction. On the other hand, for $\phi(a_{t+1} | a_t)$ to be independent of $G(a_t)$, it must be that $\frac{w(a_{t+1})}{z(a_{t+1})} F(a_{t+1}) = C(a_t)$ for every $a_{t+1} \subseteq a_t$. Since $F(a_{t+1})$ is a model ‘parameter’ (i.e., it is specified exogenously) and neither $w(a_{t+1})$ nor $z(a_{t+1})$ depends on it contemporaneously, this necessary condition for $a_t \neq a'_t$ is satisfied in a set of model parameterizations with measure zero.

The intuition behind the result is as follows: under almost every parameterization of $F(a_{t+1})$ all realizations of aggregate risk aversion can be inferred by their direct impact on state prices of future observable states. The proof of the theorem also establishes that in the few (i.e., zero measure) situations where this is not the case, state claims that pay in the following (i.e., date $t+1$) period are valued independent of current risk attitudes - see footnote 20. Thus we hasten to add that in addition to applying to almost all parameterizations of $F(a_{t+1})$, the Theorem applies to all economically compelling models for $F(a_{t+1})$.

Suppose that an equilibrium was conjectured in which the only contingent claims were those contracted on consumption states and $G(t)$. Theorem 1 can also be interpreted as providing sufficient conditions for such an equilibrium to be consistent with the requirement that claims are only available on demand and price related variables (as in Definition 2). The variable $G(t)$ itself is not observed but is consistently inferred from prices. The theorem also makes it clear that such a conjecture is not inherently benign: there are instances, albeit not economically meaningful ones, in which not every distinct realization of $G(t)$ can be said to be observable. In other words, use of the machinery developed in the beginning of this section is necessary to be sure that the state price system derived from the parametric assumptions is indeed consistent with the intuition that contingent claims should only be available for observable states.

From here on assume that for every $a_t \in \mathcal{A}_t$ there is some $a_{t+1} \subseteq a_t$ with $\frac{w(a_{t+1})}{z(a_{t+1})} F(a_{t+1}) \neq 0$. This is equivalent to requiring $\phi(a_{t+1} | a_t) = \beta F(a_{t+1})^\gamma \rho(a_{t+1})^\pi(a_{t+1} | a_t) -$ essentially a requirement of risk neutrality since the former is the expression for the pricing kernel when $G(t) = 0$. 

\footnotetext[20]{This is equivalent to requiring $\phi(a_{t+1} | a_t) = \beta F(a_{t+1})^\gamma \rho(a_{t+1})^\pi(a_{t+1} | a_t)$ - essentially a requirement of risk neutrality since the former is the expression for the pricing kernel when $G(t) = 0$.}
Thus the proof of Theorem 1 also establishes that \( \{G(t), C(t)\}_{t=1}^T \) spans a macro state filtration. Alternatively, one can interpret this to say that \( G(t) \) is observable at date \( t \) through state prices.\(^{21}\) We can therefore refer to realizations of \( G \) as \( G(a_t) \). In essence, all information about individuals’ tastes washes out upon aggregation, save for the average risk tolerance \( \frac{1}{G(a_t)} \). It is only through this factor that correlated preference states can give rise to macro-economic or priced states, and these must be part of the partition, \( \mathcal{F}_t \). For example, suppose consumption is deterministic, but agents’ tastes are random; then the macro-state partition, \( \mathcal{F}_t \), consists of all possible configurations of aggregate risk tolerance, \( \frac{1}{N} \sum_{i=1}^N \frac{1}{s_i^t} \). Clearly, this filtration need not be as large as the product filtration of individuals’ preference states. In fact, given enough agents with independent preference shocks to their risk aversion, \( \frac{1}{N} \sum_{i=1}^N \frac{1}{s_i^t} \) can be degenerate. In general, however, the market here is incomplete in that not all preference states are revealed (i.e., there can be some dispersion in individuals’ micro states that fails to show up in macro prices). Moreover, in a large economy one need not be concerned with a precise model of the evolution of individual taste shocks. The only economically relevant quantity is \( G(a_t) \).

There is one final issue to address. Can anything be said about the evolution of \( G(a_t) \)? The answer depends on the meaning ascribed to the micro-state weighting function, \( E^i \). Suppose, for instance, that the operator corresponds to an expectation using objective probabilities. Since \( \frac{1}{G(a_{t+1})} \), does not depend on micro states, it must be that

\[
\frac{1}{G(a_{t+1})} = E^i \left[ \frac{1}{G(a_{t+1})} \mid a_{t+1} \right] = E^i \left[ \frac{1}{N} \sum_{i=1}^N \frac{1}{s_i^{t+1}} \mid a_{t+1} \right] = \frac{1}{N} \sum_{i=1}^N E^i \left[ \frac{1}{s_i^{t+1}} \mid a_{t+1} \right] = \frac{1}{N} \sum_{i=1}^N \frac{1}{s_i^{t+1} F(a_{t+1})} = G(a_t) F(a_{t+1})
\]

where the equality in the second line is due to the assumption that \( E^i \) is both linear and objective (i.e., independent of the identity of agent \( i \)). Thus, the assumption of objective and linear weights in \( E^i \) completely pins down the evolution of \( G(a_t) \) in terms of existing model parameters. By contrast, if \( E^i \) is not an objective expected utility functional (i.e., if beliefs about individual future tastes are either not probabilistic, not uniform or not separable from

\(^{21}\)We have assumed that the number of macro states is finite. An extension to the case of a continuum of states or one where \( T \to \infty \) introduces additional technical difficulties that we do not want to address in this paper. If the number of agents is large (or infinite), only mild conditions need be placed on the \( s_i^t \)’s so that the number of configurations of \( G(a_t) \) is small (e.g., a ‘law of large numbers’ in the case where \( N \) is infinite).
utility) then it need not be the case that \( G(a_{t+1}) = G(a_t)F(a_{t+1}) \). Since we know that \( G(a_{t+1}) \) is \( \mathcal{F} \)-measurable, we can write, without loss of generality:

\[
G(a_{t+1}) = G(a_t)F(a_{t+1})e^{-\delta(a_{t+1})} 
\]

(20)

Where \( \delta(a_{t+1}) \) has the interpretation of measuring the degree of departure from objective expected utility in the agent’s attitudes towards micro states. We stress that there is nothing inherently wrong or irrational about non-expected utility preferences so long as they are dynamic programming consistent (which is the case here by construction). If \( \delta > 0 \), then \( \frac{1}{F(a_{t+1})} \) can be viewed as being too small to be a forecast of how taste shocks affect aggregate risk tolerance, \( \frac{1}{G(a_{t+1})} \) also represents the typical agent’s ‘preference-distorted forecast’ of how unforeseen contingencies may affect her risk tolerance, a behavioral interpretation of \( \delta > 0 \) is ‘pessimism’ or ‘bearishness’ in that the typical person underweights high risk tolerance preference states relative to expected utility. Likewise, \( \delta < 0 \) can be interpreted as an ‘aggregate state of exuberance’ or ‘bullishness’. Note that while the average agent can be said to have a distorted forecast when \( \delta \neq 0 \), this cannot be said of any particular agent. In fact, we made no specific assumptions about the evolution of any individual taste shocks.\textsuperscript{22}

We can now summarize our results in the following:

**Theorem 2.** Assume (9)-(11) and that \( G(t) \) has a finite number of possible realizations at each date \( t \). Let \( \{e^i\}_{i=1}^N \) be the set of initial endowments, \( \mathcal{F} \) be the filtration of states generated by realizations of \( G(t) \) for \( t \leq T \) and by per-capita consumption at all dates, and let \( \{\phi(\cdot)\} \) be the state price system in (17). Finally assume that for every \( a_t \in A_t \) there is some \( a_{t+1} \subseteq a_t \) with \( \frac{w(a_{t+1})}{z(a_{t+1})} F(a_{t+1}) \neq C(a_t) \). Then \( \left( \mathcal{F}, \pi(\cdot), \{\phi(\cdot)\}, \{C(\cdot)\}, \{e^i\}_{i=1}^N \right) \) is an equilibrium and for any \( t < T \), \( \frac{1}{G(a_t)} \equiv \frac{1}{N} \sum_{i=1}^N \frac{1}{a_i} \) and \( C(a_t) \) are adapted to the \( \mathcal{F}_t \) filtration of macro states. The pricing kernel is given by:

\[
\phi(a_{t+1}|a_t) = \pi(a_{t+1}|a_t)\beta \left( \frac{G(a_{t+1})C(a_{t+1}) + 1}{G(a_t)C(a_t) + 1} \frac{G(a_t)}{G(a_{t+1})} \left[ 1 - \frac{1 - e^{-\delta(a_{t+1})}}{G(a_{t+1})} \right] \right)^{-\gamma} 
\]

(21)

where \( \delta(a_{t+1}) \equiv \ln F(a_{t+1}) + \ln G(a_t) - \ln G(a_{t+1}) \) is adapted to \( \mathcal{F} \).

**Proof:** Only Eqn. (21) needs derivation. This is done in the Appendix. \( \square \)

There are several observations that remain to be made.

\textsuperscript{22}While Equation (11) may be interpreted as a constraint on the evolution of taste shocks, it can also be viewed as an assumption over preferences.
**Remark 1.** If \( \delta(a_{t+1}) = 0 \), corresponding to expected utility with objective probabilities, then the pricing kernel only features the term \( \frac{G(a_{t+1})C(a_{t+1})+1}{G(a_t)C(a_t)+1} \). This is exactly the result that would be obtained if we postulated a representative agent with time-separable and state-dependent utility function,\(^{23}\)

\[
U_t = E_t \left[ \sum_{j=0}^{\infty} \beta^j \left( \frac{C(a_{t+j}) + \frac{1}{G(a_{t+j+1})}}{1 - \gamma} \right)^{1-\gamma} \right].
\]

\(1/G(a_t)\) above is a factor that summarizes the impact of unforeseen contingencies on aggregated tastes. The impact is identical to that of a ‘substitute’ for current consumption. A similar ‘substitution’ factor can arise due to the formation of habit,\(^{24}\) or due to the presence of consumption bundle components that are not captured when constructing an aggregate consumption index.

We stress that a true representative agent (i.e., in the sense of efficient allocations) does not necessarily exist for this economy when \( \delta(a_{t+1}) = 0 \). That is because the market may still be incomplete in the sense that each agent may not be able to hedge all of her taste shocks. As far as the macro-economy is concerned, however, such incompleteness is immaterial whenever \( \delta(a_{t+1}) = 0 \).

**Remark 2.** Note that the relative risk aversion of this representative agent in the last remark (with respect to date \( t \) consumption) is

\[
R_R(a_t) = \frac{\gamma G(a_t)C(a_t)}{G(a_t)C(a_t)+1}
\]

and the pricing kernel can be written as

\[
\phi(a_{t+1}|a_t) = \pi(a_{t+1}|a_t)\beta \left( \frac{C(a_{t+1})}{C(a_t)} \right)^{-\gamma} \left( \frac{R_R(a_{t+1})}{R_R(a_t)} \right)^{\gamma}
\]

(22)

All but the last term are standard, while the last reflects the impact of time varying risk aversion. In particular, the case \( \frac{G(a_{t+1})}{G(a_t)} = F(a_{t+1}) \equiv \frac{C(a_t)}{C(a_{t+1})} \forall t \) reduces to the Lucas (1978) model. Moreover, if \( \frac{R_R(a_{t+1})}{R_R(a_t)} = \left( \frac{w(a_{t+1})}{w(a_t)} \right)^{\eta} \), for some constant \( \eta \), then the pricing kernel is equivalent to that of Epstein and Zin (1989).

\(^{23}\)Papers postulating a representative agent with state dependent preferences include Gordon and St-Amour (2000), Mehra and Sah (2002), Danthine et. al. (2002) and Melino and Yang (2003).

\(^{24}\)One can nearly recover an Abel-like (1990) external habit formation utility for the representative agent by assuming that the habit level is inversely proportional to \( G(a_t) \). If \( u_{a_i}(x) \sim (s_i x - 1)^{1-\gamma} \) instead, then one can recover the Campbell and Cochrane (1999) model.
Note that relative risk aversion for consumption may be small in level (e.g., if $G(a_t)C(a_t)$ is small in level), and yet the standard deviation of state prices can be high (if the relative volatility of $R_R(a_t)$ is high).25 Thus the volatility of relative risk aversion can contribute to the equity premium equally or even a great deal more than the level of relative risk aversion.

Remark 3. The case $\delta(a_{t+1}) \neq 0$ corresponds to non-expected utility preferences (or to expected utility without objective probabilities) with respect to micro-states. The result is a pricing kernel that is explicitly wealth dependent: both the consumption wealth ratio and the ‘term-structure’ variable, $z(a_t)$, become explicit components of the kernel. Moreover, in this case one cannot derive state-prices from the marginal utility of a representative agent.

If there are no micro states (i.e., all preference states are revealed) then the weighting operator, $E'$, is degenerate and consistent with expected utility, and therefore perforce $\delta(a_{t+1})$ must be zero. A direct implication is that the presence of a non-trivial $\delta(a_{t+1})$ is sufficient to imply the presence of micro state dispersion in agents’ preferences - i.e., markets are incomplete. Micro states, in this case, do not show up in a verifiable manner and are therefore not priced and cannot be hedged. However, their impact can be felt in the economy (through a non-zero $\delta(a_{t+1})$).

An important conclusion is that it is not merely the presence of incompleteness in the market that contributes to the more exotic aspects of the pricing kernel. Incompleteness is necessary, but an additional and crucial requirement is that un-hedgeable states enter agents’ preferences in a non-standard (i.e., non-Expected utility) manner.

Remark 4. Note that state prices increase with $\delta(a_{t+1})$. This is consistent with the interpretation of $\delta(a_{t+1}) > 0$ as signalling aggregate ‘pessimism’ and $\delta(a_{t+1}) < 0$ as corresponding to aggregate ‘exuberance’.

Remark 5. Does the state price system exist in the limit $T \to \infty$? The answer to this question depends on specific values assigned to the model parameters. Recall that even the Lucas model is not defined as $T \to \infty$ if certain parametric restrictions are not satisfied. In the next section we explicitly address this issue whenever it arises. To be sure, we consider the equilibrium well defined if state prices are stationary and key economic variables (namely, aggregate wealth and the ‘console’ bonds, $z(a_t)$ and $\hat{\rho}(a_t)$) are finite.

In summary, the price of risk in this economy is determined by the dynamics of three

\[25\text{Unless the process for } R_R(a_t) \text{ exhibited a high degree of mean reversion, there is no guarantee of maintained low relative risk aversion. With sufficiently high mean reversion, however, shocks to relative risk aversion could be large in percentage terms, while the overall level always kept low.}\]
factors: per capita aggregate consumption, a measure that aggregates the ‘utility states’ across agents (i.e., $G(a_t)$), and a variable that jointly measures the presence of market incompleteness plus the departure of preferences from expected utility theory (i.e., $\delta(a_t)$).

4 Asset Pricing

Potentially, the pricing kernel in Eqn. (21) can describe anything, so long as the random variables $G(a_t)$ and $\delta(a_t)$ are chosen appropriately. This is because one of the effects of these variables, if their innovations are not perfectly correlated with the consumption process, is to ‘add’ new priced states to the economy. Such freedom is actually a burden unless there is some economic guideline for deciding what macro-economic variables other than aggregate consumption affect prices. To deal with this difficulty, the literature has implicitly set a standard for consumption based models of time varying risk premia: innovations to the macroeconomic variables should be spanned by shocks to consumption. In other words, the only news that is priced is news about changes to aggregate consumption. This rules out injecting new states by introducing variables that are unrelated to aggregate consumption. Consumption based asset pricing, however, has disadvantages.

The observed Sharpe ratio is large. If shocks to consumption growth are the sole determinants of risk premia, even mild assumptions over heterogeneity in risk aversion would suggest the existence of profitable hedging instruments for aggregate consumption, or at least GDP. Such hedging markets do not exist, in stark contrast with the great variety of risks for which hedging markets do exist, and the hedging instrument that do exist are poorly correlated with consumption shocks. Moreover, there is no evidence that the market price of risk for *empirically priced* factors (most of which can be hedged) arises due to covariance with aggregate consumption (e.g., Chen, Roll and Ross (1986), and a more recent paper by Duffee (2002)); equivalently, there is little proof that aggregate consumption performs well in cross-sectional asset pricing tests.

In the next two subsections we explore some implications of non-consumption related factors. In particular, we assume, consistent with stylized facts, that changes to aggregate consumption have little pricing implication. At this stage, our purpose is to achieve an understanding of basic theoretical properties of the model and how they might relate to the data; given this, we can hope to elsewhere pursue a true econometric test of a sensibly conjectured set of Euler equations. Section 4.1 demonstrates circumstances under which
$G(a_t)$ can actually be proxied by a power function of the wealth consumption ratio, thus making a case for generically incorporating such a variable explicitly in the pricing kernel even if one does not know how $G(a_t)$ fundamentally arises. In Section 4.2 we study a simple two state model that captures many of the macroeconomic stylized facts using a two state non-consumption variable. An important and robust result of the analysis is that, although we may not know the source of fluctuations in $G(a_t)$, its dynamics cannot be completely arbitrary. There are two basic messages from the analysis: (i), whereas changes to consumption are not particularly relevant for pricing risk, the wealth consumption ratio - a forward looking measure of preferences for consumption - is important, and (ii) to fit the stylized facts additional factors cannot be entirely divorced from aggregate consumption. In particular, shocks to $C(a_t)G(a_t)$ must be correlated with shocks to the expected growth rate of aggregate consumption. Moreover, we argue that the stylized facts force $\delta(a_{t+1}) \neq 0$, thus both incomplete markets and non-standard preferences are required, at least in our model.

### 4.1 Aggregate Risk Aversion and the Wealth-Consumption Ratio

If $G(a_t)$ is not entirely dependent on consumption states then what does it depend on? In this subsection we argue that, at least under some circumstances, the wealth-consumption ratio, $f(a_t) \equiv \frac{w(a_t)}{C(a_t)}$, is a monotonically decreasing function of $G(a_t)C(a_t)$. Moreover, if the volatility of $G(a_t)C(a_t)$ is not too large, then aggregate risk aversion satisfies an inverse power law:

$$\frac{G(a_t)C(a_t)}{G(a_t)C(a_t) + 1} \propto f(a_t)^{-\frac{1}{\gamma}}$$

While some of the approximations and assumptions are ‘crude’ we nevertheless believe that they illustrate the punch line: although we may not be able to specify a priori non-consumption based variables, their presence are felt in derivative macroeconomic variables such as the wealth-consumption ratio.

Suppose that $\delta$ in (21) is zero. One can therefore describe the economy via a representative agent (see Remark 2) whose relative risk aversion is

$$R_R(a_t) \equiv \gamma \frac{G(a_t)C(a_t)}{G(a_t)C(a_t) + 1}$$

In what follows we wish to assume that $R_R(a_t)$ is a mean stationary Markov process that can be well approximated as AR(1). In turn, this requires that $G(a_t)C(a_t)$ is itself mean stationary and is nearly always of order 1 or less. One interesting limit is when $G(a_t)C(a_t)$
is much smaller than 1 – in which case, \( R(a_t) \approx \gamma G(a_t) C(a_t) \). Note that this limit corresponds to a representative agent that is not very risk averse with respect to variation in consumption, which is consistent with the empirical fact that there are no active markets for hedging aggregate consumption risk nor a significant risk premium associated with aggregate consumption. In this limit, the agent’s relative risk aversion with respect to changes in \( 1/G(a_t) \), is roughly \( \gamma \),\(^{26}\) and thus his hedging concerns are mostly with \( G(a_t) \).

The next result relates \( G(a_t) \) to observable variables whether or not \( G(a_t) \ll 1 \).

**Lemma 4.1.** Assume \( \delta \) in (21) is zero and let \( \beta \equiv e^{-r_0 \Delta} \) for some constant \( r_0 > 0 \). Assume that in the limit \( T \to \infty \) and \( \Delta \to 0 \), \( x(a_t) \equiv \ln R(a_t) \) can be well approximated as an Ornstein-Uhlenbeck process: \( dx_t = \kappa_x (x_t - x_{\mu_x}) dt + \sigma_x dW_t \), with \( dW_t \) a Wiener variate. Then if \( \gamma = 1 \), the wealth-consumption ratio, \( f(a_t) \equiv \frac{w(a_t)}{C(a_t)} \), can be approximated as:

\[
f(x_t) \approx \frac{1}{\kappa_x} \int_0^1 \frac{e^{\frac{r_0}{\kappa_x}} - 1}{y \kappa_x} e^{-(x_t - \mu_x)(1-y) + \sigma_x^2 \frac{1-y^2}{4 \kappa_x}} dy
\]  

**Proof:** See Appendix □

The first and most important observation follows immediately from the Lemma. Note that \( f(x_t) \) is monotonically decreasing in \( x_t \). Thus \( G(a_t) \) can be ‘backed out’ of wealth-consumption ratio data. In other words, one does not have to know the details of non-consumption based variables. These show up in aggregated quantities like the wealth-consumption ratio. Moreover, if one is willing to make parametric assumptions, as is done in the Lemma, then it is possible to ‘back out’ the underlying economic shocks.

The integral in (23) can be further approximated as

\[
f_t \approx \frac{e^{-(x_t - \mu_x)}}{r_0} \left( 1 + O\left( \frac{r_0}{\kappa_x} \right) + O\left( \frac{\sigma_x^2}{2 \kappa_x} \right) \right)
\]

where \( O(\epsilon) \) corresponds to polynomial terms in \( x_t - \mu_x \) that have coefficients of order \( \epsilon \) or smaller. Thus if the unconditional variance of \( x_t \) is small (i.e., \( \frac{\sigma_x^2}{2 \kappa_x} \ll 1 \)), and agents’ personal discount rate is small compared to the half-life of fluctuations in relative risk aversion, then \( f(a_t) \) is approximately inversely related to aggregate relative risk aversion, \( R_R(a_t) \). Note that, in principle, the validity for these approximations can be backed out of the wealth

---

\(^{26}\)The representative agent’s relative risk aversion with respect to changes in \( \Gamma(a_t) \equiv 1/G(a_t) \) is defined to be

\[
-G(a_t) \frac{\partial^2 U_t}{\partial \Gamma(a_t)^2} = \gamma \frac{1}{C(a_t)G(a_t) + 1}
\]
consumption ratio. Assuming $C(a_t)$ can be approximated by geometric Brownian motion, a similar yet more tedious calculation leads to $f_t \approx Ae^{-\gamma(x_t - \mu_x)}$, for some constant $A$. Within the approximations considered here the pricing kernel becomes,

$$
\phi(a_{t+1}|a_t) \approx \pi(a_{t+1}|a_t)\beta \left( \frac{C(a_{t+1})}{C(a_t)} \right)^{-\gamma} \frac{f(a_t)}{f(a_{t+1})}
$$

(24)

Note that a corollary of the model described above is that the wealth-consumption ratio must be time varying. If consumption shocks have little pricing implication, then the opposite will be true for shocks to the wealth-consumption ratio. Equation (24) characterizes a simple and testable asset pricing model with the same number of parameters as the standard Lucas Model with constant relative risk aversion.\footnote{We should add the following observations about stationarity of the model: although we have not explicitly demonstrated that the limiting expressions for portfolio choice, bond prices, etc., are convergent, it is a simple matter to show this for $\gamma$ in some bounded interval that includes $\gamma = 1$.} Unfortunately, the model is guaranteed to be rejected because the required high volatility in the wealth-consumption ratio is not only counterfactual, but will lead to large fluctuations in the risk-free rate. As we show in the next section, the problem can be traced to having assumed standard expected utility in how agents treat micro states. In other words, one can hope to alleviate this by setting $\delta(a_{t+1}) \neq 0$.

Regardless, the above argument is simply a motivating illustration of the fact that $G(a_t)$ need not be perfectly adapted to the aggregate consumption filtration, and yet ‘observed’ in the wealth-consumption ratio. Moreover, it lends additional theoretical support to the direct use of Lettau and Ludvigson’s (2001ab) CAY variable as a pricing factor.

### 4.2 A Simple Parameterization

In the previous section we motivated the notion that aggregate relative risk aversion is inversely related to the wealth consumption ratio. We now consider another simple model where such a relationship is true. Before proceeding, define $X(a_t) \equiv G(a_t)C(a_t)$. Note that $X(a_t)$ is a monotonic transform of aggregate relative risk aversion when a representative agent exists - see Remark 2.

**Assumption 1.** Aside from aggregate consumption, the only other macro-state variable is $X(a_t)$. Realizations of $X(a_t)$ follow a Hamilton (1989) style two-state Markov switching process and are labeled by $s_t \in \{+, -\}$. 

27
Recall that \( a_t \) corresponds to a unique path in the state-tree generated by \( C(a_t) \) and \( G(a_t) \) (or, alternatively, \( C(a_t) \) and \( X(a_t) \)). The current markov state, \( s_t \), reflects part of the information or component of the path \( a_t \). Since \( X(a_t) \) only depends on this component, we abuse notation by writing \( X(s_t) \) instead of \( X(a_t) \). We stress that \( X(s_t) \) cannot be directly observed - its variation is due to taste shocks or unforeseen contingencies that are correlated across agents. We cannot say what it is that causes such aggregate shifts, but if they exist we can infer them from prices. We will soon be led to suggest, however, that, as with other models with time varying risk aversion (e.g., habit formation), changes in \( X(s_t) \) are associated with the business cycle.

**Assumption 2.** *Consumption growth from date \( t \) to date \( t + 1 \) can be written as*

\[
\frac{C(a_{t+1})}{C(a_t)} = \varepsilon_{t+1} e^{\mu(s_t)}
\]

*where \( \varepsilon_{t+1} \) is an iid random variable with mean 1 and statistically independent of realizations of \( s_{t+1} \), and \( \mu(s_t) \) is a, potentially, state dependent trend.*

The second assumption allows for two important properties: (i) conditional on date \( t - 1 \) information, \( X(s_t) \) and \( C(a_t) \) are statistically independent, while (ii) unconditionally, \( X(s_t) \) and \( C(a_t) \) may be correlated.

**Assumption 3.** *The pricing kernel takes the form given in Eqn. (21):*

\[
\phi(a_{t+1}|a_t) = \pi(a_{t+1}|a_t) \beta \left( \frac{C(a_{t+1}) X(s_{t+1}) + 1}{C(a_t)} X(s_t) + 1 \right) \left[ 1 - \frac{1 - e^{-\delta(s_{t+1}|s_t)}}{X(s_{t+1}) f(a_{t+1}) z(a_{t+1}) + 1} \right]^{-\gamma}
\]

*with \( \delta(a_{t+1}) \) a function of \( s_{t+1} \) and \( s_t \) only.*

As in Section 4.1, \( f(a_t) \equiv \frac{w(a_t)}{C(a_t)} \) is the wealth-consumption ratio. We stress that \( w(a_t) \) is to be viewed as aggregate wealth in the sense of Lettau and Ludvigson (2001b). In particular, changes in the log of aggregate wealth are not to be confused with the return on the stock market as in some asset pricing models. To model stocks we will consider separate assumptions about dividends and calculate their present value. Assumption 3 requires the pricing kernel to be history independent, and in particular, requires \( \delta(a_{t+1}) \) to be independent of contemporaneous consumption shocks. This is both to simplify the analysis, as well as to help differentiate our approach from the literature on habit formation.
4.2.1 Basic Properties of the 2-State Model

Holding $s_t$ constant, the random walk assumption for aggregate consumption growth implies that the pricing kernel at date $t$ does not depend on the current or previous levels of consumption. In particular, it is readily verifiable from the definitions of $f(a_t)$ and $z(a_t)$ that they are functions of $s_t$ only.

The pricing kernel can be expressed as a product of two independent components

$$
\phi(a_{t+1}|a_t) = \text{Prob}(\varepsilon_{t+1} = \varepsilon)\varepsilon_{t+1}^{-\gamma} \times P(s_{t+1}|s_t)K_{s_{t+1}, s_t}
$$

where, $\text{Prob}(\varepsilon_{t+1} = \varepsilon)$ is the date-$t$ conditional probability that $\varepsilon_{t+1}$ realizes the value $\varepsilon$, $P(s_{t+1}|s_t)$ is the transition probability from the state $s_t$ to $s_{t+1}$, and

$$
K_{s_{t+1}, s_t} \equiv \beta \left( e^{\mu(s_t)} \frac{X(s_{t+1}) + 1}{X(s_t)} + 1 \frac{X(s_t)}{X(s_{t+1})} \right) X(s_{t+1}) \frac{(s_{t+1})}{X(s_{t+1})} + 1 \right)^{-\gamma}
$$

Naturally, $\pi(a_{t+1}|a_t)$ is the product of the two conditional probability distributions.

To verify that the model is well defined with an infinite horizon one has to ensure that state prices are finite and that agents’ portfolio choices correspond to interior solutions. For this it is sufficient to demonstrate that $f(s_t)$ and $z(s_t)$ are finite and strictly positive.\(^{28}\) In particular, from Eqns. (15) and (19), it is simple to show that these variables must satisfy:

$$
z(s_t) = \Delta + E \left[ \varepsilon_{t+1}^{-\gamma} \right] \sum_{s_{t+1}} P(s_{t+1}|s_t) e^{\mu(s_t)} K(s_{t+1}, s_t) \frac{X(s_t)}{X(s_{t+1})} e^{-\delta(s_{t+1}|s_t)} z(s_{t+1}) \tag{25}
$$

$$
f(s_t) = \Delta + E \left[ \varepsilon_{t+1}^{-\gamma} \right] \sum_{s_{t+1}} P(s_{t+1}|s_t) e^{\mu(s_t)} K(s_{t+1}, s_t) f(s_{t+1}) \tag{26}
$$

where $E[\cdot]$ denotes an expectation. Each of these identities describes two equations (one each for $s_t = \pm$). In particular, if $\delta \neq 0$, the equations for $f(s_t)$ and $z(s_t)$ are coupled and non-linear. We will explicitly verify the stationary equilibrium restriction on $z$ and $f$ when calibrating the model. Finally, the one-period real interest rate is given by

$$
\frac{1}{R_f(s_t)} \equiv E \left[ \varepsilon_{t+1}^{-\gamma} \right] \sum_{s_{t+1}} P(s_{t+1}|s_t) K(s_{t+1}, s_t) \tag{27}
$$

\(^{28}\)From Equation (43), $\hat{\rho}(s_t) \equiv \hat{\rho}^{\frac{1}{2}}(s_t)z(s_t)$ can be calculated as $\frac{X(s_t)f(s_t) + z(s_t)}{X(s_t)+1}$, which is finite and strictly positive if $f(s_t)$ and $z(s_t)$ are as well.
4.2.2 Stylized Facts and Implications for the Two State Model

Our intention is to examine how the simple model of Assumptions 1-3 is constrained by stylized macroeconomic facts. In so doing we hope to infer how the model can be more generally constrained and whether it is worthwhile exploring it in more detail both as a theoretical and econometric tool:\textsuperscript{29}

Assumption 4.

\textit{i) According to the NBER, the typical duration of a U.S. recession is between 4-6 quarters (post-war recessions have been shorter). Typical expansion duration is between 9-17 quarters (post-war expansions have been longer).}

\textit{ii) The unconditional mean of the annual maximum Sharpe ratio is between 0.33-0.66 (half that number per quarter). The maximum Sharpe ratio is counter-cyclical (Campbell (2002)) and is close to twice as much during recessions as during expansions (Lettau and Ludvigson (2001b)).}\textsuperscript{30}

\textit{iii) The unconditional mean for U.S. quarterly consumption growth is around 0.5\% and the standard deviation is between 0.5-1.5\% (Campbell (2002)).}

\textit{iv) The risk premium on a portfolio that mimics aggregate consumption growth is not statistically different from zero (see the argument in the introductory part of Section 4).}

Assumption 4 is not comprehensive, but it is sufficient for strong model constraints. Later, when we calibrate the model, we will consider other stylized facts as needed.

The two-state nature of the model implies that any two variables that depend only on $s_t$ will be positively or negatively \textit{perfectly} correlated. Thus, it is impossible to accommodate the fact that both real interest rates and the wealth-consumption ratio vary \textit{and} have low correlation (i.e., we cannot accommodate realistic time variation in both variables). Our compromise solution is to stick to the two state model and set the real rate to be constant\textsuperscript{29,30}

\textsuperscript{29}The ‘stylized’ facts strongly depend on the time period in which they are documented. For instance, Campbell (2002) reports that the mean and standard deviation for the real U.S. risk free rate was, respectively, 1\% (2\%) and 2\% (9\%) for the period 1947-1996 (1891-1995).

\textsuperscript{30}The factor of two change is based on, (i) Lettau and Ludvigson (2001b) who report that a one standard deviation change in their predictive variable, CAY, leads to a 220 basis point change in expected excess returns, and (ii) on Campbell (2002) who notes that conditional volatility does not vary much over quarterly horizons.
and between 0.25 - 0.75%.  

The first result establishes that, although we have said nothing about what causes variation in $X(s_t)$, time-variation in the wealth-consumption ratio implies a non-trivial relationship between $X(s_t)$ and consumption growth. In particular, the trend component in assumption 2 is different in the two $X(s_t)$ states.

**Theorem 3.** Given Assumptions 1-3, if $\mu(+) = \mu(-)$ and interest rates do not vary with $s_t$ then the wealth-consumption ratio, if an equilibrium exists, is constant.

**Proof:** Suppose an equilibrium exists. Then $f(s_t)$ is finite for $s_t = \pm$. With constant interest rates, iid assumption over $\varepsilon_\tau$, and state independent $\mu(s_\tau) \equiv \mu$, Eqn. (27) gives:

$$E\left[\varepsilon_{\tau+1}^{1-\gamma}\sum_{s_{\tau+1}} P(s_{\tau+1}|s_\tau)e^{\mu K(s_{\tau+1}, s_\tau)}\right] = \frac{\theta}{R_f}$$

for some constant $\theta$ and any date $\tau$. Eqn. (26) can be iterated to give

$$f(s_t) = \Delta + E\left[\varepsilon_{t+1}^{1-\gamma}\sum_{s_{t+1}} P(s_{t+1}|s_t)e^{\mu K(s_{t+1}, s_t)}\Deltaight]$$

$$+ E\left[\varepsilon_{t+1}^{1-\gamma}\sum_{s_{t+1}} P(s_{t+1}|s_t)e^{\mu K(s_{t+1}, s_t)}E\left[\varepsilon_{t+2}^{1-\gamma}\sum_{s_{t+2}} P(s_{t+2}|s_{t+1})e^{\mu K(s_{t+2}, s_{t+1})}\Deltaight] + \ldotsight]$$

Substituting Eqn. (28) gives,

$$f(s_t) = \Delta + \frac{\theta}{R_f}\Delta + \frac{\theta}{R_f}\frac{\theta}{R_f}\Delta + \ldots$$

Thus, $f(s_t)$ is constant if an equilibrium exists, and a necessary condition for the latter is that $\frac{\theta}{R_f} < 1$. \hfill \Box

The primary economic implication of the Theorem is that time variation in the wealth-consumption ratio must be associated with time variation in the expected consumption growth rate - this is a general property that does not depend on the two-state nature of the model. We elaborate on this and other issues below:

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31 This is also in the spirit of Campbell and Cochrane (1999). The unconditional mean and standard deviation for the U.S. quarterly real risk free rate is, respectively, between 0.25 - 0.75% and 0.5 - 4.0% (post-war rates have been low and smooth – Campbell (2002)). The correlation between the U.S. equity market and real interest rates is low and negative (Campbell (2002)).
1. The Theorem, in fact, can be generalized. First note that the result did not depend on the number of states spanned by $s_t$. Second, assuming state independent $\mu(s_t)$, the iid assumption on consumption growth implies that the wealth-consumption ratio is effectively an interest rate derivative instrument and is therefore highly correlated with some instrument with similar duration. The relatively large empirical correlation that such instruments have with each other implies that the wealth-consumption ratio will itself be highly correlated with the risk free rate (or, alternatively, some other term-structure variable). The latter observation is counterfactual; thus, in the class of models we are considering consumption growth cannot be iid. Note that this analysis has implications for other models that can be nested within ours (e.g., the Epstein-Zin pricing kernel), and gives further justification for a modeling approach that imposes additional structure on consumption growth (e.g., Bansal and Yaron (2002)).

$X(s_t)$ and consumption growth must be unconditionally correlated (although shocks to either are contemporaneously uncorrelated). Knowledge about the current expected growth rate of consumption can help to identify the current state of $X(s_t)$. Conversely, the current state of $X(s_t)$ can be used to make a more precise forecast of the current growth rate of consumption, but gives no other information about future consumption states. Perhaps the most important implication is that whatever the nature of the factor driving the pricing kernel (i.e., $X(s_t)$), it cannot be independent of the state of aggregate consumption. Examination of the proof of the Theorem reveals that this is a direct consequence of (i) the assumption of iid consumption growth, and (ii) the assumption of low (or even zero) correlation between the risk free rate and wealth consumption ratio.

2. The fact that $s_t$ corresponds to periods of high versus low consumption growth suggests that $s_t$ corresponds to the state of the economy within a business cycle. In other words, the model is consistent with Hamilton’s (1989) description of GNP growth both in form as well as in substance. Identifying $+$ (resp. $-$) with economic expansion (resp. recession), one obtains through Assumption 4(i) the state transition probability matrix. Since in a binary switching model state $j$ has average duration of $\frac{1}{1-P(s_j|s_j)}$, $P(+-)$ (continued economic expansion) is between 0.89-0.96, while $P(--)$ (continued economic recession) is between 0.75-0.83.

3. Why is it that shocks to expected consumption are so important to agents in the economy while consumption shocks are not? Our framework does not permit a direct answer to this. According to our model, the interpretation is that changes in the forecast
of consumption growth are somehow connected to changes in, $G(a_t)$, the degree of commonality of unforeseen contingencies across agents in the economy. We concede that this is not a completely satisfactory answer, but note that the point of our model is to avoid conjecture and assume that agents in the economy are just as ignorant as we are.

Our next result concerns the form of $e^{-\delta(a_{t+1})}$ in the pricing kernel. Recall that

$$e^{-\delta(a_{t+1})} \equiv \frac{G(a_{t+1})}{F(a_{t+1})G(a_t)} = \frac{X(s_{t+1})}{X(s_t)} \frac{C(a_t)}{C(a_{t+1})} \frac{1}{F(a_{t+1})}$$

where $\frac{1}{F(a_{t+1})}$ is the 'expected' change in each agent’s risk aversion parameter under the (possibly non-linear) weighting operator $E^i$. We next establish that $e^{-\delta(a_{t+1})}$ necessarily does not equal one. As discussed in Section 3, this has very important economic implications: (i) the market is necessarily incomplete, and (ii) investors do not have ‘objective’ expected utility preferences with respect to microstates. States that cannot be hedged and are not priced must exist, and these will have an economically significant impact on states that can be hedged (the economic impact arises due to the subjective or non-linear weighting scheme for microstates in investors’ preferences). The following result helps to justify our elaborate approach as opposed to one in which a representative agent with random risk aversion is assumed.

**Calibration Result:** $e^{-\delta(a_{t+1})} = 1$ is not consistent with Assumptions 1-4.

**Proof:** Assume $\delta(a_{t+1}) = 0$ and note that under this assumption, $\frac{K_{-,+}}{K_{+,+}} = \frac{K_{-,-}}{K_{+,-}}$. Let $E\left[\varepsilon_{t+1}^{-\gamma}\right] \equiv \eta$; then the risk free rate in state $s_t$ is given by $\frac{1}{R_f(s_t)} = \eta \left(P(+|s_t)K_{+,s_t} + P(-|s_t)K_{-,s_t}\right)$. The maximum Sharpe Ratio in state $s_t$ is the risk free rate times the standard deviation of the state-price deflator, conditional on $s_t$. Assumption 4(iv) implies that only a small portion of that standard deviation is due to conditional shocks to aggregate consumption; equivalently, it states that $\gamma$ cannot be much larger than 1. Given the small standard deviation of $\varepsilon_{t}^{-\gamma}$ and large observed Sharpe Ratio, one can make the following approximation (which ignores the variance contribution due to aggregate consumption shocks):

$$SR(s_t) \approx R_f(s_t) \eta \sqrt{P(s_t|s_t)(1-P(s_t|s_t))} \left| K_{+,s_t} - K_{-,s_t} \right|$$

The latter can be written exclusively in terms of $\frac{K_{-,+}}{K_{+,+}} \equiv x$ and the transition probabilities. Solving for $x$ in terms of $SR(\cdot)$ and $P(+\cdot+)$ and then plugging that into the equation for
The unconditional maximum Sharpe Ratio, $SR$, is related to $SR(\cdot)$ and $SR(-)$ via the unconditional transition probability, $P(\cdot) = \frac{1-P(-)}{2P(\cdot|-)+P(\cdot)}$: i.e., $SR \equiv P(\cdot)SR(\cdot) + (1-P(\cdot))SR(-)$. The latter relationship can be used with Eqn. (29) to write $\frac{SR(-)}{SR(+)}$ as an algebraic function of $SR, P(-)$ and $P(+)$ only.\(^{32}\) For $P(+|+) > P(-|-) > \frac{1}{2}$, this function is decreasing in $SR$ and $P(-)$, and increasing in $P(+|+)$. Thus within the range of parameters implied by Assumptions 4(i-ii), $\frac{SR(-)}{SR(+)}$ can be no larger than the value it attains at $P(+|+) = 16/17, P(-|-) = 2/3$ and $SR = 0.16$; explicitly calculating this value we conclude that $\frac{SR(-)}{SR(+)} \leq 1.44$. This is not consistent with Assumption 4(ii).\(^{\Box}\)

Note that the last result is obtained \textit{without} restricting the risk free rate to be constant. Additional restrictions, such as requiring $\beta < 1$ or insisting that $f(s_t)$ and $z(s_t)$ are finite and positive, drive the $\delta = 0$ model further from agreement with the stylized facts. Given specific assumptions over interest rates, one can back out values for $K_{++}$ and $K_{--}$, and therefore $\beta$, in terms of observed variables. We conducted an exercise in which $\beta$ and $\frac{SR(-)}{SR(+)}$ were calculated for the intervals of observed macro economic variables specified in the Assumptions. Assuming these intervals are uniform around their true values (e.g., the true quarterly unconditional Sharpe Ratio is uniformly distributed in the interval $0.16 - 0.25$), in less than 5% of cases was $\beta < 1$ and $\frac{SR(-)}{SR(+)} > 1.1$.\(^{33}\) This gives reassurance that the calibration theorem is robust to a great deal of error in measuring the degree of predictability in expected returns (i.e., errors in measuring $\frac{SR(-)}{SR(+)}$ or the probability of business cycle transitions.

4.2.3 A more comprehensive calibration of the two-state model

Our model gives little guidance as to the structural form of $e^{-\delta(s_{t+1}|s_t)}$ when it is not equal to one. We now provide an example of a pricing kernel consistent with Assumptions 1-4,

\(^{32}\)The resulting algebraic equation is too tedious to write, but can be easily derived using standard symbolic algebra software.

\(^{33}\)The uniform distribution assumption made with respect to the confidence intervals of the observed macroeconomic moments is conservative. For a point estimate, (i.e., the center of the confidence intervals) $\beta > 1$ and $\delta(a_{t+1}) = 0$ is soundly rejected; thus, assuming the confidence intervals conformed to that given by a multivariate normal distribution would tend to greatly reduce the frequency at which $\beta < 1$ and $\frac{SR(-)}{SR(+)} > 1.1$. \(^{\Box}\)

36
constant interest rates, and other stylized facts. We assume that
\[ z(s_t) \equiv X(s_t) f(s_t) \]
and look for model parameters that rationalize this assumption and the stylized facts in a stationary equilibrium. Note that under this assumption, stationarity is assured so long as \( f(s_t) \) and \( X(s_t) \) are finite. The pricing kernel simplifies to
\[
\phi(a_{t+1}|a_t) = \text{Prob}(\varepsilon_{t+1} = \varepsilon) P(s_{t+1}|s_t) \beta \varepsilon_{t+1}^{-\gamma} \left( \frac{X(s_{t+1}) + 1}{X(s_t) + 1} \frac{(1 + e^{-\delta(s_{t+1}|s_t)})}{2} \right)^{-\gamma}
\]
We assume that \( \varepsilon_{t+1} \) is log-normally distributed. There are some clear guidelines for fitting several model parameters. First, note that if the wealth-consumption ratio is time varying, the two-state Markov process admits an AR(1) representation with auto-correlation coefficient equal to \( P(+|+) + P(-|-) - 1 \). Lettau and Ludvigson’s (2001b) CAY has an auto-correlation coefficient of 0.83 with standard error of 0.04; given that the variation in \( f(s_t) \) was shown to be connected with variations in expected consumption growth, it is reasonable to assume that \( f(s_t) \) parameterizes the state of the business cycle. Hamilton (1989) estimates \( P(+|+) = 0.905, P(-|-) = 0.755 \), with respective standard errors of 0.04 and 0.1. To find a satisfactory compromise between these facts we set \( P(+|+) = 0.94 \) and \( P(-|-) = 0.84 \), giving an auto-correlation coefficient for the wealth-consumption ratio of 0.78. From these, we calculate the average numbers of recession (expansion) quarters to be \( \tau_- = 6.3 \) (\( \tau_+ = 16.7 \)) – somewhat on the high side, but within statistical tolerance.

The consumption growth parameters, \( \mu(s_t) \), can in principle be estimated as in Hamilton (1989). Instead, we roughly infer them from the known relation between consumption growth and the wealth consumption ratio.\(^{34}\) In particular, Lettau and Ludvigson (2001b) report \( R^2 \) of less than 3\% for forecasting regressions of consumption growth on the consumption to wealth ratio with horizons between 1 and 24 quarters (see their Table VI).\(^{35}\) Since \( f(s_t) \) is perfectly correlated with \( \mu(s_t) \) in the two-state model, this implies that only about \( \sim 3\% \) of the variance of consumption can be attributable to variation in \( \mu(s_t) \). To reproduce this variance decomposition while fixing the unconditional mean and standard deviation of aggregate consumption growth at approximately, 0.005 and 0.01, respectively, we calculate

\(^{34}\) It is incorrect to assume that consumption growth has the same trend parameters as estimated in Hamilton’s (1989) GNP model. The latter measures income growth, while consumption is a smoothed derivative of labor income and therefore has smaller trend fluctuations. We thank an anonymous referee for guidance on this issue.

\(^{35}\) The one-period ahead forecast \( R^2 \) is 0. This, however, may be due to the various measurement issues that plague the U.S. aggregate non-durable consumption time series (see Wilcox (1992)).
that $e^{\mu(+) = 1.006}$, while $e^{\mu(-) = 1.002}$. Note that the implied degree of predictability in consumption growth is at the limits of what may be empirically detectable.

We fix the quarterly maximum Sharpe ratio during recessions (expansions) at 0.25 (0.16) and the quarterly risk free return at 1.005 in both states. Fixing these parameters gives four restrictions that are sufficient to pin down $K_{st+1,st}$ (see the proof of the ‘Calibration Result’); the latter can be used to calculate the wealth-consumption ratio. This leads to an over-constrained system when time series properties of wealth are compared to the data. For instance, one can attempt to fit to (i) the observed ratio of the unconditional standard deviation of wealth to that of consumption, (ii) the correlation between aggregate consumption and aggregate wealth, and (iii) the correlation between the wealth-consumption ratio and returns on wealth. Shortly, we also calibrate to stock market data by introducing a dividend process.

After fixing the $K_{st+1,st}$’s, two additional restrictions come from setting $z(s_{\pm}) \equiv X(s_{\pm})f(s_{\pm})$ in Equation (25). This allows us to solve for $\delta(s_{\pm}|s_{\pm})$ and $X(s_{\pm})$ as functions of $\gamma$ and $\beta$. This part of the system of parameters is therefore under-constrained. In principle, $\gamma$ can be inferred from the risk premium on aggregate consumption shocks, since the latter are not correlated with the state variable. Empirically, however, it is unlikely that $\gamma$ can be estimated in this way with reliable accuracy. Another source of data to pin down $\gamma$ is the sensitivity of interest rates to consumption growth (the inverse elasticity of inter-temporal substitution). From this, Hall (1988) estimates $\gamma \approx 5$; although this estimate is somewhat model specific and is not uncontroversial, we adopt it to further tie down the calibration. The remaining parameter, $\beta$, is monotonically and steeply related to the relative risk aversion, $R_R(s_t) = \gamma \frac{X(s_t)}{1+X(s_t)}$. At a value of $\beta = 0.995$, corresponding to an annual rate of impatience of 2%, $R_R(s_t) \approx 0.5$ for both states. In magnitude these are behaviorally reasonable; however, $\gamma = 5$ and $\beta = 0.995$ imply that $R_R(s_t)$ contributes insignificantly to the variation in the pricing kernel, and that the dominant contributor to the latter is variation in $\delta(s_{t+1}|s_t)$. In other words, under this parameterization and consistent with the ‘Calibration Result’, nearly all the explanatory power of the model arises from deviation from objective expected utility with respect to microstates (i.e., optimism/pessimism with respect to future risk tolerance). Note that $\delta(s_{t+1}|s_t)$ is negligible for claims on sustaining the economic status quo.

36 Clearly there is not much room for increasing $\beta$, while decreasing it implies levels of relative risk aversion that are somewhat low.
37 If $\gamma$ is set to 0.5 instead, the unconditional standard deviation of $R_R(s_t)$ can be as high as 5% while maintaining $R_R \approx 0.5$; the variation of $\delta(s_{t+1}|s_t)$ must correspondingly increase, however, to compensate for the attenuated exponent (i.e., $\gamma$).
On the other hand $e^{-\delta(-|+)} \sim 1.25$, indicating that in good times agents are too ‘optimistic’ (by about 25%) about their risk tolerance should a recession befall the economy. Likewise, $e^{-\delta(+|-)} \sim 0.73$, indicating a similar magnitude of pessimism during a recession about risk attitudes in a possible recovery. Again, we stress that the terms ‘optimism’ and ‘pessimism’ correspond to preferences and not beliefs. Finally, it is easy to calculate from Equation (43) that $\rho(s_t)^{\frac{1}{2}}z(s_t)$ is between 25 and 26. Thus from (13), the constant proportion of wealth consumed by each agent per period is about 4%. In summary, the calibrated parameters are not behaviorally implausible.

A summary of the parameter restrictions discussed is in Table 1. The quarterly Sharpe Ratio for consumption shocks is 0.05 – about four times smaller than the maximum Sharpe Ratio. The four restrictions on the risk-free return and Sharpe ratio can be used to solve for the $K(s_{t+1}, s_t)$’s. In turn, these can be used to calculate $f(s_{\pm})$ – see Table 1. The level of the wealth-consumption ratio is high ($\approx 140$) due to the small difference between the risk-free rate and unconditional mean of consumption growth.$^{38}$ Table 1 also exhibits calculated squared correlation between the wealth-consumption ratio and next period’s wealth growth (i.e., $\frac{w_{t+1}}{w_t}$). This is simply the $R^2$ coefficient of a forecast regression of wealth growth on the lagged wealth-consumption ratio. Consistent with data, this coefficient is close to zero.

The standard deviation of wealth growth is only 20% larger than that of consumption growth (Lettau and Ludvigson (2001b) calculate the ratio to be roughly 2). This can be traced to the fact that interest rates are constant and thus the only variation in $f(s_t)$ is due to variation in $\mu(s_t)$ (see Theorem 3), which is necessarily very small. Moreover, a contemporary regression of wealth growth on aggregate consumption growth indicates that the variance component of consumption growth is $\sim 65\%$ of the variance of wealth growth. This is roughly twice larger than the value calculated from a regression of changes in $\ln w_t \approx 0.2781\alpha_t + 0.6126\gamma_t$ (from Lettau and Ludvigson (2001b)) on changes in aggregate consumption growth.$^{39}$ Note that both discrepancies can be caused by measurement error in Lettau and Ludvigson’s proxy for aggregate wealth.

Thus far we have focused on fitting to stylized facts concerned with aggregate wealth, consumption and the maximum Sharpe ratio. Much of the empirical and theoretical asset pricing literature attempts to additionally relate stylized facts about the stock market to the historical stream of dividends. Before attempting to fit to these facts we note that

$^{38}$Recall that in the standard model the wealth consumption ratio is $\frac{dt}{r_f + \lambda c - \mu c}$, where $dt$ is the unit of time over which the variables are measured (a quarter, in this case).

$^{39}$The data is from Martin Lettau’s website.
dividends are highly seasonal and volatile in the short run, and are a non-unique and extremely inefficient means of transferring value to shareholders. Moreover, the long-horizon ‘predictability’ of returns using the price-dividend ratio is econometrically controversial (see Valkanov (2003)). We therefore feel that, although our model is capable of capturing some of the stylized facts just noted, it is not clear that a model should do so.

To model the stock market, assume that changes to the log of its dividend stream are given by:

\[
\ln \frac{D_{t+1}}{D_t} = \bar{d} + a \ln \left( \frac{f(s_{t+1})}{\bar{f}} \right) + \sigma \nu_{t+1} \tag{30}
\]

where \( \ln \bar{f} \) is the unconditional mean of \( \ln f_t \), \( \sigma \alpha \) and \( a \) are constant, \( \nu_t \) is an iid standard normal variate independent of \( f_t \) or \( \varepsilon_t \), and \( \bar{d} \) is the unconditional mean of \( \ln D_t \).\(^{40}\) Predictability in dividend growth comes from predictability in \( f(s_{t+1}) \): the expected growth is higher during expansions if \( a > 0 \). We abstain from adding additional structure so as to preserve the two-state nature of the model. In particular, our assumptions imply that the (cum dividend) price-dividend ratio, \( p(s_t) \), depends only on the current state variable, \( s_t \), and is the solution to an equation similar to (26):

\[
p(s_t) = \Delta + E \left[ \varepsilon_{t+1}^{-1} \right] \sum_{s_{t+1}} P(s_{t+1}|s_t) e^{d+\sigma^2/2} \left( \frac{f(s_{t+1})}{\bar{f}} \right)^a K(s_{t+1}, s_t) p(s_{t+1})
\]

We set \( d = 0.006 \) and the unconditional standard deviation of \( \ln D_t \) to be 0.04.\(^{41}\) Note that \( \sigma \nu \) can be determined if \( a \) is known. The remaining parameter, \( a \), is pinned down by requiring the unconditional mean of \( p(s_t) \) to be 28.3, as in the data. The total return on the stock market is given by the cum-dividend price at date \( t+1 \) divided by the ex-dividend price at date \( t \):

\[
R_m(s_{t+1}|s_t) = \frac{p(s_{t+1})D_{t+1}}{p(s_t)D_t - \Delta D_t}
\]

Our numerical assumptions are documented in Table 2 along with results on the annualized first and second moments of the excess return. The magnitudes of excess returns are on the low side (5% per year) but well within empirical standard errors. The volatility of returns arises largely from transitions between states of the economy; because \( f(-) < \bar{f} <

\(^{40}\)Stock returns have a low but significant correlation with aggregate consumption shocks. This can be arranged by allowing \( \varepsilon_t \) and \( \nu_t \) to be correlated and has little effect on subsequent results due to the low risk premium associated with aggregate consumption shocks.

\(^{41}\)Both figures are consistent with stylized facts (Campbell (2002)). The annual standard deviation of log-dividends is 0.06 while the annualized quarterly standard deviation is 0.28. Since our model is quarterly, we use a compromise annualized figure of 0.08.
\( f(+) \), returns are usually poor during recessions and good during expansions. The larger expected return during recessions is due to the potential for change in the economic condition (i.e., \( p(-) \) changing to \( p(+) \)). Agents who invest during recessions are compensated in the transition to economic recovery.

Another commonly cited stylized fact is that stock market returns are ‘predictable’ and the predictability increases with the return horizon. Predictability is synonymous with time-variation in the expected returns, which in (30) manifests itself via the dependence of \( s_{t+1} \) on \( s_t \). We calculate the auto-correlation of excess stock returns to be \(-0.045\); this is consistent with the quarterly time series auto-correlation of \(-0.082\) between 1928-1998 and \(0.066\) between 1946-1998. Moreover, the price-dividend ratio, which is perfectly correlated with \( f(s_t) \) due to the two-state nature of the model, can forecast returns because of its marginal persistence. Intuition from other studies (see, for example, Bansal and Yaron (2002)) suggests that with auto-correlation of \(0.78\), the price-dividend ratio (or equivalently, the wealth-consumption ratio) is not persistent enough to account for the large observed predictability of post-war returns at long horizons (i.e., \( R^2 \approx 0.37 \) at 4+ years). By further expanding the state space, as do Bansal and Yaron (2002), it is possible to increase the persistence of the price-dividend ratio independently of that of the wealth-consumption ratio. Table 2 reports the results of a simulation of 5000 data sets each a time-series of 200 quarters; specifically we calculate the simulation average for the total-return forecasting power (correlation and \( R^2 \)) of the price-dividend ratio (or equivalently, the wealth-consumption ratio) at horizons of 1, 2 and 4 years. The empirical moments for the price-dividend ratio regressions are well outside the confidence intervals of the simulation, but barely within statistical tolerance of wealth-consumption ratio regressions (see Lettau and Ludvigson (2001b) Table VI). Thus, as might be suspected, additional structure should be assumed within our framework to account for long-run predictability orthogonal to that contained in the wealth-consumption ratio.

Although a 2-state model is not a ‘perfect fit’, given its parsimony it seems to do a reasonable job and does illustrate the basic properties of the model: the risk-free rate can be low and smooth while allowing for a high equity premium unrelated to consumption shocks, but this only comes about if some micro-states cannot be hedged and agents’ preferences deviate from objective expected utility. Moreover, the wealth-consumption ratio exhibits time variation and predicts the maximum Sharpe ratio only if there is some, albeit nearly imperceptible, predictability in consumption growth rates. For a complete description of the time series of real interest rates and the magnitudes of long-horizon predictability in stock...
returns, one must introduce additional state-variables if the pricing kernel of Eqn. (21) is to serve as the basic framework.

5 Conclusion

Our main goal in this paper is to argue that a sensible asset pricing theory can be constructed from decision-theoretic primitives involving factors other than pure aggregate consumption, and yet does not commit to the identification of such factors. The stylized models considered in Section 4 derive from a general equilibrium in which heterogeneous agents experience aggregate consumption risk as well as unforeseen contingencies. A large component of macro risk can arise from commonalities across unforeseen contingencies faced by agents. These shocks need not be conditionally related to innovations in aggregate consumption, but must be related to innovations in the expected growth rate of aggregate consumption. Thus, at least according to our model, changes in expected consumption growth rates play an important role in asset pricing. Moreover, our model is only consistent with stylized facts if markets are incomplete and agents’ behavior deviates from expected utility. Finally, the wealth-consumption ratio varies with non-consumption shocks and therefore can be used as a proxy for non-consumption factors in the pricing kernel. A possible further avenue to explore is a version of the pricing kernel in Eqn. (21) that incorporates the intuition gained from the simple models of Section 4 and yet is econometrically manageable.
Appendix:

Proof of Proposition 1: The argument is inductive. At macro state $a_{t+1} \in \mathcal{F}_{t+1}$ and microstate $s_{t+1}$ we conjecture the form of the indirect utility function in Eqn. (14), and assume that at date $t$ the investor maximizes a derived utility as in Eq. (7). Equations (9)-(11), lead to utility in event $a_{t+1} \in \mathcal{F}_{t+1}$ of

$$U^{s_{t}}_{t}(c_{t}^{i}, w_{t+1}^{i}, a_{t+1}) = \frac{(s_{t}^{i}c_{t}^{i} + 1)^{1-\gamma}}{s_{t}^{i}(1 - \gamma)} \Delta + \beta \rho(a_{t+1})z(a_{t+1})\left(\frac{w_{t+1}^{i}}{z(a_{t+1})} + 1\right)^{1-\gamma}$$

where $w_{t+1}^{i}$ is the total wealth with which agent $i$ begins the period in state $a_{t+1}$:

$$w_{t+1}^{i} = c_{t}^{i}(a_{t+1})\Delta + \sum_{a_{s} \subset a_{t+1}} c_{t}^{i}(a_{s})\phi(a_{s}|a_{t+1})\Delta$$

Under the model assumptions it can be readily verified that Eqn. (31) holds for date $T - 1$ by making the identification, $\rho(a_{T}) = 1, z(a_{T}) = \Delta$ and $w_{T}^{i} = c_{T}^{i}$.

The agent’s optimization program is as follows:

$$V^{s_{t}}_{t}(w_{t}^{i}, a_{t}) = \max_{\lambda, c_{t}^{i}, \{c_{t}^{i}(a_{s})\}} \left\{ \sum_{a_{t+1} \subset a_{t}} \pi(a_{t+1}|a_{t}) U^{s_{t}}_{t}(c_{t}^{i}, w_{t+1}^{i}, a_{t+1}) - \lambda \left( c_{t}^{i}\Delta + \sum_{a_{s} \subset a_{t}} c_{t}^{i}(a_{s})\phi(a_{s}|a_{t})\Delta - w_{t}^{i} \right) \right\}$$

where $\lambda$ is a Lagrange multiplier. The Lagrange multiplier can be eliminated from the first order conditions for $c_{t}^{i}(a_{t+1})$ and $c_{t}^{i}$ to give

$$\phi(a_{t+1}|a_{t}) = \pi(a_{t+1}|a_{t})\rho(a_{t+1})\phi(a_{t+1})\left(\frac{s_{t}^{i}F(a_{t+1})}{s_{t}^{i}c_{t}^{i} + 1}\right)^{1-\gamma}$$

Eq. (34) can be written as

$$\phi(a_{t+1}|a_{t}) = \beta \pi(a_{t+1}|a_{t})\rho(a_{t+1})F(a_{t+1})^{\gamma}\left(\frac{w_{t+1}^{i}}{s_{t}^{i}c_{t}^{i} + 1}\right)^{-\gamma}$$

By solving for $(s_{t}^{i}F(a_{t+1})^{\gamma}\left(\frac{w_{t+1}^{i}}{s_{t}^{i}c_{t}^{i} + 1}\right)^{-\gamma}$ one can use (31) to write the optimized utility from Eq. (33) as,

$$V^{s_{t}}_{t}(w_{t}^{i}, a_{t}) = \frac{(s_{t}^{i}c_{t}^{i} + 1)^{1-\gamma}}{s_{t}^{i}(1 - \gamma)} \Delta + \sum_{a_{t+1} \subset a_{t}} \phi(a_{t+1}|a_{t})z(a_{t+1})\left(\frac{\phi(a_{t+1}|a_{t})}{\beta \pi(a_{t+1}|a_{t})\rho(a_{t+1})}\right)^{-\frac{1}{\gamma}}$$
The left hand side is not an explicit function of $w_t^i$. To fix this, note that Eq. (35) can be manipulated to give

$$F(a_{t+1}) \left( \frac{\phi(a_{t+1}|a_t)}{\beta \pi(a_{t+1}|a_t) \rho(a_{t+1})} \right)^{-\frac{1}{2}} (s_t^i c_t^i + 1) = s_t^i F(a_{t+1}) \frac{w_{t+1}^i}{z(a_{t+1})} + 1 \quad (37)$$

Multiplying by $\frac{\phi(a_{t+1}|a_t) z(a_{t+1})}{F(a_{t+1})}$ and then summing over state prices gives (with the help of the budget constraint):

$$(c_t^i s_t^i + 1) \sum_{a_{t+1} \subseteq a_t} \phi(a_{t+1}|a_t) z(a_{t+1}) \left( \frac{\phi(a_{t+1}|a_t)}{\beta \pi(a_{t+1}|a_t) \rho(a_{t+1})} \right)^{-\frac{1}{2}} = \sum_{a_{t+1} \subseteq a_t} \frac{\phi(a_{t+1}|a_t) z(a_{t+1})}{F(a_{t+1})} s_t^i (w_t^i - c_t^i \Delta) \quad (38)$$

The above can be solved for $c_t^i$ in terms of $w_t^i$ and the results used in (36). By defining $z(a_t)$ and $\rho(a_t)$ as in (15)-(16), and after some manipulation, one derives (13) which leads directly to the conjectured form in (14).

\[ \square \]

**Proof of Proposition 2:** Using the fact that the events, $\{a_s\}$ for $s > t$ partition the state space, the first order condition for $c_t^i(a_s)$ for $s > t + 1$ in (33) can be written as

$$\phi(a_s|a_t) = \pi(a_{t+1}|a_t) \phi(a_s|a_{t+1}) \frac{\partial}{\partial c} U_t^{a_t} (c_t^i(a_t), w_{t+1}^i) \quad (39)$$

Substituting from Eq. (34) and rearranging, the last equation becomes

$$\phi(a_s|a_{t+1}) = \frac{\phi(a_s|a_t)}{\phi(a_{t+1}|a_t)} \quad (40)$$

Setting $\phi(a_s) \equiv \phi(a_s|a_1)$, these equations can be used to inductively derive Eq. (1).

It is now possible to return to directly analyze the state prices. Specifically, Eqn. (38) can be written as

$$(c_t^i + \frac{1}{s_t^i}) \sum_{a_{t+1} \subseteq a_t} \phi(a_{t+1}|a_t) z(a_{t+1}) \left( \frac{\phi(a_{t+1}|a_t)}{\beta \pi(a_{t+1}|a_t) \rho(a_{t+1})} \right)^{-\frac{1}{2}} = \frac{1}{s_t^i} \sum_{a_{t+1} \subseteq a_t} \frac{\phi(a_{t+1}|a_t) z(a_{t+1})}{F(a_{t+1})} + (w_t^i - c_t^i \Delta) \quad (41)$$

Averaging over all investors and using the market clearing conditions, Eqns. (8), yields

$$\left( C(a_t) + \frac{1}{G(t)} \right) \sum_{a_{t+1} \subseteq a_t} \phi(a_{t+1}|a_t) z(a_{t+1}) \left( \frac{\phi(a_{t+1}|a_t)}{\beta \pi(a_{t+1}|a_t) \rho(a_{t+1})} \right)^{-\frac{1}{2}} = \frac{1}{G(t)} \sum_{a_{t+1} \subseteq a_t} \frac{\phi(a_{t+1}|a_t) z(a_{t+1})}{F(a_{t+1})} + (w(a_t) - C(a_t) \Delta) \quad (42)$$
where \( w(a_t) \) is aggregate per-capita wealth and \( \frac{1}{C(t)} \equiv \frac{1}{N} \sum_{i=1}^{N} \frac{1}{s_i} \). Note that the sum on the left side can be replaced with \((\rho(a_t)^{\frac{1}{\gamma}}z(a_t) - \Delta)\) and the sum on the right side can be replaced with \((z(a_t) - \Delta)\), all of which leads to:

\[
\rho(a_t)^{\frac{1}{\gamma}} = \frac{G(t)C(a_t) + 1}{G(t)\frac{w(a_t)}{z(a_t)} + 1}
\]

(43)

In the same way, one can also manipulate Eqn. (37) to give:

\[
F(a_{t+1})(C(a_t) + \frac{1}{G(t)}(\frac{\phi(a_{t+1}|a_t)}{\beta\pi(a_{t+1}|a_t)\rho(a_{t+1})})^{-\frac{1}{\gamma}} = \frac{1}{G(t)} + \frac{w(a_{t+1})}{z(a_{t+1})}F(a_{t+1})
\]

(44)

From Equation (44), one can derive:

\[
\phi(a_{t+1}|a_t) = \beta\pi(a_{t+1}|a_t)\rho(a_{t+1})\left(\frac{G(t)\frac{w(a_{t+1})}{z(a_{t+1})} + \frac{1}{F(a_{t+1})}}{G(t)C(a_t) + 1}\right)^{-\gamma}
\]

(45)

\[\Box\]

**Proof of Theorem 2:** Taking Eqn. (43) plugging it into (45), state prices are given by:

\[
\phi(a_{t+1}|a_t) = \beta\pi(a_{t+1}|a_t)\left(\frac{G(a_{t+1})C(a_{t+1}) + 1}{G(a_t)C(a_t) + 1} + \frac{G(a_t)}{G(a_{t+1})^{\frac{w(a_{t+1})}{z(a_{t+1})} + 1}}\right)^{-\gamma}
\]

(46)

Setting \( \frac{G(a_{t+1})}{G(a_t)F(a_{t+1})} = e^{-\delta(a_{t+1})} \), the last expression gives Eq. (21):

\[
\phi(a_{t+1}|a_t) = \beta\pi(a_{t+1}|a_t)\left(\frac{G(a_{t+1})C(a_{t+1}) + 1}{G(a_t)C(a_t) + 1} - \frac{G(a_t)}{G(a_{t+1})^{\frac{w(a_{t+1})}{z(a_{t+1})} + 1}}\right)^{-\gamma}
\]

(47)

\[\Box\]

**Proof of Lemma 4.1:** Using the pricing kernel in (22) with \( \gamma = 1 \), and dividing both sides of Eqn. (19) by \( C(a_t) \), the wealth-consumption ratio is

\[
f(a_t) \equiv \Delta + \sum_{a_{t+1} \subseteq a_t} \phi(a_{t+1}|a_t)\frac{C(a_{t+1})}{C(a_t)}f(a_{t+1})
\]

\[
= \Delta + \beta E_t\left[\left(\frac{C_{t+\Delta}}{C_t} - R_{t+\Delta}R_t\right)^{-1}\frac{C_{t+\Delta}}{C_t}f_{t+\Delta}\right]
\]

\[
f_t = \Delta + \beta E_t\left[e^{x_{t+\Delta} - x_t}f_{t+\Delta}\right]
\]

45
Where we abused notation slightly in the second equation by identifying period \( t + n \) with real-time date \( t + n \Delta \). The last equation is recursive and can be written out as

\[
f_t = \Delta + \beta E_t \left[ e^{x_t + \Delta - x_t} \right] \Delta + \beta^2 E_t \left[ e^{x_t + 2\Delta - x_t} \right] \Delta + \ldots
\]

Using the continuous time limit as well as the limit \( T \to \infty \) gives,

\[
f_t \approx \int_0^\infty e^{-r_0 \tau} E_t \left[ e^{x_{t+\tau} - x_t} \right] d\tau
\]

Since \( x_{t+\tau} \) is assumed to be an Ornstein-Uhlenbeck process, one can write

\[
x_{t+\tau} - x_t = (1 - e^{-\kappa x \tau}) (\mu_x - x_t) + \frac{\sigma_x}{\sqrt{2\kappa_x}} \sqrt{1 - e^{-2\kappa_x \tau}} N(0, 1)
\]

where \( N(0, 1) \) is a standard Gaussian variate. Taking the expectation yields,

\[
f_t \approx \int_0^\infty e^{-r_0 \tau} \exp \left( (1 - e^{-\kappa x \tau}) (\mu_x - x_t) + \frac{\sigma_x^2}{4\kappa_x} (1 - e^{-2\kappa_x \tau}) \right) d\tau
\]

The change of variables, \( y = e^{-\kappa x \tau} \) gives the desired result.

References


Table 1

| \( P( s^+ | s^+ ) \) | 0.94 | \( f( s^- ) \) | 136.8 |
| \( P( s^- | s^- ) \) | 0.84 | \( f( s^+ ) \) | 140.1 |
| \( \tau(s^-) \) | 6.3 | autocorrelation of \( f( s_t ) \) | 0.78 |
| \( \tau(s^+) \) | 16.7 | stdev wealth growth | 1.19% |
| \( \exp(\mu(s^-)) \) | 1.002 | stdev of \( \ln(f( s_t )) \) | 1.06% |
| \( \exp(\mu(s^+)) \) | 1.006 | \( R^2(C_{t+1}/C_t \text{ on } f_t) \) | 3.2% |
| \( \sigma_c \) | 0.980% | \( R^2(R_{w,t+1} \text{ on } f_t) \) | 0.2% |
| \( \sigma_c \) | 1.00% | \( R^2(C_{t+1}/C_t \text{ on } R_{w,t}) \) | 65.1% |
| \( \mu_{uncond} \) | 0.48% | | |
| \( \sigma_{c \text{ uncond}} \) | 1.00% | \( K( s^+ | s^+ ) \) | 0.9534 |
| \( \gamma \) | 5.00 | \( K( s^+ | s^- ) \) | 0.4244 |
| \( \beta \) | 0.995 | \( K( s^- | s^+ ) \) | 1.6228 |
| | | \( K( s^- | s^- ) \) | 1.1020 |
| \( R_f(s^-) \) | 1.005 | \( X(s^-) \) | 0.1016 |
| \( R_f(s^+) \) | 1.005 | \( X(s^+) \) | 0.1019 |
| \( SR(s^-) \) | 25.0% | \( SR(s^+) \) | 16.0% |
| \( SR_{uncond} \) | 18.5% | "relative risk aversion" | |
| \( R_R(s^-) \) | 0.4612 | \( R_R(s^+) \) | 0.4624 |
| \( \delta(s^+ | s^+ ) \) | 0.0051 | | |
| \( \delta(s^+ | s^- ) \) | 0.3171 | | |
| \( \delta(s^- | s^+ ) \) | -0.2256 | | |
| \( \delta(s^- | s^- ) \) | -0.0454 | | |
### Table 2

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<tr>
<th>Variable</th>
<th>Mean (Annualized)</th>
<th>Return Auto-correlation</th>
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<tr>
<td>$\bar{d}$</td>
<td>2.4%</td>
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</tr>
<tr>
<td>$\sigma_v$</td>
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<tr>
<td>$a$</td>
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<td>$\sigma_{ln \ D}$</td>
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Correlations and $R^2$ between $p(s_i)$ and $\sum_{i=1}^{n} r_{i+i}$

<table>
<thead>
<tr>
<th>Variable</th>
<th>Mean (Annualized)</th>
<th># quarters (n)</th>
<th>$\rho$</th>
<th>5th percentile</th>
<th>$R^2$</th>
<th>95th percentile</th>
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<tr>
<td>$\sigma_{R_m}$</td>
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<td>16</td>
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