Robust Control and Model Uncertainty in Macroeconomics
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Chapter 1.
Introduction

Figure 1.1 reproduces John Doyle’s visualization of developments in optimal control theory since World War II.¹ Two scientists in the upper panels devise control laws or estimators assuming that their models are true, but use different mathematical methods. The person on the left uses classical methods (Euler equations, z-transforms, lag operators) and the one on the right uses modern recursive methods (Bellman equations, Kalman filtering). Both scientists in the top panel assume that their model of the transition dynamics is true. The gentleman in the lower panel shares the purposes (objective functions) of his predecessors from the 50’s, 60’s, and 70’s, but regards his model as an approximation to an unknown and unspecified model that he thinks actually generates the data. The 1980–1990’s control theorist in the lower panel seeks decision rules and estimators that work over a set of models near his approximating model, a set measured by $\theta$. The $H_\infty$ in his postmodern tattoo and the $\theta$ on his staff express his doubts about his model and alludes to a continuum of alternative models. The parameter $\theta$ is interpretable as a Lagrange multiplier on a constraint measuring the size of that set of alternative models.

Macroeconomists and rational expectations econometricians have gathered inspiration and techniques from the classical and modern control theory represented in the top panels of Fig. 1.1. Classical and modern control theory supplied ideal tools for applying Muth’s (1961) concept of rational expectations to a variety of problems in dynamic economics. The reason that rational expectations initially diffused slowly after Muth’s (1961) paper was that in 1961 most economists were not sufficiently familiar with the tools idealized in the top panel of Fig. 1.1. Rational expectations took hold only after a new generation of macroeconomists had learned those tools in the 1970’s.

Ironically, just when macroeconomists had begun applying classical and modern control theory in the late 1970’s, control theorists and applied mathematicians began to construct new methods to repair adverse outcomes they had experienced from applying classical and modern control theory to a variety of engineering and physical problems. They thought that model misspecification

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¹ John Doyle consented to let us reproduce this drawing, which appears in Zhou, Doyle, and Glover (1996). We have changed Doyle’s notation by making $\theta$ (Doyle’s $\mu$) the free parameter borne by the post-modern control theorist.
explained why outcomes were sometimes much worse than predicted and therefore sought controls and estimators that acknowledged model misspecification. That is how robust control and estimation theory came into being.

Figure 1.1: A pictorial history of control theory (courtesy John Doyle). Beware of theorists bearing a free parameter ($\theta$).
1.1. Misspecification and rational expectations

This book borrows and adapts tools from the literature on robust control and estimation to model a decision maker who regards his model as an approximation. He believes that the data come from an unknown member of a set of unspecified models near his approximating model.² Concern about model misspecification induces the decision maker to prefer decision rules that work over that set of nearby models.

If they actually resided in rational expectations models, decision makers would not have to worry about model misspecification. Decision makers inside a rational expectations model can trust their model because subjective and objective probability distributions (i.e., models) coincide. The whole point of rational expectations theorizing is to remove agents’ personal models as elements of the model.³

Our starting point is that although the artificial agents within a rational expectations model trust the model, a model’s author often doubts it, especially after performing specification tests or when calibrating it. There are several good reasons for wanting to extend rational expectations models to acknowledge fear of model misspecification.⁴ First, doing so accepts Muth’s (1961) intention of putting econometricians and the agents being modelled on the same footing: because econometricians face specification doubts, the agents inside the model

² We say unspecified because they are formulated as vague perturbations to the decision maker’s approximating model.
³ In a rational expectations model, each agent’s model (i.e., his subjective joint probability distribution over exogenous and endogenous variables) is an equilibrium outcome, not something to be specified by the model builder. Its early advocates in econometrics emphasized that the rational expectations hypothesis eliminates all free parameters associated with peoples’ beliefs. For example, see Hansen and Sargent (1980) and Sargent (1981).
⁴ In chapter 16, we explore various mappings, the fixed points of which can be used to restrict a robust decision makers’ approximating model. As in rational expectations models, we are silent about the process by which an agent arrives at that model. A qualification to the claim that rational expectations models do not model the process by which agents’ model is formed comes from the literature on learning, in which agents who use recursive least squares learning schemes eventually come to have rational expectations. Early examples of such work are Bray (1982), Marcet and Sargent (1989), and Woodford (1990). See Evans and Honkapohja (2001) for new results.
might too.\(^5\) Second, in various contexts, rational expectations models under-predict prices for risk that are revealed by asset market data. For example, relative to standard rational expectations models, asset markets put too high a premium on macroeconomic risks, the equity premium puzzle. A related finding is that rational expectations models impute low costs to business cycles.\(^6\) Agents’ cautious responses to possible model misspecification raise the theoretical values of such empirical measures of risk aversion. This reason for studying robust decisions is positive and is to be judged by how it helps explain market data. A third reason for studying robustness to model misspecification is normative. A long tradition dating back to Friedman (1953), Bailey (1971), and Brainard (1967) advocates framing macroeconomic policy rules in light of doubts about model specification, though how those doubts are formalized has varied.

1.2. Distorted expectations and robust control theory

Ordinary optimal control theory assumes that decision makers know the model in the form of a transition law linking state variables and controls. That theory associates a distinct decision rule with each specification of shock processes. Many aspects of rational expectations models flow from this association. For example, the cornerstone of the Lucas (1976) Critique is the finding that, under rational expectations, decision \textit{rules} are functionals of the serial correlations of shocks. Rational expectations econometrics achieves parameter identification by exploiting the structure of the function that maps shock serial correlation properties to decision rules.\(^7\)

Robust control theory alters the mapping from shock temporal properties to decision rules. Robust control theory treats the decision maker’s model as approximations and seeks one rule to use for a set of models that might also govern the data. The alternative models are specified vaguely in terms of possibly serially correlated shifts in the conditional means of the shock processes in the decision maker’s model. These shifts or distortions to the shocks can feed back arbitrarily on the history of the states and thereby represent quite generally misspecified dynamics.

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\(^5\) Maybe this argument offends a preference against justifying modelling assumptions on behavioral grounds.


1.1. Misspecification and entropy

The literatures on the statistical and econometric analysis of model misspecification contain essential tools for thinking about decision making in the presence of model misspecification.

1.1.1. Specification analysis in econometrics

Where \( y^* \) denotes next period’s value of a state vector \( y \), let the data truly come from a Markov process with one step transition density \( f(y^*|y) \). Let the econometrician’s model be \( f_\alpha(y^*|y) \) where \( \alpha \in A \) and \( A \) is some compact set of values for the parameter vector \( \alpha \). If there is no \( \alpha \in A \) such that \( f_\alpha = f \), we say that the econometrician’s model is misspecified. Assume that the econometrician estimates \( \alpha \) by maximum likelihood. Under some regularity conditions, the maximum likelihood estimator \( \hat{\alpha}_o \) will converge in large samples to

\[
\text{plim} \hat{\alpha}_o = \arg \min_{\alpha \in A} I(\alpha, f) \tag{1.1.1}
\]

where \( I(\alpha, f) \) is the relative entropy of model \( f \) with respect to model \( f_\alpha \) defined as

\[
I(\alpha, f) = \int \log \left( \frac{f(y^*|y)}{f_\alpha(y^*|y)} \right) f(y^*|y) dy^*. \tag{1.1.2}
\]

It can be shown that \( I(\alpha, f) \geq 0 \). When the model is misspecified, the minimized value of relative entropy on the right side of (1.1.1) is positive. Figure 1.1.1 depicts how the probability limit of the estimator \( \hat{\alpha}_o \) makes \( I(\alpha, f) \) as small as possible.
1.1.2. Acknowledging misspecification in decision making

The preceding analysis of estimation of misspecified models can be used to deduce the consequences of various types of misspecification for estimates of particular parameters. To study decision making in the presence of model misspecification, we in effect turn this analysis on its head by taking $f_{\alpha_o}$ as given and thinking of a set of possible data generating processes that surround it, one element of which is the true process $f$. See figure 1.1.2. In practice, decision makers know their model $f_{\alpha_o}$ but not $f$ and so must base their decisions on $f_{\alpha_o}$. The decision maker’s parametric class of models $f_{\alpha}(y^*|y)$ has been specified by some process of theorizing that we do not model. We also take for granted the decision maker’s parameter estimates $\alpha_o$. We impute to the decision maker some doubts about his model. In particular, the decision maker suspects that the data are actually generated by another model $f(y^*|y)$ with relative entropy $I(\alpha_o, f)$. The decision maker thinks that his model is a good approximation in the sense that $I(\alpha_o, f)$ is not too large, but wants to make decisions that will be good when $f \neq f_{\alpha_o}$. We endow the decision maker with a discount factor $\beta$ and construct the following intertemporal measure of model misspecification:

$$I = E_f \sum_{t=0}^{\infty} \beta^t I(\alpha_o, f)_t .$$

Our decision maker confronts model misspecification by seeking a decision rule that will work well across a set of models for which $I \leq \tilde{\eta}$, where $\tilde{\eta}$ measures the set of models $F$ surrounding his approximating model $f_{\alpha}$. Fig. 1.1.2 portrays the decision maker’s view of the world.

1.2. Organization

This monograph displays alternative ways to sweep a decision maker’s doubts about model specification into altered objective functions. We study both control and estimation (or filtering) problems, and both single- and multiple-agent settings. We borrow and adapt results from the robust control literature. We stay mostly but not exclusively within a linear quadratic framework (see chapter 17 for the more general case), in which a pervasive certainty equivalence principle allows a nonstochastic presentation of most of the control and filtering theory.

The monograph is organized as follows. Chapter 2 summarizes a set of practical results at the lowest possible technical level. One message of this

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Introduction

Figure 1.1.2: Robust decision making: A decision maker with model \( f_0 \) suspects that the data are actually generated by a nearby model \( f \), where \( I(\alpha_0, f) \leq \eta \).

Chapter is that, although sophisticated arguments from chapters 6 and 7 are needed fully to justify the techniques of robust control, the techniques themselves are as easy to apply as the ordinary dynamic programming techniques now widely used throughout macroeconomics and applied general equilibrium theory. Chapter 2 uses linear quadratic dynamic problems to convey this message, but the message applies more generally, as we shall illustrate in chapters 17 and 18.

Chapter 3 starts with important basic principles by summarizing some basic results from linear optimal control for solving the classic optimal linear regulator problem. This chapter culminates in a description of invariant subspace methods for solving linear optimal control and filtering problems. Later chapters apply these methods to various problems: to compute robust decision rules as solutions of zero sum two player games; to compute robust filters via another zero sum two player game; and to compute robust solutions of Stackelberg or Ramsey problems in macroeconomics. Chapter 4 shows how the Kalman filter is the dual (in a sense familiar to economists from their use of Lagrange multipliers) of the basic linear-quadratic dynamic programming problem, the so-called optimal linear regulator problem of chapter 3. We exploit duality relations often in subsequent chapters.

Within a one-period setting, chapter 5 introduces two-person zero-sum games as a way to induce robust decisions. Although the forms of model misspecifications considered in this chapter are very simple relative to those considered in subsequent chapters, the static setting of chapter 5 is a good one for addressing some important conceptual issues. In particular, in this chapter for the
first time we lay out versions of multiplier and constraint problems, alternative optimization problems that induce robust decision rules.

Chapters 6 and 7 extend and modify results in the literature to formulate robust control problems with discounted quadratic objective functions and linear transition laws. Incorporating discounting requires carefully restating the control problems used to induce robust decision rules. We describe two ways to alter the objective function for an ordinary discounted linear quadratic optimal control problem: (1) as a two-player, zero-sum game where nature chooses from a set of models to make the decision maker want robust decision rules; and (2) as the value function or indirect utility function from one of those two-player games. This value function incorporates nature’s worst choice of model specification in response to the decision maker’s decision rule. In category (1), we present a detailed account of several two-person zero-sum games that induce robust decision rules. For category (2), we present three specifications that express a preference for robust rules. Two of them are expressed in the frequency domain: the $H_\infty$ and entropy criteria. The entropy objective function summarizes model specification doubts with a single parameter. We describe how that parameter relates to a Lagrange multiplier in a two-player zero-sum game, and also to the risk-sensitivity parameter of Jacobson (1973) and Whittle (1990), as modified for discounting by Hansen and Sargent (1995). Thus, a decision maker can find rules that are less sensitive to model misspecification by replacing his true objective function with another that embeds his doubts about the constraints (his model). It will be useful for the reader recurrently to remind himself that our criteria for promoting robustness are the indirect utility functions for these games.

Chapters 6 and 7 show how robustness is induced by using local versions of min-max strategies: the decision maker maximizes while nature minimizes over models. There are alternative timing protocols in terms of which a zero-sum two player game can be cast. A main finding of chapter 6 is that the zero-sum game makes a variety of different timing protocols share outcomes and representations of equilibrium strategies. This finding lets us use recursive methods to compute our robust rules. As chapter 6 emphasizes, a key to our results is that we cast the zero-sum game in terms of a time-invariant penalty parameter or Lagrange multiplier that measures the specification error.

Chapter 9 uses the permanent income model of consumption as a laboratory for illustrating some of the concepts from chapters 6 and 7. Because he prefers smooth consumption paths, the permanent income consumer saves to attenuate the effects of income fluctuations on consumption. A robust consumer engages
in a kind of precautionary savings because he suspects error in the specification of the income process.

Chapter 12 describes how a preference for robustness affects asset pricing. A preference for robustness induces a multiplicative adjustment to the stochastic discount factor, where the adjustment is measures fear that the approximating model is misspecified. We describe the basic theory within a class of linear quadratic general equilibrium models.

Chapter 13 describes a discounted robust filtering problem that is dual to the control problem of chapter 6. We discover this problem by stating and solving a conjugate problem of a kind familiar to economists through duality theory.

Chapter 14 studies robust filtering again and, by using a different criterion than chapter 13, finds a different robust filter. We argue that the chapter 14 filter is the appropriate one for many problems. We give some examples.

Chapters 16 and 18 study versions of a macroeconomic control problem, called a Ramsey problem, where the government wants optimally to control a private economy that is forecasting its controls. We describe how to compute a robust government policy when the government can commit to a rule. We accomplish that by using a robust version of the optimal linear regulator or else one of the invariant subspace methods of chapter 3. Chapter 8 briefly describes how we think of choosing the key parameter that measures the size of the set of models against which the robust decision maker seeks robustness. Chapter 17 tells how the key ideas about robustness generalize to models that are not linear-quadratic. This chapter also describes further links between a preference for robustness and risk sensitivity.

This monograph is part of a research program to model a preference for robustness in a tractable way for macroeconomic purposes (see Hansen, Sargent, and Tallarini (1997), and Anderson, Hansen, and Sargent (1997)).
Part I

Standard control and filtering
Chapter 2.
Basic ideas and methods

There are two different drives toward exactitude that will never attain complete fulfillment, one because “natural” languages always say something more than formalized languages can – natural languages always involve a certain amount of noise that impinges on the essentiality of the information – and the other because, in representing the density and continuity of the world around us, language is revealed as defective and fragmentary, always saying something less with respect to the sum of what can be experienced.
— Italo Calvino, Six Memos for the Next Millenium, 1996, pp. 74-75

2.1. Introduction

Standard control theory tells a decision maker how to make optimal decisions when his model is correct. Robust control theory is about making good enough decisions in the presence of doubts about the model. This chapter summarizes methods for computing robust decision rules when the decision maker’s criterion function is quadratic and his approximating model is linear.¹ The Bellman equation and the Riccati equation associated with the standard linear-quadratic dynamic programming problem can readily be adapted to incorporate concerns about misspecification of the transition law. The required adjustments to the Bellman equation have alternative representations, each of which has practical uses in various contexts. While this chapter concentrates mainly on single-agent decision theory, later chapters extend the theory to environments with multiple decision makers with concern about model misspecification.² Chapter 17 shows that many of the insights of this chapter extend beyond the linear quadratic setting.

¹ Later chapters will supply technical details that justify assertions made in this chapter.
² Chapter 15 injects motives for robustness into Markov perfect equilibria for two-player dynamic games; chapter 10 discusses competitive equilibria in representative agent economies, and chapter 16 studies Stackelberg and Ramsey problems. In Ramsey problems, a government chooses among competitive equilibria of a dynamic economy. A Ramsey problem too ends up looking like a single-agent problem, the single agent being a benevolent government that faces a peculiar set of constraints that describe competitive equilibrium allocations.
2.2. Approximating models

Let \( y_t \) be a state vector and \( u_t \) a vector of controls. A decision maker’s model takes the form of a linear state transition law

\[
y_{t+1} = Ay_t + Bu_t + C\hat{\epsilon}_{t+1},
\]

(2.2.1)

where \( y_t \) is a vector of state variables, \( u_t \) is a vector of controls to be chosen at \( t \), and \( \{\hat{\epsilon}_t\} \) is an i.i.d. Gaussian vector process with mean 0 and identity contemporaneous covariance matrix. The decision maker regards the model as approximating another model that he cannot specify. How should the notion that (2.2.1) is misspecified be represented? The i.i.d. random process \( \hat{\epsilon}_{t+1} \) can represent only a very limited class of approximation errors and in particular cannot depict misspecified dynamics such as nonlinear and time-dependent feedback of \( y_{t+1} \) on past states. To represent dynamic misspecification, we surround (2.2.1) with a set of models of the form

\[
y_{t+1} = Ay_t + Bu_t + C(\epsilon_{t+1} + w_{t+1}),
\]

(2.2.2)

where \( \{\epsilon_t\} \) is another i.i.d. Gaussian process with mean zero and identity covariance matrix and \( w_{t+1} \) is a vector process that can feed back in a possibly nonlinear way on the history of \( y \):

\[
w_{t+1} = g_t(y_t, y_{t-1}, \ldots),
\]

(2.2.3)

where \( \{g_t\} \) is a sequence of measurable functions. When (2.2.2) generates the data, it is as though the errors \( \epsilon_{t+1} \) in model (2.2.1) are distributed as \( N(w_{t+1}, I) \) rather than as \( N(0, I) \). Thus, we capture the idea that the approximating model (2.2.1) is misspecified by allowing the conditional mean of the shock vector in the model (2.2.2) that actually generates the data to feed back arbitrarily on the history of the state. To express the idea that model (2.2.1) is a good

---

3 On page 28 of this chapter and in chapters 5 and 17, we allow a broader class of misspecifications. Chapter 17 represents the approximating model as a Markov transition density, and considers misspecifications that alter the assignment of probabilities over future states in a relatively flexible way. When the approximating model is Gaussian, many results of this chapter survive. However, (2.2.2) ignores an additional adjustment to the covariance of the distorted model. In many applications, this adjustment is quantitatively insignificant (it vanishes in the limiting case of continuous time). See page 333.
approximation when (2.2.2) generates the data, we restrain the approximation errors by

$$E_0 \sum_{t=0}^{\infty} \beta^{t+1} w'_{t+1} w_{t+1} \leq \eta_0, \quad (2.2.4)$$

where $E_t$ denotes mathematical expectation conditioned on $y^t = [y_t, \ldots, y_0]$, evaluated with model (2.2.2). In section 2.3 and chapter 8, we shall interpret the left side of (2.2.4) as a statistical measure of the discrepancy between the distorted and approximating models.

The decision maker believes that the data are generated by a model of the form (2.2.2) with some unknown process $w_t$ satisfying (2.2.4). The decision maker forsakes trying to improve his specification by learning because $\eta_0$ is so small that statistically it is difficult to distinguish model (2.2.2) from (2.2.1) using a time series $\{y_t\}_{t=1}^{T}$ of moderate size $T$, an idea that we develop in Chapter 8.

The decision maker’s distrust of his model (2.2.1) makes him want good decisions over a set of models (2.2.2) satisfying (2.2.4). Such decisions are said to be robust to model misspecification.

Robust decisions rules can be computed by solving one of several distinct but related two-player zero-sum games. The games are related because they share common payoffs, but they differ because they assume different timing protocols. Nevertheless, as we show in chapters 6 and 7, equilibrium outcomes and decision rules for the games coincide. Important technicalities are required to verify this claim, but their equivalence makes the games easy to solve. Computing robust decision rules comes down to solving Bellman equations for dynamic programming problems that are very similar to ones routinely used today throughout macroeconomics and applied economic dynamics. Before later chapters assemble the results needed to substantiate these claims, this chapter describes how to compute robust decision rules with standard methods.

We begin with the ordinary linear quadratic dynamic programming problem without model misspecification, called the optimal linear regulator. Then we describe how robust decision rules can be computed by solving another optimal linear regulator problem. Next we briefly describe Lagrangian (or Hamiltonian) methods. We close by highlighting material from chapter 16 that shows how those Lagrangian methods achieve robust control of forward-looking macro models and thereby solve robust Ramsey or Stackelberg problems.

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4 The main thing that generates this outcome is that all of them are zero-sum games, a feature that completely aligns the preferences of the two players. What is good for one of the players is bad for the other.
2.2.1. Dynamic programming without model misspecification

The standard dynamic programming problem assumes no model misspecification.\(^5\) Let the one-period loss function be \(r(y, u) = -(y'Qy + u'Ru)\), where the matrices \(Q\) and \(R\) are symmetric and satisfy some stabilizability and detectability assumptions set forth in chapter 3. The optimal linear regulator problem is

\[
\max_{\{u_t\}_{t=n}^\infty} E_0 \sum_{t=0}^\infty \beta^t r(y_t, u_t), \quad 0 < \beta < 1, \tag{2.2.5}
\]

where the maximization is subject to (2.2.1), \(y_0\) is given, \(E\) denotes the mathematical expectation operator evaluated with respect to the distribution of \(\tilde{\epsilon}\), and \(E_0\) denotes mathematical expectation conditional on time 0 information, namely the state \(y_0\). Let \(-y'_0Py_0 - p\) be the optimal value of problem (2.2.5), (2.2.1). Letting \(y^*\) denote next period’s value of \(y\), the linear constraints and quadratic objective function in (2.2.5), (2.2.1) imply the Bellman equation

\[
-y'P y - p = \max_u E \left[ r(y, u) - \beta y^*P y^* - \beta p \right] \bigg| y \tag{2.2.6}
\]

where the maximization is subject to

\[
y^* = Ay + Bu + C\tilde{\epsilon}, \tag{2.2.7}
\]

where \(\tilde{\epsilon}\) is a random vector with mean zero and identity variance matrix.

Subject to assumptions about \(A, B, R, Q, \beta\), to be described in Chapter 3, some salient facts about the optimal linear regulator are these:

1. The Riccati equation. The matrix \(P\) in the value function is a fixed point of a matrix Riccati equation:

\[
P = Q + \beta A'PA - \beta^2 A'PB(R + \beta B'PB)^{-1} B'PA. \tag{2.2.8}
\]

The optimal decision rule is \(u_t = -Fy_t\) where

\[
F = \beta (R + \beta B'PB)^{-1} B'PA. \tag{2.2.9}
\]

We can find the appropriate fixed point \(P\) and solve problem (2.2.5), (2.2.1) by iterating to convergence on the Riccati equation (2.2.8) starting from initial value \(P_0 = 0\).

\(^5\) Many technical results and computational methods for this problem are catalogued in chapter 3.
2. Certainty equivalence. In the Bellman equation (2.2.6), the scalar \( p = \frac{1}{\beta} \text{trace} PCC' \). The ‘volatility matrix’ \( C \) influences the value function through \( p \), but not through \( P \). It follows from (2.2.8), (2.2.9) that the optimal decision rule \( F \) is independent of the volatility matrix \( C \). In (2.2.1), we have normalized by setting \( E \tilde{\epsilon}_t \tilde{\epsilon}'_t = I \). Therefore, the matrix \( C \) determines the covariance matrix \( CC' \) of random shocks impinging on the system. The finding that \( F \) is independent of the volatility matrix \( C \) is known as the certainty equivalence principle: the same decision rule \( F \) emerges from stochastic (\( C \neq 0 \)) and nonstochastic (\( C = 0 \)) versions of the problem. This kind of certainty equivalence fails to describe problems that express a concern for model misspecification; but another useful kind of certainty equivalence does. See page 20.

3. Shadow prices. Since the value function is \(-y_t'Py_0 - p\), the vector of shadow prices of the initial state is \(-2Py_0\). Such shadow prices appear in a Lagrangian formulation of the optimum problem. Form a Lagrangian for (2.2.5), (2.2.1), letting the vector \(-2\beta^{t+1}\mu_{t+1}\) be Lagrange multipliers on the time \( t \) version of (2.2.1). First-order conditions for a saddle point of the Lagrangian can be rearranged to form a first-order vector difference equation in \((y_t, \mu_t)\). The optimal policy solves this difference equation subject to an initial condition for \( y_0 \) and a transversality or ‘detectability’ condition \( \sum_{t=0}^{\infty} \beta^t r(y_t, u_t) < +\infty \). On page 38 and in chapter 3, we show that subject to these boundary conditions, the difference equation is solved by setting \( \mu_t = Py_t \), where \( P \) solves the Riccati equation (2.2.8).

2.3. Measuring model misspecification: entropy

To help construct decision rules that are robust to model misspecification, we use entropy to measure model misspecification. We also state a modified certainty equivalence principle for linear quadratic models. Although we use a statistical interpretation of entropy, by appealing to a modified certainty equivalence result to be stated on page 20, we shall be able to drop randomness from the model but still retain a measure of model misspecification that takes the form of entropy.

Let the approximating model again be (2.2.1) and let the distorted model be (2.2.2). The approximating model asserts that \( w_{t+1} = 0 \). For convenience, we analyze the consequences of a fixed decision rule and assume that \( u_t = -Fx_t \). Let \( A_o = A - BF \) and write the approximating model as

\[
y_{t+1} = A_0 y_t + C \tilde{\epsilon}_{t+1}
\]
and a distorted model as
\[ y_{t+1} = A_0 y_t + C (\epsilon_{t+1} + w_{t+1}). \] (2.3.2)

The approximating model (2.3.1) asserts that \( \tilde{\epsilon}_{t+1} = (C' C)^{-1} C' (y_{t+1} - A_0 y_t) \).

When the distorted model generates the data, \( y_{t+1} - A_0 y_t = C \tilde{\epsilon}_{t+1} = C(\epsilon_{t+1} + w_{t+1}) \), which implies that the disturbances under the approximating model appear to be
\[ \tilde{\epsilon}_{t+1} = \epsilon_{t+1} + w_{t+1} \] (2.3.3)
so that misspecification manifests itself in a distortion to the conditional mean of innovations to the state evolution equation.

How close is the approximating model to the distorted model that actually governs the data? To measure the statistical discrepancy between the two models, we use relative entropy defined as
\[
I(f) = \int \log \left( \frac{f(y^*|y)}{f_o(y^*|y)} \right) f_o(y^*) \, dy^*.
\]
where \( f_o \) denotes the one-step transition density associated with the approximating model and \( f \) is the transition density for another model obtained by distorting the approximating model. In the present setting, the transition density for the approximating model is
\[ f_o(y^*|y) \sim \mathcal{N}(Ay + Bu, CC') , \]
while the transition density for the distorted model is assumed to be
\[ f(y^*|y) \sim \mathcal{N}(Ay + Bu + Cw, CC') , \]
where both \( u \) and \( w \) are assumed to be measurable functions of \( y^t \). To evaluate entropy, we first compute the ratio of probability densities (or likelihood functions) of \( y_{t+1} \) under the distorted and the approximating models, conditional on \( y_t \). Recall that \( w_{t+1} \) is measurable with respect to the history \( y^t \). Then conditional on \( y^t \), the log likelihood of \( y_{t+1} \) for the distorted model is
\[
\log L^d = - \log \sqrt{2\pi} - .5 \epsilon_{t+1}' \epsilon_{t+1}.
\]
Using (2.3.3), the conditional log likelihood of \( y_{t+1} \) under the approximating model is
\[
\log L^a = - \log \sqrt{2\pi} - .5 (\epsilon_{t+1} + w_{t+1})' (\epsilon_{t+1} + w_{t+1}).
\]
Therefore, the log likelihood ratio of the distorted model with respect to the approximating model is

\[
\log L^d - \log L^a = .5w'_{t+1}w_{t+1} + w'_{t+1}\epsilon_{t+1}.
\]  \tag{2.3.4}

Define entropy \(I(w_{t+1})\) as the mathematical expectation of the log likelihood ratio (2.3.4), evaluated when the data are generated by the distorted model. Because \(w_{t+1}\) is measurable with respect to the history of \(y_s\) up to \(t\), averaging (2.3.4) over \(\epsilon_{t+1}\) gives the expected log likelihood

\[
I(w_{t+1}) = .5w'_{t+1}w_{t+1}.
\]  \tag{2.3.5}

In chapter 8, we describe how measures like (2.3.5) govern the distribution of test statistics for discriminating among models. In chapter 12, we show how the log likelihood ratio (2.3.4) plays an important role in pricing risky securities when agents prefer a robust decision rule.

As an intertemporal measure of the size of model misspecification, we take

\[
R(w) = 2E_0\sum_{t=0}^{\infty} \beta^{t+1}I(w_{t+1}),
\]  \tag{2.3.6}

where the mathematical expectation conditioned on \(y_0\) is evaluated with respect to the distorted model (2.3.2). Then we impose constraint (2.2.4) on the set of models or equivalently

\[
R(w) \leq \eta_0.
\]  \tag{2.3.7}

In the next section, we seek to construct decision rules that work well enough over a set of models that satisfy (2.3.7). These rules are the best responses for a maximizing player in the equilibrium of a two-player zero sum game.
2.4. Two robust control problems

This section states two robust control problems, the constraint problem and the multiplier problem. The two problems differ in how they implement constraint (2.3.7). Under proper conditions, the two problems have identical solutions. The multiplier problem is a robust version of a stochastic optimal linear regulator. A certainty equivalence principle allows us to compute the optimal decision rule for the multiplier problem by solving a corresponding nonstochastic optimal linear regulator problem.

We state the

**Constraint problem:**

\[
\max_{\{u_t\}_{t=0}^{\infty}} \min_{\{w_{t+1}\}_{t=0}^{\infty}} E_0 \sum_{t=0}^{\infty} \beta^t r(y_t, u_t)
\]

where the extremization is subject to the distorted model (2.2.2) and the entropy constraint (2.3.7), and where \( E_0 \), the mathematical expectation conditioned on \( y_0 \), is evaluated with respect to the distorted model (2.2.2).

Next we state the

**Multiplier problem:** Given \( \theta \in (\theta, +\infty) \), a multiplier problem is

\[
\max_{\{u_t\}_{t=0}^{\infty}} \min_{\{w_{t+1}\}_{t=0}^{\infty}} E_0 \sum_{t=0}^{\infty} \beta^t \{ r(y_t, u_t) + \beta \theta w_{t+1}' w_{t+1} \}
\]

(2.4.1)

where the extremization is subject to the distorted model (2.2.2) and the mathematical expectation is also evaluated with respect to that model.

We shall discuss the lower bound \( \theta > 0 \) extensively in chapter 7. Chapters 6 and 7 state conditions on \( \theta \) and \( \eta_0 \) under which the two problems have identical solutions, namely, decision rules \( u_t = -F y_t \) and \( w_{t+1} = K y_t \). (For the rest of this chapter, we shall use the notation \( F_1 = F, F_2 = -K \), in order to facilitate a simple computational algorithm.) Chapter 6 establishes many useful facts about distinct versions of the multiplier problem that employ alternative timing protocols.\footnote{Following Whittle (1990), extremization means joint maximization and minimization. It is a useful term for describing saddle-point problems.} \( \theta \) or \( \omega \). That discussion justifies solving the multiplier problem

\footnote{For example, one timing protocol has the maximizing \( u \) player first commit at time 0 to an entire sequence, after which the minimizing \( w \) player commits to a sequence. Another timing protocol reverses the order of moves but has each player choose a sequence. Other timing protocols have each player choose sequentially.}
recursively by composing a Bellman equation. Let \(-y_0'Py_0 - p\) be the value of problem (2.4.1). It satisfies the Bellman equation\(^8\)

\[-y'Py - p = \max_u \min_w E \left\{ r(y,u) + \theta \beta w'w - \beta y^*Py^* - \beta p \right\} \quad (2.4.2)\]

where the extremization is subject to

\[y^* = Ay + Bu + C(\epsilon + w) \quad (2.4.3)\]

where * denotes next period’s value, and \(\epsilon \sim N(0, I)\). As a tool to explore the fragility of his decision rule, in (2.4.2) the decision maker pretends that a malevolent nature chooses a feedback rule for a model-misspecification process \(w\).

Thus, to represent the idea that model (2.2.1) is an approximation, the robust version of the linear regulator replaces the single model (2.2.1) with the set of models (2.2.2) that satisfy (2.2.4). We shall soon describe how robust decision rules emerge from the two-player zero-sum game (2.4.1). But first we say more about the kind of certainty equivalence that applies to the multiplier problem.

### 2.4.1. Modified certainty equivalence principle

On page 15, we stated the certainty equivalence principle that applies to the linear quadratic dynamic programming problem without concern for model misspecification. This principle fails to hold when there is concern about model misspecification. But there is another certainty equivalence principle that allows us to work with a non-stochastic version of (2.4.2), i.e., one in which \(\epsilon_t \equiv 0\) in (2.4.3). In particular, it can be verified directly that precisely the same Riccati equations and the same decision rules for \(u_t\) and for \(w_{t+1}\) emerge from solving the random version of the Bellman equation (2.4.2) as would from a version that sets \(\epsilon_{t+1} \equiv 0\). This allows us to drop \(p\) from the value function \(-y'Py - p\), without affecting formulas for the decision rules.\(^9\) Nevertheless, inspection of the Bellman equation and the formula for the decision rule for \(u_t\) show that

---

\(^8\) In chapter 6 we show that the multiplier and constraint problems are both recursive, but that they have different state variables and different Bellman equations. Nevertheless, they lead to identical decision rules for \(u_t\).

\(^9\) The certainty equivalence principle stated here shares with the one on page 15 the facts that \(P\) can be computed before \(p\); it diverges from the certainty equivalence principle without robustness on page 15 in that \(P\) and therefore \(F\) now both depend on the volatility matrix \(C\).
the ‘volatility matrix’ \( C \) does affect the decision rule. Therefore, the version of the certainty equivalence principle stated on page 15 — that the decision rule is independent of the volatility matrix — does not hold with a preference for robustness. This is interesting because of how a preference for robustness creates an avenue for the noise statistics (embedded in the volatility matrix \( C \)) to impinge on decisions even with quadratic preferences and linear transition laws.\(^{10}\) This effect is featured in the precautionary savings model of chapter 9, a simple version of which we shall sketch in section 2.8.

### 2.5. Robust linear regulator

The modified certainty equivalence principle lets us attain robust decision rules by positing the nonstochastic law of motion

\[
y_{t+1} = Ay_t + Bu_t + Cw_{t+1}
\]

with \( y_0 \) given, where the \( w \) process is constrained by the nonstochastic counterpart to (2.2.4). By working with this nonstochastic law of motion, we obtain the robust decision rule for the stochastic problem in which (2.5.1) is replaced by (2.2.2). The approximating model assumes that \( w_{t+1} \equiv 0 \). As we just mentioned, even though randomness has been eliminated, the volatility matrix \( C \) affects the robust decision rule because it influences how the specification errors \( w_{t+1} \) feed back on the state.

To induce a robust decision rule for \( u_t \), we solve the nonstochastic version of the multiplier problem:

\[
\max_{u_t} \min_{w_{t+1}} \sum_{t=0}^{\infty} \beta^t \ [r(y_t, u_t) + \theta \beta w'_{t+1}w_{t+1}]
\]

where the extremization is subject to (2.5.1) and \( y_0 \) is given. In (2.5.2), \( \theta \in [0, +\infty] \) is a penalty parameter restraining the minimizing choice of the \( w_{t+1} \) sequence.

Let \(-y'_0Py_0\) be the value of (2.5.2). It satisfies the Bellman equation\(^{11}\)

\[
-y'_0Py = \max_{u} \min_{w} \{r(y, u) + \theta \beta w'w - \beta y''Py\}
\]

\(^{10}\) The dependence of the decision rule on the volatility matrix is an aspect that attracted researchers like Jacobson (1973) and Whittle (1990) to risk-sensitive preferences (see chapter 17).

\(^{11}\) Notice how this is a special case of (2.4.2) with \( p = 0 \). The modified certainty equivalence principle implies that the same \( P \) matrix solves (2.5.3) and (2.4.2).
where the extremization is subject to
\[ y^* = Ay + Bu + Cw. \] (2.5.4)

In (2.5.3), a malevolent nature chooses a feedback rule for a model-misspecification process \( w \). The inner minimization problem in (2.5.3) induces an operator \( D(P) \) defined by
\[-y''D(P)y^* = \min_w \{ \theta w'w - y''Py^* \} \] (2.5.5)
where the minimization is subject to the transition law \( y^* = Ay + Cw \). From the minimization problem on the right of (2.5.5), it follows that
\[ D(P) = P + \theta^{-1}PC\left(I - \theta^{-1}C'PC\right)^{-1}C'P. \] (2.5.6)

The Bellman equation (2.5.3) can then be represented as
\[-y'Py = \max_u \{ r(y, u) - \beta y''D(P)y^* \} \] (2.5.7)
where now the maximization is subject to the approximating model \( y^* = Ay + Bu \) and concern for misspecification is reflected in our having replaced \( P \) with \( D(P) \) in the continuation value function. This ordinary Bellman equation conceals the activities of the minimizing agent within the operator \( D \) that distorts the continuation value function.

Define \( T(P) \) to be the operator associated with the right side of the ordinary Bellman equation (2.2.6) that we described in (2.2.8):
\[ T(P) = Q + \beta A'PA - \beta^2 A'PB\left(R + \beta B'PB\right)^{-1}B'PA. \] (2.5.8)

Then according to (2.5.7), \( P \) can be computed by iterating to convergence on the composite operator \( T \circ D \) and the robust decision rule can be computed by
\[ u_t = -Fx_t, \] where
\[ F = \beta\left(R + \beta B'D(P)B\right)^{-1}B'D(P)A. \] (2.5.9)
The worst case shock obeys the decision rule \( w_{t+1} = Ky_t \), where
\[ K = \theta^{-1}\left(I - \theta^{-1}C'PC\right)^{-1}C'P(A - BF). \] (2.5.10)

A number of comments about the solution of (2.5.3) are in order.
1. **Interpreting the solution.** The solution of problem (2.5.2), (2.5.1) has a recursive representation in terms of a pair of feedback rules\(^{12}\)

\[
\begin{align*}
  u_t &= -F y_t \quad \text{(2.5.11a)} \\
  w_{t+1} &= Ky_t \quad \text{(2.5.11b)}
\end{align*}
\]

Here \(u_t = -F y_t\) is the robust decision rule for the control \(u_t\), while \(w_{t+1} = Ky_t\) describes a worst case shock. This worst-case shock induces a distorted transition law

\[
y_{t+1} = (A + CK) y_t + Bu_t.
\]

After having discovered (2.5.12), we can regard the decision maker as devising a robust decision rule by choosing a sequence \(\{u_t\}\) to maximize

\[
- \sum_{t=0}^{\infty} \beta^t [y'_t Q y_t + u'_t R u_t]
\]

subject to (2.5.12). However, as noted above, the decision maker believes that the data are actually generated by a model with an unknown process \(w_{t+1} = \tilde{w}_{t+1} \neq 0\). By planning against the worst case process \(w_{t+1} = Ky_t\), he designs a robust decision rule. The worst-case transition law is endogenous and depends on \(\theta\). Equation (2.5.12) incorporates how the distortion \(w\) feeds back on the state vector \(y\); it permits \(w\) to feedback on endogenous components of the state, meaning that the decision maker indirectly influences future values of \(w\) through his decision rule. Allowing the distortion to depend on endogenous state variables in this way may or may not be a useful way to think about model misspecification. But there is an alternative interpretation that excludes feedback of \(w\) on endogenous state variables, which we take up next.

2. **Reinterpreting the solution.** We shall sometimes find it useful to reinterpret the solution of the robust linear regulator problem (2.5.1), (2.5.2) so that the decision maker believes that the distortions \(w\) do not depend on those endogenous components of the state vector whose motion his decisions affect. In particular, in chapter 6, we show that the robust decision rule \(u_t = -F y_t\) solves the ordinary linear regulator problem

\[
\max_{\{u_t\}} \sum_{t=0}^{\infty} \beta^t r (y_t, u_t)
\]

\(^{12}\) In sections 2.7 and 2.10, we’ll set \(F = F_1\) and \(K = -F_2\) to attain a simple algorithm for computing \(F, K\).
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subject to the distorted transition law

\[ y_{t+1} = Ay_t + By_t + Cw_{t+1} \]  \hspace{1cm} (2.5.14a)
\[ w_{t+1} = KY_t \]  \hspace{1cm} (2.5.14b)
\[ Y_{t+1} = A^*Y_t \]  \hspace{1cm} (2.5.14c)

where \( A^* = A - BF + CK \), where \((F,K)\) solve problem (2.5.2), (2.5.1), and where we impose the initial condition \( Y_0 = y_0 \). In (2.5.14), the maximizing player views \( Y_t \) as an exogenous state vector that propels the distortion \( w_{t+1} \) that twists the law of motion for state vector \( y_t \). The solution of (2.5.13), (2.5.14) has the outcome that \( Y_t = y_t \ \forall t \geq 0 \). Chapters 6 and 7 show how formulation (2.5.13), (2.5.14) emerges from a version of the multiplier problem that imposes a timing protocol in which the minimizing agent at time 0 commits to an entire sequence of distortions \( \{w_{t+1}\}_{t=0}^{\infty} \) and in which it is optimal for the minimizing agent to make \( w_{t+1} \) obey (2.5.14b), (2.5.14c). As we shall see in chapter 7, this formulation helps us to interpret frequency domain criteria for inducing robust decision rules. In addition, the transition law (2.5.14) rationalizes a Bayesian interpretation of the robust decision maker’s behavior by identifying a particular belief about the shocks for which the maximizing player’s decision rule is optimal. This observations prompts us to take up some ideas of Fellner.

3. Relation to Fellner. In the introduction to Probability and Profit, Fellner wrote:

\[ \ldots \text{the central problems of decision theory may be described in semiprobabilistic views. By this I mean to say that in my opinion the directly observable weights which reasonable and consistent individuals attach to specific types of prospects are not necessarily the genuine (undistorted) subjective probabilities of the prospects, although these decision weights of consistently acting individuals do bear an understandable relation to probabilities.} \ldots \text{the decision weights which these decision-makers attach to alternative monetary prospects need not be universally on par} \]

\[ \text{13 In contrast to formulation (2.5.1), (2.5.2), in problem (2.5.13), (2.5.14) the maximizing agent does not believe that he his decisions can influence the future position of the distortion } w. \text{ In some applications, we might actually prefer interpretation (2.5.1), (2.5.2) depending on the types of perturbations to the approximating model that the maximizing agent wants to protect against.} \]

\[ \text{14 A decision rule is said to have a Bayesian interpretation if it is undominated in the sense of being optimal for some model. See REFERENCE ???? (Blackwell-Girshick???).} \]
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with probabilities attached to head-or-tails events but may in cases be derived from such probabilities by “slanting” or “distortion.” Slanting expresses an allowance for the instability and controversial character of some types of probability judgment; the extent to which may even depend on the magnitude of the prize which is at stake when a prospect is being weighted.

Robust control theory contains concepts that embody some of Fellner’s ideas. Thus, the ‘decision weights’ implied by the ‘slanted’ transition law (2.5.14) differ from the ‘subjective probabilities’ implied by the approximating model (2.2.1). Through the dependence of $K$ on the parameters of the discounted return function, namely, $\beta, R, Q$, the distortion or slanting is context-specific.

4. Robustness bound. Let $A_o = A - BF$ for a fixed $F$ in a feedback rule $u = -Fy$. In chapter 6 on page 132, we show that equation (2.5.7) implies that

$$- (A_o y + Cw)' P (A_o y + Cw) \geq -y' A'_o D (P) A_o y - \theta w' w. \quad (2.5.15)$$

The quadratic form in $y$ on the right is a conservative estimate of the continuation value of the state $y^*$ under the approximating model $y^* = A_o y$.\(^\text{15}\) Inequality (2.5.15) says that the continuation value is at least as great as a conservative estimate of the continuation value, minus $\theta$ times the measure of model misspecification $w' w$. The parameter $\theta$ influences the conservative-adjustment operator $D$ and also determines the rate at which the bound deteriorates with misspecification. Lowering $\theta$ lowers the rate at which the bound deteriorates with misspecification. Thus, (2.5.15) provides a sense in which lower values of $\theta$ provide more conservative and also more robust estimates of continuation utility.

5. Alternative games with identical outcomes. The game (2.5.2) summarized by the Bellman equation (2.5.3) is one of several two-player zero-sum games with identical lists of players and payoffs but different timing protocols, i.e., strategy spaces and structures of commitment differ. Chapter 6 describes the relationships among these games and the remarkable fact that they have identical outcomes. The analysis of chapter 6 justifies using recursive methods to solve all of the games. That chapter also discusses senses in which the decision maker’s preferences are dynamically consistent.

\(^{15}\) That is, when $w = 0$, $-(A_o y)' D (P) A_o y$ understates the continuation value.
6. Approximating and worst-case models. The behavior of the state under the robust decision rule and the worst case model can be represented

\[ y_{t+1} = Ay_t - BFy_t + CKy_t. \]  

(2.5.16)

However, the agent does not really believe that the worst-case shock process will prevail. He designs his decision rule by using \( w_{t+1} = Ky_t \) to slant the transition law in order to acquire a rule that will be robust against a range of departures from his approximating model. We shall want to evaluate the performance of the robust decision rule under other models. In particular, we often want to evaluate that rule when the approximating model governs the data (so that the decision maker’s fears of model misspecification are in the end unfounded). Under the robust decision rule but the approximating model, the law of motion is

\[ y_{t+1} = (A - BF) y_t. \]  

(2.5.17)

We obtain (2.5.17) from (2.5.16) by replacing the worst case shock \( Ky_t \) by zero. Notice that although we set \( K = 0 \) in (2.5.16) to get (2.5.17), \( F \) in (2.5.16) embodies a best response to \( K \), and thereby reflects the agent’s 'pessimistic' forecasts of future values of the state. We call (2.5.17) the approximating model and (2.5.16) the worst-case or distorted model under the robust decision rule.\(^\text{16}\) In chapter 12, we use stochastic versions of both the approximating model (2.5.17) and the distorted model (2.5.16) to express alternative formulas for the prices of risky assets when consumers fear model misspecification.

7. Breakdown point and \( H_\infty \) control. Starting from \( \theta = +\infty \), pushing \( \theta \) lower increases the preference for robustness by lowering the shadow price on the norm of the control of the minimizing player. We shall see in chapter 7 that there is a lower bound below which \( \theta \) cannot be pushed. This lower bound is associated with the largest set of alternative models, as measured by entropy, against which it is feasible to seek a robust rule: for values of \( \theta \) below this bound, the minimizing agent is penalized so little that he finds it possible to choose a distortion that sends the criterion function to \(-\infty\). Control theorists are interested in the cutoff value of \( \theta \) because of how it is can be used to compute a rule that is robust to the biggest allowable misspecifications. Although we describe the associated \( H_\infty \) control theory in chapter 7, for applications in economics we are interested in values of

\(^\text{16}\) The model with randomness adds \( C\epsilon_{t+1} \) to the right side of (2.5.17).
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θ that usually far exceed the cutoff value. Instead, in chapter 17, we use detection error probabilities to discipline our choice of θ in applications.

8. Risk-sensitive preferences. An interesting fact is that we can suppress the doubts about model specification and instead adjust attitudes toward risk in a way that preserves the decision rule and value function computed above. Thus, the decision rule \( u_t = -Fx_t \) that solves the robust control problem also solves a particular stochastic infinite horizon discounted control problem in which the decision maker has no concern about model misspecification but instead adjusts continuation values to express an additional version of risk. The additional adjustment is a special case of Epstein and Zin’s (1989) recursive specification of utility and is governed by a parameter \( \sigma < 0 \). If we set \( \sigma = -\theta^{-1} \) from the robust control problem, we recover the same decision rule for the two problems.

The risk-sensitive decision maker has no doubt that the law of motion for the state is

\[
y_{t+1} = Ay_t + Bu_t + C\epsilon_{t+1}
\]

(2.5.18)

where \( \{\epsilon_{t+1}\} \) is a sequence of i.i.d. Gaussian random vectors with mean zero and identity covariance matrix. The utility index of the decision maker is defined recursively as the fixed point \( U_0 \) of recursions on

\[
U_t = r(y_t, u_t) + \beta R_t(U_{t+1})
\]

(2.5.19)

where

\[
R_t(U_{t+1}) = \frac{2}{\sigma} \log E \left[ \exp \left( \frac{\sigma U_{t+1}}{2} \right) \right] y^t
\]

(2.5.20)

and where \( \sigma \leq 0 \) is the risk-sensitivity parameter. When \( \sigma = 0 \), an application of l’hospital’s rule shows that \( R_t \) becomes the ordinary conditional expectation operator \( E(\cdot | y^t) \). When \( \sigma < 0 \), \( R_t \) puts an additional adjustment for risk into the assessment of continuation values.

For the quadratic \( r(y, u) \) that we have assumed, the Bellman equation for the risk-sensitive control problem of Hansen and Sargent (1995) is

\[
-y'Py - \hat{p} = \max_u \{ r(y, u) + \beta R(-y''Py^* - \hat{p}) \}
\]

(2.5.21)

where the maximization is subject to \( y^* = Ay + Bu + C\epsilon \) and where \( \epsilon \) is a Gaussian vector with mean zero and identity covariance matrix.

Using a result from Jacobson (1973), it can be shown that

\[
R(-y''Py^* - \hat{p}) = -(Ay + Bu)'D(P)(Ay + Bu) - p(P, \hat{p})
\]

(2.5.22)
where $\mathcal{D}$ is the same operator defined in (2.5.6) with $\theta = -\sigma^{-1}$, and the operator $p$ is defined by

$$p(P, \hat{p}) = p - \sigma^{-1} \log \det (I + \sigma C'PC). \quad (2.5.23)$$

Consequently, the Bellman equation for the infinite-horizon discounted risk-sensitive control problem can be expressed as

$$-y'Py - \hat{p} = \max_u \{r(y, u) - \beta (Ay + Bu)' \mathcal{D}(P) (Ay + Bu) - \beta p(P, \hat{p})\}. \quad (2.5.24)$$

Evidently, $P = T \circ \mathcal{D}(P)$, and so is the same $P$ that appears in the Bellman equation (2.4.2) for the robust control problem. The constant $\hat{p}$ that solves (2.5.24) differs from the $p$ in (2.4.2), but since they depend only on $P$ and not on $p$ or $\hat{p}$, the decision rules are the same for the two problems.

### 2.6. More general misspecifications

Thus far, we have permitted the decision maker to seek robustness against misspecifications that occur only as a distortion $w_{t+1}$ to the conditional mean of the innovation to the state $y_{t+1}$. Where the approximating model has the Gaussian form (2.2.1), this is less restrictive than may at first appear. In chapter 17, we allow a more general class of misspecifications to the linear Gaussian model (2.2.1), but nevertheless find that important parts of the preceding results survive. For convenience, express (2.2.1) in the compact notation

$$f_o(y^*|y) \sim \mathcal{N}(Ay + Bu, CC'),$$

which portrays the conditional distribution of next period’s state as Gaussian with mean $Ay + Bu$ and covariance matrix $CC'$. Let $f(y^*|y)$ be an arbitrary alternative conditional distribution that puts positive probability on the same events as does the approximating model $f_o$. Define the entropy of model $f$ relative to the approximating model $f_o$ as

$$I(f) = \int \log \left( \frac{f(y^*|y)}{f_o(y^*|y)} \right) f(y^*|y) \, dy^*. $$

Entropy $I(f)$ is thus the expectation of the log likelihood ratio evaluated with respect to the distorted model $f$. A multiplier robust control problem is associated with the following Bellman equation:

$$-y'Py - p = \max_u \min_f \{r(y, u) + 2\theta \beta I(f) - \beta y^*Py^* - \beta p\} \quad (2.6.1)$$
Let \( \sigma = -\theta^{-1} \) and consider the inner minimization problem, assuming that \( u = -Fy \). On page 333, we shall show that the extremizing \( f \) is the Gaussian distribution

\[
f (y^*|y) \sim \mathcal{N} \left( Ay - BFy + CKy, \hat{C}\hat{C}' \right)
\]  

(2.6.2)

where \((F,K)\) are the same matrices appearing in (2.5.11) that solve the multiplier robust control problem, where

\[
\hat{C}\hat{C}' = C \left( I + \sigma C'PC \right)^{-1} C',
\]

(2.6.3)

and where \( P \) is the same \( P \) that appears in the solution of the Bellman equation for the deterministic multiplier robust control problem (2.5.3). Equation (2.6.2) assures us that when we allow the minimizing player to choose a general misspecification \( f(y^*|y) \), he chooses a Gaussian distribution with the same mean distortion as when we let him distort only the mean of a Gaussian conditional distribution. However, formula (2.6.3) shows that the minimizing agent would also distort the covariance matrix of the innovations, if given a chance.

The upshot of these findings is that when the conditional distribution \( f(y^*|y) \) for the approximating model is Gaussian, even if we actually were to permit general misspecifications \( f(y^*|y) \), we could compute the worst-case \( f \) by solving a deterministic multiplier robust control problem for \( P,F,K \), and then use \( P \) to compute the appropriate adjustment to the covariance matrix (2.6.3). In chapter 12, we use some of these ideas to price assets under alternative assumptions about the set of models against which decision makers seek robustness.

### 2.7. A simple algorithm

Chapter 6 discusses several algorithms for solving (2.5.3) and relationships among them. This section describes perhaps the simplest, an adapted ordinary optimal linear regulator. Chapters 6 and 7 describe necessary technical conditions, including restrictions on the magnitude of the multiplier parameter \( \theta \).

Application of the ordinary optimal linear regulator can be justified by noting that the Riccati equation for the optimal linear regulator emerges from first-order conditions alone, and that the first-order conditions for extremizing (i.e., finding the saddle point by simultaneously minimizing with respect to \( w \) and

\[ 17 \] The Matlab program cheap9.m described in the appendix implements this algorithm; doublenx9.m implements a doubling algorithm of the kind described in chapter 3 and Hansen and Sargent (XXXXbook).
maximizing with respect to $u$) the right side of (2.5.3) match those for an ordinary (non-robust) optimal linear regulator with joint control process \{${u_t, w_{t+1}}$\}. This insight allows us to solve (2.5.3) by forming an appropriate optimal linear regulator.

Thus, put the Bellman equation (2.5.3) into a more compact form by defining

\[
\begin{align*}
\tilde{B} &= \begin{bmatrix} B & C \end{bmatrix} \\
\tilde{R} &= \begin{bmatrix} R & 0 \\ 0 & -\beta \theta I \end{bmatrix} \\
\tilde{u}_t &= \begin{bmatrix} u_t \\ w_{t+1} \end{bmatrix}.
\end{align*}
\]

Let \text{ext} denotes ‘extremization’ – maximization with respect to $u$, minimization with respect to $w$. The Bellman equation can be written

\[
-y'Py = \text{ext}_u \left\{ -y'Qy - \tilde{u}\tilde{R}\tilde{u} - \beta y^*P y^* \right\}
\]

where the extremization is subject to

\[
y^* = Ay + \tilde{B}\tilde{u}.
\]

The first-order conditions for problem (2.7.2), (2.7.3) imply the matrix Riccati equation

\[
P = Q + \beta A'PA - \beta^2 A'PB \left( \tilde{R} + \beta \tilde{B}'P\tilde{B} \right)^{-1} \tilde{B}'PA
\]

and the formula for $F$ in the decision rule $\tilde{u}_t = -Fy_t$

\[
F = \beta \left( \tilde{R} + \beta \tilde{B}'P\tilde{B} \right)^{-1} \tilde{B}'PA.
\]

Partitioning $F$, we have

\[
\begin{align*}
u_t &= -Fy_t \quad \text{(2.7.6a)} \\
w_{t+1} &= Ky_t \quad \text{(2.7.6b)}
\end{align*}
\]

The decision rule $u_t = -Fy_t$ is the robust rule. As mentioned above, $w_{t+1} = Ky_t$ provides the $\theta$-constrained worst-case specification error. We can solve the Bellman equation by iterating to convergence on the Riccati equation (2.7.4), or by using one of the faster computational methods described in chapter 3.
2.7.1. Interpretation of the simple algorithm

The adjusted Riccati equation (2.7.4) is an augmented version of the Riccati equation (2.2.8) that is associated with the ordinary optimal linear regulator. The right side of equation (2.7.4) defines one step on the composite operator $T \circ D$ where $T$ and $D$ are defined in (2.5.8) and (2.5.5).\footnote{This can be verified by unstacking the matrices in (2.7.4). See page 132 in chapter 6.} Chapter 7 connects the $D$ operator to the Hansen and Sargent’s (1995) discounted version of the risk-sensitive preferences of Jacobson (1973) and Whittle (1990).

2.8. Example: robustness and discounting in a permanent income model

This section illustrates various aspects of robust control theory in the context of a linear-quadratic version of a simple permanent income model.\footnote{See Sargent (1987) and Hansen, Roberds, and Sargent (1991) for accounts of the connection between the permanent income consumer and Barro’s (1979) model of tax smoothing. See Aiyagari, Marcet, Sargent, and Seppälä (2003) for a more extensive exploration of the connections.} In the basic permanent income model, a consumer applies a single marginal propensity to consume to the sum of his financial wealth and his human wealth, where human wealth is defined as the expected present value of his labor (or endowment) income discounted at the same risk-free rate of return that he earns on his financial assets. In the usual permanent income model without a preference for robustness, the consumer has no doubts about the probability model used to form the conditional expectation of discounted future labor income. Under a preference for robustness, the consumer doubts that model and therefore forms forecasts of future income by using a probability distribution that is twisted or slanted relative to his approximating model for his endowment. Except for this slanting, the consumer behaves as an ordinary permanent income consumer.

We show that this slanting of probabilities leads the consumer to engage in a form of precautionary savings that under the approximating model tilts his consumption profile toward the future relative to what it would be without a preference for robustness. Indeed, we shall establish that so far as his consumption and saving program is concerned, activating a preference for robustness is equivalent with making the consumer more patient. However, that is not the end of the story. In chapter 12, we shall see that attributing a preference for...
2.8.1. The LQ permanent income model

In Hall’s (1978) linear-quadratic permanent income model, a consumer receives an exogenous endowment process \( \{d_t\} \) and wants to allocate it between consumption \( c_t \) and savings \( k_t \) to maximize

\[
-E_0 \sum_{t=0}^{\infty} \beta^t (c_t - b)^2, \quad \beta \in (0, 1).
\]

We simplify the problem by assuming that the endowment process is a first-order autoregression. Thus, the household faces the state transition laws

\[
k_t + c_t = Rk_{t-1} + d_t \tag{2.8.2a}
\]
\[
d_{t+1} = \mu_d (1 - \rho) + \rho d_t + c_d (\epsilon_{t+1} + w_{t+1}), \tag{2.8.2b}
\]

where \( R > 1 \) is a time-invariant gross rate of return on financial assets \( k_{t-1} \) held at the end of period \( t - 1 \), and \(|\rho| < 1\) describes the persistence of his endowment process. In (2.8.2b), \( w_{t+1} \) is a distortion to the mean of the endowment process that represents possible model misspecification. We use \( \sigma = -\theta^{-1} \) to parameterize the consumer’s preference for robustness. Soon we’ll confirm how easily this problems maps into the robust linear regulator. But first, we’ll use classical methods to elicit some useful properties of the solution when \( \sigma = 0 \).

2.8.2. Solution when \( \sigma = 0 \)

First, we solve the household’s problem without a preference for robustness, so that \( \sigma = 0 \). The household’s Euler equation can be expressed as

\[
E_t \mu_{c,t+1} = (\beta R)^{-1} \mu_{ct}. \tag{2.8.3}
\]

Treating (2.8.2a) as a difference equation in \( k_t \), solving it forward in time, and taking conditional expectations on both sides gives

\[
k_{t-1} = \sum_{j=0}^{\infty} R^{-(j+1)} E_t (c_{t+j} - d_{t+j}). \tag{2.8.4}
\]

Solving (2.8.3) and (2.8.4) and using \( \mu_{ct} = b - c_t \) gives the following representation for \( \mu_{ct} \):

\[
\mu_{ct} = -(1 - R^{-2} \beta^{-1}) \left( Rk_{t-1} + E_t \sum_{j=0}^{\infty} R^{-j} (d_{t+j} - b) \right) \tag{2.8.5}
\]
Equations (2.8.3) and (2.8.5) can be used to deduce the following representation for $\mu_{c,t+1}$

$$\mu_{c,t+1} = (\beta R)^{-1} \mu_{c,t} + \nu \epsilon_{t+1}.$$  \hfill (2.8.6)

We shall provide a formula for the scalar $\nu$ in formula (2.8.11) below.

Given an initial condition $\mu_{c,0}$, equation (2.8.6) describes the consumer’s optimal behavior. This initial condition is determined by solving (2.8.5) at $t = 0$. Note that it is easy to use (2.8.5) to deduce an optimal consumption rule of the form

$$c_t = gy_t$$

where $g$ is a vector conformable to $y$. In the case $\beta R = 1$ that was analyzed by Hall (1978), (2.8.6) implies that the marginal utility of consumption $\mu_{ct}$ is a martingale, which because $\mu_{ct} = b - c_t$ in turn implies that consumption itself is a martingale.

2.8.3. Linear regulator for permanent income model

This problem is readily mapped into a linear regulator in which the marginal utility of consumption $b - c_t$ is the control. Express the transition law for $k_t$ as

$$k_t = R k_{t-1} + d_t - b + (c_t - b).$$

Define the state as $y_t' = [1 \ k_{t-1} \ d_t]$ and the control as $u_t = \mu_{ct} \equiv (b - c_t)$ and express the state transition law as $y_{t+1} = Ay_t + Bu_t + C(\epsilon_{t+1} + w_{t+1})$ or

$$\begin{bmatrix} 1 \\ k_t \\ d_{t+1} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -b & R & 1 \\ 0 & 0 & \rho \end{bmatrix} \begin{bmatrix} 1 \\ k_{t-1} \\ d_t \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ e_{c_d} \end{bmatrix} (b - c_t) + \begin{bmatrix} 0 \\ 0 \\ \epsilon_{t+1} + w_{t+1} \end{bmatrix} \hfill (2.8.7)$$

This equation defines the $A, B, C$ associated with a robust linear regulator. For the objective function, (2.8.1) implies that we should let $r(y, u) = -y/R y - w Qu$ where $R = 0_{3x3}$ and $Q = 1$.

We can obtain a robust rule by using the robust linear regulator and setting $\sigma < 0$. The solution of the robust linear regulator problem is a linear decision rule for the control $\mu_{ct}$:

$$\mu_{ct} = -F y_t.$$  \hfill (2.8.8)

Under the approximating model, the law of motion of the state is then

$$y_{t+1} = (A - BF) y_t + C \epsilon_{t+1}. \hfill (2.8.9)$$
Basic ideas and methods

Equations (2.8.8) and (2.8.9) imply that
\[ \mu_{c,t+1} = -F(A - BF) y_t - FC\epsilon_{t+1}. \]  
(2.8.10)

Comparing (2.8.10) and (2.8.6) shows that \(-F(A - BF) = -(\beta R)^{-1}F\) and
\[ \nu = -FC, \]  
(2.8.11)

which is the formula for \(\nu\) promised above.

2.8.4. Effects on consumption of preference for robustness

To understand the effects on consumption of a preference for robustness, we use as a benchmark Hall’s case of \(\beta R = 1\) and no preference for robustness \((\sigma = 0)\). In that case, we have seen that \(\mu_{c,t}\) and consumption are both driftless random walks. To be concrete, we set the parameters of our example to be consistent with ones calibrated from post-World War U.S. time series by Hansen, Sargent, and Tallarini (1999) for a more general permanent income model. HST set \(\beta = .9971\) and fit a two-factor model for the endowment process; each factor is a second order autoregression. To simplify their specification, we replace their estimated two-factor endowment process with the population first-order autoregression one would obtain if that two factor model actually generated the data. Thus, if use the population moments implied by Hansen, Sargent, and Tallarini’s (HST’s) estimated endowment process to fit the first-order autoregressive process (2.8.2b) with \(w_{t+1} \equiv 0\), we obtain the endowment process \(d_{t+1} = .9992d_t + 5.5819\epsilon_{t+1}\) where \(\epsilon_{t+1}\) is an i.i.d. scalar process with mean zero and unit variance.\(^{20}\) We use \(\hat{\beta}\) to denote HST’s value of \(\beta = .9971\). Throughout, we suppose that \(R = \hat{\beta}^{-1}\).

We now consider three cases.

- The \(\beta R = 1, \sigma = 0\) case studied by Hall (1978). With \(\beta = \hat{\beta}\), we compute that the marginal utility of consumption follows the law of motion
\[ \mu_{c,t+1} = \mu_{c,t} + 4.3825\epsilon_{t+1}, \]  
(2.8.12)

where we compute the coefficient 4.3825 on \(\epsilon_{t+1}\) by noting that it equals \(-FC\) by formula (2.8.11).

\(^{20}\) We computed \(\rho, c_d\) by calculating autocovariances implied by HST’s specification, then used them to calculate the implied population first-order autoregressive representation.
Basic ideas and methods

• A version of Hall’s $\beta R = 1$ specification with a preference for robustness. Retaining $\hat{\beta}R = 1$, we activate a preference for robustness by setting $\sigma = \hat{\sigma} - 2E - 7 < 0$.\textsuperscript{21} We now compute that\textsuperscript{22}

$$\mu_{c,t+1} = .9976\mu_{c,t} + 8.0473\epsilon_{t+1}. \quad (2.8.13)$$

When $b - c_t > 0$, this equation implies that $E_t(b - c_{t+1}) = .9976(b - c_t) < (b - c_t)$ which in turn implies that $E_t(c_{t+1} > c_t$. Thus, the effect of activating a preference for robustness is to put upward drift into the consumption profile, a manifestation of a kind of ‘precautionary savings’.

• A case that raises the discount factor relative to the $\beta R = 1$ benchmark prevailing in Hall’s model, but withholds a preference for robustness. In particular, while we set $\sigma = 0$ we increase $\beta$ to $\tilde{\beta} = .9995$. Remarkably, with $(\sigma, \beta) = (0, \tilde{\beta})$, we compute that $\mu_{c,t+1}$ continues to obey exactly (2.8.13).\textsuperscript{23} Thus, starting from $(\sigma, \beta) = (0, \tilde{\beta})$, in so far as the effects on consumption and saving are concerned, activating a preference for robustness by lowering $\sigma$ so that $\sigma < 0$ while keeping $\beta$ constant is evidently equivalent to keeping $\sigma = 0$ but increasing the discount factor to a particular $\tilde{\beta} > \beta$.

These numerical examples illustrate what is true more generally, that in the permanent income model an increased preference for robustness operates exactly like an increase in the discount factor $\beta$. In chapter 9, using analytical techniques, we extend these numerical examples within a broader class of permanent income models. In particular, let $\alpha^2 = \nu'\nu$ and suppose that instead of the particular pair $(\hat{\sigma}, \hat{\beta})$, where $(\hat{\sigma} < 0)$, we use the pair $(0, \hat{\beta})$, where $\hat{\beta}$ satisfies:

$$\hat{\beta}(\sigma) = \frac{\hat{\beta}(1 + \hat{\beta})}{2(1 + \sigma\alpha^2)} \left[ 1 + \sqrt{1 - 4\hat{\beta} \frac{1 + \sigma\alpha^2}{(1 + \hat{\beta})^2}} \right]. \quad (2.8.14)$$

\textsuperscript{21} We discuss how to calibrate $\sigma$ in chapters 9, 12, and 17.

\textsuperscript{22} We discover this formula computationally as follows. Use \texttt{doublex9} to solve the robust optimal linear regulator and compute representations $\mu_{c,t} = -Fy_t$ and compare it to the term $F(A - BF)y_t$ on the right side of (2.8.10) to discover that $F(A - BF) = .9976F$ i.e., the coefficients are proportional with .9976 being the factor of proportionality.

\textsuperscript{23} We discover this computationally using the method of the previous footnote.
Then the laws of motion for $\mu_{c,t}$, and therefore the decision rules for $c_t$, are identical across these two preference specifications. We establish formula (2.8.14) in appendix B of chapter 9.

### 2.8.5. Equivalence of quantities but not continuation values

We have found that, holding other parameters constant, there exists a locus of $(\sigma, \beta)$ pairs that imply the same consumption, saving programs. It can be verified that the $P$ matrices appearing in the quadratic forms in the value function are identical for the $(\hat{\sigma}, \hat{\beta})$ and $(0, \tilde{\beta})$ problems. However, in terms of their implications for pricing claims on risky future payoffs, it is significant that the $D(P)$ matrices differ across such $(\sigma, \beta)$ pairs. For the $(0, \tilde{\beta})$ pair, $P = D(P)$. However, when $\sigma < 0$, $D(P)$ differs from $P$. As we shall see in chapter 12, $D(P)$ encodes the shadow prices that are relevant for pricing uncertain claims on future consumption. Thus, although the $(\hat{\sigma}, \hat{\beta})$ and $(0, \tilde{\beta})$ parameter settings imply identical savings and consumption plans, they imply different valuations of risky future consumption payoffs. In chapter 9, we use this fact to study how a preference for robustness influences the equity premium.

#### 2.8.6. Distorted endowment process

On page 23, we described a particular distorted transition law associated with the worst case shocks $w_{t+1} = Ky_t$. If the decision maker solves an ordinary dynamic programming program without a preference for robustness but substitutes the distorted transition law for the one given by his approximating model, he attains robust decision rules. Thus, when $\sigma < 0$, instead of facing the transition law (2.8.7) that prevails under the approximating model, the household would use the distorted transition law\(^ {21} \)

$$
\begin{bmatrix}
y_{t+1} \\
y_{t+1}
\end{bmatrix} =
\begin{bmatrix}
A & CK \\
0 & (A - BF + CK)
\end{bmatrix}
\begin{bmatrix}
y_t \\
y_t
\end{bmatrix} +
\begin{bmatrix}
B \\
0
\end{bmatrix} \mu_{c,t} +
\begin{bmatrix}
C \\
C
\end{bmatrix} \epsilon_{t+1}.
$$

(2.8.15)

For our numerical example with $\sigma = -2E - 7$, we would have $A - BF + CK =
\begin{bmatrix}
1.0000 & 0 & 0 \\
15.0528 & 0.9976 & -0.4417 \\
-0.0558 & 0.0000 & 1.0016
\end{bmatrix}$ and $CK =
\begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
-0.0558 & 0.0000 & 0.0024
\end{bmatrix}$. Notice the pattern of zeros in $CK$, which shows that the distortion to the law of motion of the state affects only the $d_t$ component of the component of the state $y$. The components $Y$ of the state are information variables that account for the dynamics.

\(^ {21} \)This is not a minimal state representation because we have not eliminated the constant from the $Y$ component of the state.
in the misspecification imputed by the worst case shock $w$. In chapter 9, we shall analyze the behavior of the endowment process under the distorted model (2.8.15).

It is useful to consider our observational equivalence result in light of the distorted law of motion (2.8.15). Let $\hat{E}_t$ denote a conditional expectation with respect to the distorted transition law (2.8.15) for the endowment shock and let $E_t$ denote the expectation with respect to the approximating model. Then the observational equivalence of the pairs $(\hat{\sigma}, \hat{\beta})$ and $(0, \tilde{\beta})$ means that the following two versions of (2.8.5) imply the same \( \mu_{ct} \) processes:

$$
\mu_{ct} = -\left(1 - R^{-2}\hat{\beta}^{-1}\right) \left(Rk_{t-1} + \hat{E}_t \sum_{j=0}^{\infty} R^{-j} (d_{t+j} - b)\right)
$$

and

$$
\mu_{ct} = -\left(1 - R^{-2}\tilde{\beta}^{-1}\right) \left(Rk_{t-1} + E_t \sum_{j=0}^{\infty} R^{-j} (d_{t+j} - b)\right).
$$

For both of these expressions to be true, the effect on $\hat{E}_t$ of setting $\sigma$ less than zero must be just offset by the effect of raising $\beta$ from $\hat{\beta}$ to $\tilde{\beta}$.

2.8.7. Representing misspecification: a Stackelberg formulation

In chapters 6, 7, we show the equivalence of outcomes under different timing protocols for the two-player zero-sum games that we use to deduce robust decision rules. In appendix B of chapter 9, we shall use a Stackelberg game to establish the observational equivalence for consumption, savings plans of $(0, \tilde{\beta})$ and $(\hat{\sigma}, \hat{\beta})$ pairs. The minimizing player’s problem in the Stackelberg game can be represented in the form:

$$
\min_{\{w_{t+1}\}} \sum_{t=0}^{\infty} \hat{\beta}^t \left\{ \mu_{ct}^2 + \hat{\beta}\sigma^{-1} w_{t+1}^2 \right\}
$$

subject to

$$
\mu_{c,t+1} = \left(\hat{\beta}R\right)^{-1} \mu_{c,t} + \nu w_{t+1}.
$$

Equation (2.8.17) is the consumption Euler equation of the maximizing player (the household). Under the Stackelberg timing, the minimizing player commits to a sequence $\{w_{t+1}\}_{t=0}^{\infty}$ which the maximizing player takes as given. The minimizing player determines that sequence by solving (2.8.16), (2.8.17). The worst
case shock that emerges from this problem satisfies \( w_{t+1} = k \mu_{t+1} \) and is identical to the worst case shock \( w_{t+1} = K y_t \) that emerges from the robust linear regulator for the consumption problem.

### 2.9. Stabilizing property of shadow price \( P y_t \)

In chapter 3, we solve problem (2.5.2), (2.5.1) with Lagrangian methods. This provides a fast way to compute \( P \). It also gives insights about a recursive representation \( \mu_t = P y_t \), where \(-2 \beta^{t+1} \mu_{t+1}\) is the vector of shadow prices on the time \( t+1 \) state vector. Furthermore, the Lagrangian formulation is convenient for designing decision rules for Ramsey and Stackelberg problems, as we shall show in section 2.10 and chapter 16. We form the Lagrangian

\[
L = -\sum_{t=0}^{\infty} \beta^t \left[ y'_t Q y_t + u'_t R u_t + 2 \beta \mu'_{t+1} (Ay_t + Bu_t + Cw_{t+1} - y_{t+1}) - \theta \beta w'_{t+1} w_{t+1} \right].
\]

(2.9.1)

We want to maximize (2.9.1) with respect to sequences for \( u_t \) and \( y_{t+1} \) and minimize it with respect to a sequence for \( w_{t+1} \). The first-order conditions with respect to \( u_t, y_t, w_{t+1}, \) respectively, are:

\[
0 = Ru_t + \beta B' \mu_{t+1} \quad \text{(2.9.2a)}
\]

\[
\mu_t = Qy_t + \beta A' \mu_{t+1} \quad \text{(2.9.2b)}
\]

\[
0 = \beta \theta w_{t+1} - \beta C' \mu_{t+1}. \quad \text{(2.9.2c)}
\]

Solving (2.9.2a) and (2.9.2c) for \( u_t \) and \( w_{t+1} \) and substituting into (2.5.1) gives

\[
y_{t+1} = Ay_t - \beta (BR^{-1}B' - \beta^{-1} \theta^{-1} CC') \mu_{t+1}. \quad \text{(2.9.3)}
\]

Write (2.9.3) as

\[
y_{t+1} = Ay_t - \beta \hat{B} \tilde{R}^{-1} \hat{B}' \mu_{t+1}. \quad \text{(2.9.4)}
\]

We represent the system formed by (2.9.4) and (2.9.2b) as

\[
\begin{bmatrix}
I & \beta \hat{B} \tilde{R}^{-1} \hat{B}' \\
0 & \beta A'
\end{bmatrix}
\begin{bmatrix}
y_{t+1} \\
\mu_{t+1}
\end{bmatrix}
= \begin{bmatrix}
A & 0 \\
-\hat{Q} & I
\end{bmatrix}
\begin{bmatrix}
y_t \\
\mu_t
\end{bmatrix}
\]

(2.9.5)

or

\[
L^* \begin{bmatrix}
y_{t+1} \\
\mu_{t+1}
\end{bmatrix} = N \begin{bmatrix}
y_t \\
\mu_t
\end{bmatrix}
\]

(2.9.6)
We want to find a ‘stabilizing’ solution of (2.9.6), i.e., one that satisfies
\[ \sum_{t=0}^{\infty} \beta^t y'_t y_t < +\infty. \]

Chapter 3 shows that the stabilizing solution satisfies \( \mu_t = Py_t \), where \( P \) solves the matrix Riccati equation (2.7.4). Briefly, the generalized eigenvalues of \((L^*, N)\) occur in \( \sqrt{\beta} \)-symmetric pairs (i.e., \((\lambda_i, \lambda_{-i})\) such that if \( \lambda_i \) is an eigenvalue, another eigenvalue is \( \lambda_{-i} = \frac{1}{\lambda_i} \)). The stabilizing solution solves stable roots backward and unstable roots forward by imposing an initialization satisfying \( \mu_0 = Py_0 \). This condition replicates itself over time in the sense that

\[ \mu_t = Py_t, \quad (2.9.7) \]

and implies that \( \sum_{t=0}^{\infty} \beta^t y'_t y_t < \infty. \)

In summary, the solution of the nonstochastic multiplier problem is given by the feedback rule

\[ \begin{bmatrix} u_t \\ w_{t+1} \end{bmatrix} = -Fy_t \quad (2.9.8) \]

where \( F \) depends on \( P \) through (2.7.5). We can find \( P \) either by solving a Riccati equation or by using a method that rearranges the generalized eigenvectors of \( L^*, N \).

2.10. Forward looking models

The basic robust control problem with Bellman equation (2.5.3) pertains to a single decision maker. For macroeconomic applications with a representative agent in an economy without distortions, (2.5.3) can be used to compute equilibrium allocations and prices (for elaboration and examples see chapters 9 and 12 as well as Hansen, Sargent, and Tallarini (1999)). However, even with a representative agent, to analyze so-called Ramsey problems where there are distortions, say flat rate taxes, (2.5.3) must be modified. For a Ramsey problem, the robust decision maker is a government that wants to devise a plan to which it commits at time 0, taking into account the ‘forward looking’ behavior of private agents whose behavior is summarized by Euler equations that include the government’s policy instruments \( u_t \) as ‘forcing variables’. In chapter 16,

\[ {}^{22} \text{Chapter 3 describes the detectability and stabilizability conditions that make this restriction equivalent with } \sum_{t=0}^{\infty} \beta^t r(y_t, u_t) < +\infty. \]
we describe how to solve such robust policy design problems. We formulate
the government’s problem as a Lagrangian and note how the private sector’s
forward-looking behavior formally transforms some of the state variables in an
optimal linear regulator into ‘jump’ variables, while converting some Lagrange
multipliers into ‘state variables.’ Chapter 16 reviews the interesting intellectual
history of the Lagrangian formulation for such problems on both sides of the
Atlantic. In this section, we explain the basic idea, whose implication is that
robust Ramsey policies can be computed easily by solving and appropriately
manipulating an associated ordinary optimal linear regulator problem.

Here is the basic idea. In a forward looking model, we can partition the
state $y = \begin{bmatrix} z \\ x \end{bmatrix}$. The $z$ variables are true state variables, being inherited from
the past, but the $n_x$ variables $x$ are not truly state variables. They are ‘jump
variables’ that adjust to clear markets at $t$, e.g., prices and quantities. The
last $n_x$ equations of (2.5.1) might describe the forward-looking behavior of the
private sector, e.g., Euler equations.

We need $n_x$ additional state variables. To get them, we note that the last
$n_x$ Lagrange multipliers in (2.9.1), call them $\mu_{xt}$, adhere to ‘implementability
constraints’ that the private sector’s Euler equations impose on the Ramsey
plan. The implementability multipliers $\mu_{xt}$ are the missing state variables. These
multipliers on the promise keeping constraints encode the effects on private
agents’ past decisions of government promises about future policies.

Let $\mu_t = \begin{bmatrix} \mu_{zt} \\ \mu_{xt} \end{bmatrix}$. Here $-2\beta^{t+1}\mu_{zt}$ are shadow prices on the true state vari-
ables at $t + 1$ and $-2\beta^{t+1}\mu_{xt}$ are shadow prices on the jump variables at time
$t + 1$, being the ‘implementability multipliers’. The Ramsey problem can be
written in the form (2.5.2), (2.5.1). The first-order conditions continue to be
(2.9.6) and the solution requires that $(y_t, \mu_t)$ satisfy (2.9.7), where $P$ still solves
the Riccati equation associated with the Bellman equation (2.7.2). It is precisely
at this point that the procedure for solving the robust Ramsey problem departs
from that for the linear regulator. We must use (2.9.6) to solve for the jump
variable $x$. With this purpose, write the last $n_x$ equations of (2.9.6) as

$$\mu_{xt} = P_{21}z_t + P_{22}x_t$$

or

$$x_t = -P_{22}^{-1}P_{21}z_t + P_{22}^{-1}\mu_{xt}, \quad (2.10.1)$$

Using (2.10.1), the solution of the robust Ramsey problem is

$$\begin{bmatrix} z_{t+1} \\ \mu_{x,t+1} \end{bmatrix} = \begin{bmatrix} I & 0 \\ P_{21} & P_{22} \end{bmatrix} A_s \begin{bmatrix} I \\ -P_{22}^{-1}P_{21} & P_{22}^{-1} \end{bmatrix} \begin{bmatrix} z_t \\ \mu_{xt} \end{bmatrix}. \quad (2.10.2a)$$
where \( A_o = (A - BF_1 - CF_2) \) for the distorted or worst-case model and \( A_o = (A - BF_1) \) for the robust rule under the approximating model.

The decision rule and worst case model distortion can be represented

\[
\begin{bmatrix}
  u_t \\
  w_{t+1}
\end{bmatrix}
= \begin{bmatrix}
  -F_1 \\
  -F_2
\end{bmatrix}
\begin{bmatrix}
  I & 0 \\
  -P_{22}^{-1}P_{21} & P_{22}^{-1}
\end{bmatrix}
\begin{bmatrix}
  z_t \\
  \mu_{zt}
\end{bmatrix}.
\]

Chapter 16 shows that by eliminating \( \mu_{zt} \), the optimal rule can be represented in the form

\[
u_t = \rho u_{t-1} + \alpha_0 z_t + \alpha_1 z_{t-1}.
\]

Here the history dependence of the rule is captured through the dependence on the lagged instrument \( u_{t-1} \). Chapter 16 gives an alternative representation for the worst-case shock

\[
u_t = \nu u_{t-1} + \gamma_0 z_t + \gamma_1 z_{t-1}.
\]

### 2.11. Concluding remarks

The discounted dynamic programming problem for quadratic returns and a linear transition function is called the optimal linear regulator problem. This problem is widely used throughout macroeconomics and applied dynamics. For linear-quadratic problems, robust decision rules can be constructed by thoughtfully using the optimal linear regulator. This is true both for single-agent problems and for some Ramsey and Stackelberg problems. The optimal linear regulator has other uses too. In chapters 4, 13, and 14 we describe filtering problems. Via the concept of duality explained there, the linear regulator can also be used to solve such filtering problems, including ones with a preference for estimates that are robust to model misspecification.
A. Matlab programs

A robust optimal linear regulator is defined by the system matrices $Q, R, A, B, C$, the discount factor $\beta$, and the risk-sensitivity parameter $\sigma \equiv -\theta^{-1}$. The Matlab program `cheap9.m` implements the algorithm of section 2.7. Call the program as follows:

$$[F,K,P,P_t] = \text{cheap9}(\text{beta}, A, B, C, Q, R, \text{sig})$$

The objects returned by `cheap9` determine the decision rule $u_t = -Fy_t$, the distortion $w_{t+1} = Ky_t$, the quadratic form in the value function $-y'Py$, and the distorted continuation value function $-y''(P_t)y^*$. The program `doublex9` implements the doubling algorithm described in chapter 3 and by Hansen and Sargent (200XXXX chapter 9). To compute the robust rule with a discounted objective function, one has to trick `doublex9` into solving a discounted problem by first setting $A_d = \sqrt{\beta}A, B_d = \sqrt{\beta}B$, calling $[F,Kd,P,P_t] = \text{doublex9}(A_d, B_d, C, Q, R, \text{sig})$, then finally setting $K = K_d/\sqrt{\beta}$. 
Chapter 3.
Linear control theory

3.1. Introduction
This chapter analyzes the standard discounted linear-quadratic optimal control problem, called the optimal linear regulator. The robust decision maker to be described in later chapters adjusts this problem to reflect his doubts about the linear transition law. This chapter describes basic concepts of linear optimal control theory and efficient ways to compute solutions.\(^1\) We describe methods that are faster than direct iterations on the Bellman equation (the Riccati equation) and are more reliable than solutions based on eigenvalue-eigenvector decompositions of the state-costate evolution equation.\(^2\)

In later chapters, we use these techniques to formulate and solve various robust decision and estimation problems. Invariant subspace methods are key tools. In the present chapter, we show how they can be used to solve the Riccati equation that emerges from the Bellman equation for the linear regulator. In later chapters, we shall use invariant subspace methods in two important settings: (a) to compute robust decision rules and estimators in ‘single agent’ problems; and (b) to solve Ramsey problems in ‘forward-looking’ macroeconomic models.

Section 2 decomposes the basic linear optimal control problem into sub-problems that are more efficient to solve and describes classes of economic problems that give rise to such problems. Sections 3, 4, 5, and 6 describe recent algorithms for solving these sub-problems.

\(^1\) This chapter is based on Evan Anderson, Ellen McGrattan, and the authors (1996).
\(^2\) Our survey of these methods draws heavily on Anderson (1978), Gardiner and Laub (1986), Golub, Nash and Van Loan (1979), Laub (1979,1991), and Pappas, Laub and Sandell (1980).
3.2. Control problems

In this section, we pose three optimal control problems. We begin with a problem close to the much studied time-invariant deterministic optimal linear regulator problem. We label this problem the deterministic regulator problem. We then consider two progressively more general problems.

The first generalization introduces forcing sequences or “uncontrollable states” into the deterministic regulator problem. While this generalization is also a deterministic regulator problem, there are computational gains to exploiting the a priori knowledge that some components of the state vector are uncontrollable. We refer to this generalization as the augmented regulator problem. As we will see, a convenient first step for solving an augmented regulator problem is to solve a corresponding deterministic regulator problem in which the forcing sequence is “zeroed out.” In other words, we obtain a piece of the solution to the augmented regulator problem by initially solving a problem with a smaller number of state variables.

The second generalization introduces, among other things, discounting and uncertainty into the augmented regulator problem. We refer to the resulting problem as the discounted stochastic regulator problem. Using well known transformations of the state and control vectors, we show how to convert this problem into a corresponding undiscounted augmented regulator problem without uncertainty. Therefore, while our original problem is a discounted stochastic regulator problem, we solve it by first solving a deterministic regulator problem with a smaller number of state variables, then solving a corresponding augmented regulator problem, and finally using this latter solution to construct the solution to the original problem in the manner described below.

3.2.1. Deterministic regulator problem

Choose a control sequence \( \{v_t\} \) to maximize

\[
- \sum_{t=0}^{\infty} (v_t' R v_t + y_t' Q y_t),
\]

subject to

\[
y_{t+1} = A_{yy} y_t + B_y v_t
\]

\[
\sum_{t=0}^{\infty} (|v_t|^2 + |y_t|^2) < \infty. \tag{3.2.1} ["stable"]
\]
This control problem is a standard time-invariant, deterministic optimal linear regulator problem with one modification. We have added a stability condition, $(3.2.1)$, that is absent in the usual formulation. This stability condition plays a central role in at least one important class of dynamic economic models: permanent income models. More will be said about these models later. In these models, the stability condition can be viewed as an infinite horizon counterpart to a terminal condition on the capital stock.

Following the literature on the time-invariant optimal linear regulator problem, we impose the following:

**Definition:** The pair $(A_{yy}, B_y)$ is stabilizable if $y' B_y = 0$ and $y' A_{yy} = \lambda y'$ for some complex number $\lambda$ and some complex vector $y$ implies that $|\lambda| < 1$ or $y = 0$.

**Assumption 1:** $(A_{yy}, B_y)$ is stabilizable.

Stabilizability is equivalent to the existence of a time-invariant control law that stabilizes the state (see Anderson and Moore, 1979, Appendix C). For our applications, it can often be verified by showing that a trivial control law, such as setting investment equal to zero, achieves this stability.

In solving this problem, we are primarily interested in specifications for which all of the state variables are “endogenous,” and hence the following stronger restriction is met:

**Definition:** The pair $(A_{yy}, B_y)$ is controllable if $y' B_y = 0$ and $y' A_{yy} = \lambda y'$ for some complex number $\lambda$ and some complex vector $y$ implies that $y = 0$.

When $(A_{yy}, B_y)$ is controllable, starting from an initialization of zero, the state vector can attain any arbitrary value in a finite number of time periods by an appropriate setting of the controls (see Anderson and Moore, 1979, Appendix C).³ For this reason, we can think of a state vector sequence with evolution equation governed by a pair $(A_{yy}, B_y)$ that is controllable as being an endogenous state vector sequence.

While Assumption 1 gives us a nonempty constraint set, it is still possible that the supremum of the objective is not attained. We assume the following:

**Assumption 2:** The matrix $Q_{yy}$ is positive semidefinite, and the matrix $R$ is positive definite.

³ This is one of five equivalent characterizations of reachability given in Appendix C of Anderson and Moore (1979). However, many other control theorists take one of these characterizations as the definition of controllability. For instance, see Kwakernaak and Sivan (1972) and Caines (1988). We choose to follow this latter convention.
Among other things, this concavity assumption puts an upper bound of zero on the criterion function. Therefore, the supremum is finite (and nonpositive). We require that the supremum is attained.

**Assumption 3:** There exists a solution to the deterministic regulator problem for each initialization of \( y_0 \).

A commonly used sufficient condition in the control theory literature for there to exist a solution is **detectability**. Factor \( Q_{yy} = D_y D_y' \).

**Definition:** The pair \( (A_{yy}, D_y) \) is detectable if \( D_y' y = 0 \) and \( A_{yy} y = \lambda y \) for some complex number \( \lambda \) and some complex vector \( y \) implies that \( |\lambda| < 1 \) or \( y = 0 \).

When the pair \( (A_{yy}, D_y) \) is detectable, it is optimal to choose a control sequence that stabilizes the state vector. In this case, the solution to the control problem is the same with or without the stability constraint (3.2.1). However, as we mentioned previously, for permanent income models the stability constraint is essential for obtaining an interpretable solution to the problem. For these models, detectability is too strong of a condition to impose. Chan, Goodwin and Sin (1984) give a weaker sufficient condition for there to exist a solution (see (iii) of Theorem 3.10). In the context of a continuous-time formulation, Hansen, Heaton and Sargent (1991) proposed a very similar sufficient condition for stabilizable systems based on a spectral representation of the deterministic regulator problem. Unfortunately, these conditions may be tedious to check in practice. Some of the solution algorithms we survey below could in principle be modified to detect a violation of Assumption 3.

A sufficient condition for convergence of one of the solution algorithms that we survey below is that the pair \( (A_{yy}, D_y) \) be **observable:**

**Definition:** The pair \( (A_{yy}, D_y) \) is observable if \( D_y' y = 0 \) and \( A_{yy} y = \lambda y \) for some complex number \( \lambda \) and some complex vector \( y \) implies that \( y = 0 \).

Clearly, observability is stronger than detectability. Moreover, observability is guaranteed when the matrix \( Q_{yy} \) is nonsingular. When the pair \( (A_{yy}, D_y) \) is observable, the value function associated with the deterministic regulator problem is strictly concave in the state vector \( y \) (Caines and Mayne 1970, 1971).

The solution to the deterministic regulator problem takes the form

\[
v_t = -F_y y_t
\]

for some feedback matrix \( F_y \). The stability constraint (3.2.1) guarantees that the eigenvalues of \( A_{yy} - B_y F_y \) have absolute values that are strictly less than
one because the state evolution equation when the optimal control is imposed is given by
\[ y_{t+1} = (A_{yy} - B_y F_y) y_t. \]

### 3.2.2. Augmented regulator problem

Choose a control sequence \( \{v_t\} \) to maximize
\[
- \sum_{t=0}^{\infty} (v_t' R v_t + y_t' Q_{yy} y_t + 2y_t' Q_{yz} z_t),
\]
subject to
\[
\begin{bmatrix}
  y_{t+1} \\
  z_{t+1}
\end{bmatrix} =
\begin{bmatrix}
  A_{yy} & A_{yz} \\
  0 & A_{zz}
\end{bmatrix}
\begin{bmatrix}
  y_t \\
  z_t
\end{bmatrix} +
\begin{bmatrix}
  B_y \\
  0
\end{bmatrix} v_t
\]
\[
\sum_{t=0}^{\infty} (|v_t|^2 + |y_t|^2) < \infty.
\]

We have modified the optimal linear regulator problem by including the exogenous forcing sequence \( \{z_t\} \). The presumption here is that this partitioning may occur naturally in the specification of the original control problem. Of course, as is well known in the control theory literature, we could always transform an original state vector into controllable and uncontrollable components. Constructing this transformation, however, can be difficult to do in a numerically reliable way. In the next section we will display a class of optimal resource allocation problems associated with dynamic economies for which \( z_t \) contains a vector of taste and technology shifters. By assumption, this component of the state vector cannot be influenced by a control vector such as the level of investment.

For the augmented regulator problem to be well posed, we require that the forcing sequence be stable:

**Assumption 4:** The eigenvalues of \( A_{zz} \) have absolute values that are strictly less than one.

The solution to the deterministic regulator problem gives us a piece of the solution to the augmented regulator problem. More precisely, the solution to the augmented problem is
\[
v_t = -F_y y_t - F_z z_t,
\]
where the matrix \( F_y \) is the same as in the solution to the regulator problem for which the forcing sequence \( \{z_t\} \) is zeroed out. Consequently, our solution
methods entail first computing $F_y$ by solving a deterministic regulator problem of lower dimension and then computing $F_z$ given $F_y$.

### 3.2.3. Discounted stochastic regulator problem

Let $\{F_t : t = 0, 1, \ldots \}$ denote an increasing sequence of sigma algebras (information sets) defined on an underlying probability space. We presume the existence of a “building block” process of conditionally homoskedastic martingale differences $\{\epsilon_t : t = 1, 2, \ldots \}$, which obeys

**Assumption 5:** The process $\{\epsilon_t : t = 1, 2, \ldots \}$ satisfies

(i) $E(\epsilon_{t+1} | F_t) = 0$; 
(ii) $E(\epsilon_{t+1} \epsilon_{t+1}' | F_t) = I$.

The discounted stochastic regulator problem is to choose a control process $\{u_t\}$, adapted to $\{F_t\}$, to maximize

$$\mathbb{E} \left( \sum_{t=0}^{\infty} \beta^t \begin{bmatrix} u_t' & x_t' \end{bmatrix} \begin{bmatrix} R & W' \\ W & Q \end{bmatrix} \begin{bmatrix} u_t \\ x_t \end{bmatrix} \bigg| F_0 \right),$$

subject to

$$x_{t+1} = Ax_t + Bu_t + C\epsilon_{t+1}$$

$$\mathbb{E} \left( \sum_{t=0}^{\infty} \beta^t (|u_t|^2 + |x_t|^2) \bigg| F_0 \right) < \infty.$$

The state vector $x_t$ is taken to be the composite of the endogenous and exogenous state variables. Let $U_y = [I \ 0]$ be a matrix that selects the endogenous state vector $U_yx_t$ and $U_z = [0 \ I]$ be a matrix that selects the exogenous state vector $U_zx_t$ for an optimization problem with discounting. To justify our partitioning, the matrix $A$ is restricted to satisfy $U_z A U_y' = 0$, and the matrix $B$ is restricted to satisfy $U_z B = 0$. Notice that in addition to incorporating discounting and uncertainty, the discounted stochastic regulator includes cross-product terms between controls and states, captured with $u'W'x$, which are absent in the augmented control problem.

We now apply a standard trick for converting a discounted stochastic regulator problem to an augmented regulator problem. Using the well known certainty equivalence property of stochastic optimal linear regulator problems, we zero out the uncertainty without altering the optimal control law. That is, we are free to set the matrix $C$ to zero and instead solve the resulting deterministic control problem. We eliminate discounting and cross-product terms between states and controls by using the transformations
\[ y_t = \beta^{t/2}U_yx_t, \quad z_t = \beta^{t/2}U_zx_t, \quad v_t = \beta^{t/2}(u_t + R^{-1}W^*x_t). \]

As is evident from these formulas, we have absorbed the discounting directly into the construction of the transformed state and control vectors. In addition, the cross-product matrix \( W \) is folded into the construction of the transformed control vector. We are left with a version of the *augmented regulator problem* with the following matrices:

\[
\begin{bmatrix}
A_{yy} & A_{yz} \\
0 & A_{zz}
\end{bmatrix} = \beta^{1/2}(A - BR^{-1}W'), \quad B_y = \beta^{1/2}U_yB,
\]

\[
\begin{bmatrix}
Q_{yy} & Q_{yz} \\
Q_{yz} & Q_{zz}
\end{bmatrix} = Q - WR^{-1}W'. \tag{3.2.2} \]

Assumptions 1 - 4 are imposed on the constructed matrices on the left-hand side of the equal signs in (3.2.2).

As before, write the solution to the *augmented regulator problem* as

\[ v_t = -F_yy_t - F_zz_t. \]

Then the solution to the *discounted stochastic regulator problem* is

\[ u_t = -Fx_t, \]

where

\[ F = \begin{bmatrix} F_y \\ F_z \end{bmatrix} + R^{-1}W'. \]

Also as before, the matrix \( F_y \) can be computed by solving the corresponding *deterministic regulator problem* with the forcing sequence “zeroed out.” Subsequent sections will describe methods for computing \( F_y \) and \( F_z \).

In macroeconomics, the *discounted stochastic regulator problem* is often obtained in the fashion of Kydland and Prescott (1982), who use it to replace a nonlinear-quadratic problem. Thus consider the nonquadratic optimization problem: choose an adapted (to \( \{F_t\} \)) control process \( \{u_t\} \) to maximize

\[ -E\left(\sum_{t=0}^{\infty} \beta^t r(u_t,x_t) \mid F_0\right), \tag{3.2.3} \]

subject to

\[ x_{t+1} = Ax_t + Bu_t + C\epsilon_{t+1}. \]
Here $r$ is not required to be a quadratic function of $u_t$ and $x_t$. When the associated constraints are nonlinear, sometimes we can substitute the nonlinear constraints into the criterion function to obtain a problem of the form of (3.2.3). Kydland and Prescott (1982) simply replace the function $r$ by a quadratic form in $[u_t' \ x_t']'$ as required for the discounted stochastic regulator problem, where the quadratic function is designed to “approximate” $r$ well near a particular value for the state vector.\footnote{While Kydland and Prescott (1982) apply an ad hoc global approximation to $r$ in which the range of approximation is adapted to the amount of underlying uncertainty, many later researchers have instead simply used a local Taylor series approximation around some “nonstochastic” steady state produced by shutting down all randomness in the model. Kydland and Prescott (1982) note that for the range of uncertainty they considered, the two methods gave similar answers. In forming the linear quadratic problem, it is important to substitute the non-linear constraints into the objective function before taking a Taylor series approximation.} In chapter 5, we describe a different approach where, by design, the initial optimal resource allocation problem can be directly converted into a discounted stochastic regulator problem.

### 3.3. Solving the deterministic linear regulator problem

In this section we describe ways to solve for the matrix $F_y$. Recall that this matrix has a double role. First, it gives the control law for a particular deterministic regulator problem. More importantly for us, it also gives a piece of the solution to the discounted stochastic regulator problem.

In describing methods for computing $F_y$, it is convenient to work with the state-costate equations associated with the Lagrangian

$$L = - \sum_{t=0}^{\infty} [y_t'Q_{yy}y_t + v_t' R v_t + 2 \mu_{t+1}'(A_{yy}y_t + B_yv_t - y_{t+1})]. \quad (3.3.1)$$

First-order necessary conditions for the maximization of $L$ with respect to $\{v_t\}_{t=0}^{\infty}$ and $\{y_t\}_{t=0}^{\infty}$ are

$$v_t : \quad R v_t + B_y' \mu_{t+1} = 0, \quad t \geq 0 \quad (3.3.2)$$

$$y_t : \quad \mu_t = Q_{yy}y_t + A_{yy}' \mu_{t+1}, \quad t \geq 0. \quad (3.3.3)$$

To obtain a composite state-costate evolution equation, solve (3.3.2) for $v_t$, substitute the solution into the state evolution equation, and stack the resulting
equation and (3.3.3) and write the state-costate evolution equation as

\[ L \begin{bmatrix} y_{t+1} \\ \mu_{t+1} \end{bmatrix} = N \begin{bmatrix} y_t \\ \mu_t \end{bmatrix}, \] (3.3.4)  

where

\[ L \equiv \begin{bmatrix} I & B_y R^{-1} B_y' \\ 0 & A_{yy}' \end{bmatrix}, \quad N \equiv \begin{bmatrix} A_{yy} & 0 \\ -Q_{yy} & I \end{bmatrix}. \]

For a continuous-time system the a corresponding differential equation for states and costates is

\[ \begin{bmatrix} D y_t \\ D \mu_t \end{bmatrix} = H \begin{bmatrix} y_t \\ \mu_t \end{bmatrix}, \] (3.3.5)  

where

\[ H \equiv \begin{bmatrix} A_{yy} & -B_y R^{-1} B_y' \\ -Q_{yy} & -A_{yy}' \end{bmatrix}, \] (3.3.6)  

which assembles the first-order conditions for the problem with criterion \(- \int_0^\infty \left[ y(t)' Q_{yy} y(t) + u(t)' R u(t) \right] dt\) and law of motion \(D y(t) = A_{yy} y(t) + B_y u(t)\), where \(D\) is the time-differentiation operator. We describe several methods for solving equations (3.3.4) and (3.3.5). Formally, we will devote most of our attention to the discrete-time system (3.3.4). As we will see, methods designed for solving the continuous-time system (3.3.5) can be adapted easily to solve the discrete-time system (3.3.4), and conversely.

We want the solution of (3.3.4) that stabilizes the state-costate vector sequence for any initialization \(y_0\). Since we have transformed the state vector to eliminate discounting, we impose stability in the form of square summability:

\[ \sum_{t=0}^\infty \left| \begin{bmatrix} y_t \\ \mu_t \end{bmatrix} \right|^2 < \infty, \] (3.3.7)  

for the discrete-time system (3.3.4). (We impose the analogous square integrability restriction on the continuous time system (3.3.5)).

One way to ascertain the solution to the deterministic regulator problem is to find an initial costate vector expressed as a function of the initial state vector \(y_0\) that guarantees the stability of system (3.3.4) or (3.3.5). The initialization of the costate vector takes the form \(\mu_0 = P_y y_0\) and is replicated over time. Substituting \(P_y y_t\) for \(\mu_t\) into (3.3.4), we find that

\[ (I + B_y R^{-1} B_y' P_y) y_{t+1} = A_{yy} y_t \]
\[ A_{yy}' P_y y_{t+1} = -Q_{yy} y_t + P_y y_t. \] (3.3.8)
It is straightforward to verify that
\[(I + B_yR^{-1}B_y'y_y)^{-1} = I - B_y(R + B_y'y_yB_yB_y)^{-1}B_y'y_y.\] (*ric5*)

Solving the first equation in (3.3.8) for \(y_{t+1}\)
\[y_{t+1} = (A_{yy} - B_yF_y)y_t,\] (*ric2*)

where
\[F_y \equiv (R + B_y'y_yB_y)^{-1}B_y'y_yA_{yy},\] (*ric6*)

Premultiplying (3.3.10) by \(A_{yy}'P_y\) gives
\[A_{yy}'P_yy_{t+1} = \left(A_{yy}'P_yA_{yy} - A_{yy}'P_yB_yF_y\right)y_t.\] (*ric3*)

For the right-hand side of equation (3.3.12) to agree with the right-hand side of the second equation of (3.3.8) for any initialization \(y_0\), it must be that
\[P_y = Q_{yy} + A_{yy}'P_yA_{yy} - A_{yy}'P_yB_y\left(R + B_y'y_yB_y\right)^{-1}B_y'y_yA_{yy},\] (*ric4*)

which is the familiar Riccati equation. In other words, the matrix \(P_y\) used to set the initial condition on the costate vector is also a solution to the Riccati equation (3.3.13). With this initialization, the costate relation \(\mu_t = P_yy_t\) holds for all \(t \geq 0\). Finally, it follows from (3.3.10) that this state-costate solution is implemented by the control law \(v_t = -F_y'y_t\).

The remainder of this section is organized as follows. In the first subsection, we initially consider the case in which the matrix \(A_{yy}\) is nonsingular. While this case is studied for pedagogical simplicity, it is also of interest in its own right. In the second subsection, we then treat the more general case in which \(A_{yy}\) can be singular. As emphasized by Pappas, Laub and Sandell (1980), singularity in \(A_{yy}\) occurs naturally in dynamic systems with delays. One of our example economies used in our numerical experiments has a singular matrix \(A_{yy}\). Finally, in the third subsection we study the continuous-time counterpart to the deterministic regulator problem. We describe an alternative solution method and show how to convert a discrete-time regulator problem into a continuous-time regulator with the same relation between optimally chosen state and co-state vectors. We defer the discussion of the numerical algorithms used for implementing these methods until the next section.
3.3.1. Nonsingular $A_{yy}$

When the matrix $A_{yy}$ is nonsingular, we can solve (3.3.4) for $\begin{bmatrix} y_{t+1} \\ \mu_{t+1} \end{bmatrix}$:

$$
\begin{bmatrix}
    y_{t+1} \\
    \mu_{t+1}
\end{bmatrix} = M \begin{bmatrix} y_t \\ \mu_t \end{bmatrix}, \tag{3.3.14}$$

where

$$
M \equiv L^{-1} = \begin{bmatrix} A_{yy} + B_y R^{-1} B'_y A'^{-1}_{yy} Q_{yy} & -B_y R^{-1} B'_y A'^{-1}_{yy} \\
-A'^{-1}_{yy} Q_{yy} & A'^{-1}_{yy} \end{bmatrix}. \tag{3.3.15}
$$

We find the matrix $P_y$ by locating the stable invariant subspace of the matrix $M$.

**Definition:** An invariant subspace of a matrix $M$ is a linear space $C$ of possibly complex vectors for which $MC = C$.

Invariant subspaces are constructed by taking linear combinations of eigenvectors of $M$. A stable invariant subspace is one for which the corresponding eigenvalues have absolute values less than one. To solve the model, we seek a matrix $P_y$ such that $\begin{bmatrix} I \\ P_y \end{bmatrix} y$ is in the stable invariant subspace of $M$ for every $n$ dimensional vector $y$. We now elaborate on how to compute this subspace.

The matrix $M$ has a particular structure that we can exploit in characterizing its eigenvalues. To represent this structure, we introduce a matrix $J$ given by

$$
J \equiv \begin{bmatrix} 0 & -I \\
I & 0 \end{bmatrix}.
$$

Notice that $J^{-1} = J' = -J$.

**Definition:** A matrix $M$ is symplectic if $MJM' = J$.

It is straightforward to verify that $M$ given by (3.3.15) is symplectic. It follows that

$$
M' = J^{-1} M^{-1} J. \tag{3.3.16}
$$

Therefore, the transpose of $M$ is similar to its inverse. Recall that similar matrices define the same linear transformation but with respect to a different coordinate system. Thus $M'$ and $M^{-1}$ share the same eigenvalues. For any matrix $M$, the eigenvalues of $M^{-1}$ are the reciprocals of the eigenvalues of $M$. 

so it follows that the eigenvalues of a real symplectic matrix come in reciprocal pairs, and the number of stable eigenvalues cannot exceed the number of states \( n \). However, merely requiring \( M \) to be symplectic permits there to be eigenvalues with absolute values equal to one, and so we will need an additional argument to show that there are exactly \( n \) stable eigenvalues.

To locate the stable invariant subspace of the symplectic matrix \( M \), we follow Laub (1979) and (block) triangularize \( M \):

\[
V^{-1}MV = W
\]

\[
W = \begin{bmatrix}
    W_{11} & W_{12} \\
    0 & W_{22}
\end{bmatrix},
\]

where \( V \) is a nonsingular matrix. By construction, the matrices \( M \) and \( W \) are similar. The matrix partitions in (3.3.17) are built to coincide with the number of stable and unstable eigenvalues. In particular, the absolute values of the eigenvalues of \( W_{11} \) are stable.

A special case of this decomposition is an appropriately ordered Jordan decomposition of \( M \) as was used by Vaughan (1970) in developing an invariant subspace algorithm for computing \( P_y \). Laub (1991) traces this solution strategy back to the 19th century and credits MacFarlane (1963) and Potter (1966) with introducing it to the control literature. As emphasized by Laub (1991), it is preferable to build algorithms based on other upper triangular decompositions that are more stable numerically. The Jordan decomposition is particularly problematic when the symplectic matrix \( M \) has eigenvalues with multiplicities greater than one (see also Golub and Wilkinson 1976). In the next section, we describe alternative Schur decompositions, which are more reliable numerically.

To use this triangularization to calculate \( P_y \), apply \( V^{-1} \) to both sides of the state equation (3.3.14):

\[
\tilde{y}_{t+1} = W \tilde{y}_t,
\]

where

\[
\tilde{y}_t = V^{-1} \begin{bmatrix} y_t \\ \mu_t \end{bmatrix}.
\]

This transformation permits us to study asymptotic properties in terms of two smaller uncoupled subsystems. Partition \( \tilde{y}_t \) into two blocks with dimensions given by the number of stable and unstable eigenvalues:

\[
\tilde{y}_t \equiv \begin{bmatrix} \tilde{y}_{1,t} \\ \tilde{y}_{2,t} \end{bmatrix}.
\]

Then

\[
\tilde{y}_{2,t+1} = W_{22} \tilde{y}_{2,t},
\]
and the solution sequence \( \{\hat{y}_{2,t}\} \) fails to converge to zero unless it is initialized at zero. Setting \( \hat{y}_{2,0} \) at zero can be accomplished by an appropriate initialization of the costate vector, as we now verify.

Partition the matrices \( V \) and \( V^{-1} \) as

\[
V = \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix}, \quad V^{-1} = \begin{bmatrix} V_{11}^{-1} & V_{12}^{-1} \\ V_{21}^{-1} & V_{22}^{-1} \end{bmatrix}.
\]

Since \( V \) is nonsingular and there exists a (stable) solution to the optimal control problem, we must have

\[
V^{21} y_t + V^{22} \mu_t = 0.
\]

The rank of the matrix \( [V^{21} \quad V^{22}] \) equals the number of unstable eigenvalues of \( M \), and thus the rank of its null space must equal the number of stable eigenvalues. For a solution to exist for every initialization \( y_0 = y \), it follows from (3.3.18) that there must exist a \( \mu \) such that

\[
V^{21} y_t + V^{22} \mu_t = 0.
\]

Thus the dimensionality of the null space of \( [V^{21} \quad V^{22}] \) must also be at least \( n \). Therefore, \( M \) has exactly \( n \) stable eigenvalues, and the matrix partition \( V^{22} \) is nonsingular. Solving (3.3.18) for \( \mu_t \) gives

\[
\mu_t = - (V^{22})^{-1} V^{21} y_t.
\]

Consequently, the matrix \( P_y \) used to initialize the costate vector is given by

\[
P_y = - (V^{22})^{-1} V^{21} = V_{21} V_{11}^{-1},
\]

where the second equality follows from the fact that the rank of \( \begin{bmatrix} V_{11} \\ V_{21} \end{bmatrix} \) is \( n \), and

\[
[ V^{21} \quad V^{22} ] \begin{bmatrix} V_{11} \\ V_{21} \end{bmatrix} = 0.
\]
3.3.2. Singular $A_{yy}$

We now extend the solution method to accommodate singularity in $A_{yy}$. This method avoids inverting the $L$ matrix in (3.3.4). Instead of locating the stable invariant subspace of $M$, a deflating subspace method finds the stable deflating subspace of the pencil $\lambda L - N$.

**Definition:** A pencil $\lambda L - N$ is the family of matrices $\{\lambda L - N\}$ indexed by the complex variable $\lambda$.

**Definition:** A deflating subspace of the pencil $\lambda L - N$ is a subspace $C$ of complex vectors such that the dimension of $C$ is at least as large as the dimension of the sum of the subspaces $LC$ and $NC$.

For the matrices $L$ and $N$ of equation (3.3.4), it can be verified that the intersection of their null spaces contains only the zero vector. This ensures us that a generalized eigenvalue problem is well posed. When a subspace $C$ is deflating, there exists a vector $y$ in $C$ that solves the generalized eigenvalue problem

$$\lambda L y = N y$$

(see Theorem 2.1 in Stewart 1972). Implicitly, we are including the possibility of a solution with $\lambda = \infty$, which occurs when $y$ is in the null space of $L$ but not in the null space of $N$. As with the previous (invariant subspace) method, the deflating subspace of interest for solving the optimal control problem is the deflating subspace associated with the stable state-costate sequence. The stable deflating subspace is the subspace associated with the stable generalized eigenvectors (the eigenvectors associated with generalized eigenvalues with absolute values strictly less than one.) Hence we solve the model by finding a matrix $P_y$ such that

$$\begin{bmatrix} I & P_y \end{bmatrix} y$$

is in the stable deflating subspace of the pencil $\lambda L - N$.

Recall that when $A_{yy}$ is nonsingular, the matrix $M$ is symplectic. More generally, system (3.3.4) is associated with a symplectic pencil

**Definition:** A pencil $\lambda L - N$ is symplectic if $LJJ' = NJN'$.

---

5 See Theorem 3 of Pappas, Laub and Sandell (1980) for the case in which $(A_{yy},D_y)$ is detectable. As we noted previously, the restriction to a detectable system rules out some interesting economic models. More generally, nonexistence of a common nonzero vector in the null spaces of $N$ and $L$ can be shown by way of contradiction. Suppose there is a common nonzero vector in the null space. Then the matrix $(I + Q_{yy}B_yR^{-1}B_y')$ is singular. However, this singularity contradicts Theorem 1 of Kimura (1988).
Pappas, Laub and Sandell (1980, Theorem 4) show that the generalized eigenvalues of the symplectic pencil \((\lambda L - N)\) come in reciprocal pairs, just as the eigenvalues of \(M\) do when \(A_yy\) is nonsingular. Hence we again have that the number of stable generalized eigenvalues is no greater than \(n\). Furthermore, we can imitate our argument in the case in which \(A_yy\) is nonsingular to show that there are exactly \(n\) stable generalized eigenvalues.\(^6\)

We triangularize the state-costate system (3.3.4) using the solutions to the generalized eigenvalue problem. As in Theorem 2.1 of Stewart (1972), there exists a decomposition of the pencil \(\lambda L - N\) such that

\[
ULV = T = \begin{bmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{bmatrix},\quad UNV = W = \begin{bmatrix} W_{11} & W_{12} \\ 0 & W_{22} \end{bmatrix},
\]

(3.3.20) \(\text{["gad"]}\)

where \(U\) and \(V\) are unitary matrices and the matrix partitions have the same number, \(n\), of elements as the number of entries in the state vector \(y_t\). Premultiplication of the pencil \(\lambda L - N\) by the nonsingular matrix \(U\) preserves the solutions to the generalized eigenvalue problem, and postmultiplication by \(V\) alters the generalized eigenvectors but not the eigenvalues. A consequence of the triangularization is that the solutions to the generalized eigenvalue problem for the original system are constructed directly from the solutions to the following two smaller problems:

\[
\lambda T_{11}\tilde{y} = W_{11}\tilde{y} \\
\lambda T_{22}\tilde{y} = W_{22}\tilde{y}.
\]

(3.3.21) \(\text{["lars8"]}\)

As with the invariant subspace method, we build the blocks of the triangularization so that the generalized eigenvalues of the first problem in (3.3.21) satisfy \(|\lambda| < 1\), and for the second problem \(|\lambda| > 1\). As a consequence, the span of the first \(n\) columns of \(V\) gives the vectors of the deflating subspace we seek. The span of the remaining \(n\) columns contains the problematic initializations of the state-costate vector for which the implied sequence of state-costate vectors diverges exponentially. In addition, it includes the span of the generalized eigenvectors associated with infinite eigenvalues. Imitating the solution method when \(A_yy\) is nonsingular, we initialize the costate vector as \(\mu_t = P_yy_t\), where the matrix \(P_y\) is again given by (3.3.19).

To understand better the nature of this unstable subspace, recall that an eigenvector associated with an infinite eigenvalue is in the null space of \(T_{22}\). Suppose the triangularization of \(L\) and \(N\) is built so that we can further partition

\(^6\) Theorems 3 and 4 of Pappas, Laub and Sandell (1980) establish this result when the pair \((A_yy, D_y)\) is detectable.
the matrices:

\[
T_{22} = \begin{bmatrix}
M_{11} & M_{12} \\
0 & 0
\end{bmatrix}
\]

\[
W_{22} = \begin{bmatrix}
O_{11} & O_{12} \\
0 & O_{22}
\end{bmatrix},
\]

where the matrices \(M_{11}\) and \(O_{22}\) are nonsingular. Such a triangularization always exists. Consider solving the following equation recursively for a sequence \(\{\tilde{y}_{t+1}\}\); for each \(t\) solve for \(\tilde{y}_{t+1}\) given \(\tilde{y}_t\) by using

\[
T_{22}\tilde{y}_{t+1} = W_{22}\tilde{y}_t.
\]

For this equation to have a solution, the second component of \(\tilde{y}_t\) must be zero for all \(t\) because

\[
O_{22}\tilde{y}_{t,2} = 0,
\]

and \(O_{22}\) is nonsingular. In addition to eliminating the nonexistence problem, imposing this restriction also resolves the multiplicity problem. Note that the multiplicity problem for the triangular system is that for a given \(t\), (3.3.22) does not restrict \(\tilde{y}_{t+1,2}\). However, applying (3.3.22) at \(t + 1\) resolves the problem.

### 3.3.3. Continuous-time systems

To conclude this section, we consider solving continuous-time Hamiltonian systems of the form (3.3.5). The defining feature of a Hamiltonian matrix is:

**Definition:** A matrix \(H\) is Hamiltonian if \(JH\) is symmetric.

The matrix \(H\) in (3.3.5), (3.3.6) clearly satisfies this property. It follows that

\[
H' = -JHJ^{-1},
\]

which in turn implies that the matrix \(H'\) is similar to \(-H\). Consequently, the eigenvalues of a real Hamiltonian matrix come in pairs that are symmetric about the imaginary axis of the complex plane. The stable eigenvalues of a Hamiltonian matrix are those whose real parts are strictly negative. Similar arguments to those given above guarantee that there are exactly \(n\) stable eigenvalues of \(H\). Therefore, (3.3.5) can be solved by using an invariant subspace method and its
associated decomposition (3.3.17), provided that the classification of stable and unstable eigenvalues is modified appropriately.\(^7\)

There is an alternative approach for solving a continuous-time Hamiltonian system. Given a Hamiltonian matrix \(H\), another Hamiltonian matrix \(G\) is constructed with the same stable and unstable invariant subspaces. The matrix \(G\) is called the “sign” of the matrix \(H\), and is defined as follows. Take the Jordan decomposition of \(H\):

\[
H = V \begin{bmatrix} \Lambda_{11} & 0 \\ 0 & \Lambda_{22} \end{bmatrix} V^{-1},
\]

where \(\Lambda_{11}\) is an upper triangular matrix with the eigenvalues of \(H\) that have strictly negative real parts on the diagonals, and \(\Lambda_{22}\) is an upper triangular matrix with the eigenvalues of \(H\) that have strictly positive real parts on the diagonals. Then

\[
G = \text{sign}(H) \equiv V \begin{bmatrix} -I & 0 \\ 0 & I \end{bmatrix} V^{-1}.
\]

Thus the sign of a matrix is a new matrix with the same eigenvectors as the original matrix and with eigenvalues replaced by \(-1\) or \(1\) depending on the signs of the real parts of the original eigenvalues.

The matrix \(P_y\) can be inferred directly from \(G\). To see this, we use an insight from Roberts (1980). By construction, all of the stable eigenvalues of \(G\) are equal to \(-1\). Consequently, the matrix \(P_y\) satisfies the following eigenvalue problem:

\[
G \begin{bmatrix} I & P_y \end{bmatrix} y = - \begin{bmatrix} I & P_y \end{bmatrix} y
\]

for any \(n\) dimensional vector \(y\), and the matrix \(P_y\) solves the affine equation

\[
G \begin{bmatrix} I & P_y \end{bmatrix} + \begin{bmatrix} I & P_y \end{bmatrix} = 0.
\]  
\[\text{(3.3.23)} \quad ["lars15"]\]

This method is implemented by finding fast ways to compute the “sign” of a matrix.

While the matrix sign method is directly applicable for solving continuous-time Hamiltonian systems, Hitz and Anderson (1972) and Gardiner and Laub

\(^7\) Deflating subspace methods are not needed for solving the class of continuous-time quadratic control problems considered here because we can form directly the Hamiltonian matrix and apply an invariant subspace method. However, as we have formulated it, the continuous-time problem does not permit systems with finite gestation lags in making investment goods productive or systems for which consumption services depend on only a finite interval of past consumptions.
Linear control theory show how to use it to locate deflating subspaces of discrete-time systems. Consider the generalized eigenvalue problem for the symplectic pencil

$$\lambda Ly = N.$$ 

Then

$$(1 + \lambda) (L - N) y = (1 - \lambda) (L + N) y.$$ 

Since the only common vector in the null space of $L$ and $N$ is zero, we construct the solution to the eigenvalue problem

$$\delta y = (L - N)^{-1} (L + N) y,$$ 

where

$$\delta = \frac{1 + \lambda}{1 - \lambda}.$$ 

Consequently, the stability relations (3.2.1) carry over here as well, and we apply the matrix sign algorithm to $(L - N)^{-1}(L + N)$. It also turns out that $(L - N)^{-1}(L + N)$ is a Hamiltonian matrix, which we can exploit in computation. To verify the Hamiltonian structure, note that

$$(L - N) J (L' + N') = LJL' - N JN' - NJL' + LJN'$$

$$= -NJL' + LJN'$$

$$= NJN' - LJJ' - NJL' + LJN'$$

$$= - (L + N) J (L' - N')$$

where we have used the fact that $\lambda L - N$ is a symplectic pencil. Therefore,

$$J (L - N)^{-1} (L + N) = (L' + N') (L' + N')^{-1} J (L - N)^{-1} (L + N)$$

$$= (L' + N') [- (L - N) J (L' + N')]^{-1} (L + N)$$

$$= (L' + N') [(L + N) J (L' - N')]^{-1} (L + N)$$

$$= (L' + N') (L' - N')^{-1} J',$$

which proves that $(L - N)^{-1}(L + N)$ is a Hamiltonian matrix.

In summary, by construction, the stable (unstable) invariant subspace of the Hamiltonian matrix $(L - N)^{-1}(L + N)$ coincides with the stable (unstable) deflating subspace of the symplectic pencil $\lambda L - N$. This coincidence permits us to compute the matrix $P_\gamma$ used for initializing the costate vector for the discrete-time system (3.3.4) by applying a matrix sign algorithm to $(L - N)^{-1}(L + N)$. 
3.4. Computational techniques for solving Riccati equations

We consider three types of algorithms for computing $P_y$:

(i) Schur algorithm;
(ii) doubling algorithm;
(iii) matrix sign algorithm.

A Schur algorithm is based on locating a stable subspace using a Schur decomposition of the state-costate system. As we noted in the previous section, once a stable subspace is located, the relevant Riccati equation solution $P_y$ is easily computed. There are two versions of a Schur decomposition, depending on whether the matrix $A_{yy}$ is known to be nonsingular or not. A Schur decomposition gives a more reliable way of locating stable spaces than the familiar Jordan decomposition and its generalization for pencils.

A doubling algorithm is an iterative method for speeding up the dynamic programming Riccati equation iteration by doubling the number of time periods in each iteration.

Recall from our discussion in the previous section that the stable deflating subspace of the pencil $\{\lambda L - N\}$ coincides with the invariant subspace of the sign of the matrix $(L-N)^{-1}(L+N)$ associated with the eigenvalue $-1$. A matrix sign algorithm is an iterative method for computing the sign of $(L-N)^{-1}(L+N)$ from which we can recover $P_y$ easily. See section 3.4.4 for details of the matrix sign algorithm.

3.4.1. Schur algorithm

Suppose the matrix $A_{yy}$ is nonsingular. As we noted in section 3, the matrix $P_y$ can be found by locating the stable invariant subspace of the matrix $M$ given in (3.3.15). In some of our numerical calculations, we use what is referred to as a real Schur decomposition of $M$ to locate its invariant subspace.

Definition: The real Schur decomposition of a real matrix $M$ is an orthogonal matrix $\hat{V}$ and a real upper block triangular matrix $\hat{W}$ such that

$$\hat{V}'M\hat{V} = \hat{W} = \begin{bmatrix} \hat{W}_{11} & \hat{W}_{12} & \cdots & \hat{W}_{1m} \\ 0 & \hat{W}_{22} & \cdots & \hat{W}_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \hat{W}_{mm} \end{bmatrix}$$
where \( \hat{W}_{ii} \) is either a scalar or a \( 2 \times 2 \) matrix with complex conjugate eigenvalues.\(^8\)

A real Schur decomposition is a computationally convenient version of the block triangular decomposition (3.3.17) used to compute \( P_y \) when \( A_{yy} \) is nonsingular. Golub and Van Loan (1989) describe how to compute the real Schur decomposition (in particular, see sections 7.4 and 7.5). Recall that the block triangular matrix \( W \) in (3.3.17) results from partitioning the eigenvalues into stable and unstable eigenvalues. Algorithms that compute the real Schur decomposition of a matrix typically do not partition the diagonal blocks of \( \hat{W} \) according to stability. Instead, given an arbitrary real Schur decomposition \( M = \hat{V} \hat{W} \hat{V}^\prime \), one can use the approaches described in either Bai and Demmel (1993) or Stewart (1976) to construct a sequence of orthogonal transformations that reorder the diagonal blocks of \( \hat{W} \), while updating \( \hat{V} \) so that \( M = \hat{V} \hat{W} \hat{V}^\prime \) holds at every step.

In summary, the steps for implementing a Schur algorithm are

1. form the matrix \( M \) in (3.3.15);
2. form a real Schur decomposition of \( M \) where the first \( n \) columns of \( \hat{V} \), written in a partitioned form as
   \[
   \begin{bmatrix}
   \hat{V}_{11} & \hat{V}_{21}
   \end{bmatrix}
   \]
   are a basis for the stable invariant subspace of \( M \);
3. solve \( P_y \hat{V}_{11} = \hat{V}_{21} \) for \( P_y \).

We recommend computing the real Schur decomposition of \( M \) by using the LAPACK function DGEES; \( P_y \) in step (3) can be computed using the built-in MATLAB operator `/`, which solves a linear equation using Gaussian elimination with partial pivoting.

A deflating subspace method is required when \( A_{yy} \) is singular and likely to be more stable numerically when \( A_{yy} \) is nearly singular. To implement this approach in practice, we use an ordered real generalized Schur decomposition to find an appropriate triangularization of the state-costate dynamical system (see Van Dooren (1982)).

Definition: A *generalized real Schur decomposition* of a real matrix pencil \( \lambda L - N \) is a pair of orthogonal matrices \( \hat{U} \) and \( \hat{V} \), a real upper triangular matrix \( \hat{T} \), and

\[\text{Definition: A complex Schur decomposition of a real or complex matrix in which } \hat{V} \text{ is a unitary matrix and } \hat{W} \text{ is upper triangular.}\]
a real upper block triangular matrix \( \hat{W} \), such that

\[
\begin{bmatrix}
\hat{T}_{11} & \hat{T}_{12} & \ldots & \hat{T}_{1m} \\
0 & \hat{T}_{22} & \ldots & \hat{T}_{2m} \\
\vdots & \ddots & \ddots & \vdots \\
0 & \ldots & 0 & \hat{T}_{mm}
\end{bmatrix}
\]

\[
\begin{bmatrix}
\hat{W}_{11} & \hat{W}_{12} & \ldots & \hat{W}_{1m} \\
0 & \hat{W}_{22} & \ldots & \hat{W}_{2m} \\
\vdots & \ddots & \ddots & \vdots \\
0 & \ldots & 0 & \hat{W}_{mm}
\end{bmatrix}
\]

where the pencil \( \lambda \hat{T}_{ii} - \hat{W}_{ii} \) is either a \( 1 \times 1 \) matrix pencil or a \( 2 \times 2 \) matrix pencil with complex conjugate generalized eigenvalues.

As with the real Schur decomposition, we initially compute a generalized real Schur decomposition of \( \lambda L - N \) without regard to whether the generalized eigenvalues are stable or not. We then reorder the diagonal blocks of \( \hat{T} \) and \( \hat{W} \) so that the generalized eigenvalues are partitioned in the manner required by (3.3.20). This partitioning can be done using the algorithms described in Van Dooren (1981, 1982) or in Kågström and Poromaa (1994).

Thus the steps for implementing a generalized Schur algorithm are

1. form the matrices \( L \) and \( N \) in (3.3.4);
2. form a generalized real Schur decomposition of the pencil \( \lambda L - N \) where the first \( n \) columns of \( \hat{V} \), written in a partitioned form as \([ \hat{V}_{11} \hat{V}_{12} \ldots \hat{V}_{21} \hat{V}_{22} \ldots \hat{V}_{mm} \])', span the deflating subspace of the pencil \( \lambda L - N \);
3. solve \( P_y \hat{V}_{11} = \hat{V}_{21} \) for \( P_y \).
3.4.2. Digression: solving DGE models with distortions

Linear or log-linear approximations to the equilibrium conditions of dynamic general equilibrium (DGE) models take one of the forms

\[ Ly_{t+1} = Ny_t + \tilde{G}z_t \tag{3.4.1} \]

or, if \( L \) is nonsingular,

\[ y_{t+1} = My_t + Gz_t \tag{3.4.2} \]

where \( M = L^{-1}N \) and \( z_t \) is a vector of forcing variables governed by a law of motion

\[ z_{t+1} = A_{22}z_t, \tag{3.4.3} \]

where the eigenvalues of \( A_{22} \) are all less than or equal to unity in modulus.\(^9\)

We shall consider the case in which \( L \) is nonsingular. We assume that the eigenvalues of \( M \) split into equal numbers of stable and unstable ones so that we can obtain a real Schur decomposition of \( M = VWV^{-1} \)

where \( W_{11} \) is a stable matrix and \( W_{22} \) is an unstable matrix. The assumption that the eigenvalues split in this way is tantamount to assuming that there exists a unique stabilizing solution of (3.4.1).

Using \( M = VWV^{-1} \) in (3.4.2) and premultiplying both sides by \( V^{-1} \) gives

\[ V^{-1}y_{t+1} = WV^{-1}y_t + V^{-1}Gz_t \tag{3.4.4} \]

or

\[ y^*_t + 1 = Wy_t^* + G^*z_t \tag{3.4.5} \]

where \( y_t^* = V^{-1}y_t \) and \( G^* = V^{-1}G \). Express (3.4.5) in terms of the uncoupled dynamic system

\[ y^*_{t+1} = W_{11}y_{1t}^* + W_{12}y_{2t}^* + G_1^*z_t \tag{3.4.6a} \]

\[ y^*_{2t+1} = W_{22}y_{2t}^* + G_2^*z_t \tag{3.4.6b} \]

Where \( \tilde{L} \) is the lag operator, rewrite (3.4.6b) as \((I - W_{22}\tilde{L})y^*_{2t+1} = G_2^*z_t \) or

\[ -W_{22}L(I - W_{22}^{-1}\tilde{L}^{-1})y^*_{2t+1} = G_2^*z_t \]

or\(^10\)

\[ y^*_{2t} = -W_{22}^{-1}

This assumption can be relaxed to be that the eigenvalue of maximum modulus of \( A_{22} \) times the reciprocal of the largest eigenvalue of \( A_{22} \) is strictly less than unity.

**Tom: check the sign of this statement.**

\(^9\) These formulas can be viewed as extensions to the vector case of formulas found in Sargent (1987a, ch IX, pp. ???)

\(^10\) This assumption can be relaxed to be that the eigenvalue of maximum modulus of \( A_{22} \) times the reciprocal of the largest eigenvalue of \( A_{22} \) is strictly less than unity.
Substituting this into (3.4.6a) and rearranging gives

\[ y_{t+1}^* = W_{11}y_{t+1}^* + \left[ G_1^* - W_{12}W_{22}^{-1} \left( I - W_{22}^{-1} \tilde{L}^{-1} \right)^{-1} G_2^* \right] z_t. \] (3.4.8) \[ ^{\text{dgel8}} \]

Equations (3.4.7), (3.4.8) give the stabilizing solution for the uncoupled dynamic system cast in terms of \( y_t^* \). To retrieve the original variables, we simply use \( y_t = Vy_t^* \).

The very same solution would also be sustained as the solution of the stochastic system in which (3.4.3) is replaced by the stochastic law of motion

\[ z_{t+1} = A_{22}z_t + Cw_{t+1} \] (3.4.9) \[ ^{\text{dgel3a}} \]

where \( w_{t+1} \) is a martingale difference sequence with identity covariance matrix; and where \( y_{t+1} \) on the left side of (3.4.1) and (3.4.2) is replaced by \( E[y_{t+1} | y_t, z^t] \), where here \( E \) is the mathematical expectation operator and \( z^t \) denotes the history of the \( z \) process up to and including \( t \). Equations (3.4.7), (3.4.8) are also the heart of the solution that would obtain if were we to assume that in a stochastic system the state \( z_t \) is not observed, but that noisy signals \( Y_t \) that are linearly related to it. In that case, the solution is to replace \( y_t \) in (3.4.7), (3.4.8) with \( E[z_t | Y^t] \). The projection \( E[z_t | Y^t] \) can be computed recursively using the standard Kalman filtering formulas reported in chapter 4.

### 3.4.3. Doubling algorithm

Dynamic programming solves the infinite horizon problem by backward induction, which leads to iterations on the Riccati equation (3.3.13). A doubling algorithm can be viewed as a refinement of this approach. It preserves the idea of approximating the solution to the infinite horizon problem by a sequence of finite horizon problems, but instead of increasing the horizon by one time period in each iteration, the number of time periods gets doubled.

To see how this approach works, recall that the solution to the finite horizon problem for periods 0...\((\tau - 1)\) can be viewed as a two point boundary value problem where the initial state vector \( y_0 \) is set to some arbitrary vector \( y \) and the costate vector at the terminal date \( \mu_\tau \) is set to zero. Suppose for simplicity that \( A_{yy} \) is nonsingular. By iterating on relation (3.3.14), we find that

\[ \hat{M} \begin{bmatrix} y_\tau \\ 0 \end{bmatrix} = \begin{bmatrix} y_0 \\ \mu_0 \end{bmatrix}, \] (3.4.10) \[ ^{\text{lars10}} \]

where

\[ \hat{M} \equiv M^{-\tau}. \]
To approximate the matrix $P_y$, we solve (3.4.10) for the initial costate vector $\mu_0$ as a function of $y_0$. Partitioning $\hat{M}$ conformably to the state-costate partition, we see that

$$\hat{M}_{11} y_\tau = y_0, \quad \hat{M}_{21} y_\tau = \mu_0.$$ 

Therefore, the implicit initialization of the costate vector is

$$\mu_0 = \hat{M}_{21} \left( \hat{M}_{11} \right)^{-1} y_0,$$

and our approximation for the matrix $P_y$ is given by $\hat{M}_{21}(\hat{M}_{11})^{-1}$.

What is needed to implement this approach is a way to compute $\hat{M}$ when the horizon $\tau$ is large. Expanding the horizon one period at a time corresponds to multiplying the matrix $M^{-1}$, $\tau$ times in succession. However, when $\tau$ is chosen to be a power of two, computations can be sped up by using

$$M^{-2^{k+1}} = \left( M^{-2^k} \right) M^{-2^k}.$$ 

As a consequence, when $\tau = 2^j$, the desired matrix can be computed in $j$ iterations instead of $2^j$ iterations, which explains the name doubling algorithm.

Given that the matrix $M^{-1}$ has unstable eigenvalues, direct iterations on (3.4.11) can be very unreliable. Clearly, the sequence of matrices $\{M^{-2^k}\}$ diverges. One of the features of a doubling algorithm is to transform these computations into matrix iterations that converge. Another feature is that a doubling algorithm exploits the fact that the matrix $M$ is symplectic. Symplectic matrices have several nice properties.\(^{11}\) We have already seen that their eigenvalues come in reciprocal pairs. In addition, the product of symplectic matrices is symplectic, and the inverse of a symplectic matrix is symplectic. Moreover, for any symplectic matrix $S$, the matrices $S_{21}(S_{11})^{-1}$ and $(S_{11})^{-1}S_{12}$ are both symmetric and

$$S_{22} = (S_{11}')^{-1} + S_{21}(S_{11})^{-1} S_{12}$$

$$= (S_{11}')^{-1} + S_{21}(S_{11})^{-1} S_{11}(S_{11})^{-1} S_{12}. $$

Therefore, a $(2n \times 2n)$ symplectic matrix can be represented in terms of the three $n \times n$ matrices $\alpha = (S_{11})^{-1}$, $\beta = (S_{11})^{-1}S_{12}$, $\gamma = S_{21}(S_{11})^{-1}$, the latter two of which are symmetric.

\(^{11}\) There is a variation of the Schur algorithm that exploits the symplectic structure of $M$. See pages 431-434 of Petkov et al. (1991) for an overview of this algorithm.
The doubling algorithm described by Anderson (1978) and Anderson and Moore (1979) exploits such a representation by using the following parameterization of $M^{-2^k}$:

$$M^{-2^k} = \begin{bmatrix} (\alpha_k)^{-1} & (\alpha_k)^{-1} \beta_k \\ \gamma_k (\alpha_k)^{-1} & \alpha'_k + \gamma_k (\alpha_k)^{-1} \beta_k \end{bmatrix},$$

where the $n \times n$ matrices $\alpha_k, \beta_k, \gamma_k$ are given by the recursions

\begin{align*}
\alpha_{k+1} &= \alpha_k (I + \beta_k \gamma_k)^{-1} \alpha_k \\
\beta_{k+1} &= \beta_k + \alpha_k (I + \beta_k \gamma_k)^{-1} \beta_k \alpha'_k \\
\gamma_{k+1} &= \gamma_k + \alpha'_k \gamma_k (I + \beta_k \gamma_k)^{-1} \alpha_k.
\end{align*}

(3.4.12) \["lars12"\]

While this alternative parameterization introduces a matrix inverse into the recursions (3.4.12) that is absent in (3.4.11), the matrix $I + \beta_k \gamma_k$ being inverted is only $n$ dimensional. The nonsingularity of this matrix for all $k$ is established in Kimura (1988). To initialize the doubling algorithm, we simply deduce the implicit parameterization of $M^{-1}$ given in partitioned form by

$$M^{-1} = N^{-1}L = \begin{bmatrix} A_{yy}^{-1} & A_{yy}^{-1}B_y R^{-1}B_y' \\
Q_{yy} A_{yy}^{-1} & Q_{yy} A_{yy}^{-1}B_y R^{-1}B_y' + A_{yy}' \end{bmatrix},$$

(3.4.13) \["lars12a"\]

which leads to the initializations

$$\alpha_0 = A_{yy}, \quad \beta_0 = B_y R^{-1}B_y', \quad \gamma_0 = Q_{yy}.$$ 

While our derivation took the matrix $A_{yy}$ to be nonsingular, Anderson (1978) argues that the doubling algorithm is more generally applicable.

A convenient feature of this parameterization is that there are known conditions under which the matrix sequences $\{\alpha_k\}, \{\beta_k\}, \{\gamma_k\}$ converge. When the pair $(A_{yy}, D_y)$ is detectable, then the sequence $\{\gamma_k\}$ is nondecreasing and converges to the matrix $P_y$. (Here we are adopting the usual partial ordering for positive semidefinite matrices.) As noted by Kimura (1988, Theorem 5), under the same restrictions, the sequence $\{\beta_k\}$ is nondecreasing and converges to a positive semidefinite matrix $P_y^*$ associated with a “dual” to the deterministic regulator problem.

The convergence of the $\{\alpha_k\}$ sequence is more problematic. Unfortunately, without simultaneous convergence of $\{\alpha_k\}$, it is not evident that iterations of the form given in (3.4.12) can be used as the basis of a numerical algorithm. If this latter sequence diverges, small numerical errors may get magnified, causing the resulting algorithm to be poorly behaved. Kimura (1988) provides some
sufficient conditions for \( \{\alpha_k\} \) to converge to a matrix of zeros. His sufficient conditions are used to guarantee that either \( P_y \) or \( P_y^* \) is nonsingular.

As we noted previously, a sufficient condition for \( P_y \) to be nonsingular is that the pair \((A_{yy}, D_y)\) be observable. Sufficient conditions for the nonsingularity of the matrix \( P_y^* \) are that (i) \((A_{yy}, B_y)\) is controllable; and (ii) \((A_{yy}, D_y)\) is detectable (Kimura 1988). Recall that controllability is often achieved by our a priori partitioning of the state vector into endogenous and exogenous components. Thus for our purposes, the restrictions guaranteeing the nonsingularity of \( P_y^* \) may be of particular interest. Even so, detectability is too strong for some of our applications.

To apply a doubling algorithm more generally, we sometimes modify the control problem by adding small quadratic penalties to linear combinations of the states and controls. As long as these penalties are sufficient to guarantee that either \( P_y \) or \( P_y^* \) is nonsingular, we are assured of convergence of all three sequences. Of course, there is a danger that the penalty distorts the solution to the original control problem in a nontrivial way, which must be checked in practice.

3.4.3.1. Initialization from a positive definite matrix

Instead of adding small quadratic penalties to the objective function for each calendar date, we could add a terminal penalty to the finite horizon approximation to the control problem. From Chan, Goodwin and Sin (1984), it is known that iterations on the Riccati difference equation converge to the unique stabilizing solution whenever the Riccati equation is initialized at a positive definite matrix.\(^{12} \) Initializing the Riccati difference equation at a positive definite matrix is equivalent to imposing a terminal penalty that is a negative definite quadratic form in the state vector. We will now show how to initialize the doubling algorithm to impose a terminal penalty. This will permit us to compute \( P_y \) via a doubling algorithm for a richer class of control problems.

Consider first a finite time horizon problem with a quadratic penalty on the terminal state. We select this penalty so that the terminal multiplier \( \mu_\tau = P_0y_\tau \) for some positive definite matrix \( P_0 \). Then equation (3.4.10) is altered to be

\[
M \begin{bmatrix} I & P_0 \end{bmatrix} y_\tau = \begin{bmatrix} y_0 \\ \mu_0 \end{bmatrix}.
\]  

\[(3.4.14) \quad \text{["double2"]}\]

\(^{12} \) Here we are using the fact that the pair \((A_{yy}, B_y)\) is stabilizable and that there exists a solution to the deterministic regulator problem when constraint (3.2.1) is imposed. The result follows from (i) and (iii) of Theorem 3.1 and Theorem 4.2 of Chan, Goodwin and Sin (1984).
Build a matrix $K$

$$K \equiv \begin{bmatrix} I & 0 \\ P_o & I \end{bmatrix}.$$  

Then equation (3.4.14) can be rewritten as

$$K^{-1} \hat{M} K K^{-1} \begin{bmatrix} I \\ P_o \end{bmatrix} y_\tau = K^{-1} \begin{bmatrix} y_0 \\ \mu_0 \end{bmatrix}.$$  

Equivalently,

$$M^* \begin{bmatrix} y_\tau \\ 0 \end{bmatrix} = \begin{bmatrix} y_0 \\ \mu_0 - P_o y_0 \end{bmatrix},$$

where

$$M^* = K^{-1} \hat{M} K.$$  

Partitioning $M^*$ consistently with the state-costate vector, the implicit initialization of the costate vector is now

$$\mu_0 = P_o y_0 + M^*_{12} (M^*_{11})^{-1} y_0,$$

and our approximation for $P_y$ is given by $M^*_{11} (M^*_{11})^{-1} + P_o$.

We are now left with computing the matrix $M^*$ when the horizon $\tau$ is very large. Notice that

$$M^* = (K^{-1} M K)^{-\tau}.$$  

It is straightforward to verify that because $M$ is symplectic, so is $K^{-1} M K$. This means that doubling algorithm (3.4.12) is applicable for computing $(K^{-1} M K)^{-2^k}$; however, the initializations must be altered. The new initializations can be deduced by looking at the implicit parameterization of the symplectic matrix $K^{-1} M K$, and they are given by

$$\alpha_0 = (I + B_y R^{-1} B_y' P_o)^{-1} A_{yy},$$

$$\beta_0 = (I + B_y R^{-1} B_y' P_o)^{-1} B_y R^{-1} B_y',$$

$$\gamma_0 = Q_{yy} - P_o + A_{yy} P_o (I + B_y R^{-1} B_y' P_o)^{-1} A_{yy}.$$  

Not surprisingly, the original initializations coincide with setting $P_o$ to zero in (3.4.15).

There are two related advantages to these initializations over the previous ones. First, the sequence $\{\gamma_j\}$ converges to $P_y - P_o$ whenever $P_o$ is positive definite. This follows from the Riccati difference equation convergence described
previously and does not require that \((A_y, D_y)\) be detectable. Second, the sequence \(\{\beta_j\}\) converges and satisfies the bounds

\[
0 \leq \beta_j \leq (P_o)^{-1}
\]
even when \((A_y, D_y)\) is not detectable.\(^{13}\) Although we do not have a complete characterization of convergence of the resulting algorithm, all three matrix sequences (including \(\{\alpha_j\}\)) are guaranteed to converge with these alternative initializations if they converge with the original ones.

In summary, the steps for implementing the doubling algorithm are

1. initialize \(\alpha_0, \beta_0, \gamma_0\) according to (3.4.15);
2. iterate in accordance with (3.4.12);
3. form \(P_y\) as the limit of \(\{\gamma_k\} + P_o\).

### 3.4.3.2. Application to continuous time

As noted by Anderson (1978) and Kimura (1989), a doubling algorithm for a discrete-time symplectic system can be used to solve a continuous-time Hamiltonian system. Recall that in our discussion of solving control problems via a matrix sign algorithm, we showed how to covert a discrete-time symplectic system into a continuous-time Hamiltonian system. To apply a doubling algorithm, we want to “invert” this mapping, e.g., given a Hamiltonian matrix \(H\), we construct a symplectic pencil with the same stable deflating subspace. The symplectic pencil associated with \(H\) is given by \(\lambda(I + H) - (I - H)\). By adopting a very similar argument as before, we found it easy to show that the generalized

\[^{13}\] The convergence and bound can be established as follows. Let \(\{\beta^*_j\}\) denote the sequence starting from the original initialization. Then it is straightforward to show that

\[
\beta_j = (I + \beta^*_j P_o)^{-1} \beta^*_j.
\]
Exploiting the nonsingularity of \(P_o\), the following equivalent formula can be deduced:

\[
\beta_j = (P_o)^{-1} - (P_o + P_o \beta_j^* P_o)^{-1}.
\]

The reported bound follows immediately. The sequence \(\{\beta^*_j\}\) is monotone increasing because it is a subsequence of Riccati difference equation iterations for a dual problem initialized at zero. Therefore, the sequence \(\{\beta_j\}\) is also monotone increasing. Given the upper bound \((P_o)^{-1}\), this latter sequence must converge.
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eigenvectors for the constructed pencil coincide with the eigenvectors of the original Hamiltonian matrix $H$. Moreover, the classification of stable and unstable (generalized) eigenvalues is preserved.

3.4.4. Matrix sign algorithm

In section 3.3.3 we showed how to compute $P_y$ from the sign of the Hamiltonian matrix for a continuous-time state-costate system. To compute $P_y$ for a symplectic pencil $\lambda L - N$, we first form the Hamiltonian matrix

$$H = (L - N)^{-1}(L + N)$$

and then compute $\text{sign}(H)$. For this to be a viable solution method, we must be able to compute $\text{sign}(H)$ easily.

There are alternative matrix sign algorithms. An algorithm advocated by Roberts (1980) and Denman and Beavers (1976) is to average a matrix and its inverse:

$$G_0 = H, \quad G_{k+1} = G_k + (1/2) \left[(G_k)^{-1} - G_k\right], k = 0, 1, \ldots$$

To speed up convergence, Gardiner and Laub (1986) suggest using the recursion

$$G_0 = H, \quad G_{k+1} = \frac{1}{2\epsilon_k} \left(JG_k + \epsilon_k^2 JG_k^{-1}J\right),$$

where

$$\epsilon_k = |\det G_k|^{1/n}. \quad (3.4.17) \quad \text{["lars16"]}$$

Bierman (1984) and Byers (1987) propose a further refinement, which exploits the fact that the matrix $G_k$ is a Hamiltonian matrix for each $k$. Recall that if $H$ is a Hamiltonian matrix, then $JH$ is symmetric where

$$J = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}.$$ 

Hence

$$JG_{k+1} = \frac{1}{2\epsilon_k} \left(JG_k + \epsilon_k^2 JG_k^{-1}J\right), \quad (3.4.18) \quad \text{["lars17"]}$$

where $\epsilon_k$ is either set to one as in the original sign algorithm or set via formula (3.4.17) using $JG_k$ in place of $G_k$. Consequently, it suffices to compute the
sequence of symmetric matrices \(\{JG_k\}\) recursively via (3.4.18) starting from the initialization \(JH\).\footnote{Kenney, Laub and Papadopoulos (1993) and Lu and Lin(1993) discuss further improvements to the matrix sign algorithm.}

In summary, the steps for implementing a matrix sign algorithm are

1. form the matrices \(L\) and \(N\) in (3.3.4);
2. compute the sign of \(G = (L - N)^{-1}(L + N)\);
3. compute \(P_y\) by solving the over-determined system

\[
\begin{bmatrix}
G_{12} \\
G_{22} + I
\end{bmatrix} P_y = - \begin{bmatrix}
G_{11} + I \\
G_{21}
\end{bmatrix}
\]

(3.4.19) ["signp"]

for \(P_y\).

As noted in Anderson (1978), the original sign algorithm (3.4.16) also can be viewed as a doubling algorithm. Interpreted in this manner, it uses (at least implicitly) an alternative parameterization of the symplectic matrix \(M^{-1}\) to that used in the doubling algorithm (3.4.12). Both recursions entail inverting a matrix. While recursion (3.4.18) requires that a symmetric \((2n \times 2n)\) matrix be inverted in each iteration, the doubling algorithm (3.4.12) requires that a nonsymmetric \(n \times n\) matrix be computed at each iteration.

### 3.5. Solving the augmented regulator problem

So far, we have shown how to compute the matrix \(F_y\), which provides us with the optimal control law for the deterministic regulator problem. This matrix also gives us a piece of the solution to the augmented control problem and, hence, to the problem of interest: the discounted stochastic regulator problem. The missing ingredient is the matrix \(F_z\), where the optimal control law for the augmented regulator problem is given by \(v_t = -F_y y_t - F_z z_t\). In this section, we show that \(F_z\) can be calculated by solving a particular Sylvester equation.

We start by forming a Lagrangian modified to incorporate the exogenous state vector sequence \(\{z_t\}\):

\[
\mathcal{L} = - \sum_{t=0}^{\infty} \left[ y_t'Q_{yy}y_t + 2y_t'Q_{yz}z_t + v_t'Rv_t + 2\mu_{t+1}' (A_{yy}y_t + A_{yz}z_t + B_y v_t - y_{t+1}) \right],
\]

where the evolution of the forcing sequence is given by

\[z_{t+1} = A_{zz}z_t.\] (3.5.1) ["AUG2"]
First-order necessary conditions for the maximization of $L$ with respect to $\{v_t\}_{t=0}^\infty$ and $\{y_t\}_{t=0}^\infty$ are

\[
v_t : \quad Rv_t + B_y' \mu_{t+1} = 0, \quad t \geq 0 \tag{3.5.2} \text{["AUG3 "]}
\]

\[
y_t : \quad \mu_t = Q_{yy}y_t + Q_{yz}z_t + A_{yy}' \mu_{t+1}, \quad t \geq 0. \tag{3.5.3} \text{["AUG4 "]}
\]

Solve equation (3.5.2) for $v_t$; substitute it into the state equation; and stack the resulting equation along with (3.5.3) and (3.5.1) as composite system

\[
L^a \begin{bmatrix} y_{t+1} \\ \mu_{t+1} \\ z_{t+1} \end{bmatrix} = N^a \begin{bmatrix} y_t \\ \mu_t \\ z_t \end{bmatrix},
\]

where

\[
L^a \equiv \begin{bmatrix} I & B_y R^{-1} B_y' & 0 \\ 0 & A_{yy}' & 0 \\ 0 & 0 & I \end{bmatrix}, \quad N^a \equiv \begin{bmatrix} A_{yy} & 0 & A_{yz} \\ -Q_{yy} & I & -Q_{yz} \\ 0 & 0 & A_{zz} \end{bmatrix}. \tag{3.5.4} \text{["ASCOS "]}
\]

As with the deterministic regulator problem, the relevant solution is the one that stabilizes the state-costate vector for any initialization of $y_0$ and $z_0$. Hence we seek a characterization of the multiplier $\mu_t$ of the form

\[
\mu_t = P \begin{bmatrix} y_t \\ z_t \end{bmatrix},
\]

such that the resulting composite sequence $[y_t', \mu_t', z_t']'$ is in the stable deflating subspace of the augmented pencil $\lambda L^a - N^a$. Assuming for the moment that a solution $P$ exists, it must be the case that $P = [P_y \ P_z]$, where $P_y$ is the Riccati equation solution that was characterized in section 3.3, and $P_z$ is a matrix that has not yet been characterized. To see why this must be the case, note that the solution to the augmented regulator problem with $z_0 = 0$ coincides with the solution to the deterministic regulator problem. We have previously shown that $P_y$ is a matrix, such that all vectors in the deflating subspace of the pencil $\lambda L - N$ can be represented as $[y' \ y' P_y]'$. When the forcing sequence is initialized at zero, so that it remains there for all $t$, it must also be the case that $[y' \ y' P_y 0]'$ is in the stable deflating subspace of the augmented pencil $\lambda L^a - N^a$. This justifies our previous claim that the solution to the deterministic regulator problem is a piece of the solution to the augmented regulator problem.

To deduce the control law associated with the matrix $P$, we substitute $P$ into (3.5.4), which yields

\[
L^a \begin{bmatrix} y_{t+1} \\ P_y y_{t+1} + P_z z_{t+1} \\ z_{t+1} \end{bmatrix} = N^a \begin{bmatrix} y_t \\ P_y y_t + P_z z_t \\ z_t \end{bmatrix}.
\]
Write the three equations in this composite system separately:

\[
(I + B_y R^{-1} B_y' P_y) y_{t+1} + B_y R^{-1} B_y' P_z z_{t+1} = A_{yy} y_t + A_{yz} z_t
\]

\[
A_{yy}' P_y y_{t+1} + A_{yy}' P_z z_{t+1} = (P_y - Q_{yy}) y_t + (P_z - Q_{yz}) z_t
\]

\[
z_{t+1} = A_{zz} z_t.
\]  

(3.5) **[system]**

Substitute the last equation into the first and solve for \( y_{t+1} \):

\[
y_{t+1} = (I + B_y R^{-1} B_y' P_y)^{-1} [A_{yy} y_t + (A_{yz} - B_y R^{-1} B_y' P_z A_{zz}) z_t].
\]

(3.5) **[evolve2]**

It follows from relation (3.3.9) that this evolution equation for \( y_t \) can be rewritten as

\[
y_{t+1} = (A_{yy} - B_y F_y) y_t + (A_{yz} - B_y F_z) z_t,
\]

(3.5) **[evolve2]**

where \( F_y \) and \( F_z \) are given by

\[
F_y \equiv (R + B_y' P_y B_y)^{-1} B_y' P_y A_{yy},
\]

\[
F_z \equiv (R + B_y' P_y B_y)^{-1} B_y' (P_y A_{yz} + P_z A_{zz}.
\]

(3.5) **[cntrz]**

For the reasons given previously, our construction of \( F_y \) coincides with (3.3.11) used to represent the optimal control law for the deterministic regulator problem. Stability of the state vector sequence \( \{y_t\} \) is guaranteed by evolution equation (3.5.6) because the matrix \( A_{yy} - B_y F_y \) is the same matrix that appears in the state evolution equation for the deterministic regulator problem under the optimal control law. Since the solution to the deterministic regulator problem is stable by design, the eigenvalues of \( A_{yy} - B_y F_y \) have absolute values that are strictly less than one. The optimal control law for the augmented regulator problem is given by

\[
v_t = -F_y y_t - F_z z_t.
\]

The matrix \( F_z \) can be computed using formula (3.5.7) once we know \( P_z \). We now show that \( P_z \) is the solution to a Sylvester equation. Premultiply (3.5.6) by \( A_{yy}' P_y \):

\[
A_{yy}' P_y y_{t+1} = A_{yy}' P_y (A_{yy} - B_y F_y) y_t + A_{yy}' P_y (A_{yz} - B_y F_z) z_t.
\]

(3.5) **[sys2]**

Using formula (3.5.7), we rewrite the coefficient matrix on \( z_t \) as

\[
A_{yy}' P_y (A_{yz} - B_y F_z) = (A_{yy} - B_y F_y)' (P_y A_{yz} + P_z A_{zz}) - A_{yy}' P_z A_{zz}.
\]
To obtain an alternative formula for this coefficient, substitute the last equation of (3.5.5) into the second equation and solve for \( A_{yy}'P_yy_{t+1} \):

\[
A_{yy}'P_yy_{t+1} = (P_z - Q_{yz} - A_{yy}'P_z A_{zz}) z_t + (P_y - Q_{yy}) y_t. \tag{3.5.9} \]

Equating coefficients on \( z_t \) in (3.5.8) and (3.5.9) results in

\[
(A_{yy} - B_y F_y)' (P_y A_{yz} + P_z A_{zz}) - A_{yy}' P_z A_{zz} = P_z - Q_{yz} - A_{yy}' P_z A_{zz}. \]

Rewriting this in the form of a Sylvester equation (in the unknown matrix \( P_z \)), we have that

\[
P_z = Q_{yz} + (A_{yy} - B_y F_y)' P_y A_{yz} + (A_{yy} - B_y F_y)' P_z A_{zz}. \tag{3.5.10} \]

As already noted, the matrix \((A_{yy} - B_y F_y)\) has only stable eigenvalues. Also, we assumed that the matrix \( A_{zz} \) has only stable eigenvalues (Assumption 4). These restrictions are sufficient for there to exist a unique solution \( P_z \) to (3.5.10). Up to now, our discussion proceeded under the presumption that there exists a matrix \( P \), such that by setting \( \mu_t = P \begin{bmatrix} y_t \\ z_t \end{bmatrix} \), we stabilize the state vector sequence. We can now work backwards using the (unique) solution to the Sylvester equation to show that indeed such a matrix \( P \) does exist.

### 3.6. Computational techniques for solving Sylvester equations

A Sylvester equation is represented by

\[
M = W + SMT, \tag{3.6.1} \]

where the matrices \( W, S, \) and \( T \) are specified in advance and \( M \) is the matrix to be computed. Consistent with (3.5.10), the matrices \( S \) and \( T \) have stable eigenvalues.\(^{15}\) There is a variety of ways to depict the solution to a Sylvester equation. One is to vectorize (3.6.1) as

\[
[I - T' \otimes S] \text{vec}(M) = \text{vec}(W), \tag{3.6.2} \]

where \( \text{vec}(\cdot) \) denotes stacks of the columns of a matrix argument. (To derive (3.6.2) from (3.6.1), use the identity \( \text{vec}(SMT) = [T' \otimes S] \text{vec}(M) \)). Hence

\(^{15}\) We have recycled some of the notation used in previous sections.
vec($M$) is the solution to a linear equation system. Alternatively, $M$ is given by the infinite sum

$$M = \sum_{j=0}^{\infty} S^i W T^j.$$  \hspace{1cm} (3.6.3)  \hspace{1cm} ["sylv4 "]

This representation can be deduced by iterating on equation (3.6.1), starting from any initial matrix with the appropriate dimensions.

We consider two types of algorithms for computing $M$:

(i) Hessenburg-Schur algorithm;
(ii) doubling algorithm.

The Hessenburg-Schur algorithm uses a Schur decomposition of the matrix $T$ to convert a single Sylvester equation to a collection of much smaller Sylvester equations, each of which can be vectorized as in (3.6.2). A Hessenberg decomposition of the matrix $S$ is used further to simplify the calculations. The doubling algorithm is an iterative algorithm that approximates the infinite sum on the right-hand side of (3.6.3) by a finite sum. As with the doubling algorithm for solving a Riccati equation, the number of terms included in the finite sum approximation “doubles” at each iteration.

3.6.1. The Hessenberg-Schur algorithm

As suggested by Bartels and Stewart (1972), one strategy for solving Sylvester equations entails block triangularizing the matrices $T$ and/or $S$. We follow Golub, Nash and Van Loan (1979) by forming a Schur decomposition of the matrix $T$: $V^T V = \hat{T}$, where $V$ is an orthogonal matrix and $\hat{T}$ is upper block triangular with row and column blocks that are either one or two dimensional (see section 3.4.1 for a formal definition). Postmultiply Sylvester equation (3.6.1) by $V$ and rewrite the equation as

$$\hat{M} = \hat{W} + \hat{S} \hat{M} \hat{T},$$  \hspace{1cm} (3.6.4)  \hspace{1cm} ["sylv5 "]

where $\hat{M} = MV$, $\hat{W} = WV$, and $\hat{S} = S$. Notice that (3.6.4) is in the form of a Sylvester equation in the matrix $\hat{M}$.

The block triangularity of $\hat{T}$ can now be exploited to reduce (3.6.4) into $m$ smaller Sylvester equations, where $m$ is the number of row and column blocks of $\hat{T}$. Write the matrix $\hat{T}$ in partitioned form as

$$\hat{T} = \begin{bmatrix}
\hat{T}_{11} & \hat{T}_{12} & \cdots & \hat{T}_{1m} \\
0 & \hat{T}_{22} & \cdots & \hat{T}_{2m} \\
\vdots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & \hat{T}_{mm}
\end{bmatrix}.$$
Use the column partition of $W$ to partition $\hat{M}$ and $\hat{W}$, and let $\hat{M}_j$ and $\hat{W}_j$ denote the corresponding $j^{th}$ partitions. Decompose Sylvester equation (3.6.4):

$$\hat{M}_1 = \hat{W}_1 + \hat{S}\hat{M}_{11}$$  \hspace{1cm} (3.6.5) \hspace{1cm} ["sylv6"]

$$\hat{M}_j = \hat{W}_j + \hat{S}\sum_{k=1}^{j-1} \hat{M}_k\hat{T}_{kj} + \hat{S}\hat{M}_j\hat{T}_{jj}, \hspace{0.5cm} j = 2, ..., m.$$  \hspace{1cm} (3.6.6) \hspace{1cm} ["sylv7"]

Notice that (3.6.5) is a Sylvester equation in $\hat{M}_1$ and that (3.6.6) is a Sylvester equation in $\hat{M}_j$ as long as the matrices $\hat{M}_k$ for $k = 1, 2, ..., j - 1$ have already been computed. Thus these $m$ Sylvester equations can be solved sequentially as linear equations using vectorization (3.6.2).

An additional refinement advocated by Golub, Nash and Van Loan (1979) entails taking a Hessenberg decomposition of the matrix $S$.\footnote{Alternatively, we could take the Schur decomposition of $S$ as proposed by Bartels and Stewart (1972).}

**Definition:** The Hessenberg decomposition of the square matrix $S$ is an orthogonal matrix $U$ and a matrix $\hat{S}$ that has all zeros below the first subdiagonal, such that $S = U\hat{S}U'$.

In addition to postmultiplying equation (3.6.1) by $V$, we now also premultiply this equation by $U'$. Equation (3.6.4) continues to hold with $\hat{M} = U'\hat{M}V$, $\hat{W} = U'\hat{W}V$, and $\hat{S} = U'\hat{S}U$. This Sylvester equation can still be decomposed as in (3.6.5) and (3.6.6). With $\hat{S}$ in Hessenberg form, we can solve these latter Sylvester equations more efficiently using an equation solver designed for Hessenberg systems.\footnote{Interesting variations on the Hessenberg-Schur algorithm have been proposed by Hammarling (1982) and Gardiner et al. (1992).}

In summary, the steps for implementing a Hessenberg-Schur algorithm for computing $P_z$ are

(i) form the matrices $W = Q_{yz} + (A_{yy} - B_yF_y)'P_yA_{yz}$, $S = (A_{yy} - B_yF_y)'$, and $T = A_{zz}$;

(ii) form a Hessenberg decomposition $S = U\hat{S}U'$ and a Schur decomposition $T = V\hat{T}V'$;

(iii) compute the solution $\hat{M}$ to (3.6.5) and (3.6.6) and form $P_z = U\hat{M}V'$. 

Since the Hessenberg decomposition of a matrix can be computed faster than the real Schur decomposition, one should always arrange the Sylvester equation so
that the Hessenberg decomposition is taken of the matrix \((A_{yy} - B_y F_y)')\) or \(A_{zz}\), whichever has more entries. The steps just described should be implemented if there are more elements in the vector \(y_t\) than \(z_t\). If \(z_t\) has more elements, then the alternative Sylvester equation

\[
P_z' = Q_{yz}' + A_{yz} P_y (A_{yy} - B_y F_y) + A_{zz}' P_z' (A_{yy} - B_y F_y)'
\]

should be solved for the matrix \(P_z'\).\(^{18}\)

### 3.6.2. Doubling algorithm

The doubling algorithm for Sylvester equations iterates on

\[
\begin{align*}
\alpha_{k+1} &= \alpha_k \alpha_k \\
\beta_{k+1} &= \beta_k \beta_k \\
\gamma_{k+1} &= \gamma_k + \alpha_k \beta_k \gamma_k
\end{align*}
\]

(3.6.7) \[^{*sylvd *}\]

to convergence, where \(\alpha_0 = S\), \(\beta_0 = T\), and \(\gamma_0 = W\). By repeated substitution, it can be shown that

\[
\gamma_k = \sum_{j=0}^{2^k-1} S^j W T^j.
\]

In other words, each iteration doubles the number of terms in the sum.\(^{19}\)

To use this doubling algorithm to compute \(P_z\)

(i) initialize \(\alpha_0 = (A_{yy} - B_y F_y)', \beta_0 = A_{zz}\), and \(\gamma_0 = Q_{yz} + (A_{yy} - B_y F_y)' P_y A_{yz}\);

(ii) iterate in accordance to (3.6.7);

(iii) form \(P_z\) as the limit of \(\{\gamma_k\}\).

\(^{18}\) In numerical work in Anderson, Hansen, McGrattan, and Sargent (1996), we formed the Hessenberg decomposition of a matrix using MATLAB subroutine HESS and the Schur decomposition of a matrix with SCHUR. We solved Hessenberg systems using the routines HSFA and HSSL, which are part of the package described in Gardiner et al. (1992). See pages 364-370 of Golub and Van Loan (1989) for how to compute the Hessenberg decomposition.

\(^{19}\) This algorithm is a slight generalization of the doubling algorithm for Lyapunov equations discussed in Anderson and Moore (1979). A Lyapunov equation is a Sylvester equation in which \(S = T\).
3.7. Concluding remarks

This chapter has focused on computational details for the optimal linear regulator. Many aspects of these calculations will recur in various settings below. Indeed, key ideas and formulas in all of the subsequent chapters of this book build directly or indirectly on results in this chapter. Thus, in chapter 4, we see how the Kalman filter emerges as the dual of the optimal linear regulator. Chapter 6 uses invariant subspace methods to prove the equivalence of alternative ways of formulating a robust control problem. Chapter 16 uses a Lagrangian formulation and invariant subspace methods to construct robust decision rules for controlling forward looking models. As already indicated in chapter 2, the optimal linear regulator can be induced to do all of the hard work in computing a robust rule for such models.
Chapter 4.
The Kalman filter

... we are always searching for something hidden or merely potential or hypothetical, following its traces whenever they appear on the surface.

4.1. Introduction

The Kalman filter is a recursive method for computing linear least squares estimates of sequences of random vectors comprising hidden states and future observables. The states and observables are described by a known linear state-space system that is perturbed by Gaussian shocks with zero mean and known covariances.

Remarkably, the Kalman filter formulas correspond with those for an optimal linear regulator, a fact that reflects the duality of filtering and control, the subject of this chapter. Following Whittle (1990, 1996), we formulate a filtering problem in terms of a Lagrangian. After performing minimizations and maximizations in a particular order, there emerges an optimal linear regulator problem with the flow of time reversed. We therefore say that the linear regulator problem is dual to the Kalman filter.

The Kalman filter is a powerful tool in economics and econometrics. It accomplishes many tasks, including these: (1) It efficiently computes the Wold and autoregressive representations associated with an economic model whose equilibrium is representable as a linear state space system; (2) It facilitates computing the likelihood function of a linear model recursively; (3) By building upon (2), it can be used to infer the econometric implications of aggregation over time; and (4) It is the basic tool for estimating and forecasting hidden factors in linear models. Items (1)–(4) make the Kalman filter an essential tool in deducing the observable implications for an important class of models whose equilibria occur, or can be well approximated, in the form of a linear state space system.¹

Before getting into the details, we first state the Kalman filtering problem and its solution, then assert the associated optimal linear regulator problem for

¹ So far as first and second moments are concerned, those implications are characterized by a vector autoregression. Using the Kalman filter is the easiest way to obtain the autoregressive representation. See Hansen and Sargent (200XXX, chapter 8).
which it is the dual. The remaining sections of the chapter fill in the details required to prove the duality of the filtering and control problems.

We assume throughout this chapter that the state-space model is true, so that issues of model approximation are not in play. Chapters 13 and 14 will formulate filtering problems in settings where the decision maker suspects model misspecification and therefore wants a robust filter.

4.2. Review of Kalman filter and preview of main result

Throughout this chapter, we let $x_t$ denote a state vector at time $t$ and $y_t$ a vector of possibly noise-ridden observations on linear combinations of $x_{t-1}$. This section uses a convention for indexing time that differs from the one used in the remainder of the chapter. We temporarily use this timing convention because we shall use it again in chapter 13 and because it leads to a dual control problem in which the direction of time matches the one we used in chapters 2 and 3. To attain that familiar representation for the control problem, for the filtering problem we let larger indexes $t$ recede further into the past. We begin with a simple and famous example.

4.2.1. Muth’s problem as an example

John F. Muth (1960) applied classical filtering methods to discover a stochastic process for income for which Milton Friedman’s adaptive expectations scheme would be an optimal estimator of permanent income. Muth’s problem can be formulated recursively using the Kalman filter. Where $x_{-t}$ is a scalar state variable and $y_{-t}$ is a scalar observed variable at time $-t, t \geq 0$, consider the state space system

$$
x_{-t} = ax_{-t-1} + [c \ 0] \epsilon_{-t} \tag{4.2.1a}$$
$$y_{-t} = gx_{-t-1} + [0 \ d] \epsilon_{-t} \tag{4.2.1b}
$$

where $a, g, c, d$ are scalars and $\epsilon_{-t}$ is an i.i.d. $(2 \times 1)$ vector of Gaussian random variables with mean zero and covariance matrix $I$. To analyze Milton Friedman’s (1956) concept of permanent income, Muth set $a = 1, g = 1$ and $c > 0, d > 0$. He regarded $x_{-t}$ as a permanent component of income and $d \epsilon_{2,-t}$ as transitory income, while $y_{-t}$ is measured income at $-t$. A consumer facing an income process with this structure wants to estimate his permanent income. Thus, he wants to compute $\hat{x}_{-t} \equiv E[x_{-t}|y^{-t}]$ where $y^{-t}$ denotes the infinite
history of \([y_{-t}, y_{-t-1}, \ldots]\). That is, the consumer wants to form an estimator \(\hat{x}_{-t}\) that is a measurable function of the infinite history \(y^{-t}\) and that minimizes \(E[(x_{-t} - \hat{x}_{-t})^2|y^{-t}]\).

The Kalman filter attains Muth’s solution of this problem. The solution for the optimal estimator \(\hat{x}_{-t}\) can be represented as

\[
\hat{x}_{-t} = K \sum_{j=0}^{\infty} (a - Kg)^j y_{-t-j} \tag{4.2.2}
\]

where \(K\) is the Kalman gain. Equation (4.2.2) expresses the consumer’s estimate of the permanent component of his income as a geometric weighted sum of past income levels. The conditional variance of this estimator is \(\Sigma = E[(x_{-t} - \hat{x}_{-t})^2|y^{-t}]\). The Kalman filter gives a way to compute \(\Sigma\) and \(K\).

4.2.2. The dual to Muth’s filtering problem

The dual to Muth’s filtering problem is the optimal linear regulator

\[-\Sigma \lambda_0^2 \equiv \max_{\{\mu_t\}} - \sum_{t=0}^{\infty} (c^2 \lambda_t^2 + d^2 \mu_t^2) \tag{4.2.3}\]

where the maximization is subject to the law of motion

\[\lambda_{t+1} = a\lambda_t + g\mu_t, \tag{4.2.4}\]

with \(\lambda_0\) given, and where \(a, g, c, d\) take the same values as in Muth’s problem. Problem (4.2.3), (4.2.4) has a solution in the form of a feedback rule

\[\mu_t = -K \lambda_t \tag{4.2.5}\]

where \(K\) is the same scalar that emerges from the Kalman filter, and the matrix \(\Sigma\) in the value function \(\Sigma \lambda_0^2\) is the state covariance matrix that emerges from the Kalman filter. In this chapter, we shall interpret the \(\lambda\)’s as Lagrange multipliers associated with the Kalman filtering problem.

For particular values of \(a, g, c, d\), we invite the reader to use the Matlab program \(\text{olrp.m}\) to solve the regulator problem and \(\text{kfilter.m}\) to solve the Kalman filtering problem, and thereby to verify numerically the duality that we have asserted. In the next section, we verify duality analytically and in the process tell why the adjective ‘dual’ is appropriate, in the sense of mathematical programming, is appropriate. But first we state a more general versions of the Kalman filter problem and the associated dual optimal linear problem.

\[\text{Muth solved the problem using classical (i.e., non-recursive, methods.}\]
4.2.3. The filtering problem

Consider the following optimal filtering problem. For \( t \geq 0 \), a state vector \( x_{-t} \) and an observation vector \( y_{-t} \) satisfy

\[
\begin{align*}
x_{-t} &= Ax_{-t-1} + C\epsilon_{-t} \\
y_{-t} &= Gx_{-t-1} + D\epsilon_{-t}
\end{align*}
\]  

\(^{(4.2.6a)}\) \(^{(4.2.6b)}\)  

where \( \epsilon_{-t} \) is an i.i.d. Gaussian vector with mean zero and covariance matrix \( I \).

We want a recursive way to compute the projections \( \hat{x}_{-t} = E[x_{-t}|y^{-t}] \) \( \hat{y}_{-t} = E[y_{-t}|y^{-t}] \) where \( y^{-t} = [y_{-t}, y_{-t-1}, \ldots] \).\(^4\) Let \( \Sigma \) be the covariance matrix of the state-reconstruction errors \( \epsilon_{-t} = x_{-t} - \hat{x}_{-t} \), conditional on \( y^{-t} \). The maximum-likelihood estimator \( \hat{x}_{-t} \) maximizes \(-\epsilon_{-t}'\Sigma^{-1}\epsilon_{-t}\). The Kalman filter constructs \( \Sigma \) and gives a recursive way of computing \( \hat{x}_{-t} \) as a function of the infinite history \( y^{-t} \). In particular, the Kalman filter attains the representation

\[
\begin{align*}
\hat{x}_{-t} &= A\hat{x}_{-t-1} + K(y_{-t} - \hat{y}_{-t}) \\
\hat{y}_{-t} &= G\hat{x}_{-t-1}
\end{align*}
\]  

\(^{(4.2.7a)}\) \(^{(4.2.7b)}\)  

where \( K \) is the Kalman gain. Equations \(^{(4.2.6)}\), \(^{(4.2.7)}\) imply that the prediction errors satisfy \( y_{-t} - \hat{y}_{-t} = G(x_{-t-1} - \hat{x}_{-t-1}) + D\epsilon_{-t} \). Define the errors in estimating \( x_{-t} \) as \( e_{-t} = x_{-t} - \hat{x}_{-t} \). Substitute \(^{(4.2.7)}\) into \(^{(4.2.6)}\) to deduce

\[
e_{-t} = (A - KG)\epsilon_{-t-1} + (C - KD)\epsilon_{-t}.
\]  

\(^{(4.2.8)}\)  

Define the error covariance matrix \( \Sigma_{-t} = Ee_{-t}e_{-t}' \). Then for a fixed, not necessarily optimal \( K \), \(^{(4.2.8)}\) implies

\[
\Sigma_{-t} = (A - KG)\Sigma_{-t-1}(A - KG)' + (C - KD)(C - KD)'.
\]  

\(^{(4.2.9)}\)  

\(^3\) The text of this section assumes an infinite history \( y^t \). Alternatively, let \( s \) denote a finite horizon. Then for the filtering problem with the timing convention of this section, we would have an initial condition stating that \( e_{-s} \) has a Gaussian distribution with mean zero and covariance matrix \( \Sigma_0 \). This corresponds to setting a terminal value function for the dual control problem with the quadratic form \( \lambda_0'\Sigma_0\lambda_0 \). Under the different convention about time indexes that we shall use in section 4.3 and the rest of this chapter, for the horizon \( s \) version of the problem, the initial condition for the filtering problem is stated in terms of a quadratic form \( e_0'\Sigma_0^{-1}e_0 \). That corresponds to a terminal condition stated in terms of \( \lambda_0'\Sigma_0\lambda_0 \). It is a terminal condition because the flow of time is reversed.

\(^4\) Note the different conditioning information denoted by \( \hat{x}_{-t} \) and \( \hat{y}_{-t} \).
The limit of iterations on (4.2.9) satisfies
\[ \Sigma = (A - KG) \Sigma (A - KG)' + (C - KD) (C - KD)' . \]  \hspace{1cm} (4.2.10) \[ "kback5" \]

The value of \( K \) that minimizes \( \Sigma \) in (4.2.10) satisfies
\[ K = (CD' + A \Sigma G') (DD' + G \Sigma G')^{-1} . \]  \hspace{1cm} (4.2.11) \[ "kback6" \]

Formulas (4.2.10), (4.2.11) implement the steady state Kalman filter. An efficient algorithm for computing \((K, \Sigma)\) iterates on (4.2.11), (4.2.10), starting from the initial value \( \Sigma = 0 \). This is a version of the Howard policy improvement algorithm.

Equations (4.2.11), (4.2.10) also implement the policy improvement algorithm for solving a particular optimal linear regulator that is defined in terms of a state vector \( \lambda_t \) and a control vector \( \mu_t \). Given the initial value of the state, \( \lambda_0 \), the dual problem is
\[ \max_{\{\mu_t\}} \left\{ -0.5 \sum_{t=0}^{\infty} \tilde{z}_t' \tilde{z}_t \right\} \]  \hspace{1cm} (4.2.12) \[ "olrpdual" \]

where the maximization is subject to \( \lambda_0 \) given and
\[ \tilde{z}_t = C' \lambda_t + D' \mu_t \]  \hspace{1cm} (4.2.13a) \[ "kback7;a" \]
\[ \lambda_{t+1} = A' \lambda_t + G' \mu_t . \]  \hspace{1cm} (4.2.13b) \[ "kback7;b" \]

Equation (4.2.13a) defines the objective function. The solution of the optimal linear regulator is a policy rule
\[ \mu_t = -K' \lambda_t \]  \hspace{1cm} (4.2.14) \[ "kback7" \]
that attains the optimal value function
\[ v (\lambda_0) = -0.5 \lambda_0' \Sigma \lambda_0 . \]  \hspace{1cm} (4.2.15) \[ "kback8" \]

We shall show that \( \lambda_0 = \Sigma^{-1} e_0 \) and that therefore the optimized value \(-0.5 \lambda_0' \Sigma \lambda_0 \) in (4.2.12) equals the quadratic term \(-0.5 e_0' \Sigma^{-1} e_0 \) in a log-likelihood function.

The key practical insight of these findings is that we can compute the \((\Sigma, K)\) for the filtering problem by solving the associated optimal linear regulator (4.2.12), (4.2.14). The reversal in time and the transposition of matrices as we move from the filtering problem to the optimal linear regulator problem are manifestations of duality, as subsequent sections show.
The Kalman filter

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The duality of optimal filtering and control brings substantial insights and computational advantages. In chapters 13 and 14, we shall use these insights to pose and solve robust filtering problems.

The remainder of this chapter substantiates our claims about duality. The reader who is willing to accept the preceding assertions about duality on faith can proceed immediately to subsequent chapters. Though it can be skipped, we think that the subsequent arguments convey some of the magic associated with the duality of filtering and control.

4.3. Sequence version of primal and dual problems

This section substantiates various assertions in the previous section. We show how the Kalman filtering problem leads to an augmented optimal linear regulator in terms of dual variables. We now let the time index $t$ flow forward. (A reversal of time will occur in the dual problem.) We consider the state space system for $t \geq 1$:

\[ x_t = Ax_{t-1} + C\epsilon_t \]  
(4.3.1a) \[ y_t = Gx_{t-1} + D\epsilon_t. \]  
(4.3.1b)

Here $\epsilon_t$, $t \geq 1$, is an i.i.d. Gaussian disturbance vector with mean zero and covariance matrix $I$. We take the initial condition $x_0$ to be unknown with prior distribution described by

\[ x_0 = \hat{x}_0 + e_0 \]  
(4.3.2)

where $e_0$ is a Gaussian vector with mean zero and covariance matrix $Ee_0e_0' = \Sigma_0$. We assume that $e_0$ is distributed independently of the $\epsilon_t$'s for $t \geq 0$. For any variable $z$, let $z^*$ be the vector of observations on $\{z_t, t = 1, \ldots, s\}$. The joint density of $(y^*, x^*)$ is Gaussian. Therefore it can be represented

\[ f(x^*, y^*) \propto \exp \left( -D_s \right), \]

where

\[ D_s = \frac{1}{2} e_0' \Sigma_0^{-1} e_0 + \frac{1}{2} \sum_{t=1}^{s} \epsilon_t' \epsilon_t. \]  
(4.3.3)

Whittle (1990, 1996) calls $D_s$ the 'discrepancy'. To see that the time $t$ contribution to $D_s$ is $(1/2)\epsilon_t'\epsilon_t$, note that by (4.3.1)

\[ \begin{bmatrix} x_t \\ y_t \end{bmatrix} = \begin{bmatrix} A \\ G \end{bmatrix} x_{t-1} + C^*\epsilon_t, \]
where $C^* = \begin{bmatrix} C \\ D \end{bmatrix}$. The covariance matrix of $C^* \epsilon_t$ is $C^* C^{*\prime}$. Then the time $t$ contribution to the discrepancy is \footnote{The matrix $(C^* C^{*\prime})^{-1} C^*$ is the Moore-Penrose generalized inverse of $C^*$.}

$$\frac{1}{2} \epsilon_t^\prime \epsilon_t (C^* C^{*\prime})^{-1} C^* \epsilon_t = \frac{1}{2} \epsilon_t^\prime \epsilon_t.$$ 

4.3.1. Sequence version of Kalman filtering problem

Given $y^s$, we seek estimators of the hidden state $x_t$ for $t = 1, \ldots, s - 1$. We observe $y^s$ and estimate the hidden states by maximizing the log likelihood $-D_s$ with respect to the unobserved states and shocks $\epsilon_s$. In particular, we seek values of $e_0, \{\epsilon_t, x_{t-1}\}_{t=1}^s$ that minimize (4.3.3) subject to (4.3.1), (4.3.2). Following Whittle, we formulate this minimization problem in terms of a Lagrangian. Letting $\{\lambda_t, \mu_{t+1}\}_{t=0}^s$ be sequences of vectors of Lagrange multipliers, we form

$$J_1 = \frac{1}{2} e_0^\prime \Sigma_0^{-1} e_0 + \frac{1}{2} \sum_{t=1}^s \epsilon_t^\prime \epsilon_t + \lambda_0^\prime (x_0 - \hat{x}_0 - e_0)$$

$$+ \sum_{t=1}^s \lambda_t^\prime (x_t - Ax_{t-1} - C \epsilon_t) + \sum_{t=1}^s \mu_t^\prime (y_t - G x_{t-1} - D \epsilon_t).$$

4.3.2. Sequence version of dual problem

We want to minimize $J_1$ with respect to $e_0$, $\epsilon_t$ for $t = 1, \ldots, s$, and $x_t$ for $t = 0, \ldots, s - 1$ and to maximize with respect to $\lambda_t, t = 0, \ldots, s$, and $\mu_t, t = 1, \ldots, s$. To illuminate how the Kalman filter is the dual of a linear regulator, we optimize in a particular order, thereby eventually arriving at a reduced Lagrangian that takes the form of an augmented linear regulator problem.
4.3.2.1. Minimizing over $e_0, \epsilon_t$

Following Whittle (1990, 1996), we first minimize with respect to $e_0, \epsilon_t, t = 1, \ldots, s$. The first order conditions with respect to $\epsilon_t$ and $e_0$ can be written

$$
\epsilon_t = C' \lambda_t + D' \mu_t \tag{4.3.5a}
$$

$$
e_0 = \Sigma_0 \lambda_0. \tag{4.3.5b}
$$

Condition (4.3.5a) implies that

$$
\epsilon_t' \epsilon_t = \begin{bmatrix} \lambda_t' \\ \mu_t \end{bmatrix}' \begin{bmatrix} CC' & CD' \\ DC' & DD' \end{bmatrix} \begin{bmatrix} \lambda_t \\ \mu_t \end{bmatrix}. \tag{4.3.6}
$$

A quick calculation also shows that

$$
\lambda_t' C \epsilon_t + \mu_t' D \epsilon_t = \begin{bmatrix} \lambda_t' \\ \mu_t \end{bmatrix}' \begin{bmatrix} CC' & CD' \\ DC' & DD' \end{bmatrix} \begin{bmatrix} \lambda_t \\ \mu_t \end{bmatrix}. \tag{4.3.7}
$$

Condition (4.3.5b) implies that

$$
\epsilon_0' \Sigma_0^{-1} e_0 = \lambda_0' \Sigma_0 \lambda_0 \tag{4.3.8}
$$

and that

$$
\lambda_0' (x_0 - \hat{x}_0 - e_0) = \lambda_0' (x_0 - \hat{x}_0 - \Sigma_0 \lambda_0). \tag{4.3.9}
$$

Note the presence of $\Sigma_0$ rather than $\Sigma_0^{-1}$ on the right side of (4.3.8). Substituting (4.3.6), (4.3.7), (4.3.8), and (4.3.9) into (4.3.4) gives $J_1 = J_2$ where

$$
J_2 = -\frac{1}{2} \lambda_0' \Sigma_0 \lambda_0 - \frac{1}{2} \sum_{t=1}^{s} \left[ \begin{bmatrix} \lambda_t' \\ \mu_t \end{bmatrix}' \begin{bmatrix} CC' & CD' \\ DC' & DD' \end{bmatrix} \begin{bmatrix} \lambda_t \\ \mu_t \end{bmatrix} + \lambda_0' (x_0 - \hat{x}_0) 
+ \sum_{i=1}^{s} \lambda_i' (x_i - Ax_{i-1}) + \sum_{i=1}^{s} \mu_i' (y_i - Gx_{i-1}). \tag{4.3.10}
$$

By expressing the objective in terms of the dual variables (i.e., the multipliers $\mu_t, \lambda_t$), through equation (4.3.8) the objective function in (4.3.10) involves a quadratic form in $\Sigma_0$ rather than $\Sigma_0^{-1}$. This feature is important for understanding the duality of filtering and control.
4.3.2.2. Extremizing over \(\lambda_t, \mu_t; x_t\)

We want to maximize \(J_2\) with respect to \(\lambda_t, t = 0, \ldots, s\) and \(\mu_t, t = 1, \ldots, s\), and to minimize it with respect to \(x_t, t = 0, \ldots, s - 1\). Minimizing (4.3.10) with respect to \(x_t, t = 0, \ldots, s - 1\) yields the first-order condition

\[
\lambda_{t-1} = A'\lambda_t + G'\mu_t. \tag{4.3.11} \text{["lawlam"]}
\]

Having minimized out the \(x_t\)'s, we are left with the problem of choosing \(\lambda_t, t = 0, \ldots, s\) and \(\mu_t, t = 1, \ldots, s\) to maximize

\[
J_3 = -\frac{1}{2}\lambda_0'\Sigma_0\lambda_0 - \frac{1}{2}\sum_{t=1}^{s} \left[ \begin{array}{c} \lambda_t \\ \mu_t \end{array} \right]' \left[ \begin{array}{cc} CC' & CD' \\ DC' & DD' \end{array} \right] \left[ \begin{array}{c} \lambda_t \\ \mu_t \end{array} \right] - \lambda_0'\hat{x}_0 + \sum_{t=1}^{s} \mu_t'y_t \tag{4.3.12} \text{["lagrange3"]}
\]

subject to (4.3.11) and the boundary conditions \(\lambda_t = 0, \mu_t = 0\) for \(t > s\). Here \(J_3 = J_2\). Notice how this resembles a finite horizon augmented linear regulator problem (see page 44) with state vector \(\lambda_t\) and control vector \(\mu_t\). However, the direction of time is reversed. The term \(-\frac{1}{2}(\lambda_0'\Sigma_0\lambda_0 + 2\lambda_0'\hat{x}_0)\) plays the role of a ‘terminal’ value function once time is reversed. The optimal control takes the form of a feedback rule

\[
\mu_t = -K_t'\lambda_t + g_t y_t + f_t'\hat{x}_0, \tag{4.3.13} \text{["feedbackk"]}
\]

where \(K_t\) is a version of the Kalman gain, as we shall see in detail below.

4.4. Digression: reversing the direction of time

We briefly return to a formulation of the filtering problem in which time recedes into the past with increases in \(t\), as in section 4.2. Supposing that \(s > 0\) and letting \(t = 0, \ldots, s\), the state space system is (4.2.6) where the initial condition at time \(-s - 1\) is

\[
x_{-s-1} = \hat{x}_{-s-1} + e_{-s-1}
\]

where \(e_{-s-1}\) is a Gaussian random vector with mean zero and covariance matrix \(\Sigma_{-s-1}\). Define the discrepancy at horizon \(s\) as

\[
D_s = \frac{1}{2}e_{-s-1}'\Sigma_{-s-1}^{-1}e_{-s-1} + \frac{1}{2}\sum_{t=0}^{s} \epsilon_{-t}'\epsilon_{-t}. \tag{4.4.1} \text{["discrep2"]}
\]
We could follow the steps in the previous section to derive the dual problem with these timing conventions. In the limit as $s \to +\infty$, the dual problem would assume the form of the optimal linear regulator (4.2.12), (4.2.14).

For the remainder of this chapter, we shall use the timing conventions of section 4.3. However, in chapter 13, we shall again use the timing convention of section 4.2.

### 4.5. Recursive version of dual problem

We are sometimes interested in versions of problem (4.3.12) that condition on infinite histories of observations, in which case there is a recursive formulation of the problem. We seek a time invariant $K$, which we attain by studying the problem as $s \to \infty$ and then taking the limit of $K_t$ as $t \to \infty$. The recursive version of problem (4.3.12) is associated with the Bellman equation

$$\frac{1}{2} \lambda \Sigma \lambda - \lambda' \hat{x} - \frac{1}{2} \psi = \max_{\mu, \lambda} \left\{ \left( \frac{1}{2} \lambda^* \Sigma^* \lambda^* - \frac{1}{2} \mu' \left[ \begin{array}{c} \lambda \\ \mu \end{array} \right] \left[ \begin{array}{cc} CC' & CD' \\ DC' & DD' \end{array} \right] \left[ \begin{array}{c} \lambda \\ \mu \end{array} \right] \right) + \mu' y - \lambda' \hat{x}_0 \right\},$$

(4.5.1) ["bellmank"]

where the maximization on the right is subject to the law of motion

$$\lambda^* = A' \lambda + G' \mu$$

(4.5.2) ["lawlam2"]

and where $\lambda^*$ now denotes last period’s value of $\lambda$ and $\Sigma^*$ is last period’s value of $\Sigma$. The term $\psi$ is a constant that we’ll explain later. This Bellman equation induces a mapping from $\Sigma^*$ to $\Sigma$. The unique positive semi-definite matrix fixed point $\Sigma$ and the matrix $K$ associated with the optimal feedback rule supply the ingredients ($\Sigma, K$) that solve the infinite-history Kalman filtering problem.

Letting $\psi$ be a vector of Lagrange multipliers on (4.5.2), the first-order conditions with respect to $\lambda^*, \mu$ for maximizing (4.5.1) subject to (4.5.2) are

$$0 = - \Sigma^* \lambda^* - \hat{x}_0 - \psi$$
$$0 = -DC' \lambda + y + G\psi - DD' \mu.$$

Eliminate $\psi$ and rearrange to get the feedback rule

$$\mu = -K' \lambda + (G \Sigma^* G' + DD')^{-1} (y - G \hat{x}_0),$$

(4.5.3) ["lawmu3"]

where

$$K = (CD' + A \Sigma^* G')(DD' + G \Sigma^* G')^{-1}.$$
The matrix $K$ is the Kalman gain. When (4.5.4) is evaluated at the stationary solution $\Sigma = \Sigma^*$ of the Riccati equation implied by the Bellman equation (4.5.1), (4.5.3) solves the infinite-history, time-invariant filtering problem. We now indicate how (4.5.1) implies a Riccati equation mapping $\Sigma^*$ into $\Sigma$.

Use (4.5.2) and (4.5.3) to express $\lambda^*$ as

$$
\lambda^* = (A - KG)'\lambda + G'\left(G\Sigma^*G' + DD'\right)^{-1}(y - G\hat{x}_0).
$$

(4.5.5) "lambda0 "

Using (4.5.3) and (4.5.5) to evaluate the quadratic forms in $\lambda_0$ on the first line of the right side of (4.5.1) shows

$$
\left\{\lambda^*\Sigma^*\lambda^* + \begin{bmatrix} \lambda \\ \mu \end{bmatrix}' \begin{bmatrix} CC' & CD' \\ DC' & DD' \end{bmatrix} \begin{bmatrix} \lambda \\ \mu \end{bmatrix} \right\} = \lambda^*\Sigma\lambda + \text{terms in } (y - G\hat{x}_0)
$$

where

$$
\Sigma = (A - KG)(A - KG)' + (C - KD)(C - KD)'.
$$

(4.5.6) "riccati4 "

Formula (4.5.6) in conjunction with formula (4.5.4) is one form of the Riccati equation for the conditional covariance matrix $\Sigma$ for the hidden state next period.

For the next step of the argument, we temporarily ignore the term in $y - G\hat{x}_0$ appearing in (4.5.3). Then, using (4.5.5) and $\mu = -K^*\lambda$, we can calculate that

$$
\mu'y - \lambda^*\hat{x}_0 = -\lambda'(A\hat{x}_0 + K(y - G\hat{x}_0)) \equiv -\lambda\hat{x}
$$

(4.5.7) "xhat "

where

$$
\hat{x} = A\hat{x}_0 + K(y - G\hat{x}_0)
$$

(4.5.8) "xhat0 "

is the estimator of the state for next period. Formulas (4.5.8) and (4.5.4), evaluated at the fixed point of (4.5.6) are the standard time-invariant Kalman filtering formulas.

Finally, we have to complete and collect the terms coming from $(G\Sigma^*G' + DD')^{-1}(y - G\hat{x}_0)$ in (4.5.3). Tedious algebra verifies that they contribute the term

$$
i = (y - G\hat{x}_0)'(G\Sigma^*G' + DD')^{-1}(y - G\hat{x}_0)
$$

that appears on the left side of (4.5.1). The matrix $G\Sigma^*G' + DD'$ is the covariance matrix of the innovations $y - G\hat{x}_0$. 


4.6. Recursive version of Kalman filtering problem

For some of our future work, it is convenient to study a recursive version of the filtering problem using the dual variables again but to embrace a somewhat different perspective.

We return to the original problem and give a recursive representation of it. In a recursive spirit, we formulate a one-period filtering problem and seek a recursion in the optimized value function. The state-space system is

\[
\begin{align*}
    x &= Ax_0 + C\epsilon \\
    y &= Gx_0 + D\epsilon \\
    x_0 &= \hat{x}_0 + e_0,
\end{align*}
\]

where \( \epsilon \) is a Gaussian random vector with mean zero and identity covariance matrix and \( e_0 \) is a Gaussian random vector distributed independently of \( \epsilon \) with mean 0 and covariance matrix \( \Sigma_0 \). The joint density of \( (x, y) \) is

\[
f(x, y) \propto \exp(-D)
\]

where

\[
D = \frac{1}{2} (e'_1 \Sigma^{-1}_1 e_1 + \epsilon' \epsilon).
\]

Given \( y, \hat{x}_0 \), we want to choose \((\epsilon, x)\) to maximize the log likelihood, or equivalently, to minimize discrepancy \(D\) subject to (4.6.1). We will show that the optimized value of the discrepancy (4.6.2) takes the form

\[
\frac{1}{2} e'_1 \Sigma^{-1}_1 e_1 + \frac{1}{2} \iota
\]

where \( e_1 = x - \hat{x}_1 \), \( \hat{x}_1 = A\hat{x}_0 + K(y - G\hat{x}_0) \), \( K \) is the Kalman gain, \( \Sigma_1 \) is related to \( \Sigma_0 \) by a matrix Riccati difference equation, and \( \iota \), defined in our discussion of (4.5.1), is the contribution to the log-likelihood function (entropy) that cannot be influenced by the filter. Thus, we have the Bellman equation

\[
\frac{1}{2} e'_1 \Sigma^{-1}_1 e_1 + \frac{1}{2} \iota = \min_{\epsilon, x} \{ 0.5e'_0 \Sigma^{-1}_0 e_0 + \epsilon' \epsilon \}
\]

where the minimization is subject to (4.6.1). Further, the quadratic form \( e'_1 \Sigma^{-1}_1 e_1 \) on the left equals the quadratic form \( \lambda'_1 \Sigma_1 \lambda_1 \) that appears on the left side of the Bellman equation for the dual problem (4.5.1).
To solve the filtering problem for an additional period, we would use $\Sigma_1$ to update the criterion (4.6.2) to be $\frac{1}{2}(e_1^\prime \Sigma_1^{-1} e_1 + \epsilon^2 \epsilon)$ and continue as before with next period’s observation on $y$ and $e_1 = x - \hat{x}_1$.

It is useful to solve the recursive version of the filtering problem using Lagrangian methods. Form the Lagrangian

$$J = \frac{1}{2}(e_0^\prime \Sigma_0^{-1} e_0 + \epsilon^2 \epsilon) + \lambda_0(x_0 - \hat{x}_0 - e_0) + \lambda'(x - Ax_0 - C\epsilon) + \mu'(y - Gx_0 - D\epsilon).$$

The first-order conditions for minimizing $J$ with respect to $e, e_0$ imply

$$e = C^\prime \lambda + D^\prime \mu,$$  

$$e_0 = \Sigma_0 (A^\prime \lambda + G^\prime \mu),$$

where we are using the first-order condition $\lambda_0 = A^\prime \lambda + G^\prime \mu$ to get (4.6.5b).

The equality $e_0 = x_0 - \hat{x}_0$ and (4.6.1) imply

$$x - Ax_0 = C\epsilon + Ae_0,$$  

$$y - G\hat{x}_0 = D\epsilon + Ge_0.$$

Substitute (4.6.5) into (4.6.6) and rearrange to get

$$\begin{bmatrix} y - G\hat{x}_0 \\ x - Ax_0 \end{bmatrix} = \Lambda \begin{bmatrix} \mu \\ \lambda \end{bmatrix},$$

where

$$\Lambda = \begin{bmatrix} G\Sigma_0 G' + DD' & DC' + G\Sigma_0 A' \\ CC' + A\Sigma_0 G' & A\Sigma_0 A' + CC' \end{bmatrix}.$$  

Then

$$\begin{bmatrix} \mu \\ \lambda \end{bmatrix} = \Lambda^{-1} \begin{bmatrix} y - G\hat{x}_0 \\ x - Ax_0 \end{bmatrix}.$$

For reasons to be explained in chapter 14, we call the optimized value of $\epsilon^2 \epsilon + e_0 \Sigma_0^{-1} e_0$ the conditional entropy of $(y, x)$ and denote it $\text{ent}(y, x)$. It is the maximized value of the log likelihood function. Using (4.6.5), we can evaluate $\text{ent}(y, x)$ to be

$$\text{ent}(y, x) \equiv \epsilon^2 \epsilon + e_0 \Sigma_0^{-1} e_0 = \begin{bmatrix} \mu \\ \lambda \end{bmatrix}^\prime \Lambda \begin{bmatrix} \mu \\ \lambda \end{bmatrix} \equiv \begin{bmatrix} y - G\hat{x}_0 \\ x - Ax_0 \end{bmatrix}^\prime \Lambda^{-1} \begin{bmatrix} y - G\hat{x}_0 \\ x - Ax_0 \end{bmatrix}.$$
Let
\[
L = \begin{bmatrix} I & 0 \\ -K & I \end{bmatrix}
\]
where
\[
K = \Lambda_{22}^{-1} = (A\Sigma_0G' + CD')(DD' + G\Sigma_0G')^{-1}. \tag{4.6.10}
\]
We recognize \(K\) to be the Kalman gain. It can be verified that
\[
LAL' = \begin{bmatrix} \Lambda_{11} & 0 \\ 0 & \Lambda_{22} - \Lambda_{22}\Lambda_{11}^{-1}\Lambda_{21}' \end{bmatrix}, \tag{4.6.11}
\]
where
\[
\Lambda_{11} = G\Sigma_0G' + DD' \tag{4.6.12}
\]
and
\[
\Sigma_1 \equiv \Lambda_{22} - \Lambda_{21}\Lambda_{11}^{-1}\Lambda_{21}'
= CC' + A\Sigma_0A' - (A\Sigma_0G' + CD')(DD' + G\Sigma_0G')^{-1}(A\Sigma_0G' + CD)'. \tag{4.6.13}
\]
It turns out that \(\Lambda_{11}\) is the covariance matrix of the innovations \(y - G\hat{x}_0\) and \(\Lambda_{22} - \Lambda_{21}\Lambda_{11}^{-1}\Lambda_{21}'\) is the covariance matrix of \(x - \hat{x}_1\) where \(\hat{x}_1\) is the estimator of the state \(x\). In particular, notice that
\[
L \begin{bmatrix} y - G\hat{x}_0 \\ x - A\hat{x}_0 \end{bmatrix} = \begin{bmatrix} y - G\hat{x}_0 \\ x - A\hat{x}_0 - K(y - G\hat{x}_0) \end{bmatrix} = \begin{bmatrix} y - G\hat{x}_0 \\ x - \hat{x}_1 \end{bmatrix}
\]
where
\[
\hat{x}_1 = A\hat{x}_0 + K(y - G\hat{x}_0). \tag{4.6.14}
\]
Here \(\hat{x}_1\) is the estimate of the state next period, based on the observed value of \(y\). Thus, returning to (4.6.9), we have
\[
\text{ent}(y, x) = \begin{bmatrix} y - G\hat{x}_0 \\ x - A\hat{x}_0 \end{bmatrix}' L' (LAL')^{-1} L \begin{bmatrix} y - G\hat{x}_0 \\ x - A\hat{x}_0 \end{bmatrix}
= \begin{bmatrix} y - G\hat{x}_0 \\ x - \hat{x}_1 \end{bmatrix}' \begin{bmatrix} \Lambda_{11} & 0 \\ 0 & \Lambda_{22} - \Lambda_{22}\Lambda_{11}^{-1}\Lambda_{21}' \end{bmatrix}^{-1} \begin{bmatrix} y - G\hat{x}_0 \\ x - \hat{x}_1 \end{bmatrix}
= (y - G\hat{x}_0)' \Lambda_{11}^{-1}(y - G\hat{x}_0) + (x - \hat{x}_1)' (\Lambda_{22} - \Lambda_{22}\Lambda_{11}^{-1}\Lambda_{21}')^{-1}(x - \hat{x}_1)
= (y - G\hat{x}_0)' \Lambda_{11}^{-1}(y - G\hat{x}_0) + e_1\Sigma_1^{-1}e_1. \tag{4.6.15}
\]
Formula (4.6.15) inspires the updating formula (4.6.13) for the covariance matrix of \(x - \hat{x}_1\). The entropy-minimizing choice of \(x\) is evidently \(\hat{x}_1\); the value of \(y\) is observed, and the value \(\hat{x}_0\) is given, so the first term on the last line of (4.6.15) cannot be influenced by the filter. It contributes \(e_1\) in (4.6.3).
4.7. Concluding remarks

In the filtering and control problems of this chapter, the decision maker assumes that his state-space model is correctly specified. Later chapters extend the duality between filtering and control to filtering problems in which the decision maker fears that the model (4.2.6) is misspecified. Chapters 6 and 7 formulate and solve a robust control problem. Chapter 13 then exploits duality to get a corresponding robust filtering problem. Effectively, that chapter works backwards from a robust version of the optimal linear regulator problem (4.2.12),(4.2.14) to get a corresponding filtering problem. Not surprisingly in view of the time-reversal between the dual and original problems, the objective function of the decision maker in the dual problem is backward-looking. While interesting, that is not always the most natural formulation for economic problems. Therefore, in chapter 14 we alter the objective function of the decision maker to be forward-looking. That leads us to another robust filtering problem. We cast that forward-looking robust filtering problem as a model-approximation problem using entropy to measure model misspecification. This forward-looking robust filtering problem has the same mathematical structure as the one studied in section 4.6. In chapters 6 and 7, we shall also use duality theory extensively to formulate our basic robust dynamic decision problem.
Part II

Robust control and applications
Chapter 5.
Multiplier and constraint games

5.1. Introduction
This chapter formulates static zero-sum two-player games whose equilibria induce robust decisions for the maximizing player within a one-period setting. In particular, we formulate what we call a multiplier game and a constraint game. We begin with a simple static Phillips curve example in section 5.2. Subsequent sections then focus on another simple example with the aim of exposing the role of technical assumptions that reconcile outcomes from alternative games.

This chapter describes two classes of possible misspecifications to a Gaussian approximating model that might concern the decision maker. We begin with a more restricted class that allows misspecifications only in the mean of a Gaussian random variable. We then extend the class of misspecifications to include arbitrary alternative distributions. For a Gaussian approximating model, the worse case model from this class remains Gaussian, but has distortions to both the mean and the variance.  

5.2. Phillips curve example
To illustrate basic ideas, this section adapts Kydland and Prescott’s (1977) model of a policy maker who sets inflation in view of an expectational Phillips curve. We modify Sargent’s (1999, chapter 3) formulation of Kydland and Prescott’s model by assuming that the policy maker views his model as an approximation. The policy maker solves a multiplier game as a way to compute a decision that is robust to model misspecification. Let $U, \pi, \pi_e$ be the unemployment rate, the inflation rate, and the public’s expected rate of inflation, respectively. The government’s approximating model is

$$U = U^* - \gamma(\pi - \pi_e) + \hat{\epsilon}$$

(5.2.1)

where $\gamma > 0$ and $\hat{\epsilon}$ is $\mathcal{N}(0, 1)$. Here $U^*$ is the natural rate of unemployment, the unemployment rate that on average prevails when $\pi = \pi_e$.  

1 Chapter 2 described two related such classes of distortions for dynamic models.

2 To bring the setup closer to that used in dynamic settings in chapters 2 and 6, we could have added a parameter $c$ and expressed (5.2.2) as $U = U^* - \gamma(\pi - \pi_e) + c(\epsilon + w)$,
π, the public sets πe, and nature draws ē. The government views (5.2.1) as an approximation in the sense that it suspects that U might actually be governed by

\[
U = U^* - \gamma (\pi - \pi_e) + (\epsilon + w),
\]

where \(\epsilon\) is another random variable that is distributed \(\mathcal{N}(0, 1)\) and \(w\) can be regarded as an unknown distortion to the mean \(\epsilon\). It is as if the government suspects that the natural unemployment rate might be \(U^* + w\) for some unknown \(w\). The government does know that

\[
w^2 \leq \eta^2. 
\]

The parameter \(\eta^2\) bounds the square of the government’s specification error \(w^2\).

5.2.1. The government’s problem

The government values outcomes \((U, \pi)\) according to the same utility function assigned by Kydland and Prescott, namely,

\[
-E(U^2 + \pi^2)
\]

where \(E\) denotes the mathematical expectation. Because it does not trust the approximating model, the government wants to evaluate the mathematical expectation over multiple models indexed by \(w\)’s that satisfy (5.2.3).

We proceed in the spirit of Stokey’s (1989) analysis of credible government policies. We derive the government’s robust best response to the private sector’s setting of \(\pi_e\). The appendix then uses that robust best response function to formulate a rational expectations equilibrium. The government’s best response function takes \(\pi_e\) as fixed. Given \(\pi_e\), the government wants to set \(\pi\) so that it attains satisfactory outcomes for all \(w^2 \leq \eta^2\). The government therefore sets \(\pi\) equal to the equilibrium \(\pi\)-component of the following two-player zero-sum multiplier game

\[
\max_{\pi} \min_w -E\{U^2 + \pi^2\} + \theta w^2
\]

where both the minimization and maximization are subject to (5.2.2) and \(\theta > 1\) is a fixed penalty parameter. We shall soon explain how the penalty parameter \(\theta\) relates to \(\eta\) in (5.2.3) and why we impose \(\theta > 1\). We shall also discuss conditions where \(c\) is used to scale the volatility of the noise \(\epsilon\). We have set \(c = 1\) to simplify some formulas in this chapter.
that let us exchange the order of maximization and minimization in (5.2.5). The first order conditions for \( \pi \) and \( w \), respectively, for problem (5.2.5) are

\[
\begin{align*}
(1 + \gamma^2) \pi - \gamma^2 \pi_e - \gamma (U^* + w) &= 0 \\
U^* - \gamma \pi + \gamma \pi_e + w (1 - \theta) &= 0.
\end{align*}
\]

Solving these equations jointly for \( \pi, w \) as functions of \( \pi_e \) gives:

\[
\begin{align*}
\pi(\theta) &= \left( \frac{\gamma}{1 - \theta^{-1} + \gamma^2} \right) (U^* + \gamma \pi_e) \\
w(\theta) &= \left( \frac{\theta^{-1}}{1 - \theta^{-1} + \gamma^2} \right) (U^* + \gamma \pi_e).
\end{align*}
\]

Here \( \pi(\theta) \) gives the government’s (robust) best response function for setting \( \pi \) as a function of \( \pi_e \), while \( w(\theta) \) determines the worst case model, given \( \pi_e \) and the government’s setting \( \pi(\theta) \).

Note that when there is no concern for model misspecification (i.e., when \( \theta = +\infty \)),

\[
\begin{align*}
\pi(\infty) &= \left( \frac{\gamma}{1 + \gamma^2} \right) (U^* + \gamma \pi_e) \\
w(\infty) &= 0.
\end{align*}
\]

Note also that (5.2.6a) says that \( \pi(\theta) \) satisfies

\[
\pi(\theta) = \left( \frac{\gamma}{1 + \gamma^2} \right) ((U^* + w(\theta)) + \gamma \pi_e).
\]

This equation defines a function

\[
\pi(\theta) = B(\pi_e; \theta),
\]

which is the government’s robust best response function to the state of expectations \( \pi_e \). Evidently the robust rule can be obtained by replacing the estimate of the natural unemployment rate \( U^* \) under the approximating model in (5.2.9) with the worst case estimate of the natural rate \( U^* + w(\theta) \). Thus, the way to achieve robustness is to distort estimates of exogenous variables relative to the approximating model, then to proceed with ordinary decision making procedures.\(^3\) A related characterization of robust decision making procedures will

\(^3\) See the citation attributed to Fellner on page 25.
prevail in the dynamic settings to be studied in subsequent chapters. However, because the models there are dynamic, the distortions become more interesting and involve misspecifications in how state vectors feed back on their own histories.

It is useful to compute the limiting decision \( \pi(\theta) \) and worst case distortion \( w(\theta) \) as \( \theta \searrow 1 \).

\[
\begin{align*}
\pi(1) &= \gamma^{-1}U^* + \pi_e \quad (5.2.12) \\
w(1) &= \gamma^{-2}(U^* + \gamma \pi_e). \quad (5.2.13)
\end{align*}
\]

In the appendix to this chapter we show how the unit slope of the government's best response to \( \pi_e \) in (5.2.12) will cause a rational expectations equilibrium inflation rate to approach \(+\infty\) as \( \theta \searrow 1 \). That rational expectations inflation rate satisfies \( \pi = \pi_e \), as well as having \( \pi \) be a robust best government response to \( \pi_e \).

Given \( \pi_e \), we can now tell how the penalty parameter \( \theta \) is related to the constraint \( \eta \). We can relate the multiplier game to the Lagrangian associated with a closely related constraint game:

\[
\sup_{\pi} \inf_{|w| \leq \eta} -E(U^2 + \pi^2),
\]

where \( \theta \) will turn out to be the Lagrange multiplier on the constraint \( w^2 \leq \eta^2 \).

The associated Lagrangian is

\[
\sup_{\pi} \sup_{\theta \geq 0} \inf_w -E(U^2 + \pi^2) + \theta (w^2 - \eta^2).
\]

If \( \theta > 1 \), for the inner minimization part of this problem it is evidently optimal to set \( w \) so that the constraint \( w^2 \leq \eta^2 \) holds with equality. Then set \( w = \pm \eta \) and solve (5.2.8) for \( \theta \):

\[
\theta = 1 + \frac{|U^* - \gamma (\pi - \pi_e)|}{\eta}. \quad (5.2.14)
\]

Equation (5.2.14) shows how to map \( \eta \) into an associated \( \theta \). As described by equation (5.2.14), the parameter \( \theta \) thus measures the set of alternative models over which the decision maker seeks a satisfactory outcome. We shall discuss the connection between the constraint game and the multiplier game further in the following sections. Before that, we briefly describe the sense in which (5.2.7) gives a decision for \( \pi \) that is robust to model misspecification.

\footnote{The value \( \theta = 1 \) is the breakdown point to be discussed later. In the generalization of the model where \( c(\epsilon + w) \) replaces \( (\epsilon + w) \), the breakdown point is \( \theta = c^2 \).}
5.2.2. Robustness of robust decisions

For convenience, we define $\sigma = -\theta^{-1}$; $\sigma$ is the risk-sensitivity parameter of Jacobson (1973) and Whittle (1990). Fig. 5.2.1 illustrates the sense in which a robust decision for $\pi$ is robust. Let $J(\sigma_1, \sigma_2)$ be the value of $-E(U^2 + \pi^2)$ associated with setting $\pi = \pi(\sigma_1)$ when $w = w(\sigma_2)$. Assuming $\gamma = 1, U^* = 5$, for three settings of inflation $\pi(\sigma_1)$, Fig. 5.2.1 plots $J(\sigma_1, \cdot)$ as a function of $\sigma_2$, where the worst case $w = w(\sigma_2)$ varies along the ordinate axis. Notice how the three payoff functions $J(\sigma_1, \cdot)$ cross. The $\sigma = \sigma_1 = 0$ rule gives the highest value for the government’s objective when there is no specification error (i.e., $\sigma_2 = 0$ implies that $w = 0$), but its performance deteriorates more quickly than the robust ($\sigma_1 = -0.25, \sigma_1 = -0.5$) rules as $w$ increases along the $\sigma_2$ axis. The robust rules sacrifice performance when the approximating model is correct. However, they experience lower rates of deterioration in the objective $J$ as the specification error increases.

![Figure 5.2.1](image)

**Figure 5.2.1:** Values of $J(\sigma_1, \sigma_2) = -E(U^2 + \pi^2)$ for three decision rules $\pi(\sigma_1)$ for $\sigma_1 = 0, -0.25, -0.5$ for the worst-case $w(\sigma_2)$ for values of $\sigma_2$ on the ordinate axis. The $\sigma_1 = 0$ rule works best when $w = 0$, but its performance deteriorates more rapidly as $|w|$ increases than do the robust rules.

Because our principal focus in this chapter is single-agent robust control theory, we have taken $\pi_e$ as given. To complete the analysis of the Kydland-Prescott model, we should describe how $\pi_e$ is set. Appendix A applies the notion
of a rational expectations equilibrium to make $\pi_e$ equal to the $\pi(\sigma)$ chosen by the robust monetary authority. We postpone that material to the appendix because it involves issues that would interrupt our main line of argument. We now turn to important technical details about our single agent decision model.

5.3. Basic setup with a correct model

This section describes in more detail the relationship between a static constraint game and a static multiplier game. Let $x$ be an endogenous variable and $u$ a scalar control variable. The variables $u$ and $x$ are linked by the approximating model

$$x = u + \hat{\epsilon}$$

(5.3.1)

where $\hat{\epsilon}$ is a random variable with mean zero and variance 1. Letting $E$ denote the mathematical expectation and $b$ be a scalar, a decision maker wants $(u, x)$ to maximize

$$-\frac{u^2}{2} - \frac{1}{2}E(x - b)^2$$

(5.3.2)

or

$$-\frac{u^2}{2} - \frac{(u - b)^2}{2} - \frac{1}{2}.$$  

(5.3.3)

The maximizing choice is $u = \frac{b}{2}$.

We want to think about the situation where the decision maker treats the model (5.3.1) not as true but as a good approximation. To represent specification error, the decision maker replaces the approximating model (5.3.1) with the distorted model

$$x = u + (\epsilon + w),$$

(5.3.4)

where $\epsilon$ is another random variable with mean zero and variance 1. The distorted model thus has a random term with unknown mean $w$ and known variance 1, rather than known mean 0 and variance 1 as under the approximating model (5.3.1). The decision maker formulates the idea that his model is a good approximation by assuming that $|w| \leq \eta$ where $\eta > 0$. Substituting (5.3.4) into (5.3.2), the criterion function becomes

$$-\frac{u^2}{2} - \frac{(u + w - b)^2}{2} - \frac{1}{2}.$$  

(5.3.5)

The decision maker seeks a $u$ that works well for any $|w| \leq \eta$. Since the variance 1 is constant, we can replace (5.3.5) with

$$-\frac{u^2}{2} - \frac{(u + w - b)^2}{2}.$$  

(5.3.6)
Within this simple setting, we consider two types of zero-sum two-person games that can be used to design a choice of \( u \) that is robust to misspecifications that take the form of alternative values of \( w \). The two games are: (1) a ‘constraint game’ that constrains the choices of \( u, v \) in (5.3.6) by \(|w| \leq \eta\); and (2) a ‘multiplier game’ that appends to the right side of (5.3.6) a penalty term \( \theta (w^2 - \eta^2) \). For an appropriate choice of \( \theta \), these two formulations are equivalent under conditions identified by the Lagrange multiplier theorem (see Luenberger (1969), pp. 216-221). However, that equivalence breaks down when \( \eta > |b| \). As a vehicle for exploring conditions for the equivalence between the two approaches, we start by analyzing the pathological \( b = 0 \) case. Later parts of this chapter shed further light on the pathological case by allowing a larger class of misspecifications.

5.4. The constraint game with \( b = 0 \)

To create an apparent pathology, we temporarily set \( b = 0 \). To induce a robust decision \( u \) we formulate a ‘constraint game’:\footnote{We thank Dirk Bergemann for suggesting this example and its consequences.}

\[
\max_u \min_{|w| \leq \eta} \left[ -\frac{u^2}{2} - \frac{(u + w)^2}{2} \right].
\] (5.4.1)

Notice that the objective is concave and not convex in \( w \) (this is also true when \( b \neq 0 \)). Also notice the timing protocol implicit in the order of maximization and minimization in (5.4.1).

The equilibrium of this zero-sum two-person game can be computed by considering three possible sets of values for \( u \). If \( u = 0 \), \( w = \pm \eta \) solves the inner minimization problem, with a minimized value of \(-\frac{\eta^2}{2}\). If \( u > 0 \), the solution of the inner problem is to set \( w = \eta \), which makes the objective smaller than \(-\frac{\eta^2}{2}\). Similarly, if \( u \leq 0 \), the solution of the inner problem is to set \( w = -\eta \), and the objective (5.4.1) is again smaller than \(-\frac{\eta^2}{2}\). Thus the ‘robust’ decision is to set \( u \) to zero; this decision is supported by the expectation that \( w \) will respond to \( u \) by the rule \( w = \frac{u}{|u|} \eta \) for \( u \neq 0 \) and \( w = \pm \eta \) when \( u \) is zero.

A strange feature of (5.4.1) is that a preference for robustness to model misspecification has no effect on the decision \( u \). The equilibrium outcome for \( u \) is 0, independently of the value of \( \eta \).

For various reasons to be explained below, we would like to be able to exchange the order of minimization and maximization in (5.4.1). However, another
peculiarity of (5.4.1) is that we cannot exchange orders of the minimization and maximization operations; neither \( u = 0, w = \eta \) nor \( u = 0, w = -\eta \) is a Nash equilibrium of the game with the order of maximization and minimization exchanged. There is no pure strategy Nash equilibrium. We will explore mixed strategy equilibria later.

5.5. Multiplier game with \( b = 0 \)

We want to understand the connection between the constraint game (5.4.1) and an associated ‘multiplier game’. To do so, in this section we study a Lagrangian formulation of the constraint game. This will eventually lead us to a multiplier game. We reformulate the constraint in (5.4.1) as \( w^2 \leq \eta^2 \) and form a Lagrangian:

\[
\max_u \inf_w \sup_{\theta \geq 0} \left( -\frac{u^2}{2} - \frac{(u + w)^2}{2} + \frac{\theta}{2} \left( w^2 - \eta^2 \right) \right)
\]

or

\[
\max_u \sup_{\theta \geq 0} \inf_w \left( -\frac{u^2}{2} - \frac{(u + w)^2}{2} + \frac{\theta}{2} \left( w^2 - \eta^2 \right) \right).
\]

The standard sufficient conditions for the Lagrange Multiplier Theorem are not applicable here. While the constraint set for \( w \) is convex, the objective is also convex in \( w \). As we will see in later chapters, with appropriate qualifications, a version of the Lagrange Multiplier Theorem does apply.

Consider the inner most minimization problem of (5.5.1), holding fixed \( \theta \) and \( u \). Suppose \( \theta \leq 1 \). Then the objective is convex in \( w \), (affine for \( \theta = 1 \)) and the infimum over \( w \) is \(-\infty\). Therefore, we need consider only \( \theta > 1 \). For \( \theta > 1 \), the first-order conditions for \( w \) are:

\[
(\theta - 1) w - u = 0,
\]

or

\[
w = \frac{u}{\theta - 1}.
\]

Consider next the second inner-most maximization problem in (5.5.1). Provided that \( u \neq 0 \), the supremum over \( \theta \) is attained by setting \( \theta \) so that the constraint is satisfied. Thus

\[
\theta = 1 + \frac{|u|}{\eta}
\]

and

\[
w = \frac{u}{|u|} \eta.
\]
At these values of $\theta, v$, the objective for the outer maximization problem in (5.5.1) becomes

$$L(u) = -\frac{u^2}{2} - \frac{u^2}{2} \left( \left| u \right| + \eta \right)^2 = -\frac{u^2}{2} - \frac{(|u| + \eta)^2}{2} < -\frac{\eta^2}{2}.$$ 

By making $u$ arbitrarily close to zero, we see that the right side of the above inequality is a least upper bound. In fact if $u = 0$, then $w = 0$ and

$$\sup_{\theta \geq 1} \inf_w -\frac{u^2}{2} - \frac{(u + w)^2}{2} + \frac{\theta}{2} (w^2 - \eta^2) = -\frac{1}{2}\eta^2.$$ 

This gives the correct value of the objective of the constraint game (5.4.1), and $u = 0$ is the correct robust action for that game. Since the solution is not attained at $\theta = 1$, the solution of (5.5.1) must be computed as a limit of the solution as $\theta \searrow 1$. The value $\theta = 1$ corresponds to what we shall refer to in chapter 6 as a ‘breakdown point’ for $\theta$.

Games (5.4.1) and (5.5.1) are pathological because neither $\eta$ in the constraint game nor $\theta$ in the multiplier game affects the equilibrium decision $u$. We show below how this pathology reflects that $|b| < \eta$.

### 5.6. The model with $b \neq 0$

By setting $b \neq 0$, we can repair the pathology that variations in the multiplier $\theta$ in (5.5.1) do not change the action $u$. We alter (5.4.1) to be:

$$\max_u \min_{|w| \leq \eta} -\frac{u^2}{2} - \frac{(u + w - b)^2}{2}. \quad (5.6.1)$$

The Lagrangian is:

$$\max_u \inf_w \sup_{\theta \geq 0} -\frac{u^2}{2} - \frac{(u + w - b)^2}{2} + \frac{\theta}{2} (w^2 - \eta^2)$$

or

$$\max_u \sup_{\theta \geq 0} \inf_w -\frac{u^2}{2} - \frac{(u + w - b)^2}{2} + \frac{\theta}{2} (w^2 - \eta^2).$$

It is again true that for $\theta \leq 1$, the inner-most minimization problem has a criterion equal to $-\infty$ for any $u$. Thus $\theta = 1$ remains a ‘breakdown point’.
Variations of $\theta$ for $\theta > 1$ will now affect the decision $u$, thereby capturing the sense in which a concern for robustness affects the decision.

For $\theta > 1$, consider the multiplier game:

$$\max_u \min_w \frac{u^2}{2} - \frac{(u + w - b)^2}{2} + \frac{\theta (w^2 - \eta^2)}{2}. \quad (5.6.2)$$

The objective is concave in $u$ and convex in $w$. It can be verified directly that the order of maximization and minimization does not matter, and that the Nash equilibrium of the game defined by (5.6.2) can be obtained by stacking and solving first-order conditions for the minimizing and maximizing players.$^6$

The first-order conditions are:

$$u + (u + w - b) = 0$$
$$u + w - b - \theta w = 0.$$

The equilibrium outcomes are:

$$u = \frac{\theta b}{2\theta - 1}$$
$$w = \frac{-b}{2\theta - 1}. \quad (5.6.3)$$

We have thus established:

**Theorem 5.6.1.** For $\eta$ in the interval $(0, |b|)$ we can find a value of $\theta > 1$ for which the solution to the multiplier game (5.6.2) is the same as that of the constraint game (5.6.1) and conversely. This mapping breaks down when $\theta = 1$ and $\eta \geq |b|$.

**Proof.** From (5.6.3), as $\theta$ ranges from $+\infty$ to one, the solution for $w$ ranges from zero to $-b$. \[ \square \]

Notice that $u = \frac{b}{2}$ for the limiting $\theta = +\infty$ case, and that $u$ converges to $b$ as $\theta$ declines to one.

---

$^6$ This is a version of von Neumann’s Minimax Theorem. For example, see Dantzig (1998, pp. 286–287).
5.6.1. Analysis of pathology

Consider now the constraint game when $\eta > |b|$. Form two quadratic functions:

\[ p_-(u) = -\frac{u^2}{2} - \frac{(u - \eta - b)^2}{2} \]
\[ p_+(u) = -\frac{u^2}{2} - \frac{(u + \eta - b)^2}{2}. \]

The robust choice of $u$ solves:

\[ \max_u \min \{ p_-(u), p_+(u) \}. \]

Notice that $p_-(b) = p_+(b)$. Moreover, $dp_-(0)/du = b + \eta$ and $dp_+(0)/du = b - \eta$. Because $\eta > |b|$, these derivatives have opposite signs, implying that $u = b$ remains the robust solution for large enough values of $\eta$.

![Figure 5.6.1: The functions $p_-(u), p_+(u), \min \{ p_-(u), p_+(u) \}$ for $\eta = .3, b = 0$. The maximum of $\min \{ p_-(u), p_+(u) \}$ occurs at $u = b = 0$, a kink point of the function.](image)

Figures 5.6.1 and 5.6.2 help reveal what is going on in the two cases $\eta > |b|$ and $\eta < |b|$. Fig. 5.6.1 plots the function $\min \{ p_-(u), p_+(u) \}$ for $\eta = .3, b = 0$ while Fig. 5.6.2 plots it for $\eta = .3, b = .5$. In Fig. 5.6.1, which corresponds to a pathological case in which $\eta > |b|$, $\min \{ p_-(u), p_+(u) \}$ has a maximum.
Figure 5.6.2: The functions $p_-(u), p_+(u), \min \{ p_-(u), p_+(u) \}$ for $\eta = .3, b = .5$. The maximum of $\min \{ p_-(u), p_+(u) \}$ occurs at $u = \frac{b - \eta}{2} = .4$, where the function is differentiable.

at $u = b = 0$, a nondifferentiable point formed by the intersection of the two component functions. In Fig. 5.6.2, for which $\eta < |b|$, the maximum of $\min \{ p_-(u), p_+(u) \}$ occurs at $u = \frac{2b + \eta}{2} = .4$, a point where the function is differentiable. Here $u$ depends on $\eta$, reflecting a concern for robustness that was absent in the pathological $\eta > |b|$ case.

5.7. Probabilistic formulation ($b = 0$)

We alter game (5.4.1) by allowing random perturbations to the approximating model. The approximating model is now

$$x = u + \epsilon$$

where $\epsilon \sim f_\epsilon(\epsilon)$ and $f_\epsilon$ is the standard normal density. The distorted models have $\epsilon \sim f(\epsilon)$ for some density $f \neq f_\epsilon$. Corresponding to the $b = 0$ case above, we now make the objective in our zero-sum two-player games be

$$- \frac{u^2}{2} - \int \frac{(u + \epsilon)^2 f(\epsilon)}{2} d\epsilon.$$  (5.7.1)

To measure model misspecification we use relative entropy defined to be the expected log likelihood ratio where the expectation is valued at the distorted
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model:

$$I(f) = \int [\log f(\epsilon) - \log f_o(\epsilon)] f(\epsilon) \, d\epsilon.$$  \hfill (5.7.2)

This entropy measure is convex in \( f \). We study the game:

$$\max_u \min_{f, I(f) \leq \xi} -\frac{u^2}{2} - \frac{\int (u + \epsilon^2 f(\epsilon) \, d\epsilon}{2}. \hfill (5.7.3)$$

The objective in (5.7.3) is linear in the density \( f \) and the constraint set is convex, so Lagrangian methods apply.

5.7.1. Gaussian perturbations

Before relating game (5.7.3) to game (5.4.1), we calculate entropy measure (5.7.2) where \( f \) is a normal density with mean \( w \) and variance \( \sigma^2 \). Then\footnote{Simple calculations show that \( I(f) \) is the expectation of \( \log(\sigma^{-1}) - (2\sigma^{-1})(\epsilon - w)^2 + (2)^{-1}\epsilon^2 \) evaluated with respect to \( f(\epsilon) \).}

$$I(f) = \frac{w^2}{2} + \sigma^2 - 1 - \frac{\log \sigma^2}{2}. \hfill (5.7.4)$$

Thus entropy decomposes into a part \( \frac{w^2}{2} \) due to a mean distortion and a part \( \frac{\sigma^2 - 1}{2} - \frac{\log \sigma^2}{2} \) due to a variance distortion. Because the logarithm is a concave function, the variance distortion is nonnegative:

$$\frac{\sigma^2 - 1}{2} - \frac{\log \sigma^2}{2} \geq 0.$$

To understand how game (5.7.3) relates to game (5.4.1), consider a perturbed density \( f \) that is normal with mean \( w \) and unit variance \( \sigma^2 = 1 \) so that the distortion consists solely of a mean shift. Then \( I(f) = w^2/2 \) and the objective (5.7.1) becomes

$$-\frac{u^2}{2} - \frac{(u + w)^2 + 1}{2}.$$  

This agrees with (5.3.5) when \( b = 0 \). With the Gaussian \( f(\epsilon) \), we can view (5.7.3) as extending (5.4.1) to a larger set of perturbations. In effect, (5.4.1) admits only perturbations that are equivalent to mean shifts in a standard normal
distribution. The \( \eta \) in (5.4.1) relates to the parameter \( \xi \) in (5.7.3) through the formula:

\[
\eta^2 = \frac{\xi}{2}
\]

In shifting the distortions from numbers \( w \)'s to densities \( f \) we have made the objective function linear in the distortion. The family of normal distributions with a unit variance and mean \( w \) is not convex, however. We could imagine mixing \( w \) actions by entertaining finite mixtures of normal distributions. We have chosen to go even further by entertaining more than just finite normal mixtures. We allow other densities, but constrain their relative entropy.

### 5.7.2. Letting the minimizing agent make random perturbations when \( b = 0 \)

We now extend the set of perturbations to include shifts in the variance as well as mixtures of normals. By appropriately choosing \( f \), which is now the counterpart to \( w \) in (5.4.1), the minimizing player can implement a mixed strategy. This changes the solution to the problem in a substantial way.

The Lagrange saddle-point problem is:

\[
\max_u \min_f \sup_{\theta \geq 0} - \frac{u^2}{2} - \frac{\int (u + \epsilon)^2 f(\epsilon) d\epsilon}{2} + \theta [\mathcal{I}(f) - \xi]
\]

or

\[
\max_u \max_{\theta \geq 0} \inf_f - \frac{u^2}{2} - \frac{\int (u + \epsilon)^2 f(\epsilon) d\epsilon}{2} + \theta [\mathcal{I}(f) - \xi]. \tag{5.7.5}
\]

The first-order conditions for the inner-most minimization problem of (5.7.5) are

\[
\theta [\log f(\epsilon) - \log f_0(\epsilon) + 1] + \kappa = \frac{(u + \epsilon)^2}{2}\tag{5.7.6}
\]

where \( \kappa \) is a constant introduced by the constraint \( \int f = 1 \). The solution to this problem is:

\[
f_\theta(\epsilon) \propto \exp \left[ \frac{(u + \epsilon)^2}{2\theta} \right] f_0(\epsilon) \tag{5.7.7}
\]

where the constant of proportionality is chosen so that \( f_\theta(\epsilon) \) integrates to unity. Such a constant will exist only when

\[
\int \exp \left[ \frac{(u + \epsilon)^2}{2\theta} \right] f_0(\epsilon) d\epsilon < \infty.
\]
The integral is finite provided that $\theta > 1$. When $\theta > 1$, the density $f_\theta$ defined by (5.7.7) is normal since it is the product of exponentials with quadratic terms in $\epsilon$. It is easy to verify that the density $f_\theta$ is proportional to the exponential of the following term:

$$
\frac{(u + \epsilon)^2}{2\theta} - \frac{\epsilon^2}{2} = -\frac{(\theta - 1) \epsilon^2}{2\theta} + \frac{ue}{\theta} + \frac{u^2}{2\theta} = -\frac{(\epsilon - \mu_\theta)^2}{2\sigma^2_\theta} + c
$$

where $c$ does not depend on $\epsilon$ and where

$$
\mu_\theta = \frac{u}{\theta - 1}
$$

$$
\sigma^2_\theta = \frac{\theta}{\theta - 1}.
$$

Thus $f_\theta$ is normal with mean $\mu_\theta$ and variance $\sigma^2_\theta$. Notice that the variance $\sigma^2_\theta$ becomes arbitrarily large as $\theta$ approaches unity. As a consequence, the relative entropy associated with $\theta$ becomes arbitrarily large. For instance, when $u = 0$ (5.7.4) implies

$$
\mathcal{I} (f_\theta) = \frac{\sigma^2_\theta - 1}{2} - \frac{\log \sigma^2_\theta}{2}.
$$

There is a multiplier $\theta$ associated with each positive $\xi$. The optimized choice of $u$ remains zero in this example, and the worst case distribution $f$ has an increased variance (relative to the standard normal) that depends on the magnitude of $\xi$. Thus, in contrast to the deterministic game, values of $\theta > 1$ correspond to specific values of $\xi$. Moreover, every value of $\xi$ is associated with a multiplier $\theta$ that is greater than one. Finally, we can exchange the order of the min and max, which implies that $u = 0, f = f_\theta$ is a Nash equilibrium as well, where $\theta$ is chosen to satisfy the entropy constraint for a given value of $\xi$.

Thus, by expanding the set of admissible perturbations from mean shifts to arbitrary (absolutely continuous) density shifts, we have been able to avoid some of the complications of game (5.4.1). But we will still be led to study limiting decision rules as $\theta$ decreases to some critical value, namely $\theta = 1$ in this example. The breakdown point for $\theta$ will no longer be associated with a finite value of $\xi$. The limiting solution as $\theta \downarrow 1$ corresponds to the $H_\infty$ control in chapter 7.

Introducing a translation term $b$ into the objective as in

$$
-\frac{u^2}{2} - \int (u - b + \epsilon)^2 f(\epsilon) d\epsilon
$$
will cause the worst-case distribution to have a nonzero mean, but there will still be a variance enhancement. The quadratic objective makes the worst-case distribution remain normal. The enhanced variance will not alter the decision for $u$. Thus the multiplier solution for $u$ in (5.5.1) will be a constraint solution to the stochastic game (5.7.3). However, the implied variance enhancement is needed to match multipliers and constraints for the stochastic game.

5.8. Concluding remarks
This chapter has displayed two types of zero-sum two-player games that can be used to induce decisions that are robust to model misspecification. Each game has a malevolent nature choose a model misspecification to frustrate the decision maker. The ‘constraint game’ directly constrains the distortions to the approximating model that the malevolent agent can make. The ‘multiplier game’ penalizes those distortions. The two games are equivalent under conditions that allow us to invoke the Lagrange multiplier theorem. For our simple static example, we displayed conditions under which the two games are equivalent, and explored conditions under which they capture concern for model misspecification.

We have considered two classes of misspecifications, one that allows distortions only in the mean of a Gaussian random variable, the other than allows arbitrary alternative density functions that satisfy a constraint on entropy. In the static setting of this chapter, for the first class of mean misspecifications only, misspecification is confined to not knowing the mean of a random shock or a constant term in a linear equation. Subsequent chapters take up models where the decision maker fears misspecified dynamics, which he captures by allowing $w$ to be the conditional mean of a shock vector. By allowing that conditional mean to feed back on the history of the state, a large variety of misspecifications can be modelled. **Add more stuff about the sets of perturbations and link back to chapter 2.** Thus, the following two chapters return to our main theme of dynamic games that can be used to design robust decision rules. The conceptual issues connecting the constraint game and the multiplier game will carry over to the richer settings of chapters 6 and 7.
A. Rational expectations equilibrium

The Phillips curve example of section 5.2 took $\pi_e$ as given. This appendix constructs a rational expectations version of the model and shows how to compute a time-consistent or Nash equilibrium rate of inflation. We proceed by adapting some concepts of Stokey (1989) to this example. Thus, we define a Nash equilibrium (with robustness) for the model as follows:

**Definition 5.A.1.** Given multiplier $\theta > 1$, a Nash equilibrium is a pair $(\pi, \pi_e)$ such that (a) $\pi = B(\pi_e; \theta)$, and (b) $\pi = \pi_e$. Here $B$ is the government’s best response map (5.2.7).

Condition (a) says that given $\pi_e$, the government is choosing a robust rule associated with multiplier $\theta$. Condition (b) imposes rational expectations. It is easy to compute a rational expectations equilibrium by solving (5.2.7) and $\pi = \pi_e$ for $\pi_e$:

$$\pi_e (\theta) = \frac{\theta}{\theta - 1} U^* \gamma.$$  

Notice that $\pi'_e (\theta) < 0$, $\lim_{\theta \to \infty} \pi_e (\theta) = U^* \gamma$, and $\lim_{\theta \downarrow 1} \pi_e (\theta) = +\infty$. If the approximating model is true, so that the government’s concern about misspecification is misplaced, the government’s ignorance of the model causes it to set inflation higher than if it knew the model for sure.

Notice that Definition 5.A.1 imputes a concern for model misspecification to the government, but not to the private forecasters, who are assumed to know the $\pi$ chosen by the government. In chapters 16 and 18, we shall return to discuss an alternative version of rational expectations that imposes more symmetry between the government and private agents.
Chapter 6.
Time domain games

6.1. Introduction

This chapter and chapter 7 focus on a decision maker who copes with his fear of misspecified dynamics. We appeal to the modified certainty equivalence principle stated on page 20 and use a nonstochastic specification both for the decision maker’s approximating model and for perturbations to it. In particular, we model misspecification by allowing shocks to feed back on the history of the state in ways that the approximating model excludes.

We proceed by formulating dynamic versions of the constraint and multiplier problems that we encountered in a static context in chapter 5. The dynamic setting means that there are alternative timing protocols for a two-player zero-sum game, and that potentially the equilibrium outcomes may depend on the timing protocol. In this chapter, we focus mainly on dynamic versions of a multiplier game. We describe three different multiplier games distinguished by their timing protocols. We identify a set of conditions under which, despite their different timing protocols, the three games have identical outcomes and identical recursive representations of their equilibria. An equilibrium induces a robust decision rule. We derive some useful formulas for rapidly computing equilibria of the games and the associated robust decision rules. Chapter 7 analyzes multiplier games in the frequency domain.

6.2. Three games

A decision maker uses a set of models surrounding a single explicitly specified approximating model. One parameter, either θ or η, measures a set of perturbations around the approximating model. Three models within the set are especially important: the decision maker’s approximating model; an unknown true model that generates the data; and a worst case model that emerges as a by-product of his robust planning procedure. Each model specifies that an $n \times 1$ state vector evolves according to

$$x_{t+1} = A_0 x_t + B u_t + C w_{t+1}$$  \hspace{1cm} (6.2.1)
where $x_0$ is given, $u_t$ is a vector of controls, and $w_{t+1}$ is a vector of specification errors with $w_t = 0 \ \forall t < 0$. The approximating model also specifies that $w_t = 0 \ \forall t \geq 0$. The other models have $w_t \neq 0$ for some $t \geq 0$.\footnote{In chapter 8, we will consider stochastic models formed by replacing $w_{t+1}$ by the sum of an i.i.d. Gaussian vector $\tilde{\epsilon}_{t+1}$ with mean zero and identity covariance matrix $I$ and a distortion $w_{t+1}$ that is measurable with respect to the history of $x_t$. The presence of $\epsilon_{t+1}$ obscures the model misspecification with noise. This setting lets us use model detection error probabilities to restrict the value of $\theta$.} We assume that the matrix $A_o$ has all of its eigenvalues inside the circle $\Gamma$ in the complex plane, where $\Gamma = \{ \zeta : |\zeta| = \frac{1}{\sqrt{\beta}} \}$. We assume that the pair $(\sqrt{\beta}A_o, B)$ is\footnote{See chapter 3, page 45 for a definition of stabilizable.} stabilizable,\footnote{See chapter 3, page 45 for a definition of stabilizable.} where $\beta \in (0, 1]$ is a discount factor. Here and in chapter 7, we shall often evaluate a discounted infinite-horizon criterion under alternative time-invariant decision rules for $u_t$ and $w_{t+1}$. The rule for $u_t$ takes the form

$$u_t = -Fx_t, \tag{6.2.2}$$

where $F$ is restricted to be in an admissible set:

$${\mathcal{F}} = \{ F : A - BF \text{ has eigenvalues with moduli strictly less than } \frac{1}{\sqrt{\beta}} \}. \tag{6.2.2.1}$$

The rule for $w_{t+1}$ takes the form

$$w_{t+1} = Kx_t, \tag{6.2.3}$$

where $K$ is restricted to be in the admissible set

$${\mathcal{K}} = \{ K : A + CK \text{ has eigenvalues with modulii strictly less than } \frac{1}{\sqrt{\beta}} \}. \tag{6.2.3.1}$$

Define a target vector:

$$z_t = H_0x_t + Ju_t. \tag{6.2.4}$$

The decision maker wants to maximize the objective function

$$- \sum_{t=0}^{\infty} \beta^t z_t'z_t. \tag{6.2.5}$$

To attain robust decisions for $u_t$, we use the following measure of model misspecification:

$$R(w) = \sum_{t=0}^{\infty} \beta^{t+1} w_{t+1}'w_{t+1}. \tag{6.2.6}$$
Hansen, Sargent, Turmuhambetova, and Williams (2001) refer to $R(w)$ as entropy. The decision maker believes that the data are generated by a model that satisfies $R(w) \leq \eta$ but is otherwise ignorant about $\{w_{t+1}\}$. The decision maker wants a decision rule that works well for any model satisfying $R(w) \leq \eta$.

### 6.2.1. Constraint and multiplier problems

We now describe two types of games that induce robust decisions. Let $u$ denote the sequence $\{u_t\}_{t=0}^\infty$ and $w$ the sequence $\{w_{t+1}\}_{t=0}^\infty$. Consider the following two problems. First, for $\tilde{\eta} \geq \eta \geq 0$, we have:

**Definition 6.2.1.** The constraint robust control problem is

$$
\sup_u \inf_w \left[ \sum_{t=0}^{\infty} \beta^t z'_t z_t \right] 
$$

subject to (6.2.1) and $R(w) \leq \eta$.

For $\theta$ belonging to a set $\Theta = \{\theta : 0 < \tilde{\theta} < \theta \leq +\infty\}$, we also define

**Definition 6.2.2.** The multiplier robust control problem is

$$
\sup_u \inf_w \left[ \sum_{t=0}^{\infty} \beta^t z'_t z_t - \theta R(w) \right] 
$$

subject to (6.2.1).

The Lagrange multiplier theorem (see Luenberger (1969), pp. 216-221) connects the solutions of these two problems, as we shall discuss in detail in chapter 7. Hansen, Sargent, Turmuhambetova, and Williams (2001) discuss the connection between these problems in a continuous time random context. Chapter 7 also describes the lower bound $\tilde{\theta}$ and the upper bound $\tilde{\eta}$. These bounds assure that the problems have finite values.

In this chapter, we focus on the multiplier problem (6.2.8). Section 6.6 briefly discusses the relation between the constraint problem and the multiplier problem, while section 6.7 briefly formulates a recursive version of the constraint game and links the multiplier in the constraint game to a derivative of the value function for the recursive formulation of the constraint game.
6.2.2. Three versions of a multiplier game

Three versions of the zero-sum two-player multiplier game (6.2.8) are distinguished by the spaces to which the maximizing player’s choice of the sequence for \( u \) and the minimizing player’s choice of the sequence for \( w \) are confined. We restrict the choice of \( u \) or \( w \) to one of the following spaces:

\[
W = \{ w : \sum_{t=1}^{\infty} \beta^t w_t w_t' < +\infty \}
\]

\[
U = \{ u : \sum_{t=0}^{\infty} \beta^t u_t u_t' < +\infty \}
\]

\[
W_K = \{ w : w_{t+1} = K x_t, K \in \mathcal{K} \}
\]

\[
U_F = \{ u : u_t = -F x_t, F \in \mathcal{F} \}
\]

where \( \mathcal{F} \) and \( \mathcal{K} \) are defined below (6.2.2) and (6.2.3), respectively.

We shall study versions of the multiplier problem (6.2.8) determined by the following three specifications:

Game 1 (SEQ): a multiplier game where both players choose sequences: \( u \in U, w \in W \).

Game 2 (STACK): a Stackelberg game where the minimizing player commits to a sequence but the maximizing player chooses sequentially: \( u \in U_F, w \in W \).

Game 3 (MARKOV): a Markov game where both players choose sequentially: \( u \in U_F, w \in W_K \).

We apply the Nash equilibrium concept to each of these three games. The games share payoffs and players but differ in their timing protocols. In game 1, both players commit to sequences at time 0. In the Stackelberg game 2, the minimizing player commits to a sequence at time 0 but the maximizing player can set its control as function of the state at \( t \). In game 3, both players can let their control respond to the state at \( t \). Despite their diverse timing protocols, equilibria share common outcomes and recursive equilibrium representations.
6.3. Game 1: The multiplier game in sequences

In game 1 (SEQ), the objective of the two players can be written:

\[ C = \sum_{t=0}^{\infty} \beta^t \left( -z'_t z_t + \beta \theta w'_{t+1} w_{t+1} \right). \] (6.3.1)

It is subject to the state-evolution equation (6.2.1) and the target vector relation (6.2.4). The initial state vector \( x_0 \) is given. A maximizing player chooses \( u \in U \) and a minimizing player chooses \( w \in W \). Elements \( w \in W \) and \( u \in U \) are said to be stabilizing. When \( u \) and \( w \) are stabilizing and the matrix \( A_o \) has all of its eigenvalues inside \( \Gamma \), the state vector sequence is stable, implying that \( \sum_{t=0}^{\infty} \beta^t x'_t x_t < +\infty \).

**Definition 6.3.1.** An equilibrium of the multiplier game in sequences (SEQ) is a pair of sequences \( \{u_t^*\}, \{w_{t+1}^*\} \) that solve both players’ problems.

To find an equilibrium of game SEQ, we begin by substituting from (6.2.4) for \( z_t \) in the objective function. We can then form a Lagrangian for each player. These Lagrangians have the special feature that the first-order conditions for the two players’ problems impart the same laws of motion to their co-state variables.\(^3\) This feature reflects the zero-sum payoffs and allows us to equate the co-state sequences of the two players and to analyze the game by forming a single Lagrangian:\(^4\)

\[ L = -\sum_{t=0}^{\infty} \beta^t \left( x'_t H_0 x_t + u'_t J' u_t + 2u'_t J' H_0 x_t \right) \]
\[ 2\beta \mu'_{t+1} (A_o x_t + B u_t + C w_{t+1} - x_{t+1}) - \beta \theta w_{t+1} w_{t+1}. \]

First-order conditions with respect to \( u_t, w_{t+1}, x_{t+1} \), respectively, are:

\[ J' u_t + J' H_0 x_t + \beta B' \mu_{t+1} = 0 \]
\[ -\theta w_{t+1} + C' \mu_{t+1} = 0 \]
\[ \beta A'_o \mu_{t+1} + H_0 H_0 x_t + H'_0 J u_t - \mu_t = 0 \] (6.3.2)

Assume that \( J' J \) is nonsingular, and solve for \( u_t \) and \( w_{t+1} \):

\[ u_t = - (J' J)^{-1} J' H_0 x_t - \beta (J' J)^{-1} B' \mu_{t+1} \] (6.3.3)
\[ w_{t+1} = \frac{1}{\theta} C' \mu_{t+1}. \] (6.3.4)

\(^3\) Two times the co-state vector \( \mu_t \) equals the gradient of the value function \(-x' P x\).

\(^4\) We do not impose \( x_0 = C w_0 \).
Substitute these expressions for $u_t$ and $w_{t+1}$ into the state equation to get

$$x_{t+1} = \left[ A_0 - B (J'J)^{-1} J' H_0 \right] x_t - \left[ \beta B (J'J)^{-1} B' - \frac{1}{\theta} CC' \right] \mu_{t+1}.$$ 

Substituting the same expressions into (6.3.2) implies

$$\beta \left[ A_0' - H_0' J (J'J)^{-1} B' \right] \mu_{t+1} + \left[ H_0' H_0 - H_0' J (J'J)^{-1} J'H_0 \right] x_t - \mu_t = 0.$$ 

Write the system as

$$L \begin{bmatrix} x_{t+1} \\ \mu_{t+1} \end{bmatrix} = N \begin{bmatrix} x_t \\ \mu_t \end{bmatrix} \quad (6.3.5)$$

where

$$L = \begin{pmatrix} I & \beta B (J'J)^{-1} B' - \frac{1}{\theta} CC' \\ 0 & \beta \left[ A_0' - H_0' J (J'J)^{-1} B' \right] \end{pmatrix}$$

and

$$N = \begin{pmatrix} \left[ A_0 - B (J'J)^{-1} J' H_0 \right] & 0 \\ - \left[ H_0' H_0 - H_0' J (J'J)^{-1} J'H_0 \right] & I \end{pmatrix}.$$ 

It can be verified that the matrix pencil $(\frac{\lambda}{\sqrt{\beta}} L - N)$ is *symplectic*.\(^5\) It follows that the generalized eigenvalues of $(L, N)$ come in $\sqrt{\beta}$-symmetric pairs: for every eigenvalue $\lambda_i$, there is another eigenvalue $\lambda_{-i}$ such that $\lambda_i \lambda_{-i} = \beta^{-1}$.

To guarantee that an equilibrium of game SEQ exists, we must rule out generalized eigenvalues of $(L, N)$ on $\Gamma = \{ \zeta : |\zeta| \leq \frac{1}{\sqrt{\beta}} \}$. Then half of the generalized eigenvalues are inside the circle $\Gamma$ and the other half are outside this circle. The generalized eigenvectors associated with the eigenvalues inside $\Gamma$ generate what we refer to as the $(\sqrt{\beta})$ *stable* deflating subspace. The dimension of this subspace equals the number of entries in the state vector $x_t$. We assume that there exists a positive semidefinite matrix $P$ such that the stable deflating subspace can be represented as $\begin{pmatrix} I \\ P \end{pmatrix} x$.

Under these restrictions, we can construct an equilibrium of game SEQ for which $\mu_t = Px_t$ and an equilibrium state vector sequence satisfies

$$L \begin{pmatrix} I \\ P \end{pmatrix} x_{t+1} = N \begin{pmatrix} I \\ P \end{pmatrix} x_t. \quad (6.3.6)$$

Thus we can state:

\(^5\) See chapter 3 for the definition and properties of symplectic pencils.
Theorem 6.3.1. Suppose
(i) $J' J$ is nonsingular and $J + H_0 \zeta (I - \zeta A)^{-1} B$ has full column rank on $\Gamma$;\(^6\)
(ii) $(L, N)$ has no generalized eigenvalues on $\Gamma$;
(iii) Any element of the $(\sqrt{\beta})$ deflating subspace of $(L, N)$ can be represented as $(I P) x$ for some vector $x$ where $P$ is a symmetric matrix;
(iv) $\theta I - G_0' G_0$ is nonsingular on $\Gamma$ where $G_0 = H_0 (I - \zeta A_o)^{-1} C$.

Then there exists an equilibrium of game SEQ of the form $u_{t} = -F^* (A^*)^t x_0$ and $w_{t+1} = K^* (A^*)^t x_0$ for some $K^*$ and $F^*$, where $A^* = A_o - BF^* + K^* C$ has eigenvalues that are inside $\Gamma$.

Proof. The first player maximizes (6.3.1) by choice of $\{u_t\}$ subject to the state evolution:

\[
x_{t+1} = A_o x_t + B u_t + C K^* \hat{x}_t
\]

\[
\hat{x}_{t+1} = A^* \hat{x}_t.
\]

The second player minimizes (6.3.1) by choice of $\{w_{t+1}\}$ subject to the state evolution:

\[
x_{t+1} = A_o x_t + C w_{t+1} - B F^* \hat{x}_t
\]

\[
\hat{x}_{t+1} = A^* \hat{x}_t.
\]

Both problems are special cases of what Anderson et al (1996) and chapter 2 call an augmented regulator problem. Condition (i) guarantees that the objective is strictly concave in the decision sequence of the first player and condition (ii) guarantees that the objective is strictly convex in the decision sequence of the second player.

We compute the equilibrium by stacking the state-costate equations of the two players and solving the resulting difference equation. We impose the equilibrium conditions that $\hat{x}_t = x_t$ and seek a solution in which the costate sequences for both players coincide. This leads to the linear system (6.3.6) that contains stacked first-order conditions and incorporates $\hat{x}_t = x_t$ for all $t$. Thus we find an equilibrium by constructing a $\sqrt{\beta}$ stable sequence of state vectors that satisfies (6.3.6). From the first partition of (6.3.6), we see that

\[(I + GP) x_{t+1} = \left[ A_o - B (J' J)^{-1} J' H_0 \right] x_t \]

---

\(^6\) This condition is weaker than the corresponding detectability condition for the regulator problem without robustness. It guarantees the existence of a solution to this problem under $\sqrt{\beta}$ stability.
where

\[ G = \beta B (J'J)^{-1} B' - \frac{1}{\theta} CC'. \]

It follows from Theorem 21.7 of Zhou, Doyle and Glover (1996) that \( P \) is symmetric and \( I + GP \) is nonsingular. Hence we have the state evolution:

\[ x_{t+1} = A^* x_t \]

where

\[ A^* = (I + GP)^{-1} \left[ A_o - B (J'J)^{-1} J'H_0 \right]. \]

By (iii), the matrix \( A^* \) has eigenvalues that are inside \( \Gamma \). From the first-order conditions, we have

\[
\begin{align*}
w_{t+1} &= \frac{1}{\theta} C'' P A^* x_t \\
&= K^* x_t \\
u_t &= -(J'J)^{-1} (J'H_0 + \beta B' P A^*) x_t \\
&= -F^* x_t.
\end{align*}
\]

Thus the equilibrium of game SEQ is

\[
\begin{align*}
u_{t+1}^* &= K^* (A^*)^t x_0, \\
u_t^* &= -F^* (A^*)^t x_0.
\end{align*}
\]

AAAAA Lars: let’s check to see where assumption (iv) is used in the preceding proof.
6.4. Game 2: The Stackelberg multiplier game

This section analyzes the Stackelberg game 2 (STACK) and how it is related to game 1 (SEQ). Game SEQ is useful as a tool for discovering components of the equilibrium of the Stackelberg game. As we have seen, the equilibrium conditions of game SEQ give rise to a linear difference equation system. However, relative to the Stackelberg multiplier game STACK, game SEQ imposes too much commitment about the choice of the control sequence \( \{u_t\} \): recall the commitment to a sequence for \( u_t \) in game 1 (SEQ) with the commitment only to a feedback rule \( u_t = Fx_t \) in game 2 (STACK).

Theorem 6.4.1. Where \( A = A_0 - BF \) and \( H = H_0 - JF \), let \( K = K(F) \) be induced by

\[
K = (\theta^* I - C''\Sigma C)^{-1} C''\Sigma A,
\]

where \( \Sigma \) is the positive semidefinite solution to the Riccati equation

\[
\Sigma = H' H + \beta A'' \Sigma A + \beta A'' \Sigma C (\theta^* I - C''\Sigma C)^{-1} C''\Sigma A
\]

for which \( A + CK \) has eigenvalues that are inside the circle \( \Gamma \). Suppose

(i) \( J' J \) is nonsingular and \( J + H_0 \zeta (I - \zeta A)^{-1} B \) has full column rank on \( \Gamma \);
(ii) \( (L, N) \) has no generalized eigenvalues on \( \Gamma \);
(iii) Any element of the \( (\sqrt{\beta}) \) deflating subspace of \( (L, N) \) can be represented as \( (I \ P) x \) for some vector \( x \) where \( P \) is a symmetric, positive semidefinite matrix;
(iv) \( \theta I - C''PC \) is positive definite.

Then there exists an equilibrium of the Stackelberg multiplier game in which \( F = F^* \) and \( K^* = K(F^*) \); consequently \( w_{t+1}(F^*) = -F^*(A^*)' x_0 \), where \( F^* \), \( K^* \) and \( A^* \) are the same matrices that represent the equilibrium of game SEQ.

The regularity conditions imposed in Theorem 6.4.1 differ from those in Theorem 6.3.1. As we will see, this difference is due to the change in the vantage point of the player who chooses \( \{w_{t+1}\} \) when we move from game 1 (SEQ) to the Stackelberg game 2 (STACK). Nevertheless, the formulas for the equilibrium of game SEQ describe the equilibrium of the Stackelberg multiplier game, and the same notion of stability prevails.

---

6 In effect, the optimization over \( \{u_t\} \) in the Stackelberg multiplier game (7.4.3) allows the \( u \)-decision maker to contemplate that the disturbances will differ from the \( \{w_{t+1}\} \) computed from the game SEQ equilibrium. Formally, this connects to issues of perfection in game theory.
Proof. For the Stackelberg multiplier game, the maximizing player submits a decision rule $u_t = -Fx_t$. The minimizing player chooses a sequence $\{w_{t+1}(F)\}$ to minimize (6.3.1). For some $F$'s, the infimum may not be attained. We can form the criterion $C(F, x_0)$, noting that it may be $-\infty$ for some choices of $F$. We wish to show that

$$C(F^*, x_0) \geq C(F, x_0)$$

for any $F \in \mathcal{F}$.

To verify this inequality, we first show that $\{w_{t+1}(F^*)\}$ coincides with $\{w_{t+1}^*\}$ of the equilibrium of game SEQ. Thus, we study the problem of minimizing (6.3.1) by choice of $\{w_{t+1}\}$ subject to

$$x_{t+1} = (A_0 - BF^*)x_t + Cw_{t+1}.$$ 

This differs from the optimum problem of the malevolent agent (over $w$) within a SEQ equilibrium. In the present problem, the malevolent agent no longer takes the control sequence $\{u_t\}$ as exogenous, and we do not enter $\hat{x}_t$ into the perceived state evolution equation. Here the malevolent agent knows that $x_t = \hat{x}_t$ when solving his optimization problem.

To show that $\{w_{t+1}(F^*)\}$ coincides with $\{w_{t+1}^*\}$ from the SEQ equilibrium, we form the discrete-time Hamiltonian system for the choice of $\{w_{t+1}\}$ as a function of $F^*$, as required in the Stackelberg multiplier equilibrium. Let $H = H_0 - JF$. From (6.3.2), the first-order conditions for $w_{t+1}$ collapse to:

$$-\theta w_{t+1} + C' \mu_{t+1} = 0$$

$$\beta A_0' \mu_{t+1} + H_0' H x_t - \mu_t = 0.$$ 

Next impose that $u_t = -F^* x_t$. Then from (6.3.3) $BF^* x_t = -B(J'J)^{-1} J'H_0 x_t - \beta B(J'J)^{-1} B' \mu_{t+1}$. The state equation becomes $x_{t+1} = Ax_t + \theta^{-1}(C'C) \mu_{t+1}$ for $A = A_0 - BF^*$. In addition, note that

$$H_0 x_t + J u_t = H x_t$$

for $H = H_0 - JF^*$. Note also from (6.3.3) that

$$-F^* J'J u_t = F^* J'H_0 x_t + \beta F^* B' \mu_{t+1}.$$ 

The modified co-state equation becomes $\beta A' \mu_{t+1} + H'H x_t - \mu_t = 0$, so that

$$\begin{pmatrix}
    I & -\frac{1}{\theta} CC' \\
    0 & \beta A'
\end{pmatrix}
\begin{pmatrix}
    I \\
    P
\end{pmatrix}
\begin{pmatrix}
    x_{t+1} \\
    \mu_{t+1}
\end{pmatrix}
= 
\begin{pmatrix}
    A & 0 \\
    -H'H & I
\end{pmatrix}
\begin{pmatrix}
    x_t \\
    \mu_t
\end{pmatrix}.$$
It follows that $P = \Sigma$ satisfies Riccati equation
\[
\beta A'\Sigma \left( I - \frac{1}{\theta} CC'\Sigma \right)^{-1} A - \Sigma + H'H = 0.
\] (6.4.3)
and hence also satisfies (7.5.17) (see the proof of Theorem 7.5.4 for more details).

It is the unique solution that implies that the state vector sequence is $\sqrt{\beta}$ stable.
From the proof of Theorem 6.3.1 it follows that $K = K^*$ and the positive
definiteness of $\theta^* I - G'G$ follows from the restriction that $\theta^* I - C'PC$ is positive
definite. From this result, we can compute $C(F^*, x_0)$ by simply evaluating the
objective in game SEQ.

Consider now the evaluation $C(F, x_0)$ for some other choice of $F$ in $\mathcal{F}$. We can bound this criterion as follows. First, recursively generate the SEQ
equilibrium $\{w_{t+1}^*\}$ sequence as
\[
\hat{x}_{t+1} = A^* \hat{x}_t
\]
\[
w_{t+1}^* = K^* \hat{x}_t
\]
where $\hat{x}_0 = x_0$. Then form the state equation
\[
x_{t+1} = (A_o - BF) x_t + CK \hat{x}_t
\]
\[
z_t = (H_0 - JF) x_t.
\]
Using these recursions to evaluate (6.3.1), we obtain an upper bound $\hat{C}(F, x_0)$ on $C(F, x_0)$.

A convenient feature of this upper bound is that we can dominate $\hat{C}(F, x_0)$
by solving the following augmented regulator problem: maximize (6.3.1) by
choice of a stabilizing control sequence $\{u_t\}$ for the state evolution
\[
x_{t+1} = A_o x_t + Bu_t + CK^* \hat{x}_t
\]
\[
\hat{x}_{t+1} = A^* \hat{x}_t
\]
with $w_{t+1}^* = K^* \hat{x}_t$. But this is just the problem of the player who sets $u_t$ in
game SEQ. As in chapter 3, we solve this problem by stacking a state-costate
system with the composite state $(x_t, \hat{x}_t)$ and the costate corresponding to $x_t$.
The costate for $\hat{x}_t$ can be omitted because $\hat{x}_t$ is an uncontrollable state vector.
Thus we form a system
\[
L^n \begin{pmatrix} x_{t+1} \\ \mu_{t+1} \\ \hat{x}_{t+1} \end{pmatrix} = N^n \begin{pmatrix} x_t \\ \mu_t \\ \hat{x}_t \end{pmatrix}
\]
where:

\[
L^a = \begin{pmatrix}
I & \beta B (J'J)^{-1} B' \\
0 & \beta \left[ A'_o - H'_0 J (J'J)^{-1} B' \right] \\
0 & 1
\end{pmatrix},
\]

\[
N^a = \begin{pmatrix}
A_o - B (J'J)^{-1} J' H_0 & 0 & CK^* \\
0 & CK^* & I & 0 \\
0 & 0 & A^*
\end{pmatrix}.
\]

To solve the problem we now look for the \(\sqrt{\beta}\) deflating subspace of \((L^a, N^a)\) parameterized as

\[
\begin{pmatrix}
x \\
P_2 x + \hat{P} \hat{x}
\end{pmatrix} = \begin{pmatrix}
I \\
P_2 \\
0
\end{pmatrix} (x - \hat{x}) + \begin{pmatrix}
I \\
P_2 + \hat{P}
\end{pmatrix} \hat{x}.
\]

We will show that we can reduce the problem to that of locating two smaller dimensional \(\sqrt{\beta}\) deflating subspaces. The first is for the pair \((L^a, N^a)\) with

\[
L_2 = \begin{pmatrix}
I & \beta B (J'J)^{-1} B' \\
0 & \beta \left[ A'_o - H'_0 J (J'J)^{-1} B' \right]
\end{pmatrix},
\]

\[
N_2 = \begin{pmatrix}
A_o - B (J'J)^{-1} J' H_0 & 0 \\
0 & [H'_0 H_0 - H'_0 J (J'J)^{-1} J H_0] & I
\end{pmatrix}.
\]

This is the subspace associated with the component of \(x_t - \hat{x}_t\) that must be set to zero to solve the control problem. Notice that \((L_2, N_2)\) defines the state-costate system for the \(H_2\) control problem. Thus we can restrict \(x_t - \hat{x}_t\) to reside in the \(\sqrt{\beta}\) stable deflating subspace of \((L_2, N_2)\) using the matrix \(P\) for the \(H_2\) problem.

To study the second subspace, we next seek a solution to:

\[
L^a \begin{pmatrix}
I \\
P_2 + \hat{P}
\end{pmatrix} \hat{x}_{t+1} = N^a \begin{pmatrix}
I \\
P_2 + \hat{P}
\end{pmatrix} \hat{x}_t.
\]

It is more convenient to pose this problem as being (i) to find a matrix \(\hat{P}\) such that we can represent the \(\sqrt{\beta}\) deflating subspace of \((\hat{L}, \hat{N})\) as parameterized by:

\[
\begin{pmatrix}
I \\
(P_2 + \hat{P})
\end{pmatrix} \hat{x}.
where
\[
\hat{L} = \begin{pmatrix}
I & \beta B (J'J)^{-1} B' \\
0 & \beta \left[ A_0' - H_0' J (J'J)^{-1} B' \right]
\end{pmatrix}
\]
\[
\hat{N} = \begin{pmatrix}
A_0' - B (J'J)^{-1} J' H_0 + C K^* & 0 \\
- \left[ H_0' H_0 - H_0' J (J'J)^{-1} J H_0 \right] I
\end{pmatrix},
\]
and (ii) to show that the implied law of motion for \( \hat{x}_{t+1} \) agrees with
\[
\hat{x}_{t+1} = A^* \hat{x}_t. \tag{6.4.4}
\]

In constructing the deflating subspace in (i), we will show that
\[
P_2 + \hat{P} = P.
\]
This can be done by imitating the argument that \( \{w_{t+1}(F^*)\} = \{w^*_t\} \) but reversing the roles of \( \{w_{t+1}\} \) and \( \{u_t\} \). So we impose \( w_{t+1} = K^* x_t \). It follows that \( P \) can indeed be used to represent the \( \sqrt{\beta} \) deflating subspace and that the implied evolution for \( \{\hat{x}_{t+1}\} \) is given by (6.4.4) as required by (ii).

Thus we have shown that the \( \sqrt{\beta} \) deflating subspace can indeed be uncoupled. By initializing \( \hat{x}_0 = x_0 \), it follows that \( \hat{x}_t = x_t \). Moreover, the optimized objective coincides with \( C(F^*, x_0) \). Thus
\[
C(F, x_0) \leq \hat{C}(F, x_0) \leq C(F^*, x_0).
\]

Conditions (ii) and (iii) are assured when \( \beta B(J'J)^{-1} B' - \frac{1}{\theta} C C' \) is positive semidefinite. However, this condition is too strong for many applications (unless of course \( \theta = \infty \)).

Let’s expand the explanation of what is going on in the preceding section. Explain more in the proof.
6.5. Game 3: Markov multiplier game (MARKOV)

The equilibrium outcome of the Stackelberg multiplier game matches the outcome from yet another game, the Markov multiplier game (MARKOV).\footnote{As we shall see, this game connects directly to the discounted risk-sensitivity criterion of Hansen and Sargent (1995) that is described in chapter 7 on page 152.}

**Definition 6.5.1.** An equilibrium of the Markov multiplier game (MARKOV) is a pair of strategies \( u_t = -F^*x_t, \ w_{t+1} = K^*x_t \) such that

(a) Given \( K^* \), \( u_t = -F^*x_t \) maximizes (6.3.1), subject to
\[
x_{t+1} = A_o x_t + B u_t + C w_{t+1}.
\]

(b) Given \( F^* \), \( w_{t+1} = -K^*x_t \) minimizes (6.3.1) subject to (6.5.1).

Associated with a MARKOV game is the following pair of Bellman equations

\[
-x'Px = \max_u \left[ - (H_0 x + Ju)' (H_0 x + Ju) + \beta \theta w^*w^* - \beta y'Py \right] (6.5.2a)
\]

\[
w^* = K^*x
\]

\[
-x'Px = \min_w \left[ - (H_0 x + Ju^*)' (H_0 x + Ju^*) + \beta \theta w^*w - \beta y'Py \right] (6.5.3a)
\]

\[
u^* = -F^*x
\]

where both extremizations are subject to the common law of motion

\[
y = A_o x + B u + C w.
\]

The \((P, K^*, F^*)\) that form an equilibrium of the MARKOV game also solve the following closely related zero-sum game:

\[
-x'Px = \max_u \min_w \left[ - (H_0 x + Ju)' (H_0 x + Ju) + \beta \theta w^*w - \beta y'Py \right] (6.5.4)
\]

where the maximization is again subject to

\[
y = A_o x + B u + C w.
\]

The outcome is

\[
u = -F^*x,
\]
and
\[ w = K^*x. \]

Though it yields the same equilibrium strategies and outcomes, notice that game (6.5.4) has a slightly different timing protocol from game (6.5.2)–(6.5.3). In (6.5.4), within each period, the \( w \)-player moves after the \( u \)-player, while (6.5.2)–(6.5.3) incorporates simultaneous moves within periods.

The \((P, K^*, F^*)\) associated with the equilibrium of the Stackelberg multiplier game solves (6.5.2) and (6.5.3), and thereby determines an equilibrium of the MARKOV game. We summarize the connections between an equilibrium of game MARKOV and an equilibrium of the STACK game in

**Theorem 6.5.1.** The \((P, K^*, F^*)\) associated with the equilibrium of the Stackelberg multiplier game also describe the equilibrium of the Markov multiplier game (MARKOV).

*Proof.* The required marginal conditions match. □

The functional equation (6.5.4) leads directly to computing the equilibrium by iterating to convergence on Hansen and Sargent’s (1995) composite operator \( T \circ D \). The \( T \) piece represents the maximization over \( u \) and the \( D \) piece the minimization over \( w \) in (6.5.4).

### 6.6. Relation between multiplier and constraint problems

Confront Pierre’s doubts about the following argument: \( \text{trace} A \geq \text{trace} B \) does not in general imply \( \log \det A \geq \log \det B \). See Pierre’s notes.

We have established that our three robust multiplier games have identical outcomes and equilibrium representations. The following two propositions link the multiplier game to the constraint formulation. The propositions follow from the Lagrange multiplier theorem (Luenberger (1969), pp. 216-221). Chapter 7 develops this connection in more detail in the context of a frequency domain specification of Stackelberg multiplier and constraint games.\(^8\)

**Theorem 6.6.1.** Suppose that there exists a solution \( u^*, w^* \) to the robust multiplier problem. Then \( u^* \) also solves the constraint robust control problem with \( \eta = \eta^* = R(w^*) \), where \( R(w) \) is defined by (6.2.6).

---

\(^8\) Also see Hansen, Sargent, Turmuhambetova, and Williams (2001).
Theorem 6.6.2. Suppose that $u^*, w^*$ solve the constraint robust control problem for $\eta = \eta^*$. Then there exists a $\theta^*$ such that the robust multiplier and constraint problems have the same solution.

Theorem 6.6.1 shows how to construct the specification error $\eta$ associated with a given multiplier. In the next section, we use this finding to describe a sense in which the constraint problem is recursive.

6.7. Recursivity of the constraint game

The Bellman equation (6.5.4) indicates directly that the solution of the multiplier problem is time consistent. It requires more of an argument to verify a sense in which the solution of the constraint problem is time consistent because there is a need to update a continuation value of $\eta$. This section describes a recursive formulation of the constraint problem stated in Definition 6.2.1. \footnote{See Hansen, Sargent, Turmuhambetova, and Williams (2001) for an extended discussion of the subject of this section.}

We define a time $t$ version of continuation entropy (6.2.6) as an additional state variable:

$$R_t (w) = \sum_{\tau=1}^{\infty} \beta^\tau w_{t+\tau}^{t+\tau}.$$ 

Evidently, $R_t(w)$ satisfies the recursion

$$R_t (w) = \beta w_{t+1}^{t+1} + \beta R_{t+1} (w).$$ 

Let $V(x, \eta)$ be the value function for the constraint problem (6.2.7) starting from initial state $x = x_0$ and initial value of entropy $\eta$. For the constraint problem, the counterpart to Bellman equation (6.5.4) is the Bellman equation \footnote{If a random vector $\epsilon$ is present in the transition law, the Bellman equation becomes}

$$V (x, \eta) = \sup_u \inf_{w, \eta^*(\epsilon)} \left[ -z'z + \beta V (x^*, \eta^*) \right]$$ \hspace{1cm} (6.7.2)

$$V (x, \eta) = \sup_u \inf_{w, \eta^*(\epsilon)} \left[ -z'z + \beta E V (x^*, \eta^*) \right]$$ \hspace{1cm} (6.7.1)

where $*$ the extremization is subject to

$$x^* = Ax + Bu + C (\epsilon + w)$$

$$\eta = \beta w^* + \beta \eta^*(\epsilon).$$

where $E$ is the mathematical expectation with respect to the distribution of $\epsilon$ and continuation entropy $\eta^*(\epsilon)$ is now a function of $\epsilon$. 

See Hansen, Sargent, Turmuhambetova, and Williams (2001) for an extended discussion of the subject of this section.
where * denotes next period’s value and the extremization is subject to

\[ z = H_0 x + Ju \]
\[ x^* = Ax + Bu + Cw \]  \hspace{1cm} (6.7.3)
\[ \eta = \beta w' w + \beta \eta^*. \]

The last equation of (6.7.3) is a ‘promise keeping’ constraint on the allocation of entropy \( \eta \) between today’s distortion \( w' w \) and the distortion from tomorrow on \( \eta^* \). The minimizing agent can allocate \( \eta \) over time, but must respect constraint (6.7.3).

The first order necessary condition with respect to \( \eta^* \) and the envelope condition for \( \eta \) imply that

\[ V_\eta(x, \eta) = V_\eta(x^*, \eta^*) . \]  \hspace{1cm} (6.7.4)

Further, \( V_\eta(x, \eta) \) equals minus the Lagrange multiplier on the last constraint in (6.7.3). Equation (6.7.4) implies that there is a time-invariant relationship between \( x \) and \( \eta \), which in turn implies that the extremizing choices \( (u, w) \) for the right side of (6.7.3) can be expressed as functions of \( x \) alone. These equal the functions \( u = -Fx_t \) and \( w = Kx_t \) that we computed for the multiplier games for \( \theta \) being set equal to \( V_\eta(x, \eta) \).

6.8. Useful formulas

This section uses game (6.5.4) to provide two sets of convenient formulas for computing the robust decision rule. For the purpose of displaying these formulas, notice that the one-period loss function in (6.5.4) can be represented as

\[ r(x, u) \equiv (H_0 x + Ju)' (H_0 + Ju) \]
\[ = \begin{bmatrix} x' \end{bmatrix} \begin{bmatrix} H_0' H_0 & H_0' J \\ J' H_0 & J' J \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} \]
\[ = \begin{bmatrix} x' \end{bmatrix} \begin{bmatrix} \bar{Q} & W' \\ W & R \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix}, \]

where \( \bar{Q}, W, R \) are defined implicitly by the above equality. As in chapter 3, we transform the problem to one that eliminates cross-products between states and controls. Define

\[ Q = \bar{Q} - WR^{-1} W' \]
\[ \tilde{A}_o = A_o - BR^{-1} W' \]
\[ \tilde{u} = u + R^{-1} W' x. \]  \hspace{1cm} (6.8.1)
Then
\[ x_{t+1} = \tilde{A}_o x_t + B \tilde{u}_t + C w_{t+1} \] (6.8.2)
and
\[ \tilde{r}(x, \tilde{u}) = r(x, u) = \begin{bmatrix} x' \\ \tilde{u} \end{bmatrix}' \begin{bmatrix} Q & 0 \\ 0 & R \end{bmatrix} \begin{bmatrix} x \\ \tilde{u} \end{bmatrix}. \] (6.8.3)
The Bellman equation (6.5.4) then is equivalent with
\[ -x'Px = \max_{\tilde{u}} \left\{ -\tilde{r}(x, \tilde{u}) + \beta \theta w'w - \beta y'Py \right\} \] (6.8.4)
where
\[ y = \tilde{A}_o x + B \tilde{u} + C w. \] (6.8.5)
In the problem on the right of (6.8.4), the minimizing agent moves second, taking as given the feedback rule \( \tilde{u} = -Fx \) chosen by the maximizing agent. By working backwards, we break the problem on the right of (6.8.4) into these two parts:

1. The problem for the minimizing agent reduces to
\[ J = \min_w [\theta w'w - y'Py] \] (6.8.6)
subject to
\[ y = Ax + Cw \] (6.8.7)
where \( A = \tilde{A}_o - BF \) and \( F \) is to be chosen in the problem in part 2. The minimizing \( w \) is
\[ w = \theta^{-1} (I - \theta^{-1}C'PC)^{-1} C'PAx. \] (6.8.8)
Let
\[ D(P) = P + PC(\theta I - C'PC)^{-1} C'P. \] (6.8.9)
The minimized value of the problem can be expressed as
\[ J = -x'A'D(P)Ax \]
or as
\[ J = -y'D(P)y \] (6.8.10)
where in (6.8.10), \( y \) is to be evaluated under the approximating model \( y = Ax \), not under the distorted model (6.8.7). Under the approximating model, (6.8.10) is a conservative continuation value for the problem of
the maximizing agent. Part 2 of the problem, the minimizing part, hands this conservative valuation function and the approximating model to the maximizing agent.

2. Working backwards, the problem of the maximizing agent can be expressed as

$$\max_{\tilde{u}} \left[ -x'Qx - \tilde{u}'Ru - \beta y'D(P)y \right] \quad (6.8.11)$$

subject to

$$y = \tilde{A}_o x + B\tilde{u}. \quad (6.8.12)$$

Notice that (6.8.12) is the approximating model and that allowance for distortions occurs only through the presence of $D(P)$ on the right side of (6.8.11). The solution to this problem is found by taking one step on the usual Riccati equation, with $D(P)$ as the terminal value function. Thus, define the operators

$$F(\Omega) = \beta \left[ R + \beta B'\Omega B \right]^{-1} B'\Omega \tilde{A}_o \quad (6.8.13)$$

$$T(P) = Q + \beta \tilde{A}_o' \left( P - \beta PB \left( R + \beta B'PB \right)^{-1} B'P \right) \tilde{A}_o. \quad (6.8.14)$$

Then the solution of problem (6.8.11) is $\tilde{u} = -Fx$ where $F = F \circ D(P)$. The maximizing value of (6.8.11) is $-x'T \circ D(P)x$.

We can iterate on these two sub problems to find the solution to (6.8.4). Let $P$ be the fixed point of iterations on $T \circ D$:

$$P = T \circ D(P). \quad (6.8.15)$$

Then the solution of (6.8.4), (6.8.5) is

$$\tilde{u} = -Fx \quad (6.8.16)$$

$$w = Kx, \quad (6.8.17)$$

where

$$F = F \circ D(P) \quad (6.8.18)$$

$$K = \theta^{-1} \left( I - \theta^{-1} C'PC \right)^{-1} C'P \left[ \tilde{A}_o - B\tilde{F} \right]. \quad (6.8.19)$$

\[11\] In chapter 7, the operator $D$ is used again to characterize risk-sensitive preferences. See page 152.

\[12\] In chapter 7 we show how the two operators are related to the discounted risk-sensitivity criterion of Hansen and Sargent (1995).
Here $T$ is the usual operator associated with taking one-step on the Bellman equation without a preference for robustness; it represents optimization with respect to $u$. The operator $D$ reflects minimization with respect to $w$. When $\theta = +\infty$, $D(P) = P$, and we get the usual optimal rule for a linear-quadratic dynamic program. When $\theta^* \leq \theta < \infty$, we get a robust decision rule, where $\theta^*$ is a lower bound on admissible parameters $\theta$, the source of which we shall describe in detail below.

### 6.8.1. A single Riccati equation

A robust decision rule can also be computed simply by solving an ordinary optimal linear regulator problem. This can be established in the following way.

By writing iterations $P_{k+1} = T \circ D(P_k)$ and rearranging, the matrix $P$ in the value function $-x'Px$ can be expressed as the fixed point of iterations on the Riccati equation

$$P_{k+1} = \tilde{A}_o' \left( (\beta P_k)^{-1} + BR^{-1}B' - \theta^{-1} \beta^{-1} CC' \right) \tilde{A}_o + Q. \quad (6.8.20)$$

This equation can also be represented as

$$P_{k+1} = Q + \tilde{A}_o' \left( P_k^{-1} + J \right)^{-1} \tilde{A}_o, \quad (6.8.21)$$

where $\tilde{J} = B^* R^{-1} B'^* - \theta^{-1} CC', B^* = \beta^5 B, \tilde{A}_o^* = \beta^5 \tilde{A}_o$. Equation (6.8.21) is in a form to which the doubling algorithm described in chapter 3 applies. Notice that (6.8.20) is the Riccati equation associated with an ordinary optimal linear regulator problem with controls $\begin{bmatrix} u \\ w \end{bmatrix}$ and penalty matrix on those controls appearing in the criterion function of $\begin{bmatrix} R & 0 \\ 0 & -\beta \theta I \end{bmatrix}$. Therefore, the robust rules for $u_t$ and the associated worst case shock can be computed directly from the associated ordinary linear regulator problem. It can be checked that the right side of (6.8.20) implements one step on $T \circ D$. The Riccati equation (6.8.20) is the one associated with the modified linear regulator used in chapter 2 on page 29 to compute a robust rule and the worst case shock.

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13 See chapter 2, page 29.
15 The Matlab program `doublex9.m` computes the solution using the doubling algorithm.
6.9. Robustness bound

The inner problem (6.8.6) inspires a robustness bound for continuation values. Thus, (6.8.6) implies

\[-x'A'D(P)Ax = \min_w [\theta w'w - y'Py] \leq \theta w'w - y'Py\]  \hspace{1cm} (6.9.1)

where \( y \) is evaluated under the distorted model \( y = Ax + Cw \). Inequalities (6.9.1) imply

\[-y'Py \geq -x'A'D(P)Ax - \theta w'w.\]  \hspace{1cm} (6.9.2)

The left side is evaluated under a distorted model \( y = Ax + Cw \) while the quadratic form in \( x \) on the right is a conservative estimate of the continuation value of the state \( y \) under the approximating model \( y = Ax \). Inequality (6.9.2) says that the continuation value is at least as great as a conservative estimate of the continuation value under the approximating (\( w = 0 \)) model, minus \( \theta \) times the measure of model misspecification \( w'w \). The parameter \( \theta \) affects influences the conservative-adjustment operator \( D \) and also determines the rate at which the bound deteriorates with misspecification. Lowering \( \theta \) lowers the rate at which the bound deteriorates with misspecification. Thus, (6.9.2) provides a sense in which lower values of \( \theta \) provide more conservative and also more robust estimates of continuation utility.

6.10. Special case: a pure forecasting problem

Here is an example of a pure forecasting problem in which the absence of a control eliminates the maximization part of (6.8.4). The following state space system governs consumption and bliss consumption:

\[x_{t+1} = Ax_t + Cw_{t+1}\]
\[c_t = H_c x_t\]
\[b_t = H_b x_t\]  \hspace{1cm} (6.10.1)

where \( c_t \) is an exogenous scalar consumption process, \( b_t \) is a bliss level of consumption, and \( w_{t+1} \) is a specification error sequence. To attain a conservative way of evaluating \( -\sum_{t=0}^{\infty} \beta^t (c_t - b_t)^2 \), we compute

\[-x_0'P x_0 = \min_{w_{t+1}} -\sum_{t=0}^{\infty} \beta^t [x_t'H'Hx_t - \beta w_{t+1}'w_{t+1}]\]  \hspace{1cm} (6.10.2)

\(^{16}\) That is, when \( w = 0 \), \(-y'D(P)y\) understates the continuation value.
subject to (6.10.1), where $H = H_c - H_b$. For this special case, the absence of a control causes the operator $T$ defined in (6.8.14) to simplify to

$$T(P) = H'H + \beta A'PA.$$  (6.10.3)

The matrix $P$ in (6.10.2) is the fixed point of iterations on $T \circ D$. The minimizer of (6.10.2) is given by (6.8.8), or $w = Kx$, where $K$ is defined implicitly by (6.8.8). It follows from our earlier characterizations of $K$ and $P = T \circ D(P)$ that

$$-x'Px = -\sum_{t=0}^{\infty} \beta^t x'Hx_t$$

where the right side is computed using the distorted law of motion

$$x_{t+1} = (A + KC)x_t.$$

### 6.11. Concluding remarks

A robust decision maker fears that an approximating model is misspecified. He assumes that misspecification takes the form of nonzero shocks $\{w_{t+1}\}$. To represent a preference for robustness to specification errors, the decision maker modifies the usual Bellman equation by adding another player (‘nature’) who chooses a nearby model to hurt the decision maker. The decision maker devises a robust decision rule by finding a value function $v(x)$ that solves:

$$v(x) = \max_u \min_w \left\{ - (H_0x + Ju)'(H_0x + Ju) + \beta \theta w'w + \beta v(y) \right\}$$  (6.11.1)

where

$$y = A_0x + Bu + Cw.$$  (6.11.2)

and $\theta$ satisfies $0 < \tilde{\theta} < \theta < \infty$. When $\theta = \infty$, there is no preference for robustness and we are back with the ordinary control problem. When $\theta < +\infty$, there is a preference for a robust rule. The optimum value function is $v(x) = -x'Px$ and is attained by a pair of decision rules $u_t = -F^*x_t$, $w_{t+1} = K^*x_t$, where $P$ solves an adjusted Riccati equation and $F^*$ and $K^*$ depend on $P$. The robust decision maker uses $F^*$ rather than $F$. He thus chooses $u$ by behaving ‘as if’ the shocks $w_{t+1}$ will be $K^*x_t$, where $K^*$ depends partly on his own choice $F^*$ through the Riccati equation.

The robust rule $F^*$ is as easy to compute as the ordinary optimal rule $F$. As mentioned in chapter 2, one way to compute $F^*$, $K^*$ is to formulate (6.11.1),
(6.11.2) as an ordinary linear regulator problem with a stacked control vector $[u' \ w']'$. The Bellman equation (6.11.1) represents one of three two-player zero-sum games with identical payoffs but differing timing protocols. The zero-sum feature of these games is a key element in making all three games have the same equilibrium outcomes and the same recursive representations $u_t = -Fx_t$, $w_{t+1} = Kx_t$, and $x_{t+1} = (A - BF + KC)x_t$. The different games justify alternative algorithms for computing $F$ and $K$. 
Chapter 7.
Frequency domain games and criteria for robustness

7.1. Introduction

This chapter describes a preference for robustness in the frequency domain. Economists express concerns about model misspecification in the frequency domain when they use seasonally adjusted or trend adjusted data. In addition to providing insights into how a preference for robustness manifests itself across frequencies, we make contact with important ideas in the literature, including the $H_\infty$ criterion.

The spirit of the $H_\infty$ criterion is to represent a decision maker’s concern about misspecification of his constraint set by altering his objective function. This chapter shows how robust decision rules can be attained by altering the decision maker’s objective function in a way that depends on a single positive parameter $\theta$. This parameter $\theta$ has the same interpretation imputed to it in chapter 6 as measuring the size of the set of models to which the decision maker wants to be robust. The robust criterion functions are the value functions or indirect intertemporal utility functions of a fictitious agent who chooses a model to minimize the decision maker’s original objective function. We show two such criteria, one known as the $H_\infty$ criterion, the other called the ‘entropy’ criterion. Undiscounted versions of both of these reside in the control literature. We derive appropriately discounted versions of both. Accommodating discounting requires that we pay special attention to initial conditions. Our derivation of the entropy criterion will also provide a link to the discounted risk-sensitivity criterion of Hansen and Sargent (1995).

The entropy criterion provides another way of viewing equilibria of the games from chapter 6. The $H_\infty$ and entropy criteria are expressed in the frequency domain. Accordingly, this chapter proceeds by studying within the frequency domain a Stackelberg constraint problem from chapter 6. We also consider a robust multiplier problem in the frequency domain and state its relation to the constraint problem by applying the Lagrange multiplier theorem.

We characterize restrictions on the robustness parameter $\theta$, but postpone discussing how to calibrate it in practical situations until chapter 8. Chapter 8 describes the stochastic counterpart to the approximating model used here,
and how a decision maker can select $\theta$ in a context specific way to express his confidence in the probabilities assigned by his approximating model. Various technical details are reported in five appendices to this chapter.

### 7.2. The Stackelberg game in the time domain

We begin with a Stackelberg robust multiplier problem from chapter 6. After recalling this game in the time domain, we shall describe a frequency domain version. The game requires the maximizing player to set $u_t = -Fx_t$. To get representations that build in $u_t = -Fx_t$, we first substitute (6.2.2) into (6.2.1) to get the closed-loop law of motion for the state:

$$x_{t+1} = Ax_t + Cw_{t+1}, \quad (7.2.1)$$

where

$$A = A_o - BF. \quad (7.2.2)$$

Under $u_t = -Fx_t$, the target becomes

$$z_t = Hx_t$$

where $H = H_0 - JF$.

**Definition 7.2.1.** The *Stackelberg robust constraint problem* is to find $(F, \{w_t\}_{t=1}^{\infty})$ that attain

$$\max_{F \in F} \inf_{w \in W} - \sum_{t=0}^{\infty} \beta^t z_t'z_t \quad (7.2.3)$$

subject to (7.2.1) and

$$\sum_{t=0}^{\infty} \beta^t w_t'w_t \leq \eta^2 + w_0'w_0 \quad (7.2.4a)$$

$$x_0 = Cw_0. \quad (7.2.4b)$$

We use $\max_F$ as a shorthand for $\max_{F \in F}$, and so on. This game is indexed by two parameters $(w_0, \eta)$.

We consider three versions of the Stackelberg robust constraint problem that correspond to different settings of $\eta, w_0$:

1. The $H_2$ problem: set $\eta = 0$, with arbitrary $w_0$. 
2. The $H_\infty$ problem: set $w_0 = 0$, but let $\eta > 0$ be arbitrary.

3. The entropy problem: set arbitrary $w_0 \neq 0$ and arbitrary $\eta > 0$.

The first version makes the inf part trivial and turns the game into a standard single-person linear-quadratic optimum problem and leads to the so-called $H_2$ criterion in the frequency domain. The second and third versions induce robust decision rules.

To enable using the frequency domain, (7.2.4b) restricts the initial condition. The solution of the game under this restriction can be represented recursively as a pair of feedback rules $w_{t+1} = K x_t, u_t = -F x_t$.\footnote{As can be verified by inspecting the formulas for $K, F$ that we derive later, the solution also solves the multiplier games from chapter 6. The time domain representation of the solution of this multiplier game is therefore valid for an arbitrary initial $x_0$.}

\section*{7.3. Stackelberg game in frequency domain}

\subsection*{7.3.1. Fourier transforms}

To formulate the game in the frequency domain, define one-sided Fourier transforms:

$$X(\zeta) \equiv \sum_{t=0}^{\infty} x_t \zeta^t,$$

$$W(\zeta) \equiv \sum_{t=0}^{\infty} w_t \zeta^t,$$

$$Z(\zeta) \equiv \sum_{t=0}^{\infty} z_t \zeta^t,$$

(7.3.1)

where $\zeta$ is a complex variable. Then (7.2.1) and (7.3.1) imply that $\zeta^{-1}[X(\zeta) - x_0] = AX(\zeta) + \zeta^{-1}C[W(\zeta) - w_0]$. Using (7.2.4b) and solving for $X(\zeta)$ gives $X(\zeta) = (I - \zeta A)^{-1} CW(\zeta)$, and hence

$$Z(\zeta) = G(\zeta) W(\zeta)$$

(7.3.2)

where

$$G(\zeta) \equiv H(I - \zeta A)^{-1} C$$

is the transfer function from shocks to targets.
Applying Parseval’s equality to (7.3.2) gives the following representation:

$$\sum_{t=0}^{\infty} \beta^t z_t' z_t = \int_{\Gamma} W(\zeta)' G(\zeta)' G(\zeta) W(\zeta) \, d\lambda(\zeta), \quad (7.3.3)$$

where the operation $'$ denotes both matrix transposition and complex conjugation, where the measure $\lambda$ has a density given by

$$d\lambda(\zeta) \equiv \frac{1}{2\pi i \sqrt{\beta} \zeta} d\zeta,$$

and where the region of integration is the following circle in the complex plane

$$\Gamma \equiv \{ \zeta : |\zeta| = \sqrt{\beta} \}.$$

The region $\Gamma$ can be parameterized conveniently in terms of $\zeta = \sqrt{\beta} \exp(i\omega)$ for $\omega$ in the interval $(-\pi, \pi]$. Here the measure $\lambda$ satisfies

$$d\lambda(\zeta) = \frac{1}{2\pi} d\omega.$$

Thus the contour integral on the right side of (7.3.3) can be expressed as:

$$\int_{\Gamma} W(\zeta)' G(\zeta)' G(\zeta) W(\zeta) \, d\lambda(\zeta)$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} W\left(\sqrt{\beta} \exp(i\omega)\right)' \left\{ G\left(\sqrt{\beta} \exp(i\omega)\right)' G\left(\sqrt{\beta} \exp(i\omega)\right) \right\} W\left(\sqrt{\beta} \exp(i\omega)\right) \, d\omega. \quad (7.3.4)$$

We use the contour integral on the left of (7.3.4) to simplify notation. Parseval’s equality also implies

$$\sum_{t=0}^{\infty} \beta^t w_t' w_t = \int_{\Gamma} W(\zeta)' W(\zeta) \, d\lambda(\zeta). \quad (7.3.5)$$
7.3.2. $H_2$ criterion

When $\eta = 0$ in (7.2.4a), $W(\zeta) = w_0$ and

$$- \sum_{t=0}^{\infty} \beta^t z'_t z_t = w_0 \left[ \int_\Gamma G(\zeta)'G(\zeta) d\lambda(\zeta) \right] w_0.$$  

For an arbitrary $w_0$, the $H_2$ problem is to maximize this expression by choosing a feedback rule $F$. The $H_2$ criterion can be expressed as

$$H_2 \equiv - \int_\Gamma \text{trace} \left[ G(\zeta)' G(\zeta) \right] d\lambda(\zeta). \quad (7.3.6)$$

The same $F$ that maximizes $H_2$ also solves the standard optimal linear regulator problem. Thus, the $H_2$ criterion gives a frequency domain expression to the preferences embodied in the optimal linear regulator. We turn next to frequency domain criteria that express a concern about model misspecification.

7.3.3. The Stackelberg game in the frequency game

To represent the Stackelberg game in the frequency domain, we define the following two sets of admissible $W(\zeta)$'s:

$$W = \{ W(\zeta) \in W^a : \sum_{t=0}^{\infty} \beta^t w'_t w_t < \infty \}$$

We use (7.3.3) and (7.3.5) to represent the time-domain Stackelberg robust constraint problem defined in Definition 7.2.1 as:

Three frequency domain games: Find $(F, W(\zeta))$ that attain

$$\max_F \inf_W - \int_\Gamma W(\zeta)' G(\zeta)' G(\zeta) W(\zeta) d\lambda(\zeta) \quad (7.3.7)$$

subject to

$$\int_\Gamma W(\zeta)' W(\zeta) d\lambda(\zeta) \leq \eta^2 + w'_0 w_0. \quad (7.3.8)$$

In the frequency domain, our three versions of the Stackelberg multiplier game are:
1. $H_2$: set $\eta = 0$, with $W(0) = w_0$ arbitrary.
2. $H_\infty$: set arbitrary $\eta > 0$ but $W(0) = w_0 = 0$.
3. Entropy: set $W(0) = w_0 \neq 0$ and arbitrary $\eta > 0$.

We have just seen how version one leads to the $H_2$ criterion (7.3.6) and how under it the best feedback rule $F$ is independent of the initial condition $x_0 = Cw_0$.

Version 2 has the side condition that $W(0) = 0$, but otherwise leaves $W(\zeta)$ free. Version 3 requires $W(0) = w_0 \neq 0$ and also restricts $W(\zeta)$ to keep the associated $\{w_t\}$ sequence zero for $t < 0$.

### 7.3.4. Version 2: the $H_\infty$ Criterion

Let $\rho(\zeta)$ denote the eigenvalues of $G(\zeta)'G(\zeta)$. The following theorem tells how version 2 of the game leads to the $H_\infty$ criterion defined as:

$$H_\infty \equiv -\sup_{\zeta \in \Gamma} [\rho(\zeta)]^{1/2}.$$  \ (7.3.9)

Lars: since $\rho$ is a vector in the above equation, is the above notation unambiguous?

**Theorem 7.3.1.** For any $F \in \mathcal{F}$,

$$\inf_{\mathcal{W}} - \int_{\Gamma} W(\zeta)' G(\zeta)' G(\zeta) W(\zeta) d\lambda(\zeta) = -H_\infty^2 \eta^2 \quad (7.3.10)$$

where the infimization is subject to (7.3.8).

**Proof.** Given $G(\zeta)$, for each $\zeta = \sqrt{\beta} \exp(i\omega)$ solve the following eigenvalue problem\(^2\)

$$G(\zeta)' G(\zeta) v = \rho(\zeta) v$$

\(^2\) It may be useful to remind the reader of the principal components problem. Let $a$ be an $(n \times 1)$ random vector with covariance matrix $V$. The first principal component of $a$ is a scalar $b = p'a$ where $p$ is an $(n \times 1)$ vector with unit norm (i.e., $p'p = 1$), for which the variance of $b$ is maximal. Thus, the first principal component solves the problem:

$$\max_p \ p'Vp$$

subject to

$$p'p = 1.$$
for the largest eigenvalue $\rho(\zeta)$. This problem has a well defined solution with eigenvalue $\rho(\omega)$ for each $\zeta = \sqrt{3}\exp(i\omega)$. Then

$$
\int_{\Gamma} W(\zeta)' G(\zeta)' G(\zeta) W(\zeta) d\lambda(\zeta) \leq \int_{\Gamma} \rho(\zeta) W(\zeta)' W(\zeta) d\lambda(\zeta)
$$

$$
\leq \sup_{\zeta \in \Gamma} \rho(\zeta) \int_{\Gamma} W(\zeta)' W(\zeta) d\lambda(\zeta)
$$

$$
\leq \sup_{\zeta \in \Gamma} \rho(\zeta) \eta^2.
$$

The bound on the right side is attained by the limit of a sequence of approximating $\{w_t\}$ sequences described in appendix A.

The square of the value of the optimized $H_\infty$ criterion plays an important role below. We denote it by

$$
\bar{\theta} = \left(\inf_F H_\infty(F)\right)^2.
$$

(7.3.13)

For technical reasons described in appendix A, the infimum in (7.3.10) is not necessarily attained by an analytic function $W \in \mathcal{W}$.

If version 2 has a maximizer $F$, that same $F$ maximizes (7.3.9). We can drop $\eta$ from the performance criterion (7.3.9) because it becomes a positive scale factor that is independent of the control law $F$. This feature emerges from the initial condition that $w_0 = 0$.

Putting a Lagrange multiplier $\lambda$ on the constraint, the first order conditions for this problem are

$$
(V - \lambda I) p = 0,
$$

(7.3.11)

with the value of the variance of $p'b$ evidently from (7.3.11) being

$$
p'Vp = \lambda p'p = \lambda.
$$

(7.3.12)

Thus (7.3.11) and (7.3.12) indicate that $p$ is the eigenvector of $V$ associated with the largest eigenvalue; and that the variance of $b$ equals the largest eigenvalue $\lambda$. 

7.4. A multiplier game for version 3

We have seen how the $H_2$ criterion emerges from ignoring concern about model misspecification by setting $\eta = 0$. Under discounting, the $H_\infty$ control problem comes from allowing model misspecification while setting $w_0$ to zero. However, there is also a way to formulate the $H_\infty$ problem that allows the infimizer to set $w_0 \neq 0$.

We now consider an intermediate case where misspecification is allowed but the malevolent agent must respect the initial condition $w_0$. To analyze this case, we formulate the multiplier version of the Stackelberg game in the frequency domain. We let $\theta$ be a Lagrange multiplier on the constraint and obtain:

\[ Lars: \text{in chapter 5 the order was } \sum_w \sup_{\theta} \inf_{W} \. \]

How do we render them consistent?

Definition 7.4.1. Lagrangian formulation of Stackelberg constraint game: Find $(\theta, F, W(\zeta))$ that attain

\[ \sup_{\theta} \sup_{F} \inf_{W} \left[ \int_{\Gamma} W' (\theta I - G' G) W d\lambda - \theta \left( \eta^2 + w'_0 w_0 \right) \right]. \tag{7.4.1} \]

Here $\eta > 0$ and $w_0 \neq 0$.\(^3\)

In appendix C, we establish the following things about (7.4.1).

i. Let $\theta^*$ be the optimal multiplier for (7.4.1). It satisfies:

\[ \theta^* > \tilde{\theta}. \tag{7.4.2} \]

If (7.4.2) does not hold, the inner $\inf_{W \in W}$ in (7.4.1) is $-\infty$ independently of the control law $F$.

ii. When the optimal multiplier $\theta^*$ satisfies $\theta^* > \tilde{\theta}$, we are led to study the inner two-player zero-sum Stackelberg multiplier game:

\[ \sup_{\theta} \inf_{W} \int_{\Gamma} W' (\theta^* I - G' G) W d\lambda \tag{7.4.3} \]

This game connects to the single agent decision problem

\[ \sup_{\theta} \int_{\Gamma} \log \det \left[ \theta^* I - G (\zeta)' G (\zeta) \right] d\lambda (\zeta), \tag{7.4.4} \]

because the $F^*$ that attains (7.4.4) is the $F$ component of the solution of the two-player multiplier game (7.4.3).

\[^{3}\text{We have already studied the } \eta = 0 (H_2) \text{ and } w_0 = 0 (H_\infty) \text{ cases.}\]
7.5. The multiplier problem and the entropy criterion

To study the inf$_W$ part of game (7.4.3), we take $\theta^*$, $F$ and therefore $G$ as given. We refer to the resulting optimization problem as the multiplier problem and state it as:

*The multiplier problem:*

$$\inf_{W:W(0)=\omega_0} \int_{\Gamma} W(\zeta) \left[ \theta^* I - G(\zeta) G'(\zeta) \right] W(\zeta) d\lambda(\zeta).$$  \hfill (7.5.1)

For this problem to have an optimized value that exceeds $-\infty$, we require that $\theta^* I - G'G$ be positive semidefinite. As a consequence,

$$\theta^* \geq [H_\infty(F)]^2,$$

which is a sharper restriction than

$$\theta^* \geq \tilde{\theta} = \left[ \inf_F H_\infty(F) \right]^2.$$  \hfill (7.5.2)

In what follows we strengthen the restriction that $\theta^* I - G'G$ be positive semidefinite by requiring entropy to be finite:

$$\int_{\Gamma} \log \det (\theta^* I - G'G) d\lambda(\zeta) > -\infty.$$  \hfill (7.5.2)

Of course, it is only necessary to check this condition at

$$\theta^* = [H_\infty(F)]^2.$$  \hfill (7.5.2)

For larger values of $\theta^*$, (7.5.2) is satisfied automatically. As we will see, for any value of $\theta^*$ that exceeds the threshold $[H_\infty(F)]^2$, the entropy measure is closely related to the minimized value of the multiplier problem.

Provided that condition (7.5.2) holds, we can associate choices of $\theta^*$ with restrictions on the specification errors. That is, consider the following constrained worst case minimization problem:

*Constrained worst case problem:*

$$\min_W - \int_{\Gamma} W'G'GW d\lambda$$
subject to
\[ \int_{\Gamma} W'Wd\lambda \leq w'_0w_0 + \eta^2. \]

**Theorem 7.5.1.** For any \( \theta^* > [H_\infty(F)]^2 \), there exists an \( \eta \) such that the multiplier problem and the constrained worst case problem have the same solution.

**Proof.** See Appendix C.  

If the infimum of the multiplier problem is attained for \( \theta^* = [H_\infty(F)]^2 \), then there is a finite \( \eta \) such that the two problems continue to have the same solution. If the infimum is not attained, then any finite \( \eta \) is associated with a multiplier \( \theta^* \) that exceeds \([H_\infty(F)]^2 \). Thus we can think of the \( \theta^* \)'s in the multiplier problem as measuring the size of allowable specification errors.

### 7.5.1. A robustness bound

For a given decision rule \( F \), the multiplier problem yields an inequality that bounds the rate at which the criterion function deteriorates as specification errors increase. Let \( J \) denote the minimized value of the objective (7.5.1) for the multiplier problem. Then

\[ -\int_{\Gamma} W'G'GWd\lambda \geq J - \theta^* \int_{\Gamma} W'Wd\lambda. \]  

(7.5.3)

Inequality (7.5.3) shows that in the absence of specification errors, \( J \) understates the performance of the policy. It also shows how \( \theta^* \) governs the rate at which the objective function \(-\int_{\Gamma} W'G'GWd\lambda\) deteriorates with model misspecification as measured by \( \int W'Wd\lambda \). Note how lowering \( \theta^* \) gives more robustness in the sense of less sensitivity of the objective function to misspecifications \( W \).

In the remainder of this section, we study the existence of a solution to the multiplier problem and its relation to the entropy criterion. We return to the Stackelberg multiplier game in the following section.
7.5.2. Entropy is the indirect utility function of the multiplier problem

For establishing our next result, it is convenient to rewrite the multiplier problem as

$$\inf_{W(\zeta) \in W} \int_\Gamma W(\zeta) [\theta^* I - G(\zeta)' G(\zeta)] W(\zeta) d\lambda(\zeta)$$  \hspace{1cm} (7.5.4)

subject to

$$\int_\Gamma W(\zeta) d\lambda(\zeta) = w_0 \neq 0,$$  \hspace{1cm} (7.5.5)

and

$$\int_\Gamma W(\zeta) \zeta^j d\lambda(\zeta) = 0,$$  \hspace{1cm} (7.5.6)

for \( j = 1, 2, \ldots \). Constraint (7.5.5) can be restated as \( W(0) = w_0 \). Constraint (7.5.6) states that \( w_j = 0 \) for \( j < 0 \). From the definition of \( W \), the infimum in (7.5.4) is over \( W(\zeta) \) that have coefficients such that \( \sum_{t=-\infty}^{\infty} \beta^t w_t w_t' < \infty \).

**Theorem 7.5.2.** Assume that \( F \) and \( \theta^* \) are such that \( \int_\Gamma \log \det(\theta^* I - G'G) d\lambda > -\infty \).\(^4\) Then multiplier problem (7.5.1) has an optimized value function \( w_0^* D(0)' D(0) w_0 \), where \( D(0) \) is nonsingular and independent of \( w_0 \). The minimized value is attained if \( \theta^* I - G'G \) is nonsingular on \( \Gamma \).

**Proof.** The solution to the multiplier problem can be found using techniques from linear prediction theory.\(^4\) We must factor a spectral density like matrix:

$$[\theta^* I - G(\zeta)' G(\zeta)] = D(\zeta)' D(\zeta)$$  \hspace{1cm} (7.5.7)

where \( D \) is rational in \( \zeta \), has no poles inside or on the circle \( \Gamma \), is invertible inside \( \Gamma \), and the matrix coefficients of its power series expansion inside \( \Gamma \) can be chosen to be real. The matrix analytic function \( D \) is unique only up to premultiplication by an orthogonal matrix but can be chosen to be independent of \( w_0 \). The existence of this factorization follows from results about the linear extrapolation of covariance stationary stochastic processes. In particular, it is known from Theorems 4.2, 6.2 and 6.3 of Rozanov (1967) that the infimum of the objective is given by:

$$w_0^* D(0)' D(0) w_0.$$  \hspace{1cm} (7.5.8)

\(^4\) Under case (ii) or (iii), such an \( F \) exists.

\(^4\) Appendix B displays a linear prediction problem that leads to the spectral factorization problem here.
When $\theta^* I - G'G$ is nonsingular on $\Gamma$, the infimum is attained. To verify this, write the first-order conditions for maximizing (7.5.4) subject to (7.5.5) and (7.5.6) as

$$[\theta^* I - G(\zeta)'G(\zeta)] W(\zeta) = \mathcal{L}(\zeta)' ,$$

(7.5.9)

where $\mathcal{L}$ is the Lagrange multiplier on (7.5.5) and (7.5.6). Then the matrix $D$ in the factorization (7.5.7) is nonsingular with an inverse that is rational and well defined on and inside the circle $\Gamma$. Substituting the factorization (7.5.7) into (7.5.9) gives

$$D(\zeta)'D(\zeta) W(\zeta) = \mathcal{L}(\zeta)',$$

(7.5.10)

where $D(\zeta), W(\zeta)$, being analytic inside $\Gamma$, have expansions in nonnegative powers of $\zeta$, and $D(\zeta)'$ and $\mathcal{L}(\zeta)'$ have expansions in nonpositive powers of $\zeta$ in the interior of $\Gamma$. If $D(\zeta)'$ is invertible, then following Whittle (1983, p. 100), $W(\zeta)$ satisfies

$$D(\zeta) W(\zeta) = [D(\zeta)'^{-1} \mathcal{L}(\zeta)']_+,$$

where $[\cdot]_+$ is the annihilation operator that sets negative powers of $\zeta$ to zero. Because $D(\zeta)'^{-1}$ and $\mathcal{L}(\zeta)'$ are both one-sided in nonpositive powers of $\zeta$, $[D(\zeta)'^{-1} \mathcal{L}(\zeta)']_+ = D(0)'^{-1} \mathcal{L}(0)'$. Therefore, the solution is

$$D(\zeta) W(\zeta) = D(0)'^{-1} \mathcal{L}(0)'.$$

(7.5.11)

Then from (7.5.10), $\mathcal{L}(0)' = D(0)'D(0)W(0)$. Substituting into (7.5.11) gives

$$D(\zeta) W(\zeta) = D(0) w_0.$$

(7.5.12)

In addition, the infimum is attained by:

$$W^*(\zeta) = D(\zeta)^{-1} D(0) w_0.$$

(7.5.13)

Substituting into (7.5.4), we confirm that the minimized solution is (7.5.8).

As is evident from the proof, the infimum in (7.5.4) may not be attained when $\theta^* I - G'G$ is singular somewhere on $\Gamma$. But this problem can be remedied by enlarging the space from $\mathcal{W}$ to $\mathcal{W}^n$.

---

5 The factorization is also the key for calculating (see Whittle (1983, pp. 99-100)) the projection of $y_t$ on the semi-infinite history $x_s, s \leq t$ where $\{y_t, x_s\}$ is a covariance stationary process. Condition (7.5.10) corresponds to the solution of Whittle’s projection problem where $D(\zeta)'D(\zeta)$ is interpreted as the spectral density of $x$ and $\mathcal{L}(\zeta)$ is interpreted as the cross-spectral density between $y$ and $x$. 
Corollary 7.5.1. Assume that \( F \) is such that \( \int_{\Gamma} \log \det(\theta I - G^\prime G) d\lambda > -\infty \). Then the problem

\[
\min_{W} \int_{\Gamma} W(\zeta)^\prime \left[ \theta^* I - G(\zeta)^\prime G(\zeta) \right] W(\zeta) d\lambda(\zeta)
\]

has a solution and the minimized value is \( w_0^\prime D(0)^\prime D(0) w_0 \).

Proof. Solution (7.5.13) is in \( W^a \) even when \( \theta^* I - G^\prime G \) is singular somewhere on \( \Gamma \).

From Corollary 7.5.1, we see that a solution exists for the multiplier problem, provided that the entropy restriction (7.5.2) is satisfied. But unless the matrix \((\theta^* I - G^\prime G)\) is nonsingular at all frequencies, the minimizing shock sequence may not be stable and may not stabilize the state vector sequence. Problems occur when \( W^\ast(\zeta) = D(\zeta)^{-1} D(0) w_0 \) has a pole on \( \Gamma \), or equivalently when \( D(\zeta)^{-1} \) has a pole on \( \Gamma \) that is not annihilated by \( D(0) w_0 \). Nevertheless, even these destabilizing solutions for \( W^\ast \) can be approximated by a sequence of \( W^\ast \)’s, each of which is in \( W \) and hence each of which stabilizes the state vector sequence.

The multiplier problem depends on the choice of initialization \( W(0) = w_0 \). In what follows we seek to replace this multiplier problem by an entropy criterion that does not depend on the initialization. To justify this, we will eventually have to show that for a given \( \theta^* \), the control law that solves the multiplier game does not depend on the initialization \( w_0 \) and is the same control law that solves the entropy control problem.

The criterion for the entropy control problem is motivated by the following representation:

Theorem 7.5.3. Assume that \( \theta^* \) and \( F \) are such that \( \int_{\Gamma} \log \det(\theta^* I - G^\prime G) d\lambda > -\infty \). The criterion \( \log \det[D(0)^\prime D(0)] \) can be represented

\[
\log \det \left[ D(0)^\prime D(0) \right] = \int_{\Gamma} \log \det \left[ \theta^* I - G(\zeta)^\prime G(\zeta) \right] d\lambda(\zeta). \quad (7.5.14)
\]

Proof. \( D(0)^\prime D(0) \) can be regarded as a ‘one-step’ prediction error covariance matrix for a vector process \( D(L)\epsilon_t \), where \( L \) is the lag operator and \( \epsilon_t \) is an i.i.d. random process with mean zero and identity contemporaneous covariance matrix, and \( D(\zeta) \) originates in the spectral factorization (7.5.7). We can use a result from linear prediction theory to verify the representation (7.5.14). See Theorem 6.2 of Rozanov (1967, page 76).

Theorem 7.5.2 and Theorem 7.5.3 both require that \( \int_{\Gamma} \log \det(\theta^* I - G^\prime G) d\lambda > -\infty \) but permit \( \theta^* I - G^\prime G \) to be singular at isolated points in \( \Gamma \).
Evaluating the right-hand side of (7.5.14) requires no spectral factorization, just integration over frequencies. The contour integral on the right side of (7.5.14) is our criterion. In the undiscounted case, it coincides with the measure of entropy used by Mustafa and Glover (1988).\(^6\) When \(\beta = 1\), the \(F\) that maximizes (7.5.14) is often motivated as an approximator of the \(F\) that maximizes the \(H_\infty\) criterion, one that maintains analyticity of \(W\).

Next we show that when \(W^*\) stabilizes the state vector sequence, the solution has a Markov representation (i.e., the solution \(w_{t+1}\) can be represented as a function of the time \(t\) state \(x_t\)).

**Theorem 7.5.4.** Assume \(\theta^*\) and \(F\) are such that \(\theta^* I - G'G\) is nonsingular on \(\Gamma\). Then the solution to the multiplier problem can be represented recursively as:

\[ w_{t+1} = K x_t \]

where

\[ K = (\theta^* I - C'\Sigma C)^{-1} C'\Sigma A, \]

and \(\Sigma\) is the positive semidefinite solution to the Riccati equation

\[ \Sigma = H' H + \beta A' \Sigma A + \beta A' \Sigma C (\theta^* I - C'\Sigma C)^{-1} C' \Sigma A \]

for which \(A + CK\) has eigenvalues that are inside the circle \(\Gamma\). Moreover,

\[ \int_{\Gamma} \log \det [\theta^* I - G'G] \, d\lambda = \log \det (\theta^* I - C'\Sigma C). \]

**Proof.** We use a recursive formulation and solution of the spectral factorization problem (7.5.7) to prove the theorem. To compute \(D\) in the spectral factorization \(\theta^* - G'G = D'D\), we apply the factorization result given by Zhou, Doyle and Glover (1996). Recall that \(G = H(I - \zeta A)^{-1}C\). The spectral density matrix to be factored is:

\[ \theta^* I - G'G = \theta^* I - C' \left[ I - \sqrt{\beta} \exp(-i\omega) A' \right]^{-1} H' H \left[ I - \sqrt{\beta} \exp(i\omega) A \right]^{-1} C \]

\[ = \theta^* I - C' \left[ \exp(i\omega) I - \sqrt{\beta} A' \right]^{-1} H' H \left[ \exp(-i\omega) I - \sqrt{\beta} A \right]^{-1} C. \]

\(^6\) It coincides with their measure of entropy at \(s_0 = \infty\).
where we have used the parameterization: \( \zeta = \sqrt{\beta} \exp(i\omega) \). From Theorem 21.26 of Zhou, Doyle and Glover (1996, pages 557 and 558), we obtain the factorization:

\[
\begin{align*}
\theta^* I - C^* \left[ \exp(i\omega) I - \sqrt{\beta} A \right]^{-1} H' H \left[ \exp(-i\omega) I - \sqrt{\beta} A \right]^{-1} C &= \\
= \left( I - C^* \left[ \exp(i\omega) I - \sqrt{\beta} A \right]^{-1} \sqrt{\beta} K' \right) R \left( I - \sqrt{\beta} K \left[ \exp(-i\omega) I - \sqrt{\beta} A \right]^{-1} C \right) \\
= \left( I - \zeta' C' \left[ I - \zeta A \right]^{-1} K' \right) R \left( I - \zeta K \left[ I - \zeta A \right]^{-1} C \right)
\end{align*}
\] (7.5.19)

where

\[
R = \theta^* I - C' \Sigma C,
\]

(7.5.20)

\[
K = R^{-1} C' \Sigma A,
\]

(7.5.21)

and \( \Sigma \geq 0 \) is the stabilizing solution of the Riccati equation

\[
\beta A' \Sigma \left( I - \frac{1}{\theta^*} C C' \Sigma \right)^{-1} A - \Sigma + H' H = 0.
\] (7.5.22)

We establish that formula (7.5.22) is equivalent with (7.5.17) by showing that

\[
\left( I - \frac{1}{\theta^*} C C' \Sigma \right)^{-1} = I + C \left( \theta^* I - C' \Sigma C \right)^{-1} C' \Sigma.
\]

We verify this result by post multiplying the matrix \( I - \frac{1}{\theta^*} C C' \Sigma \) by the matrix \( I + C \left( \theta^* I - C' \Sigma C \right)^{-1} C' \Sigma \):

\[
\begin{align*}
\left( I - \frac{1}{\theta^*} C C' \Sigma \right) \left[ I + C \left( \theta I - C' \Sigma C \right)^{-1} C' \Sigma \right] &= I - \frac{1}{\theta^*} C C' \Sigma + C \left( I - \frac{1}{\theta^*} C C' \Sigma \right) \left( \theta I - C' \Sigma C \right)^{-1} C' \Sigma \\
&= I - \frac{1}{\theta^*} C C' \Sigma + \frac{1}{\theta^*} C C' \Sigma \\
&= I.
\end{align*}
\]

For the stabilizing solution, \( K \) from (7.5.21) is such that \( I - \zeta \sqrt{\beta} K \left[ I - \sqrt{\beta} \zeta A \right]^{-1} C \) has zeros outside the unit circle of the complex plane (Zhou, Doyle and Glover (1996)). As a consequence, \( I - \zeta K \left[ I - \zeta A \right] C \) has zeros outside of the circle \( \Gamma \). Therefore, (7.5.19) and (7.5.7) imply

\[
D^*(\zeta) = R^{1/2} \left( I - \zeta K \left[ I - \zeta A \right]^{-1} C \right)
\] (7.5.23)
has zeros outside $\Gamma$, and
\[ \theta^* I - G'G = D^*D*. \]
Furthermore,
\[ D^* (0) D^* (0) = R = \theta^* I - C'\Sigma C. \]
The entropy criterion (7.4.4) can thus be represented as $\log \det(\theta^* I - C'\Sigma C)$.

From formula (7.5.12), the solution for $W(\zeta)$ can be represented as
\[ D^* (\zeta) W (\zeta) = D^* (0) w_0. \]
Using (7.5.23) gives
\[ \zeta^{-1} [W (\zeta) - w_0] = K (I - \zeta A)^{-1} C W (\zeta) \]
and using $X(\zeta) = (I - \zeta A)^{-1} C W (\zeta)$ gives the recursive formula
\[ w_{t+1} = K x_t. \]

Theorem 7.5.4 can be extended to allow for isolated singularities. In Appendix E we show that the entropy formula (7.5.18) of Theorem 7.5.4 continues to hold if $\theta^* I - G'G$ is positive semidefinite and nonsingular at either $\sqrt{\beta}$ or $-\sqrt{\beta}$.

7.6. Relation of Stackelberg multiplier game to entropy criterion

One step remains to show that the Stackelberg multiplier game justifies the entropy criterion (7.4.4). The extra step is needed because criterion (7.5.8) depends on $w_0$ while (7.4.4) does not. But Theorem 6.4.1 showed that the $F$ that solves (7.4.3) is independent of $w_0$. Therefore, we will attain the same decision rule $F$ by maximizing a criterion defined in terms of $D(0)'D(0)$ alone, ignoring $w_0$. Thus, let $w'_0 \hat{D}(0)'\hat{D}(0)w_0$ denote criterion (7.5.8) for another control law, say $\hat{F}$. If
\[ w'_0D (0)' D (0) w_0 \geq w'_0 \hat{D} (0)' \hat{D} (0) w_0 \]
for all $w_0$, then
\[ D (0)' D (0) \geq \hat{D} (0)' \hat{D} (0) \]
where ‘$\geq$’ is the standard partial ordering of positive semidefinite matrices. As a consequence,

$$\text{trace} \left[ D(0)' D(0) \right] \geq \text{trace} \left[ \hat{D}(0)' \hat{D}(0) \right],$$

or alternatively:

$$\log \det \left[ D(0)' D(0) \right] \geq \log \det \left[ \hat{D}(0)' \hat{D}(0) \right].$$

Because we want the criterion to apply for all initial conditions $w_0$, we take our criterion to be

$$\log \det D(0)' D(0).$$

Theorem 7.5.3 shows that this is the entropy criterion used to define (7.4.4).

### 7.7. Risk sensitivity

This section briefly brings out the connection between the entropy criterion (7.5.18) and the discounted risk-sensitive criterion described by Hansen and Sargent (1995). Hansen and Sargent consider a situation where a decision maker is interested in evaluating fixed rules $u_t = -Fx_t$ from the point of view of minimizing a cost-criterion defined recursively as

$$C(x) = x' H' H x + \beta R(C(y)|x)$$

(7.7.1)

where

$$R(Y|x) = -\left( \frac{2}{\sigma} \right) \log E \left( \exp \left( -\frac{\sigma Y}{2} \right) | x \right),$$

(7.7.2)

where $H = H_0 - JF$ as above, and

$$y = Ax + Cw$$

(7.7.3)

where $w$ is now an i.i.d. Gaussian sequence with mean zero and covariance matrix $I$. In (7.7.2), $\sigma$ is the ‘risk-sensitivity’ parameter. When $\sigma < 0$, $R$ adds an additional aversion to risk beyond that embodied in the cost function $C(y)$.

Let

$$\mathcal{D}(V) = V - \sigma V C (I + \sigma CC')^{-1} C' V.$$

(7.7.4)

---

Note: This operator also appears in chapters 2 and 6.
Drawing on results of Jacobson (1973), Hansen and Sargent (1995) show that
\[ \mathcal{R}(Y|x) = x'A'D(Y)Ax \] (7.7.5)
and that the cost function \( C(x) \) that is the fixed point of (7.7.1) has the form
\[ C(x) = x'V^*x + c^* \] (7.7.6)
where \( V^* \) is the fixed point of recursions on the operator
\[ S(V) = H'H + \beta A'D(V)A \] (7.7.7)
and
\[ c^* = \frac{\beta}{1 - \beta\sigma} \log \det(I + \sigma C'V^*C). \] (7.7.8)

The connection between risk-sensitive preferences and a preference for robustness can be seen by letting \( \sigma = -\theta^{-1} \) and noting that
\[ D(V) = V + \theta^{-1}VC(I - \theta^{-1}C'VC)^{-1}C'V \] (7.7.9)
which makes (7.7.7) the same operator that appears on the right of (7.5.17). Also, \( c^* \) is
\[ c^* = \frac{1}{\theta} \frac{\beta}{1 - \beta} \log \det(I - \theta^{-1}C'V^*C). \] (7.7.10)

This can also be written
\[ c^* = -\frac{1}{\theta} \frac{\beta}{1 - \beta} \left[ -n \log \theta + \log \det(\theta I - C'V^*C) \right]. \] (7.7.11)

Consider representation (7.7.6) for the cost function. Here \( x'V^*x \) is the part of the cost function corresponding to the initial condition, and the \( \log \det \) term corresponds to the ‘stationary steady state’. Consider minimizing the cost function starting from \( x = 0 \), so that only the \( \log \det \) term is relevant. The \( \log \det(\theta I - C'V^*C) \) term can be interpreted as the log determinant of a one-step-ahead prediction error covariance matrix and so can be expressed as the right hand side of (7.5.14) for some stationary process with the proper associated spectral density matrix. It is evident from (7.5.17) and the definition of the \( \mathcal{S}(V), D(V) \) operators that the appropriate spectral density is identical with that used in defining (7.5.14). In the case that \( x = 0 \), minimizing cost comes down to minimizing \( c^* \). Because \( \theta > 0 \), this comes down to maximizing
\[ \log \det(\theta I - C'V^*C). \] (7.7.12)

This is equivalent with maximizing entropy defined by (7.5.14).
7.8. \( H_2 \), the entropy criterion, and concavity

In this section we show how the entropy criterion adjusts the \( H_2 \) criterion to express a concern about model misspecification. We show how entropy puts additional concavity into a utility function. In effect, the entropy criterion represents model misspecification by inducing \textit{risk aversion} across frequencies.

The \( H_2 \) criterion is

\[
H_2 = - \int_{\Gamma} \text{trace} \left[ G(\zeta)' G(\zeta) \right] d\lambda(\zeta),
\]

and the entropy criterion is

\[
\text{ent} = \int_{\Gamma} \log \det \left[ \theta I - G(\zeta)' G(\zeta) \right] d\lambda(\zeta).
\]

Take a symmetric negative semidefinite matrix \( V \) with eigenvalues \(-\delta_1, \ldots, -\delta_n\). Let \( \theta > \max_i -\delta_i \). Then \( \text{trace}(V) = \sum_j -\delta_j \) and

\[
\log \det (\theta I + V) = \sum_j \log (\theta - \delta_j).
\]

Note that \( \log(\theta - \delta) \) is a concave function of \(-\delta\).

Associated with each \( \zeta \) is a set of eigenvalues of \( G(\zeta)' G(\zeta) \) that we denote \( \delta_1(\zeta), \ldots, \delta_n(\zeta) \). Let them be ordered according to their magnitude. Then we can write the \( H_2 \) criterion as

\[
H_2 = \sum_j \int_{\Gamma} -\delta_j(\zeta) d\lambda(\zeta).
\]

The entropy criterion is formed from \( H_2 \) by putting a concave transformation inside the integration:

\[
\text{ent} = \sum_j \int_{\Gamma} \log [\theta - \delta_j(\zeta)] d\lambda(\zeta). \tag{7.8.1}
\]

Thus the entropy criterion puts more curvature in the return function. This has effects that could also be represented as enhanced risk aversion. Notice that here the ‘risk aversion’ seems to be across frequencies: in (7.8.1) we average over eigenvalues and frequencies instead of states of nature. Big eigenvalues have relatively more weight in the entropy criterion because of the concavity of the log function.
7.9. Concluding remarks

The decision maker’s approximating model asserts that the Fourier transform of a target vector $Z(\zeta)$ is

$$Z(\zeta) = G(\zeta) w_0$$

where $G(\zeta)$ is the transfer function $G(\zeta) = H(I - (A_o - BF)\zeta)^{-1}C$ and $F$ is the decision maker’s feedback rule. The approximating model sets $W(\zeta) = w_0$, but the misspecified models assert that

$$Z(\zeta) = G(\zeta) W(\zeta).$$

Deviations of $W(\zeta)$ from $w_0$ represent the approximating model’s misspecification of the temporal properties of the shock process.\(^8\)

Without fear of model misspecification, the decision maker would choose $F$ to maximize $H_2$ defined in equation (7.3.6). A preference for robustness to model misspecification can be expressed by having the decision maker replace $H_2$ by either $H_\infty$ or an entropy criterion. The $H_\infty$ criterion induces a robust decision rule via the following thought process. The decision maker considers perturbations to the temporal properties of the shocks and wants decisions that will work well across a broad set of such patterns. To promote robustness, the decision maker investigates the consequences of his rule under the worst shock process. But what is worst depends on his decision rule. Given his decision rule, the worst serial correlation pattern focuses spectral power at the frequency that attains the highest weight in the frequency domain representation of $Z(\zeta)'Z(\zeta)$. The contribution of that frequency to discounted costs is measured by the maximal eigenvalue of $G(\zeta)'G(\zeta)$. The decision maker achieves a robust rule by optimizing against that worst serial correlation pattern, in particular by selecting the feedback rule that minimizes the maximum eigenvalue across all frequencies. Under the entropy criterion the decision maker responds in a similar but less severe way by flattening the response $G(\zeta)$ across $\zeta$’s. We shall study an example of such behavior in chapter 9, where we shall use such insights from the frequency domain to interpret how a form of precautionary savings is called for by a robust decision rule for a permanent income model of consumption.

---

\(^8\) See appendix E for an interpretation of $W(\zeta)$ in terms of the spectral density matrix of a random vector of shocks.
A. Infimization of $H_\infty$

To verify that we have found the infimum of version 2 of (7.3.7)-(7.3.8), let $\omega^*$ be the frequency associated with the maximum value of $\rho$ and let $v(\omega^*)$ denote the corresponding eigenvector. This eigenvector can be complex. We can find a $W^*(\cdot)$ with all real coefficients, with an initial coefficient zero, and that coincides with $v(\omega^*)$ for $\zeta = \sqrt{\beta}\exp(i\omega^*)$. We accomplish this while setting all values of $w_t$ to zero except possibly those for $w_1$ and $w_2$. In particular, that the coefficients of $W^*(\zeta)$ be real requires symmetry, i.e., $W^*(\sqrt{\beta}\exp(i\omega)) = W^*(\sqrt{\beta}\exp(-i\omega))^\top$, where $^\top$ denotes transposition. This leads to two equations of the form $W^*(\zeta^*) = w_1\zeta^* + w_2\zeta^{*2}$, $W^*(\zeta^*) = w_1\zeta^* + w_2\zeta^{*2}$, where here $'$ denotes the complex conjugate, and $\zeta^* = \sqrt{\beta}\exp(i\omega)$. These two equations determine real valued vectors $w_1, w_2$. To form the infimizing $W(\cdot)$, we shall construct an approximating sequence of ‘distributed lags’ of $W^*(\cdot)$ that converge to it.

To get distributed lags of the desired form, create a sequence of continuous positive scalar functions $\{g_n\}$ such that:

(i) $g_n(\omega) = g_n(-\omega)$;
(ii) $\int_{-\pi}^{\pi} g_n(\omega) d\omega = 1$;
(iii) $\{g_n(\omega^*)\}$ diverges;
(iv) $\{g_n\}$ converges uniformly to zero outside any open interval containing $\omega^*$;
(v) $\int_{-\pi}^{\pi} \log g_n(\omega) d\omega > 0$.

Then associated with each $g_n$ is a real scalar (one-sided) sequence with transform $b_n(\zeta)$ such that $b_n(\zeta)^*b_n(\zeta) = g_n(\omega)$ for $\zeta = \sqrt{\beta}\exp(i\omega)$.

Construct $W_n(\cdot) \propto b_n(\cdot)W^*(\cdot)$, where the constant of proportionality makes the resulting $W_n$ satisfy constraint (7.3.8). We have designed the sequence $\{W_n\}$ to approximate the direction $v(\omega^*)$. The sequence of transforms $\{g_n\}$ converges to a generalized function, namely a Dirac–delta function with mass concentrated at frequency $\omega^*$. It is straightforward to show that:

$$\lim_{n \to \infty} \int_{\Gamma} W_n(\zeta)^\top G(\zeta)^\top G(\zeta) W_n(\zeta) d\lambda(\zeta) = \eta^2 (H_\infty)^2.$$  

B. A dual prediction problem

A prediction problem is dual to maximizing (7.5.4) subject to (7.5.5)-(7.5.6). Let $[\theta I - G(\zeta)^\top G(\zeta)]$ for $\zeta = \sqrt{\beta}\exp(i\omega)$ denote a spectral density matrix for a covariance stationary process $\{y_t\}$. The purpose is to predict $(w_0)^\top y_t$ linearly from past values of $y_t$. A candidate forecast rule of the form

$$- \sum_{j=1}^{\infty} (w_j)^\top y_{t-j}$$  \hspace{1cm} (7.1B.1)

has forecast error

$$\sum_{j=0}^{\infty} (w_j)^\top y_{t-j}.$$  

Then criterion (7.5.4) is interpretable as the forecast-error variance associated with this prediction problem. The constraints (7.5.6) prevent the forecast from depending on $y_{t+j}$ for $j \geq 1$. 

C. Duality

Evaluating a Given Control Law

For a given control law $F$ form the corresponding $G$ and define:

$$\theta_F = H^2_\infty (F).$$

It follows that for all $W(\zeta)$:

$$\theta_F \int_{\Gamma} W'W d\lambda \geq \int_{\Gamma} W'G'GW d\lambda.$$

Therefore, for all $\theta \geq \theta_F$, $\int_{\Gamma} W' \left[ \theta I - G'G \right] W d\lambda$ is well defined for all $\theta \geq \theta_F$ but not for $\theta < \theta_F$.

For fixed $F$, consider the inf part of Game 2 (7.3.7):

*Original (Worst Case) Minimization Problem* Lars: let’s double check that we mean $W$ and not $W^0$ as the set over which we min. See the definitions on page 140.

Problem 1

$$\min_W \int_{\Gamma} W'G'GW d\lambda$$

subject to

$$\int_{\Gamma} W'W d\lambda \leq w_0^t w_0 + \eta^2.$$

This problem minimizes a concave function subject to a convex constraint set, so standard duality theory does not apply. In the interests of applying duality theory, we study the following alternative problem:

*A Related Constrained Problem:*

Problem 2

$$\min_W \int_{\Gamma} W' \left( \theta_F I - G'G \right) W d\lambda$$

subject to:

$$\int_{\Gamma} W'W d\lambda \leq \eta^2 + w_0^t w_0.$$

This problem is to minimize a convex function subject to a convex constraint set, so duality theory applies to it. We shall first show that a solution of Problem 2 with binding constraint also solves Problem 1. Then we shall apply standard duality theory to problem 2.

Proof. Let \( W^* \) solve Problem 2 with the magnitude constraint binding:

\[
\int_{\Gamma} W^* W^* \, d\lambda = \eta^2 + w_0^t w_0
\]

and

\( W^*(0) = w_0. \)

Consider any other \( W \) such that

\[
\int_{\Gamma} W W \, d\lambda \leq \eta^2 + w_0^t w_0.
\]

and

\( W(0) = w_0. \)

Then

\[
\int_{\Gamma} W' \left( \theta_F I - G'G \right) W \, d\lambda \geq \int_{\Gamma} W^{*\prime} \left( \theta_F I - G'G \right) W^* \, d\lambda,
\]

and

\[
\theta_F \int_{\Gamma} W' W \, d\lambda \leq \theta_F \int_{\Gamma} W^{*\prime} W^* \, d\lambda.
\]

Therefore

\[
- \int_{\Gamma} W' G' G W \, d\lambda \geq - \int_{\Gamma} W^{*\prime} G' G W^* \, d\lambda,
\]

which implies that \( W^* \) also solves Problem 1.  

Thus a way to solve Problem 1 is to solve Problem 2 and verify the solution satisfies the magnitude constraint with equality.

We now apply duality theory to problem 2 by forming:

Saddle Point Version of Problem 2:

\[
\inf_{W} \sup_{\theta \geq \theta_F} \left[ \int_{\Gamma} W' \left( \theta I - G'G \right) W \, d\lambda - (\theta - \theta_F) \left( \eta^2 + w_0^t w_0 \right) \right].
\]

We interpret \( \theta - \theta_F \) as the Lagrange multiplier for Problem 2 and \( \theta \) as the Lagrange multiplier for Problem 1. Because Problem 2 entails minimizing a convex function subject to a convex constraint set, standard duality theory applies to it. The conjugate problem is obtained by switching the order of the inf and sup operations:

\[
\sup_{\theta \geq \theta_F} \inf_W \left[ \int_{\Gamma} W' \left( \theta I - G'G \right) W \, d\lambda - (\theta - \theta_F) \left( \eta^2 + w_0^t w_0 \right) \right].
\]

(7.C.1)
We can use this problem to construct the Lagrange multiplier \( \theta \) for each \( \eta > 0 \).

By construction the saddle-point value for the conjugate problem coincides with the optimized value for Problem 2. When the specification-error constraint is binding for Problem 2, we can obtain the optimized value for Problem 1 by subtracting the constant \( \theta_F(\eta^2 + w_0^0 w_0) \) from (7.C.1). The resulting conjugate problem is

\[
\sup_{\theta \geq \theta_F} \inf_{W} \left[ \int_{\Gamma} W' \left( \theta I - G'G \right) W d\lambda - \theta \left( \eta^2 + w_0^0 w_0 \right) \right]. \tag{7.C.2}
\]

Thus we have eliminated the influence of \( \theta_F \) on the objective of the saddle-point problem. But \( \theta_F \) still affects the constraint set limiting the choice of \( \theta \) (through the appearance of \( \theta_F \) under the sup operator). This dependence can also be removed by virtue of the following theorem.

**Theorem 7.C.2.** If the value of (7.C.2) is finite, then \( \theta \geq \theta_F \).

**Proof.** Suppose that \( \theta < \theta_F \), and consider the inner infimum part of the saddle-point problem (7.C.2):

\[
\inf_{W} \int_{\Gamma} W' \left( \theta I - G'G \right) W d\lambda. \tag{7.C.3}
\]

Given the construction of \( \theta_F \), \( (\theta I - G'G) \) has negative eigenvalues for some \( |\zeta*| = \sqrt{\beta} \). Parameterize \( \Gamma \) by forming \( \zeta = \sqrt{\beta} \exp(i\omega) \), and let \( \omega^* \) be the frequency associated with \( \zeta* \). Thus there exists a complex vector \( v \) such that

\[
v' \left( \theta I - G'G \right) v < 0
\]

on a nondegenerate interval of \( \omega \)'s containing \( \omega^* \). Imitating the argument in Appendix A, we can form a \( W^*(\zeta) = w_1 \zeta + w_2 \zeta \) such that \( W^*(\zeta*) = v \). We can then use the Appendix A construction to form: \( W_n(\zeta) \sim b_n(\zeta)W^*(\zeta) \). Then it is straightforward to show that:

\[
\lim_{n \to \infty} \int_{\Gamma} W'_n \left( \theta I - G'G \right) W_n d\lambda = v' \left[ \theta I - G \left( \zeta* \right)'G \left( \zeta* \right) \right] v < 0.
\]

By construction \( W_n(0) = 0 \) and hence fails to satisfy the constraint for problem (7.C.3). Also problem (7.C.3) does not constrain the magnitude of \( W \). We now form the sequence:

\[
\tilde{W}_n = nW_n + w_0,
\]

which by construction satisfies \( \tilde{W}_n(0) = w_0 \). Given our multiplication of \( W_n \) by \( n \), it clearly follows that

\[
\lim_{n \to \infty} \int_{\Gamma} W'_n \left( \theta I - G'G \right) W_n d\lambda = -\infty.
\]

Therefore, the optimized value of problem (7.C.3) is \( -\infty \) whenever \( \theta < \theta_F \). \( \blacksquare \)

Given what the theorem establishes about the behavior of the inner infimum part of saddle-point problem (7.C.2) when \( \theta < \theta_F \), we can state that (7.C.2) equals (7.C.3) defined as:
Frequency domain games and criteria for robustness

Conjugate Saddle Point Version of Problem 1

\[
\sup_{\theta} \inf_{W} \left[ \int_{\Gamma} W' \left( \theta I - G' G \right) W d\lambda - \theta \left( \eta^2 + w_0' w_0 \right) \right].
\] (7.C.4)

Whenever this problem has a solution for \( W \) that satisfies the specification-error constraint with equality, the resulting \( W \) also solves Problem 1 and the value of the conjugate saddle-point problem coincides with that of Problem 1. This conjugate problem provides the Lagrange multiplier \( \theta \geq \theta_F \) associated with Problem 1. Armed with this multiplier, consider the inner infimum problem, which we call the multiplier problem:

(Problem 3) \[
\inf_{W} \left[ \int_{\Gamma} W' \left( \theta I - G' G \right) W d\lambda \right].
\]

The solution of Problem 3 coincides with that of the prediction problem described in Appendix B and analyzed in the text.

Given any \( \eta \), we have just shown how to find the multiplier \( \theta \). We now suppose that the multiplier \( \theta \geq \theta_F \) is given and want to deduce the corresponding value of \( \eta \). Thus, suppose that we have a solution of the multiplier problem (Problem 3). It is sufficient for this problem to have a solution with \( \theta > \theta_F \). (Later we shall discuss the case in which \( \theta = \theta_F \).) We assume that:

\[
\int \log \det (\theta_F I - G' G) d\lambda > -\infty.
\] (7.C.5)

Later we will describe what happens when this condition is violated.

**Theorem 7.C.3.** Suppose that \( \theta > \theta_F \) and that \( W(\zeta) \) solves the multiplier Problem 3. Then there exists \( \eta > 0 \) such that \( W(\zeta) \) solves Problem 1.

**Proof.** From the dual prediction problem of Appendix B, we know that when \( \theta > \theta_F \), the solution to the multiplier problem is:

\[
W(\zeta) = D(\zeta)^{-1} D(0) w_0
\] (7.C.6)

where

\[
D' D = (\theta I - G' G)
\]

and \( D \) is continuous and nonsingular on the region \( |\zeta| \leq \sqrt{\eta} \). Notice that \( D \) depends implicitly on \( \theta \) The resulting objective function is:

\[
w_0' D(0)' D(0) w_0.
\]

The \( \eta \) corresponding to this choice of \( \theta \) satisfies:

\[
\eta^2 = \int w_0' D(0)' (\theta I - G' G)^{-1} D(0) w_0 d\lambda - w_0' w_0.
\] (7.C.7)
7.C.1. When $\theta = \theta_F$

Next consider the possibilities when $\theta$ is equal to the lower threshold value $\theta_F$. Condition (7.C.5) implies that we can still obtain the factorization:

$$D' D = \theta_F I - G' G,$$

where $D$ is nonsingular on the region $|\zeta| < \sqrt{\beta}$, but now it is singular at some points $|\zeta| = \sqrt{\beta}$. Thus the candidate solution for $W$ given by (7.C.6) may not be well defined, and the infimum in the multiplier Problem 3 may not be attained. Nevertheless, the infimum is still given by the quadratic form: $w_0' D(0)' D(0) w_0$ and the implied $\eta_F$ satisfies (7.C.7), and will typically be infinite.

When $\eta_F = \infty$, we can find a $\theta > \theta_F$ that yields any positive $\eta$. Sometimes $\eta_F$ is finite for a small (Lebesgue measure zero) set of initializations $w_0$. When this happens, we may only find $\theta \geq \theta_F$ for values of $\eta \leq \eta_F$.

7.C.2. Failure of entropy condition

Finally, we consider what happens when

$$\int \log \det (\theta_F I - G' G) \, d\lambda = -\infty.$$

Since $G$ is a rational function of $\zeta$ with no poles in the region $|\zeta| \leq \sqrt{\beta}$, $\theta_F I - G' G$ is singular for all $|\zeta| = \sqrt{\beta}$. Factorizations still exist now take the form:

$$D' D = \theta_F I - G' G$$

where $D$ has fewer rows than columns and has full rank on the region $|\zeta| < \sqrt{\beta}$ (see Rozanov (1967) pages 43–50). This makes it possible to have a variety of solutions to Problem 2, including solutions for which the specification-error is slack.

To understand better the multiplicity, note that it is now possible to find a $\tilde{W}$ such that:

$$D \tilde{W} = 0 \quad (7.C.8)$$

and for which $\tilde{W}(0) = 0$. Given any solution $W^\ast$ to Problem 2, we may form $W^\ast + r\tilde{W}$ for any real number $r$ without altering the objective of Problem 2. The value of $r$ is restrained by the specification-error constraint, but it possible for this range to be nondegenerate.

When the specification-error constraint for Problem 2 can be slack at an optimum, the Lagrange multiplier, $\theta - \theta_F$, is zero, or equivalently $\theta = \theta_F$. Problem 2 will then have solutions in which the specification-error constraint is binding (but with a zero multiplier), and it is only these solutions that also solve Problem 1. As a consequence, solving the multiplier problem (Problem 3) for choices of $\theta$ greater than $\theta_F$ may not correspond to fixing an $\eta$ for Problem 1. We illustrate this possibility in the following example.

Exceptional Example
In this example, we construct a \( \tilde{W} \) satisfying (7.3.8) and \( \tilde{W} > 0 \) \( \forall \zeta \in \Gamma \). Suppose that \( A - BF = 0 \) and hence \( G = HC \), which is constant across frequencies. Then \( \theta_F \) is the largest eigenvalue of the symmetric matrix \( C'H'HC \), and \( \det(\theta_F I - G'G) = 0 \) for all \( \zeta \in \Gamma \). Let \( \mu \) be an eigenvector associated with \( \theta_F \) with norm one. Solutions \( W^* \) to Problem 2 are given by:

\[
\begin{align*}
W^*_0 &= w_0 \\
W^*_t &= \alpha_t \mu
\end{align*}
\]

for \( t > 0 \) and the real numbers \( \alpha_t \) chosen so that the magnitude constraint is satisfied. The resulting objective for Problem 2 is:

\[
w_0' (\theta_F I - C'H'HC) w_0.
\]

Provided that \( \eta^2 > 0 \), the magnitude constraint can be made slack (say by letting \( \alpha_t \) be zero).

A solution to Problem 1 is obtained by setting \( \alpha_t \) to make the magnitude constraint be satisfied with equality. Then the objective for Problem 1 is:

\[
-\theta_F \eta^2 - w^*_0 C'H'HC w_0.
\]

Finally, the Lagrange multiplier obtained from the conjugate problem is given by its lower threshold \( \theta_F \).

**Optimizing the Control Law**

We next study what happens when the control law is chosen among a family of admissible laws. The choice of \( F \) alters the transfer function \( G \), and we are led to study the game:

\[
\max_{\mathcal{F}} \inf_{\mathcal{W}} \left[ \int_{\Gamma} W'G'GWd\lambda - \theta \left( \eta^2 + w_0'^2 w_0 \right) \right]
\]

subject to

\[
\int_{\Gamma} W'Wd\lambda \leq \eta^2 + w_0'^2 w_0.
\]

Again it is fruitful to analyze a conjugate formulation. With this in mind, first solve:

\[
C(\theta, F) = \inf_{\mathcal{W}} \left[ \int_{\Gamma} W' \left( \theta I - G'G \right) Wd\lambda - \theta \left( \eta^2 + w_0'^2 w_0 \right) \right]
\]

for a given \((\theta, F)\) pair. Then solve the conjugate game:

\[
\max_{\mathcal{F}, \theta} C(\theta, F) = \max_{\mathcal{F}} \sup_{\theta} C(\theta, F) = \max_{\theta} \sup_{\mathcal{F}} C(\theta, F).
\]

Therefore given a solution \( F^* \) to the original game we can find a corresponding \( \theta^* \) such that \((F^*, \theta^*)\) solves the conjugate game. Moreover, if \( F^* \) is optimal for all nonzero initializations \( w_0 \), then \( F^* \) solves the entropy criterion associated with this \( \theta^* \).

We want to show the converse.
Theorem 7.C.4. Fix a $\theta^*$. Find the $F^*$ that solves the entropy problem for $\theta^*$. Compute $\hat{\theta} = H_\infty(F^*)^2$ and verify that the control law $F^*$ satisfies:

$$\int_{\Gamma} \log \det (\hat{\theta}I - G^*G^*) \, d\lambda > -\infty \quad (7.C.9)$$

where $G^*$ is the transfer function associated with the control law $F^*$. Then there exists $W^*$ and an $\eta^* > 0$ such that $F^*, W^*$ solves Game 2.

Proof. If inequality (7.C.9) is satisfied, factor $\theta^* I - G^*G^*$

$$\theta^* (I - G^*G^*) = D^*D^*,$$

and construct the $W^*$:

$$W^*(\zeta) = D^*(\zeta)^{-1}D^*(0)w_0.$$ 

Finally, find $\eta^*$ that solves

$$\eta^2 = \int_{\Gamma} W^*W^* \, d\lambda = -w_0^Tw_0.$$ 

D. Proof of theorem 7.5.4

This appendix restates a version of Theorem 7.5.4 under weaker assumptions about the nonsingularity of $[\theta I - G(\zeta)'G(\zeta)]$.

Theorem 7.D.1. Suppose that

i) $A$ has eigenvalues that are inside the circle $\Gamma$;

ii) $\theta I - G^*G \geq 0$ on $\Gamma$;

iii) Either $\theta I - G(-\sqrt{\beta})G(-\sqrt{\beta})$ or $\theta I - G(\sqrt{\beta})G(\sqrt{\beta})$ is nonsingular.

Then the $H_{\text{entropy}}(\theta)$ criterion can be represented as

$$\log \det D(0)'D(0) = \log \det (\theta I - C^*\Sigma C)$$

where $\Sigma$ is defined implicitly by equation (7.D.3) below.

Proof. We prove this theorem by referring to results from Zhou, Doyle and Glover (1996). We outline the proof in four steps.

Step One: Transform the discrete discounted formulation into continuous undiscounted formulation. Suppose that $\theta I - G(-\sqrt{\beta})G(-\sqrt{\beta})$ is nonsingular. Define the linear fractional transformation:

$$\zeta = -\sqrt{\beta} \left( \frac{s + \sqrt{\beta}}{s - \sqrt{\beta}} \right). \quad (7.D.1)$$
This transformation maps \( s = -\sqrt{\beta} \) into \( \zeta = 0 \), \( s = 0 \) into \( \sqrt{\beta} \), \( s = \infty \) into \( -\sqrt{\beta} \). The transformation maps the imaginary axis into the circle \( \Gamma \) and points on the left side of the complex plane into points inside the circle.

Note also that

\[
\beta \zeta^{-1} = -\sqrt{\beta} \left( -\frac{s + \sqrt{\beta}}{-s - \sqrt{\beta}} \right).
\]

In the case that \( \theta I - G(\sqrt{\beta})' G(\sqrt{\beta}) \) is singular, we replace linear fractional transformation (7.D.1) with:

\[
\zeta = \sqrt{\beta} \left( \frac{s + \sqrt{\beta}}{s - \sqrt{\beta}} \right).
\]

(7.D.2)

In what follows we will use (7.D.1) but the argument for (7.D.2) is entirely similar.

**Step Two:** Use parameterization (7.D.1) to write:

\[
G(\zeta) = \left( s - \sqrt{\beta} \right) H \left[ \left( s - \sqrt{\beta} \right) I + \left( s + \sqrt{\beta} \right) \sqrt{\beta} A \right]^{-1} C
\]

\[
= \left( s - \sqrt{\beta} \right) H \left[ s \left( I + \sqrt{\beta} A \right) - \sqrt{\beta} \left( I - \sqrt{\beta} A \right) \right] C
\]

\[
= \left( s - \sqrt{\beta} \right) H (sI - \hat{A})^{-1} \hat{C}
\]

\[
= \tilde{G}(s)
\]

where

\[
\hat{A} = \sqrt{\beta} \left( I + \sqrt{\beta} A \right)^{-1} \left( I - \sqrt{\beta} A \right)
\]

\[
\hat{C} = \left( I + \sqrt{\beta} A \right)^{-1} C
\]

Rewrite \( \tilde{G} \) as

\[
\tilde{G}(s) = s H \left( sI - \hat{A} \right)^{-1} \hat{C} - \sqrt{\beta} H \left( sI - \hat{A} \right)^{-1} \hat{C}
\]

\[
= H \left( sI - \hat{A} \right) \left( sI - \hat{A} \right)^{-1} \hat{C} - \sqrt{\beta} H \left( sI - \hat{A} \right)^{-1} \hat{C} = H \hat{C} - \hat{H} \left( sI - \hat{A} \right)^{-1} \hat{C},
\]

where

\[
\hat{H} = H \left( \sqrt{\beta} I - \hat{A} \right).
\]

Notice that

\[
H \hat{C} = H \left( I + \sqrt{\beta} A \right)^{-1} C = \tilde{G}(\infty) = G \left( -\sqrt{\beta} \right).
\]

**Step Three:** Write for \( s \) imaginary

\[
\theta I - \tilde{G}' \tilde{G} = \left( \hat{C}' \left(-sI - \hat{A}'\right)^{-1} I \right) \left( \begin{array}{ccc}
-\hat{H}' \hat{H} & \hat{H}' \hat{H}\hat{C} \\
\hat{C}' H' \hat{H} & \theta I - \hat{C}' H' \hat{H}\hat{C}
\end{array} \right) \left( \begin{array}{c}
\left(-sI - \hat{A}\right)^{-1} \hat{C}
\end{array} \right).
\]
Notice that
\[ \theta I - \mathcal{C}'H'H\mathcal{C} = \theta I - G \left( -\sqrt{\beta} \right)' G \left( -\sqrt{\beta} \right) \]
is nonsingular and in fact positive definite.

**Step Four:** Apply Corollary 13.20 of Zhou, Glover and Doyle (1996) to conclude that there exists a matrix \( F \) such that:
\[ \theta I - \mathcal{C}'G = \left[ I - \mathcal{C}' \left( -sI - \hat{A}' \right)^{-1} F' \right] \left( \theta I - \mathcal{C}'H'H\mathcal{C} \right) \left[ I - F \left( sI - \hat{A} \right)^{-1} \mathcal{C} \right]. \]

Now inverse transform from \( s \) to \( \zeta \). The following are useful formulas for carrying out this transformation.
\[ \hat{A} = \sqrt{\beta} \left( I + \sqrt{\beta} \hat{A} \right)^{-1} \left( I - \sqrt{\beta} \hat{A} \right), \]
We invert this relation to find that:
\[ \left( I + \sqrt{\beta} \hat{A} \right) \hat{A} = \sqrt{\beta} I - \beta \hat{A} \]
or
\[ \sqrt{\beta} \left( \hat{A} + \sqrt{\beta} I \right) \hat{A} = - \left( \hat{A} - \sqrt{\beta} I \right) \]
or
\[ A = \frac{1}{\sqrt{\beta}} \left( \sqrt{\beta} I + \hat{A} \right)^{-1} \left( \sqrt{\beta} I - \hat{A} \right). \]
Similarly,
\[ \left( s - \sqrt{\beta} \right) \zeta = - \sqrt{\beta} \left( s + \sqrt{\beta} \right) \]
or
\[ \left( \zeta + \sqrt{\beta} \right) s = \sqrt{\beta} \left( \zeta - \sqrt{\beta} \right) \]
or
\[ s = \sqrt{\beta} \left( \frac{\zeta - \sqrt{\beta}}{\zeta + \sqrt{\beta}} \right) \]
Write:
\[ I - F \left( sI - \hat{A} \right)^{-1} \mathcal{C} = I - \left( \zeta + \sqrt{\beta} \right) F \left[ \sqrt{\beta} \left( \zeta - \sqrt{\beta} \right) I - \left( \zeta + \sqrt{\beta} \right) \hat{A} \right]^{-1} \mathcal{C} \]
\[ = I - \left( \zeta + \sqrt{\beta} \right) F \left[ \zeta \left( \sqrt{\beta} I - \hat{A} \right) - \sqrt{\beta} \left( \sqrt{\beta} I + \hat{A} \right) \right]^{-1} \mathcal{C} \]
\[ = I + \left( \zeta + \sqrt{\beta} \right) F \left( I - \zeta A \right)^{-1} \frac{1}{\sqrt{\beta}} \left( \sqrt{\beta} I + \hat{A} \right)^{-1} \mathcal{C} \]
\[ = I + \frac{\left( \zeta + \sqrt{\beta} \right)}{2\sqrt{\beta}} F \left( I - \zeta A \right)^{-1} \mathcal{C} \]
\[ = \tilde{G} (\zeta) \]
Note that
\[ I + \frac{(\zeta + \sqrt{\beta})}{2\sqrt{\beta}} F (I - \zeta A)^{-1} C = I + \frac{1}{2} FC + \frac{\zeta}{2\sqrt{\beta}} F \left( I + \sqrt{\beta} A \right) (I - \zeta A)^{-1} C. \]
Define \( \Sigma \) implicitly by:
\[ \theta I - C' \Sigma C = \left( I + \frac{1}{2} C' F' \right) \left[ \theta I - C' \left( I + \sqrt{\beta} A' \right)^{-1} H' H \left( I + \sqrt{\beta} A \right)^{-1} C \right] \left( I + \frac{1}{2} FC \right). \]

\[ (7.D.3) \]

### E. Stochastic interpretation of \( H_2 \)

This appendix displays another game that implies \( H_2 \) where the shocks \( w_t \) are permitted to be nonzero for \( t > 0 \). Recall that \( w_t \) is \( m \times 1 \), where \( m \) is the number of shocks. We continue to assume that \( w_t = 0 \) for all \( t < 0 \). We state

**Game 1a**: Choose \((F, \{w_t\})\) to attain
\[
\max_{F} \inf_{\{w_t\}} - \sum_{t=0}^{\infty} \beta^t z_t' z_t \quad (7.E.1)
\]
subject to
\[
\begin{align*}
\kappa_0 &= C w_0 \\
\sum_{t=0}^{\infty} \beta^t w_t w_t' &= \sigma^2 I \quad (7.E.2a) \\
\sum_{t=0}^{\infty} \left( \beta^t w_t \right) \left( \beta^t w_{t-j} \right)' &= 0 \quad \forall j \neq 0 \quad (7.E.2b) \\
\sigma^2 &\leq \eta^2 \quad (7.E.2d)
\end{align*}
\]

Equations (7.E.2b), (7.E.2c) imply that
\[ W(\zeta)W(\zeta)' = \sigma^2 I, \quad |\zeta| = \sqrt{\beta}, \quad (7.E.3) \]
Further, (7.E.3) implies (7.E.2b), (7.E.2c).

Game 1a has the following counterpart in the frequency domain:

**Game 1b**: Find \((F, \sigma^2)\) that attain
\[
\max_{F} \inf_{\sigma^2} - \sigma^2 \int_{\Gamma} \text{trace} \left[ G(\zeta)' G(\zeta) \right] d\lambda(\zeta), \quad (7.E.4)
\]
subject to
\[
\sigma^2 \leq \eta^2. \quad (7.E.5)
\]

We have substituted (7.E.3) into (3.7) to obtain (7.E.4). The solution of game 1b sets \( \sigma^2 \) at its upper bound \( \eta^2 \), and sets \( F \) to maximize the \( H_2 \) criterion (3.6).
7.E.1. Stochastic counterpart

Criterion (7.3.6) emerges when the shock process \( \{w_t\}_{t=1}^{\infty} \) is taken to be a martingale difference sequence adapted to \( J_t \), the sigma algebra generated by \( x_0 \) and the history of \( w \), where \( Ew_{t+1}w'_{t+1} | J_t = 1 \). The martingale difference specification implies

\[
E \sum_{t=0}^{\infty} \left( \beta^j w_t \right) \left( \beta^j w_{t-j} \right)' = \begin{cases} \sigma^2 (1 - \beta)^{-1} I & \text{if } j = 0; \\ 0 & \text{otherwise.} \end{cases}
\]  

(7.E.6)

Equation (7.E.6) is equivalent with \( EW(\zeta)W(\zeta)' = \sigma^2 (1 - \beta)^{-1} I \) for \( \zeta \in \Gamma \). With this representation, (7.3.6) is proportional to \(-(1 - \beta)^{-1} E \sum_{t=0}^{\infty} \beta^j z_t z'_t\). 

---

9 See Whiteman (1985b).
Chapter 8.
Choosing $\theta$: detection probabilities

8.1. The role of randomness

Though we are really interested in random processes, most of our calculations have been cast in terms of deterministic models. This has been true even when we studied filtering problems. In omitting explicit mention of randomness, we have exploited the mathematical structure of models with quadratic objective functions, linear transition laws, and Gaussian disturbances. Control and filtering of such models involve, after mathematical expectations have been taken appropriately, deterministic manipulations of moment matrices. Thus, the certainty equivalence principle stated on page 20 implies that we would derive the same decision rules had we included i.i.d. Gaussian shocks in the models.

In this chapter we explicitly include randomness in order to characterize models that are difficult to distinguish from the approximating model using moderate amounts of data. The presence of randomness in the transition law conceals the distortion of the alternative model relative to the approximating model and makes it statistically difficult to detect if the distortion is not too big.

We use a statistical theory of detection to define a mapping from $\theta$ to a detection error probability for discriminating between the approximating model and the endogenous worst case model associated with that $\theta$. We use that detection error probability to determine a context-specific $\theta$ that is associated with a set of alternative models against which it is reasonable to want to be robust.\(^1\)

---

\(^1\) In the context of continuous time models, Anderson, Hansen, and Sargent (2001) investigate the connection among detection error probabilities, a preference for robustness, and alterations of market prices for risk.
Choosing $\theta$: detection probabilities

8.1.1. Approximating and distorting models

For a given decision rule $u_t = -Fx_t$, we assume that the approximating model makes the state evolve according to the stochastic difference equation

$$x_{t+1} = A_o x_t + C \tilde{\epsilon}_{t+1},$$

(8.1.1)

where now $\tilde{\epsilon}_{t+1}$ is an i.i.d. sequence of Gaussian disturbances with mean zero and identity contemporaneous covariance matrix. We’ll represent a distorted model as

$$x_{t+1} = A_o x_t + C (\epsilon_{t+1} + w_{t+1}),$$

$$\hat{A} x_t + C \epsilon_{t+1},$$

(8.1.2)

where $\hat{A} = A_o + C \kappa(\theta)$, $w_{t+1} = \kappa(\theta) x_t$, and $\epsilon_{t+1}$ is another i.i.d. Gaussian vector with mean 0 and identity covariance matrix. The transition densities associated with models (8.1.1) and (8.1.2) are absolutely continuous with respect to each other, i.e., they put positive probabilities on the same events. Models that are not absolutely continuous with respect to each other are easy to distinguish empirically.

8.2. Detection error probabilities

Detection error probabilities can be calculated using likelihood ratio tests. Thus, consider two alternative models. Model A is the approximating model (8.1.1), and model B is the distorted model (8.1.2) associated with the context specific worst case shock implied by $\theta$. Consider a fixed sample of observations on the state $x_t, t = 0, \ldots, T - 1$. Let $L_{ij}$ be the likelihood of that sample for model $j$ assuming that model $i$ generates the data. Define the log likelihood ratio

$$r_i \equiv \log \frac{L_{ii}}{L_{ij}},$$

where $j \neq i$ and $i = A, B$. When model $i$ generates the data, $r_i$ should be positive. Now consider the probabilities of two kinds of mistakes. First, assume that model A generates the data and calculate

$$p_A = \text{Prob (mistake|A)} = \text{freq } (r_A \leq 0).$$

---

2 The two models (i.e., the two infinite-horizon stochastic processes) are locally absolutely continuous in the sense defined in Hansen, Sargent, Turuhambetova, and Williams (2001). The stochastic processes are not mutually absolutely continuous.
Thus, $p_A$ is the frequency of negative log likelihood ratios $r_A$ when model A is true. Similarly, $p_B = \text{Prob}(\text{mistake}|B) = \text{freq}(r_B < 0)$ is the frequency of negative log likelihood ratios $r_B$ when model B is true. Call the probability of a detection error

$$p(\theta) = \frac{1}{2}(p_A + p_B).$$

Here, $\theta$ is the robustness parameter used to generate a particular model B by taking the associated worst case perturbation of model A in light of a particular objective function for a decision maker. The following section shows in detail how to estimate the detection error probability by using simulations. In a given context, we propose to set $p(\theta)$ to a reasonable number, then invert $p(\theta)$ to find a plausible value of $\theta$.

### 8.3. Details

We now describe how to compute detection error probabilities in some detail.

#### 8.3.1. Likelihood ratio under the approximating model

Define $w^A$ as the worst case shock assuming that the underlying data generating process is the approximating model, i.e., $w^A = \kappa x^A$ where $x^A$ is generated under (8.1.1)). Define $\hat{A} = A_0 + C\kappa$. Then we can express the innovation under the worst case model as:

$$\epsilon_{t+1} = (C'C)^{-1} C' \left(x_{t+1} - \hat{A}x_t\right),$$

$$= \epsilon_{t+1} - \kappa x_t,$$

$$= \epsilon_{t+1} - w_{t+1}^A. \tag{8.3.1}$$

The log likelihood function under the approximating model is

$$\log L_{AA} = -\frac{1}{T} \sum_{t=0}^{T-1} \left\{ \log \sqrt{2\pi} + \frac{1}{2} (\epsilon_{t+1} \cdot \epsilon_{t+1}) \right\}$$

The log likelihood function for the distorted model, given that the approximating model (8.1.1) is the data generating process, is

$$\log L_{AB} = -\frac{1}{T} \sum_{t=0}^{T-1} \left\{ \log \sqrt{2\pi} + \frac{1}{2} (\epsilon_{t+1} \cdot \epsilon_{t+1}) \right\},$$

$$= -\frac{1}{T} \sum_{t=0}^{T-1} \left\{ \log \sqrt{2\pi} + \frac{1}{2} (\epsilon_{t+1} - w_{t+1}^A)' (\epsilon_{t+1} - w_{t+1}^A) \right\}. \tag{8.3.2}$$
Choosing \( \theta \): detection probabilities

Hence, assuming that the approximating model is the data generating process, the likelihood ratio \( r_A \) is:

\[
r_A \equiv \log L_{AA} - \log L_{AB},
\]

\[
= \frac{1}{T} \sum_{t=0}^{T-1} \left\{ \frac{1}{2} w_{t+1} A w_{t+1} A - w_{t+1} A \epsilon_{t+1} \right\}.
\]

(8.3.3)

The second term can be expected to vanish as \( T \to \infty \), so the log likelihood ratio converges to the average value of the one-step measure of entropy \( .5w_{t+1} A w_{t+1} A \).

8.3.2. Likelihood ratio under the distorted model

Now suppose that the data generating process is the distorted model (8.1.2). The innovations in the approximating model are linked to those in the distorted model by \( \epsilon_{t+1} = \epsilon_{t+1} + w_{t+1} B \), where \( w_{t+1} B = \kappa x_{t+1} B \) and \( x_{t+1} B \) is generated under (8.1.2).

Assuming that the distorted model generates the data, the log likelihood function \( \log L_{BB} \) for the distorted model is

\[
\log L_{BB} = -\frac{1}{T} \sum_{t=0}^{T-1} \left\{ \log \sqrt{2\pi} + \frac{1}{2} (\epsilon_{t+1} \epsilon_{t+1}) \right\}.
\]

(8.3.4)

The log likelihood function \( \log L_{BA} \) for the approximating model, assuming that the distorted model (8.1.2) generates the data is,

\[
\log L_{BA} = -\frac{1}{T} \sum_{t=0}^{T-1} \left\{ \log \sqrt{2\pi} + \frac{1}{2} (\epsilon_{t+1} \epsilon_{t+1}) \right\},
\]

(8.3.5)

\[
= -\frac{1}{T} \sum_{t=0}^{T-1} \left\{ \log \sqrt{2\pi} + \frac{1}{2} (\epsilon_{t+1} + w_{t+1} B) (\epsilon_{t+1} + w_{t+1} B) \right\}.
\]

Hence, the likelihood ratio \( r_B \), assuming that the distorted model is the data generating process is

\[
r_B \equiv \log L_{BB} - \log L_{BA},
\]

\[
= \frac{1}{T} \sum_{t=0}^{T-1} \left\{ \frac{1}{2} w_{t+1} B w_{t+1} B + w_{t+1} B \epsilon_{t+1} \right\}.
\]

(8.3.6)

As \( T \to \infty \), this converges to the average value of one-period entropy \( .5w_{t+1} B w_{t+1} B \).
8.3.3. The detection error probability

Attach equal prior weights to model A and B. Then the detection error probability is
\[ p(\theta) = \frac{1}{2} (p_A + p_B), \]
where \( p_i = \text{freq}(r_i \leq 0), i = A, B. \) To compute \( p(\theta) \), we simulate a large number of trajectories and calculate the empirical detection error probability.

8.3.4. Ball’s model

We now illustrate the use of detection error probabilities to discipline the choice of \( \theta \) in the context of the simple dynamic model that Ball (1999) designed to study alternative rules by which a monetary policy authority might set an interest rate.\(^3\) Ball’s is a ‘backward looking’ macro model with the structure

\[ y_t = -\beta r_{t-1} - \delta e_{t-1} + \epsilon_t \]  
\[ \pi_t = \pi_{t-1} + \alpha y_{t-1} - \gamma (e_{t-1} - e_{t-2}) + \eta_t \]  
\[ e_t = \theta r_t + \nu_t, \]

where \( y \) is the log of real output, \( r \) is the real interest rate, \( e \) is the log of the real exchange rate, \( \pi \) is the inflation rate, and \( \epsilon, \eta, \nu \) are serially uncorrelated and mutually orthogonal disturbances. As an objective, Ball assumed that the monetary authority wants to maximize
\[ C = -E (\pi^2_t + y^2_t). \]

The government sets the interest rate \( r_t \) as a function of the current state at \( t \), which Ball shows can be reduced to \( y_t, e_t \).

Ball motivates (8.3.8) as an open-economy IS curve and (8.3.9) as an open-economy Phillips curve; he uses (8.3.10) to capture effects of the interest rate on the exchange rate. Ball set the parameters \( \gamma, \theta, \beta, \delta \) at the values \(.2, .2, .6, .2. \)

Following Ball, we set the innovation shock standard deviations equal to 1, 1, \( \sqrt{2} \).

To discipline the choice of the parameter expressing a preference for robustness, we calculated the detection error probabilities for distinguishing Ball’s model from the worst-case models associated with various values of \( \sigma \equiv -\theta^{-1}. \)

We calculated these taking Ball’s parameter values as the approximating model and assuming that \( T = 142 \) observations are available, which corresponds to 35.5

\(^3\) See Sargent (1999) for further discussion of Ball’s model from the perspective of robust decision theory.
Choosing $\theta$: detection probabilities

years of annual data for Ball’s quarterly model. Fig. 8.3.1 shows these detection error probabilities $p(\sigma)$ as a function of $\sigma$. Notice that the detection error probability is .5 for $\sigma = 0$, as it should be, because then the approximating model and the worst case model are identical. The detection error probability falls to .1 for $\sigma \approx -0.085$. If we think that a reasonable preference for robustness is to want rules that work well for alternative models whose detection error probabilities are .1 or greater, then $\sigma = -0.085$ is a reasonable choice of this parameter. In the next section, we’ll compute a robust decision rule for Ball’s model with $\sigma = -0.085$ and compare its performance to the $\sigma = 0$ rule that expresses no preference for robustness.

Figure 8.3.1: Detection error probability (coordinate axis) as a function of $\sigma = -\theta^{-1}$ for Ball’s model.
8.3.5. Robustness in a simple macroeconomic model

We briefly illustrate how the detection error probabilities for Ball’s model from Fig. 8.3.1 can be used to guide plausible the selection of defensible values of $\theta$. We show a graph that quantifies the robustness attained by different settings of $\theta$.

We use Ball’s model to illustrate the robustness attained by alternative settings of the parameter $\theta$. For Ball’s model, we present Fig. 8.3.2 to show that while robust rules do less well when the approximating model actually generates the data, their performance deteriorates more slowly with departures of the data generating mechanism from the approximating model.

Fig. 8.3.2 plots the value $C = -E(\pi^2 + y^2)$ attained by three rules under the alternative data generating model associated with the worst case model for the value of $\sigma$ on the ordinate axis. The rules are those for the three values $\sigma = 0, -.04, -.085$. Recall how the detection error probabilities computed above associate a value of $\theta = -0.085$ with a detection error probability of about .1. Notice how the robust rules (those computed with preference parameter $\sigma = -.04$ or $-.085$) have values that deteriorate at a lower rate with model misspecification (they are flatter). Notice that the rule for $\sigma = -.085$ does worse than the $\sigma = 0$ or $\sigma = -.04$ rules when $\sigma = 0$, but is more robust in deteriorating less when the model is misspecified.

---

4 Following the risk-sensitive control literature, we transform $\theta$ into the risk-sensitivity parameter $\sigma \equiv -\theta^{-1}$. 
Figure 8.3.2: Value of $C = -E(\pi^2 + y^2)$ for three decision rules when the data are generated by the worst-case model associated with the value of $\sigma$ on the horizontal axis: $\sigma = 0$ rule (solid line), $\sigma = -0.04$ rule (dashed-dotted line), $\sigma = -0.085$ (dashed) line.
Chapter 9.
A permanent income model

9.1. Introduction
Because economists have so much intuition about it, the permanent income model is a good laboratory for exploring the consequences of fears about model misspecification. A preference for robustness activates a kind of precautionary savings that comes from slanting the probability law for the endowment process.\footnote{This context-specific slanting corresponds to that mentioned by Fellner in the passage cited on page 25 of chapter 1.}

We use insights from the Stackelberg multiplier game of chapter 6 to interpret how this probability slanting manifests itself in the permanent income model.

The permanent income model is also a good vehicle for gathering intuitions from the frequency domain approach of chapter 7. To maximize an expected utility functional, a permanent income consumer is patient enough to smooth high frequency fluctuations in income. But to smooth low frequency (i.e., very persistent) income fluctuations requires substantially more patience, and the permanent income consumer has limited patience because his discount factor $\beta$ is less than unity. Because he recognizes that low frequency income fluctuations cause the consumer the most trouble, the minimizing agent makes the worst case shocks persistent. Recognizing the fragility of his decision rule to low frequency misspecifications of the income process, the robust permanent income consumer responds to those more persistent worst case shocks by saving more than he would if he had no doubts about his endowment process. He thus engages in a form of precautionary savings that prevails even when he has quadratic preferences, which distinguishes it from the ordinary form of precautionary savings that emerges only with preferences that have convex marginal utilities.

We apply the label ‘precautionary’ because the effect increases with the volatility of innovations to endowments under the consumer’s approximating model and because it also depends on the parameter $\theta$ that indexes his preference for robustness. Our version of precautionary savings exhibits the usual symptom that it modifies the certainty equivalence present in the linear-quadratic permanent income model. Also, our model keeps the marginal propensity to save out of financial wealth equal to that out of human wealth, in contrast to models like...
those of Cabellero (XXXX) and BLANK (XXXX), where precautionary saving makes the marginal propensity to save out of human wealth exceeds that out of financial wealth.

To explore these issues, this chapter uses an equilibrium version of a permanent income model that Hansen, Sargent, and Tallarini (1999) (HST) estimated for U.S. consumption and investment data.\(^2\) We restate an observational equivalence result of HST who showed that the preference for robustness increases saving just as would increasing the discount factor. Therefore, an appropriate alteration of the discount factor can precisely offset the effect on the consumption and investment allocation of a change in the robustness parameter \(\theta\). HST thereby established that consumption and investment data alone are insufficient to identify both the robustness parameter \(\theta\) and the subjective discount factor \(\beta\).\(^3\) We use the Stackelberg multiplier game from chapter 6 to shed more light on the observational equivalence proposition and the impact on decision rules of distortions in the conditional expectations under the worst case model. We also state an observational equivalence result for a new baseline model and use it to show that activating a preference for robustness still equalizes the marginal propensities to save out of human and nonhuman wealth.

This chapter also illustrates how the detection error probabilities of chapter 8 can discipline plausible choices of \(\theta\) and provides some numerical examples of how much robustness can be achieved by rules designed with various settings of \(\theta\).

---

\(^2\) Hall (1978), Campbell (1987), Heaton (1993), and Hansen, Roberds, and Sargent (1991) have applied versions of this model to aggregate U.S. time series data on consumption and investment.

\(^3\) Despite their failure to affect the consumption allocation, HST showed that such variations in \((\sigma, \beta)\) do affect the relevant stochastic discount factor and therefore the valuation of risky assets.
9.2. A robust permanent income theory

HST’s model has a planner with preferences over consumption streams \( \{c_t\}_{t=0}^{\infty} \), intermeditated through service streams \( \{s_t\}_{t=0}^{\infty} \). Let \( b \) be a preference shifter in the form of a utility bliss point. The Bellman equation for the robust household is

\[
-x'Px - p = \sup_{c} \inf_{w} \left\{ -(s-b)^2 + \beta (\theta w' w - Ey'Py - p) \right\}
\]

where \( E \) is the expectation operator, \( c \) is consumption; \( s \) denotes a scalar service measure, and the law of motion mapping this period’s \( x \) into next period’s state \( y \) will be defined below. As usual, the penalty parameter \( \theta > 0 \) controls the preference for robustness to misspecification. We transform \( \theta \) to the risk-sensitivity parameter \( \sigma = -\theta^{-1} \). In (9.2.1), a scalar household service \( s_t \) is produced by the scalar consumption \( c_t \) via the household technology

\[
s_t = (1 + \lambda) c_t - \lambda h_{t-1} \tag{9.2a}
\]

\[
h_t = \delta h_{t-1} + (1 - \delta_h) c_t \tag{9.2b}
\]

where \( \lambda > 0 \) and \( \delta_h \in (0,1) \). System (9.2.2) accommodates habit persistence or durability as in Ryder and Heal (1973), Becker and Murphy (1988), Sundaresan (1989), Constantinides (1990) and Heaton (1993). By construction, \( h_t \) is a geometric weighted average of current and past consumption. Setting \( \lambda > 0 \) induces intertemporal complementarities. Consumption services depend positively on current consumption, but negatively on a weighted average of past consumption, a reflection of ‘habit persistence’.

There is a linear production technology

\[
c_t + i_t = \gamma k_{t-1} + d_t
\]

where the capital stock \( k_t \) at the end of period \( t \) evolves according to

\[
k_t = \delta k_{t-1} + i_t,
\]

\( i_t \) is time \( t \) gross investment, and \( \{d_t\} \) is an exogenously specified endowment process. The parameter \( \gamma \) is the (constant) marginal product of capital, and \( \delta_k \) is the depreciation factor for capital. HST specified a bivariate (‘two-factor’) stochastic endowment process: \( d_t = \mu_d + d^*_t + \hat{d}_t \). They assumed that the two endowment processes are orthogonal and obey second order autoregressions:

\[
(1 - \phi_1 L)(1 - \phi_2 L) d^*_t = c.d^*_t \left( c^*_{t-1} + w^*_{t-1} \right)
\]

\(^4\) The model fits within the framework described in chapter 10.

\(^5\) For two observed time series \((c_t, i_t)\), HST’s econometric specification needed at least two shock processes to avoid ‘stochastic singularity’.
\[
(1 - \alpha_1 L) (1 - \alpha_2 L) \hat{d}_t = c_d \left( \epsilon_d^d + w_d^d \right)
\]

where the vector \(\epsilon_t\) is i.i.d. Gaussian with mean zero and identity covariance matrix, and \(w_d^{d^*}, w_d^d\) are distortions to the means of \(\epsilon_d^{d^*}, \epsilon_d^d\). HST estimated values of the \(\phi_j\)'s and \(\alpha_j\)'s that imply that the \(d_t\) process is relatively more persistent, as we see below.

Solving the capital evolution equation for investment and substituting into the linear production technology gives:

\[
c_t + k_t = R k_{t-1} + d_t.
\]

Where 

\[
R \equiv \delta_k + \gamma
\]

which is the physical (gross) return on capital, taking into account that capital depreciates over time.\(^6\)

The state vector can be taken to be 

\[
x_t' = [h_{t-1} \ k_{t-1} \ d_{t-1} \ 1 \ d_t \ d_t^* \ d_{t-1}^*]'
\]

(see Hansen, Sargent, and Wang (2000)). There is a set of state transition equations indexed by a \(\{w_{t+1}\}\) process:

\[
x_{t+1} = Ax_t + Bu_t + C(w_{t+1} + \epsilon_{t+1})
\]

where \(u_t = c_t\) and \(w_{t+1}' = [w_{t+1}^{d^*} \ w_{t+1}^d]'\) is the distortion to the conditional mean of \(\epsilon_{t+1}\). Let \(J_t\) be the sigma algebra induced by \(\{x_s, \epsilon_s, 0 \leq s \leq t\}\). We impose that the components of the solution for \(\{c_t, h_t, k_t\}\) belong to \(L_2^2\), the space of stochastic processes \(\{y_t\}\) defined as:

\[
L_2^2 = \{y : y_t \text{ is in } J_t \text{ for } t = 0, 1, \ldots \text{ and } E \sum_{t=0}^{\infty} R^{-t} (y_t)^2 \mid J_0 < +\infty\}.
\]

HST suppose that the endowment and preference shocks \((d_t, b_t)\) are governed by 

\[
b_t = U_b z_t, \ d_t = U_d z_t \text{ where } z_t = [d_{t-1} \ 1 \ d_t \ d_t^* \ d_{t-1}^*] \text{ and }^7
\]

\[
z_{t+1} = A_{22} z_t + C_2 \epsilon_{t+1}.
\]

Here the eigenvalues of \(A_{22}\) are bounded in modulus by unity.

\(^6\) For HST’s decentralized economy, \(R\) coincided with the gross return on a risk free asset.

\(^7\) Under the approximating model, there is no distortion to the mean of the shocks \((w_{t+1} = 0)\).
Given \( x_0 \), the planner chooses a process \( \{c_t, k_t\} \) with components in \( L^2_0 \) to solve the Bellman equation (9.2.1) subject to versions of (9.2.2), (9.2.3). Soon we’ll discuss HST’s parameter values and some properties of their numerical solution. But first we show that a larger preference for robustness works just like an increase in the discount factor in terms of its effects on consumption and investment.\(^8\)

9.3. Solution when \( \sigma = 0 \)

Results from chapter 6 can be applied to show that the robust decision rule for \( \sigma < 0 \) also solves a \( \sigma = 0 \) version of the model in which the maximizing agent in (9.2.1) replaces the approximating model with a particular distorted model for \([d_t, b_t]\). We shall eventually use that insight to study the identification of \( \sigma \) and \( \beta \). First, this section solves the \( \sigma = 0 \) model.

9.3.1. The \( \sigma = 0 \) benchmark case

This subsection computes a representation of the solution of the planning problem in the \( \sigma = 0 \) case. Though we shall soon focus on the case when \( \beta R = 1 \), we also want the solution when \( \beta R \neq 1 \). Thus, for now we allow \( \beta R \neq 1 \). When \( \sigma = 0 \), the decision maker’s objective reduces to

\[
E_0 \sum_{t=0}^{\infty} \beta^t \{-(s_t - b_t)^2\}. \tag{9.3.1}
\]

Formulate the planning problem as a Lagrangian by putting random Lagrange multiplier processes of \( 2\beta \mu_{st} \) on (9.2.2a), \( 2\beta \mu_{ht} \) on (9.2.2b), and \( 2\beta \mu_{ct} \) on (9.2.3). First-order necessary conditions are

\[
\begin{align*}
\mu_{st} &= b_t - s_t \tag{9.3.2a} \\
\mu_{ct} &= (1 + \lambda) \mu_{st} + (1 - \delta_h) \mu_{ht} \tag{9.3.2b} \\
\mu_{ht} &= \beta E_t [\delta_h \mu_{ht+1} - \lambda \mu_{st+1}] \tag{9.3.2c} \\
\mu_{ct} &= \beta RE_t \mu_{ct+1} \tag{9.3.2d}
\end{align*}
\]

\( ^8 \) We can convert this problem into a special case of the control problem posed in chapter 6 as follows. Form a composite state vector \( x_t \) as described above, and let the control be given by \( s_t - b_t \). Solve (9.2.2a) for \( c_t \) as a function of \( s_t - b_t \), \( b_t \) and \( h_{t-1} \) and substitute into equations (9.2.2b) and (9.2.3). Stack the resulting two equations along with the state evolution equation for \( x_t \) to form the evolution equation for \( x_{t+1} \).

\( ^9 \) However, in chapter 12, we show that these \( \beta, \sigma \) pairs that are observationally equivalent for consumption and investment nevertheless imply different prices for risky assets.
and also (9.2.2), (9.2.3). Equation (9.3.2d) implies that \( E_t \mu_{ct+1} = (\beta R)^{-1} \mu_{ct} \); then (9.3.2b) and (9.3.2c) solved forward imply that \( \mu_{st}, \mu_{ht} \) must satisfy \( E_t \mu_{st+1} = (\beta R)^{-1} \mu_{st} \) and \( E_t \mu_{ht+1} = (\beta R)^{-1} \mu_{ht} \). Therefore \( \mu_{st} \) has the representation

\[
\mu_{st} = (\beta R)^{-1} \mu_{st-1} + \nu^t \epsilon_t \tag{9.3.3}
\]

for some vector \( \nu \). The endogenous volatility vector \( \nu \) will play an important role below, and we shall soon tell how to compute it.

Use (9.3.2a) to write \( s_t = b_t - \mu_{st} \), substitute this into the household technology (9.2.2), and rearrange to get the system

\[
c_t = \frac{1}{1+\lambda} (b_t - \mu_{st}) + \frac{\lambda}{1+\lambda} h_{t-1} \tag{9.3.4a}
\]

\[
h_t = \tilde{\delta}_h h_{t-1} + \left(1 - \tilde{\delta}_h\right) (b_t - \mu_{st}) \tag{9.3.4b}
\]

where \( \tilde{\delta}_h = \frac{\delta_h + \lambda}{1+\lambda} \). Equation (9.3.4b) can be used to compute

\[
E_t \sum_{j=0}^{\infty} R^{-j} h_{t+j-1} = (1 - R^{-1} \tilde{\delta}_h)^{-1} h_{t-1} + \frac{R^{-1} (1 - \tilde{\delta}_h)}{1 - R^{-1} \tilde{\delta}_h} \sum_{j=0}^{\infty} R^{-j} (b_{t+j} - \mu_{st+j}) . \tag{9.3.5}
\]

For the purpose of solving the first-order conditions (9.3.2), (9.2.2), (9.2.3) subject to the side condition that \( \{c_t, k_t\} \in L_0^2 \), treat the technology (9.2.3) as a difference equation in \( \{k_t\} \), solve forward, and take conditional expectations on both sides to get

\[
k_{t-1} = \sum_{j=0}^{\infty} R^{-(j+1)} E_t (c_{t+j} - d_{t+j}) . \tag{9.3.6}
\]

Use (9.3.4a) to eliminate \( \{c_{t+j}\} \) from (9.3.6), then use (9.3.3) and (9.3.5). Solve the resulting system for \( \mu_{st} \), to get

\[
\mu_{st} = \Psi_1 k_{t-1} + \Psi_2 h_{t-1} + \Psi_3 \sum_{j=0}^{\infty} R^{-j} E_t b_{t+j} + \Psi_4 \sum_{j=0}^{\infty} R^{-j} E_t d_{t+j} . \tag{9.3.7}
\]

where

\[
\Psi_1 = -(1 + \lambda) R \left(1 - R^{-2} \beta^{-1}\right) \left[\frac{1 - R^{-1} \tilde{\delta}_h}{1 - R^{-1} \tilde{\delta}_h + \lambda \left(1 - \tilde{\delta}_h\right)}\right] \]

\[
\Psi_2 = \frac{\lambda (1 - R^{-2} \beta^{-1})}{1 - R^{-1} \tilde{\delta}_h + \lambda \left(1 - \tilde{\delta}_h\right)} \tag{9.3.8}
\]

\[
\Psi_3 = (1 - R^{-2} \beta^{-1})
\]

\[
\Psi_4 = R^{-1} \Psi_1 .
\]
Equations (9.3.7), (9.3.4), and (9.2.3) represent the solution of the planning problem when $\sigma = 0$.\footnote{When $\beta R = 1$, (9.3.7) makes $\mu_{st}$ depend on a geometric average of current and future values of $b_t$. Therefore, both the optimal consumption service process and optimal consumption depend on the difference between $b_t$ and a geometric average of current and expected future values of $b$. So there is no ‘level effect’ of the preference shock on the optimal decision rules for consumption and investment. However, the level of $b_t$ will affect equilibrium asset prices.}

Formula (9.3.7) for $\mu_{st}$ can be represented in matrix notation as

\begin{align*}
\mu_{st} &= M_s x_t \quad \text{(9.3.9)} \\
x_t &= A_s x_{t-1} + C \epsilon_t \quad \text{(9.3.10)}
\end{align*}

where $x_t$ is the state vector $k_{t-1}, h_{t-1}, z_t$, the matrix $M_s$ is determined by equations (9.3.7) and the laws of motion for $b_t, d_t$, and $A, C$ tell the law of motion for the entire state under the optimal rule for $c_t$.\footnote{Here $C$ is the matrix that appears in (9.2.4) above. See Hansen and Sargent (20XX) for fast ways to compute $A, M_s, C$ for a class of models that includes that of this chapter as a special case.} It follows that $\mu_{st} = M_s A_s x_{t-1} + M_s C \epsilon_t$ which must agree with (9.3.3) so that $\mu_{s,t-1} \equiv M_s A_s x_{t-1}$ and $\nu' \equiv M_s C$. Below, the scalar $\alpha = \sqrt{\nu' \nu}$ plays an important role. It obeys

\begin{equation}
\alpha = \sqrt{M_s C' M_s'}. \tag{9.3.11}
\end{equation}

In the widely studied special case that $\lambda = \delta_h = 0$, so that $s_t = c_t$ and $\mu_{st} = b_t - c_t$, (9.3.7), (9.3.8) imply that the marginal propensity to consume out of non-human wealth $R k_{t-1}$ and the marginal propensity to consume out of non-human wealth defined as $\sum_{j=0}^{\infty} R^{-j} E_t d_{t+j}$ both equal $-\Psi_1$. Identity between these two marginal propensities to consume is a well known feature of the linear-quadratic model. Notice that human wealth discounts expected future endowments at the risk-free rate.
9.3.2. Observational equivalence (for quantities) of $\sigma = 0$ and $\sigma \neq 0$

To produce a permanent income model in the $\sigma = 0$ case, HST followed Hall (1978) and imposed that $\beta R = 1$. HST showed that for fixed values of other parameters, the following pairs of $(\beta, \sigma)$ leave decision rules for $(c_t, i_t)$ unaffected:

$$\hat{\beta}(\sigma) = \frac{1}{R} + \frac{\sigma \alpha^2}{R - 1}. \quad (9.3.12)$$

Here $\alpha^2 = \nu'\nu$, where $\nu$ is a vector in the following martingale representation for the marginal utility of services $\mu_{st}$ that prevails (as a special case of (9.3.3)) when $\beta R = 1$:

$$\mu_{st} = \mu_{st-1} + \nu' \epsilon_t. \quad (9.3.13)$$

This section explains how HST constructed this locus.

Following HST, we now assume that $\beta R = 1$. We state

**Theorem 9.3.1. (Observational Equivalence, I)** Fix all parameters except $(\beta, \sigma)$. Suppose $\beta R = 1$. There exists a $\sigma < 0$ such that for any $\sigma \in (\sigma, 0)$, the optimal consumption-investment plan for $(\beta, 0)$ is also optimal for $(\hat{\beta}(\sigma), \sigma)$ where $\hat{\beta}(\sigma) < \beta$ satisfies (9.3.12).

This proposition means that, so far as the quantities $\{c_t, k_t\}$ are concerned, the robust $(\sigma < 0)$ version of the permanent income model is observationally equivalent to the benchmark $(\sigma = 0)$ version.\textsuperscript{12} The proof of the proposition is constructive.

**Proof.** Begin with a solution $\{\bar{s}_t, \bar{c}_t, \bar{k}_t, \bar{h}_t\}$ for a benchmark $\sigma = 0, \beta R = 1$ economy, then form a comparison economy with a $\sigma \in [\bar{\sigma}, 0]$, where $\bar{\sigma}$ is the lowest value for which the solution of (9.3.16) reported below is real. The comparison economy fixes all parameters except $(\sigma, \beta)$ at their values for the benchmark economy. We then construct a discount factor $\hat{\beta} < \beta$ for which $\{\bar{s}_t, \bar{c}_t, \bar{k}_t, \bar{h}_t\}$ is also the allocation for the $\sigma < 0$ economy.

When $\beta R = 1$, (9.3.3) becomes

$$\mu_{st} = \mu_{st-1} + \nu' \epsilon_t. \quad (9.3.13)$$

\textsuperscript{12} The asset pricing theory developed by HST and (9.3.14) imply that the price of a sure claim on consumption one period ahead is $R^{-1}$ for all $t$ and all $(\sigma, \hat{\beta})$ in the locus (9.3.12). Therefore, these different parameter pairs are also observationally equivalent with respect to the risk-free rate. In this model, the technology (9.2.3) ties down the risk-free rate. For a version of the model with quadratic costs of adjusting capital, the risk-free rate comes to depend on $\sigma$, even though the observations on quantities are nearly independent of $\sigma$. See Hansen and Sargent (1996).
The optimality of the allocation under the original $(0, \beta)$ implies that (9.3.13) is satisfied, which in turn implies that $E_t \mu_{ct+1} = \mu_{ct}$ and (9.3.7) are satisfied where $E_t$ is the expectation operator under the approximating model. We consider a new value $\sigma < 0$ and seek an associated value $\hat{\beta}(\sigma)$ for which: (1) (9.3.13) remains satisfied under the approximating model; (2) the robust decision maker chooses the $(\cdot)$ allocation, which requires that $\hat{\beta} R E_t \mu_{ct+1} = \mu_{ct}$, where $\hat{E}$ is the expectation with respect to the worst case model associated with $(\sigma, \hat{\beta})$ when the approximating model obeys (9.3.13). However, when the approximating model satisfies (9.3.13), the worst case model associated with $\sigma$, $\hat{\beta}$ implies that $\hat{E}_t \mu_{ct+1} = \hat{\zeta}(\hat{\beta}) \mu_{ct}$, where $\hat{\zeta} > 1$ can be found by solving the pure forecasting problem associated with law of motion (9.3.13), one-period return function $-\mu_{st}^2 = -(b_t - s_t)^2$, and discount factor $\hat{\beta}$. If the $\sigma$-robust decision maker is to choose a decision rule that sustains (9.3.13) under the approximating model, so that (1) and (2) both prevail, $\hat{\beta}$ it must be that

$$
\hat{\beta} R \hat{\zeta}(\hat{\beta}) = 1.
$$

(9.3.14)

To complete the argument, we need to compute $\hat{\zeta}(\hat{\beta})$ by solving a pure forecasting problem to find the distorted expectation operator $\hat{E}_t$. We use the recipe given in formulas (6.8.9) on page 130 and (6.10.1) and (6.10.2) on page 134. Taking (9.3.13) as given under the approximating model and noting that $\mu_{st}^2 = (b_t - s_t)^2$, the evil agent in the pure forecasting problem seeks to minimize $-\sum_{t=0}^{\infty} \beta^t (\mu_{st}^2 + \beta \mu_{w_t}^2(w_{t+1}))$ under the distorted law $\mu_{st} = \mu_{st-1} + \alpha w_t$, where $\alpha = \nu' \nu$ (see (9.3.13)). Taking $\mu_s$ as the state, the evil agent’s Bellman equation (6.10.2) is

$$
-P \mu_s^2 = -\mu_s^2 + \beta \min_w \left( -\frac{1}{\sigma} w^2 - P (\mu_s + \alpha w)^2 \right).
$$

(9.3.15)

The scalar $P$ that solves (9.3.15) is

$$
P(\beta) = \frac{\beta - 1 + \sigma \alpha^2 + \sqrt{(\beta - 1 + \sigma \alpha^2)^2 + 4 \sigma \alpha^2}}{-2 \sigma \alpha^2}.
$$

(9.3.16)

Let $\hat{\zeta} = A + CK = 1 + \alpha K$, where $w = K \mu_s$ is the formula for the worst case shock and $A + CK$ is the state transition matrix for the distorted law of

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13 This is the robust decision maker’s Euler equation for capital.
14 See page 134 for the definition of a pure forecasting problem.
15 We exploit certainty equivalence and ignore the stochastic parts of the Bellman equation and the law of motion for $\mu_s$. 

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motion in chapter 6. Applying formula (6.8.19) for $K$ in chapter 6 to the current problem gives

$$\hat{E}_{t+1} = \hat{\zeta}_{t+1}$$

(9.3.17)

where

$$\hat{\zeta} = \hat{\zeta} (\beta) = 1 + \frac{\sigma \alpha^2 P (\beta)}{1 - \sigma \alpha^2 P (\beta)} = \frac{1}{1 - \sigma \alpha^2 P (\beta)}.$$ (9.3.18)

Hansen, Sargent, and Wang (2000) solve (9.3.14), (9.3.16), and (9.3.18) to obtain

$$\hat{\beta} (\sigma) = \frac{1}{R} + \frac{\sigma \alpha^2}{R - 1}.$$ (9.3.19)

For $\sigma \in [\sigma, 0]$, equation (9.3.19) defines a locus of $(\sigma, \hat{\beta})$'s, each point of which is observationally equivalent to $(0, \beta)$ for observations on $(c_t, k_t)$ because each supports the benchmark $(\sigma = 0)$ allocation.

9.3.3. Precautionary savings interpretation

Along the locus of observationally equivalent $(\sigma, \beta)$ pairs from formula (9.3.19), lowering the discount factor makes investment less attractive and thereby offsets the precautionary savings motive. The following experiment highlights the precautionary motive for savings. Take the base model with $\sigma = 0$ used in our proof of Theorem 9.3.1. Then activate a preference for robustness by setting $\sigma < 0$ and offset its affect on consumption by finding the observationally equivalent $\hat{\beta} (\sigma)$. For that $(\sigma, \hat{\beta} (\sigma))$ pair, if future endowments and preference shifters could be forecast perfectly, then the consumer would choose to make his capital stock, and therefore also his consumption, drift downward because discounting is large relative to the marginal productivity of capital. Investment would be sufficiently unattractive that the optimal linear rule would eventually send both consumption and capital below zero.\(^{16,17}\) However, when randomness is activated (i.e., the innovation variances are positive), this downward drift is arrested or

\(^{16}\) Introducing nonnegativity constraints in capital and/or consumption would induce nonlinearities into the consumption and savings rules, especially near zero capital. But investment would remain unattractive in the presence of those constraints for experiments like the one we are describing here. See Deaton (1991) for a survey and quantitative assessment of consumption models with binding borrowing constraints.

\(^{17}\) As emphasized by Carroll (1992), even when the discount factor is small relative to the interest rate, precautionary savings can emerge when there is a severe utility cost for zero consumption. Such a utility cost is absent in our formulation.
even completely offset, as in our observation equivalence proposition. Thus our robust control interpretation of the permanent income decision rule delivers a form of precautionary savings that occurs even when utility is quadratic.

The usual model of precautionary savings emerges when a positive variance of the innovations to the endowment process interacts with a convex derivative of the marginal utility of consumption. In contrast, the precautionary savings induced by a preference for robustness emerges because the consumer wants to protect himself against mistakes in specifying conditional means of shocks to the endowment. Thus a concern for robustness inspires precautionary savings because of fear of misspecifications that are expressed in conditional first moments of shocks. This form of precautionary saving does not require that the marginal utility of consumption be convex, and occurs even in models with quadratic preferences.

A preference for robustness affects consumption by slanting probabilities in the way described by Fellner on page 25. The household saves more for a given $\beta$ because it makes pessimistic forecasts of future endowments. Precisely how pessimism manifests itself depends on the detailed structure of the permanent income model and the temporal properties of the endowment process, as we shall discuss in the next section.

9.4. Observational equivalence and distorted expectations

In this section, we use insights from the Stackelberg multiplier game on page 121 to interpret Theorem 9.3.1. In the Stackelberg multiplier game, decisions for the maximizing player can be computed by solving his Euler equations using a particular distorted law of motion to form conditional expectations of the shocks.\footnote{Take the Euler equation $E_t \beta Ru'(c_{t+1}) = u'(c_t)$ and assume that $\beta R = 1$ so that $E_t u'(c_{t+1}) = u'(c_t)$. If $u'$ is a convex function, then applying Jensen’s inequality implies $E_t c_{t+1} > c_t$, so that consumption is expected to grow when the conditional distribution of $c_{t+1}$ is not concentrated at a point. Such consumption growth reflects precautionary savings. See Ljungqvist and Sargent (2000, chapter 13) for a brief survey and analysis of such precautionary savings models.}

\footnote{While the timing protocol for the Stackelberg multiplier game differs from the Nash-Markov timing embedded in game (9.2.1), chapter 6 showed that identical outcomes and recursive representations of them would prevail under the two timing protocols.}
A permanent income model

In the benchmark $\sigma = 0, \beta R = 1$ case that is contemplated in Theorem 9.3.1, the solution of the planning problem is determined by equations (9.3.4), (9.2.3), and (9.3.7) where the $\Psi_j$’s satisfy (9.3.8) with $\beta R = 1$. For a $\sigma \in [\sigma, 0)$ and a $\beta = \hat{\beta}(\sigma)$, the decision rule for the robust planner is characterized by equations (9.3.4), (9.2.3), and the following modified version of (9.3.7):

$$
\mu_{st} = \hat{\Psi}_1 k_{t-1} + \hat{\Psi}_2 h_{t-1} + \hat{\Psi}_3 \sum_{j=0}^{\infty} R^{-j} \hat{E}_t b_{t+j} + \hat{\Psi}_4 \sum_{j=0}^{\infty} R^{-j} \hat{E}_t d_{t+j}, \quad (9.4.1)
$$

where $\hat{\Psi}_j$ are determined by (9.3.8) with $\beta = \hat{\beta}(\sigma)$; and $\hat{E}_t$ is the conditional expectation operator with respect to the distorted law of motion for the state $x_t$. The observational equivalence Theorem 9.3.1 implies that (9.4.1) and (9.3.7) are identical solutions for $\mu_{st}$. By eliminating the terms in expected future values, the solutions (9.3.7) and (9.4.1) can also be expressed as $\mu_{st} = M_s x_t$ and $\mu_{st} = \hat{M}_s x_t$. Observational equivalence requires that $M_s = \hat{M}_s$. This requires that the $\hat{\Psi}_j$’s and $\hat{E}$ mutually adjust to keep $M_s$ fixed.

To expand on this point, consider the special case that $\lambda = \delta_h = 0$, so that we need not retain $h_{t-1}$ as a state variable. Also, assume for simplicity that $b_t = b$, so that the preference shock is constant. Shutting down the volatility of $b$ removes distortions in it from the robust decision rule. Then equating the right sides of (9.3.7) and (9.4.1) gives

$$
0 = \left( \Psi_4 - \hat{\Psi}_4 \right) R k_{t-1} + \left( \Psi_3 - \hat{\Psi}_3 \right) (1 - R^{-1})^{-1} b \\
+ \Psi_4 \sum_{j=0}^{\infty} R^{-j} E_t d_{t+j} - \hat{\Psi}_4 \sum_{j=0}^{\infty} R^{-j} \hat{E}_t d_{t+j}, \quad (9.4.2)
$$

where $\Psi_j$ without hats denotes values of $\Psi_j$ that satisfy (9.3.8) and those with hats satisfy (9.3.8) evaluated at $\beta = \hat{\beta}(\sigma)$. Equation (9.4.2) shows how the observational equivalence result asserts offsetting alterations in the coefficients $\Psi_j$ and the distorted expectations operator $\hat{E}_t$ used to form the expected sum of discounted future endowments that defines human wealth.20

The distorted expectations operator is to be interpreted in terms of game 2 of chapter 6, the Stackelberg multiplier game (STACK) (see pages 121–125). The Euler equation approach used to derive (9.3.7) or (9.4.1) presumes a timing protocol corresponding to a Stackelberg multiplier game. Following chapter 6, after the minimizing agent has committed to an entire path for the $w_{t+1}$ process,

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20 XXXXX We confirmed this in the program hst4.m in the subdirectory hst.
the maximizing agent faces the following law of motion for the endowment and preference shocks:

\[ X_{t+1} = (A - BF(\sigma) + CK(\sigma))X_t + C\tilde{\epsilon}_{t+1} \]  
-(9.4.3a)

\[ \begin{bmatrix} b_t \\ d_t \end{bmatrix} = SX_t \]  
-(9.4.3b)

where \( \tilde{\epsilon}_{t+1} \) is an i.i.d. shock identical in distribution to that of \( \epsilon_{t+1} \).\(^{21}\) In the Euler equation formulation, given a belief about the maximizing agent’s decision rule \( F(\sigma) \), the minimizing agent commits to a stochastic process for the shock that leads to the recursive representation (9.4.3) of the endowment and preference shock processes. The maximizing agent takes the \( X_t \) process as exogenous and uses the forecasting rule \( \hat{E}_t X_{t+j} = (A - BF(\sigma) + CK(\sigma))X_t \) to form forecasts of \( (b_{t+j}, d_{t+j}) \) in (9.4.1). These forecasts, together with (9.4.1), (9.3.4) and (9.2.3) can be solved as in chapter 6 for a decision rule \( c_t = -F \begin{bmatrix} x_t \\ X_t \end{bmatrix} \). After computing the decision rule as a function of \( x_t, X_t \), we equate \( x_t = X_t \); that gives the maximizing agent’s decision rule in the form \( c_t = -Fx_t \).\(^{22}\)

9.4.1. Distorted endowment process

Fig. 9.4.1 and Fig. 9.4.2, illustrate the probability slanting that leads to precautionary savings. The figures assume HST’s parameter values, which we report in Appendix A.\(^{23}\) Fig. 9.4.1 and Fig. 9.4.2 record impulse response functions for the total endowment \( d_t \) under the approximating model and a worst-case model associated with \( \sigma = -0.0001 \), where \( \beta \) is adjusted according to (9.3.19) as required under our observational equivalence proposition to preserve the same decision rule \( F(\sigma) \) for different \( \sigma \)'s.\(^{24}\)

For the approximating and the worst case model for \( \sigma = -0.0001 \), the figures report the response of the total endowment \( d_t \) to innovations \( \epsilon_t^* \) and \( \hat{\epsilon}_t \) in the relatively permanent and transitory components of the endowment, \( d_t^*, \hat{d}_t \).

\(^{21}\) In (9.4.3), \( X_t \) corresponds to \( \hat{x}_t \) in the Stackelberg multiplier game on page 123; \( X_t \) is used to attain a recursive representation of the distorted endowment and preference shock process and to keep it exogenous to the maximizer’s decisions.

\(^{22}\) The procedure of first optimizing, then setting \( x_t = X_t \) to eliminate \( X_t \) is a common way of formulating rational expectations equilibria in macroeconomics, where it is sometimes called the ‘Big K, little k’ method.

\(^{23}\) XXX These figures are computed by hst4.m.

\(^{24}\) The observational equivalence proposition makes the decision rules equivalent under the approximating model.
respectively. Under the distorted model, the impulse response functions diverge and the eigenvalue of \( A - BF(\sigma) + CK(\sigma) \) that has maximum modulus increases from its value of unity under the approximating model to 1.0016.

The distorted endowment processes respond to innovations with more persistence than they do under the approximating model. With a fixed \( \beta \), the increased persistence makes the agent save more than under the approximating model, which the observational equivalence proposition offsets by increasing the household’s impatience via (9.3.19).

![Figure 9.4.1: Response of total endowment \( d_t \) to innovation in ‘permanent’ component \( d^*_t \) under the approximating model (dotted line) and the distorted model associated with the worst case shock (dashed line) for the \( \sigma = -.0001, \beta = \beta(\sigma) \) model.](image)

Fig. 9.5.1 and Fig. 9.5.2 record impulse response functions for the total endowment \( d_t \) under the approximating model and a worst case model associated with \( \sigma = -.0001, \text{ where } \beta \text{ is held fixed at HST’s benchmark value} \). Because these figures do not adjust the discount factor according to (9.3.19) as was done for Fig. 9.4.1 and Fig. 9.4.2, the distorted impulse response functions deviate from those of the approximating model even more than those of these earlier figures. The reduction in \( \beta \) from (9.3.19) works through two channels to make the \( \sigma < 0 \) decision rule equal to that for a \( \sigma = 0 \) rule: (1) it brings the distorted impulse response functions closer to those of the approximating model, and (2) it turns on a precautionary savings motive.
9.5. Another view of precautionary savings

As an aid to interpret the precautionary savings motive inherent in our model, appendix B asserts another observational equivalence proposition. Theorem 9.B.1 takes a baseline case where $\beta R = 1$ and shows that in its effects on $(c_i)$, activating a preference in robustness operates just like an increase in the discount factor. This result is useful because the $\beta R = 1$ case forms a benchmark in the permanent income literature. It was focused on, for example, by Hall (1978). The proposition shows that the effects of raising a concern for robustness by putting $\sigma < 0$ are replicated by simply raising $\beta$ so that $\beta R > 1$.

To use this result to shed more light on how the precautionary motive manifests itself in the decision rule for consumption, we consider the important special case that $\delta = \lambda = \tilde{\delta} = 0$. In this popular special case, $\mu_{x_t} = \mu_{c_t} = b - c_t$ and the consumption Euler equation (9.3.2d) without a preference for robustness becomes

$$b - c_t = E_t \left[ (\beta R) \left( b - c_{t+1} \right) \right].$$

If $\beta R > 1$, this equation implies that $b - c_t > E_t(b - c_{t+1})$ or

$$c_t < E_t c_{t+1}, \quad (9.5.1)$$
so that the optimal policy is to make consumption grow on average.

Theorem 9.B.1 shows that when $\beta R = 1$, a preference for robustness ($\sigma < 0$) has the same effect on $c_t, i_t$ as setting $\sigma = 0$ but $\beta R > 1$. Therefore, when $\beta R = 1$, the precautionary saving that occurs when $\sigma < 0$ is follows from (9.5.1). Activating a preference for robustness imparts an upward drift to the expected consumption profile.

We can also use Theorem 9.B.1 to say some things about the decision rule for consumption in our special case that $\lambda = \delta = \tilde{\delta} = 0$. The solution (9.3.8) for $\sigma = 0$ implies the consumption rule

$$c_t = (1 - R^{-2}\beta^{-1}) \left[Rk_{t-1} + E_t \sum_{j=0}^{\infty} R^{-j} d_{t+j}\right] + \left(\frac{(R\beta)^{-1} - 1}{R - 1}\right) b. \quad (9.5.2)$$

Notice that the marginal propensity to consume out financial wealth $Rk_{t-1}$ equals that out of human wealth $E_t \sum_{j=0}^{\infty} R^{-j} d_{t+j}$. Further, an increase in $\beta$ decreases the constant $+ \left(\frac{(R\beta)^{-1} - 1}{R - 1}\right) b$ and increases the marginal propensity to consume $1 - R^{-2}\beta$. Relative to the baseline $\beta R = 1$ case, raising $\beta$ raises the marginal propensity to consume out of wealth by $R^{-1}(1 - (R\beta)^{-1})$. This increase in the marginal propensity to consume still allows wealth to have an upward trajectory because of the reduction in the second term $\frac{(R\beta)^{-1} - 1}{R - 1}b$.

The following section views the precautionary savings motive from the frequency domain.

\[25\] This implication of precautionary savings coming from robustness differs from that coming from convex marginal utility functions, where precautionary savings reduces the marginal propensity to consume out of endowment income relative to that from financial wealth. See Wang (2002XXX).
A permanent income model

Figure 9.5.1: Response of total endowment \( d_t \) to innovation in 'permanent' component \( d^*_t \) under the approximating model (solid line) and the distorted model associated with the worst case shock (dotted line) for \( \sigma = -0.0001 \), with \( \beta \) at benchmark value.

9.6. Frequency domain representation

This section uses HST’s estimated permanent income model to illustrate features of the frequency domain decompositions of the consumer’s objective function and of the worst case shocks for different values of \( \sigma \).

Denote the transfer function from shocks \( \epsilon_t \) to the ‘target’ \( s_t - b_t \) as \( G(\zeta) \). For the baseline model with habit persistence, recall from formula (7.3.6) the frequency decomposition of \( H_2 \)

\[
H_2 = -\frac{1}{2\pi} \int_{-\pi}^{\pi} \text{trace} \left[ G\left(\sqrt{\beta} \exp(i\omega)\right) G'\left(\sqrt{\beta} \exp(i\omega)\right) \right] d\omega
\]

A reinterpretation of formula (7.3.5) also gives us the frequency domain representation

\[
E \sum_{t=0}^{\infty} \beta^t w'_t w_1 = \frac{1}{2\pi} \int_{-\pi}^{\pi} W\left(\sqrt{\beta} \exp(i\omega)\right) W'\left(\sqrt{\beta} \exp(i\omega)\right) d\omega.
\]

For the baseline (\( \sigma = 0 \)) line,\(^{26}\) Fig. 12.5.1 shows \( G(\sqrt{\beta} \exp(i\omega))'G(\sqrt{\beta} \exp(i\omega)) \) as a function of frequency \( \omega \); \( GG' \) is larger at lower frequencies. Remember

\(^{26}\) XXXX These figures were computed by hst3.m in the hst directory.
that $G(\zeta) = (I - (A_o - BF)\zeta)^{-1}C$ embodies the consumer’s optimal decision rule $F$. The noise process $\epsilon_t$ upon which $G(\zeta)$ operates is i.i.d. under the approximating model, so that the spectral density matrix of $\epsilon_t$ is constant across frequencies. But seeing that the consumer’s policy makes him most vulnerable to the low frequency components of $\epsilon_t$, the minimizing player makes the conditional mean of the worst-case shock $w_{t+1}$ highly serially correlated. For two values of $\sigma$, Fig. 12.5.2 shows frequency decompositions of trace $W(\zeta)'W(\zeta)$ for $\zeta = \sqrt{\beta} \exp(i\omega)$. Notice how most of the power is at the lowest frequencies. As we varied $\sigma$ from zero to the two values in Fig. 12.5.2, we adjusted $\beta$ according to (9.3.19), so that the robust decision rule for consumption equals that for the baseline model. Notice that $[\text{trace } W(\zeta)'W(\zeta)]$ varies directly with the absolute value of $\sigma$. 

**Figure 9.5.2:** Response of total endowment $d_t$ to innovation in ‘permanent’ component $d^*_t$ under the approximating model (solid line) and the distorted model associated with the worst case shock (dotted line) for $\sigma = -.0001$ with $\beta$ at benchmark value.
9.7. Detection error probabilities
For HST’s parameter values, Fig. 9.6.3 reports detection error probabilities associated with various values of $\sigma$, adjusting $\beta$ according to (9.3.19) to keep the decision rule fixed. These detection error probabilities were calculated by the method of chapter 8 for a sample of the same length that HST used to estimate their model and for HST’s initial condition. To calculate the detection error probabilities, all other parameter values were frozen at the values in Table 9.A.1. Then the formula for the worst-case distortions $w_{t+1} = K(\sigma)x_t$ was used to compute an alternative law of motion for the endowment process.

For different values of $\sigma$, Fig. 9.6.3 records the detection error probabilities for distinguishing an approximating model from a worst-case model associated with that value of $\sigma$. The approximating model is

$$x_{t+1} = (A - BF(0))x_t + C\epsilon_{t+1}$$

while the distorted model associated with $\sigma$ is

$$x_{t+1} = (A - BF(0) + CK(\sigma))x_t + C\tilde{\epsilon}_{t+1}$$

where both $\epsilon_t$ and $\tilde{\epsilon}_t$ are i.i.d. processes with mean zero and identity covariance matrix, and where $F(0) = F(\sigma)$ by the observational equivalence proposition.

The detection error probability equals .5 for $\sigma = 0$ because then the models are identical and so cannot be distinguished. The detection error probability falls with $\sigma$ because the models spread out. In the following section, we use
9.8. Robustness of decision rules

For \( \sigma = -\theta^{-1} \), express the equilibrium decision rules of game (9.2.1) as

\[
\begin{align*}
    c_t &= -F(\sigma) x_t \quad (9.8.1a) \\
    w_{t+1} &= K(\sigma) x_t \quad (9.8.1b)
\end{align*}
\]

and express \( s_t - b \) as \( H(\sigma) x_t \). For possibly different values \( \sigma_1, \sigma_2 \), consider the law of motion of the state under the consumption plan \( F(\sigma_2) x_t \) and the worst case shock process \( K(\sigma_1) x_t \):

\[
x_{t+1} = (A - BF(\sigma_2) + CK(\sigma_1)) x_t + C \epsilon_{t+1}. \tag{9.8.2}
\]

For \( x_0 \) given, we evaluate the expected payoff

\[
\pi(\sigma_1; \sigma_2) = -E_{0,\sigma_1} \sum_{t=0}^{\infty} \beta^t x'_t H(\sigma_2)' H(\sigma_2) x_t \tag{9.8.3}
\]

under the law of motion (9.8.2). That is, we want to evaluate the performance of the rule designed by setting \( \sigma_2 \) when the data are generated by the distorted model associated with \( \sigma_1 \). For three values of \( \sigma_2 \), Fig. 9.6.3 plots \( \pi(\sigma_1; \sigma_2) \) as a function of the parameter \( \sigma_1 \) that indexes the magnitude of the distortion in the model generating the data. By construction, the \( \sigma_2 = 0 \) does better than the other rules when \( \sigma_1 = 0 \). But its performance deteriorates faster with decreases in \( \sigma_1 \) below zero than do the more robust \( \sigma_1 = -0.00004, \sigma_1 = -0.00008 \) rules.

From Fig. 9.6.3, \( \sigma = -0.00004 \) is associated with a detection error probability of over .3, and \( \sigma = -0.00008 \) with a detection error probability about .2. It is plausible for the consumer to want decisions that are robust against alternative models as close as the worst case models associated with those values of \( \sigma \).
Figure 9.8.1: $\pi(\sigma_1; \sigma_2) = -E_{0,\sigma_1} \sum_{t=0}^{\infty} \beta^t x'_t H(\sigma_2)'H(\sigma_2)x_t$ as a function of $\sigma_1$ on the ordinate axis for decision rules $F(\sigma_2)$ associated with three values of $\sigma_2$.

9.9. Concluding remarks

Different observationally equivalent $(\sigma, \beta)$ pairs identified by Theorem 9.3.1 bear different implications about (i) the pricing risky assets; (ii) the amounts required to compensate the planner for confronting different amounts of risk; (iii) the amount of model misspecification used to justify the planner’s decisions if risk sensitivity is reinterpreted as aversion to Knightian uncertainty. Hansen, Sargent, and Tallarini (1999) and Hansen, Sargent, and Wang (2000) have extracted some asset pricing implications of the model in this chapter. They show that although movements along the observational equivalence locus laid out by (9.3.19) don’t affect consumption and investment, they put an adjustment for fear of model misspecification into asset prices and boost measured market prices of risk. In chapter 12, we shall describe how standard asset pricing formulas are altered when a representative agent has a preference for robustness. There we shall describe an asset pricing theory under a preference for robustness in the context of a class of general equilibrium models of which the model of this chapter can be viewed as a special case.
Table 9.A.1: HST’s parameter estimates

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A. Parameter values

HST calibrated a σ = 0 version of their permanent income model by maximizing a likelihood function conditioned only on U.S. post-war quarterly consumption and investment data. They used U.S. quarterly data on consumption and investment for the period 1970I–1996III. They measured consumption by nondurables plus services and investment by the sum of durable consumption and gross private investment. They estimated the model from data on (c_t, i_t), setting σ = 0, then deduced pairs (σ, β) that are observationally equivalent, using formula (9.3.19).

The forcing processes are governed by seven free parameters: (α1, α2, ĉd, φ1, φ2, cd∗, µd). The parameter µd set a bliss point. While µb alters the marginal utilities, it does not influence the decision rules for consumption and investment. HST fixed µb at an arbitrary number, namely 32, for estimation.

Four parameters govern the endogenous dynamics: (γ, δh, β, λ). HST set δh = .975, and imposed the permanent income restriction, βR = 1. The restrictions that βR = 1, δh = .975 pin down γ once β is estimated. HST imposed β = .9971, which after adjustment for the effects of the geometric growth factor of 1.0033 implies an annual real interest rate of 2.5%.

Table 9.A.1 reports HST’s estimates for the parameters governing the endogenous and exogenous dynamics. Fig. 9.A.1 and Fig. 9.A.2 report impulse response functions for consumption and investment to innovations in both components of the endowment process. For comparison, Table 9.A.1 reports estimates from a no habit persistence (λ = 0) model.

Notice that the persistent endowment shock process contributes much more to consumption and investment fluctuations than does the transitory endowment shock process.

---

27 They estimated the model from data that had been scaled through multiplication by 1.0033^{-t}.
B. Another observational equivalence result

To shed more light on the form of precautionary savings, we state another observational equivalence result that takes as its benchmark an initial allocation associated with parameter settings.
\( \beta R = 1 \) and \( \sigma < 0 \). Then we find another value of \( \beta \) that implies the same decisions for \( c_t, i_t \) as the base model when \( \sigma = 0 \), so that the decision maker fears model misspecification. This entails working backwards from the worst case model that is reflected in the \( \sigma < 0 \) decision rule to the associated approximating model.

**Theorem 9.B.1.** (Observational Equivalence, II) Fix all parameters except \( (\beta, \sigma) \). Consider a consumption-investment allocation for \( (\tilde{\beta}, \tilde{\sigma}) \) where \( \tilde{\beta} \) satisfies \( \tilde{\beta} R = 1 \) and \( \tilde{\sigma} < 0 \) and \( \tilde{\sigma} < \hat{\sigma} \). Then there exists a \( \tilde{\beta} > \hat{\beta} \) such that the \( (\hat{\beta}, \hat{\sigma}) \) allocation also solves the \( (\tilde{\beta}, 0) \) problem.

**Proof.** We suppose that \( \hat{\sigma} < 0 \), so that the worst case model differs from the approximating model. We want to find the approximating model and a value \( \tilde{\beta} \) of \( \beta \) for which a \( \sigma = 0 \) decision maker would choose the \( (\hat{\beta}, \hat{\sigma}) \) allocation. Under the model with \( \hat{\sigma} < 0 \), where \( \hat{\mathbb{E}}_t \) denotes a conditional expectation under the worst case model, we have

\[
\hat{\mathbb{E}}_t \left[ c_{t+1} \right] = \mu_{c,t+1} \tag{9.B.1}
\]

because \( \hat{\beta} R = 1 \). Let

\[
\hat{\mathbb{E}}_t \mu_{s,t+1} = \xi (\tilde{\beta}) \mu_{s,t} \tag{9.B.2}
\]

Equation (9.B.1) implies that we want

\[
1 = \xi (\tilde{\beta}) \tag{9.B.3}
\]

where the projection coefficient \( \xi(\tilde{\beta}) \) emerges from the multiplier problem for the evil agent for \( \tilde{\sigma} < 0 \), which can be cast as

\[
\min_{\{w_{t+1}\}} \left[ -\sum_{t=0}^{\infty} \beta^t \left\{ \mu_{s,t}^2 + \frac{1}{\sigma} w_{t+1}^2 \right\} \right]
\]

subject to the law of motion

\[
\mu_{s,t} = \delta(\tilde{\beta}) \mu_{s,t-1} + \alpha \epsilon_t \tag{9.B.4}
\]

where \( \delta(\tilde{\beta}) = \frac{1}{\beta R} \) and \( \alpha \) is given by (9.3.11), (9.3.9), (9.3.10) under the \( (\tilde{\beta}, \tilde{\sigma}) \) model. (Remember that the decision rule for \( c_t \) and therefore the law for \( \mu_{s,t} \) will be the same under our two observationally equivalent \( \beta, \sigma \) pairs, so we can use the benchmark case to compute \( \alpha \).) We freeze all parameters except \( \beta, \sigma \). The approximating model would be \( \mu_{s,t} = \delta \mu_{s,t-1} + \alpha \epsilon_t \), so that (9.B.4) adds a perturbation \( \alpha \epsilon_t \) to the law of motion of \( \mu_{s,t} \) under a deterministic version of the approximating model. The Bellman equation for the minimizing agent is evidently

\[
-P \mu_{s,t}^2 = -\mu_{s,t}^2 + \beta \min_w \left[ \frac{1}{\sigma} w^2 - P (\delta \mu_s + \alpha \epsilon)^2 \right]. \tag{9.B.5}
\]

Notice the presence of both \( \hat{\beta} \) and \( \tilde{\beta} \), via \( \delta \) and \( \alpha \). The first-order condition is

\[
w = K \mu_s,
\]
where

\[ K = -\frac{\alpha \delta \hat{\sigma} P}{1 + \alpha^2 \sigma P}. \]

Notice that

\[ \xi (\hat{\beta}) = A + KC = \delta + K\alpha = 1 \]

which implies that

\[ 1 = \xi (\hat{\beta}) = \delta + K\alpha = \frac{\delta}{1 + \alpha^2 \sigma P}. \]

Therefore,

\[ \delta = 1 + \sigma^2 P < 1. \quad (9.B.6) \]

Equation (9.B.5) implies that

\[ -P = -1 + \hat{\beta} \left[-\frac{1}{\sigma} K^2 - P (\delta + K\alpha)^2 \right]. \]

Simplifying the above identity leaves

\[ P = \frac{1}{1-\hat{\beta}} \left[1 + \frac{\hat{\beta}}{\sigma} \left(\frac{1-\delta}{\alpha}\right)^2 \right]. \quad (9.B.7) \]

Equations (9.B.6) and (9.B.7) together imply that

\[ 0 = \hat{\beta} \left(1 - \delta \left(\frac{\hat{\beta}}{\sigma}\right)^2 \right) + (1 - \hat{\beta}) \left(1 - \delta \left(\frac{\hat{\beta}}{\sigma}\right)^2 \right) + \alpha \left(\frac{\hat{\beta}}{\sigma}\right)^2. \]

A solution of this equation determines \( \hat{\beta} \). The solution of this quadratic equation is

\[ \delta = 1 - \frac{1 - \hat{\beta}}{2\hat{\beta}} \pm \frac{\sqrt{(1 - \hat{\beta})^2 - 4\hat{\beta} \sigma^2}}{2\hat{\beta}}. \]

If \( \sigma = 0 \), this equation implies \( \delta = 1 \). When \( \sigma < 0 \), the appropriate root is

\[ \delta = 1 - \frac{1 - \hat{\beta}}{2\hat{\beta}} + \frac{\sqrt{(1 - \hat{\beta})^2 - 4\hat{\beta} \sigma^2}}{2\hat{\beta}}. \]

Using \( \hat{\beta} R = 1 \), this is equivalent to

\[ \hat{\beta} (\sigma) = \frac{\hat{\beta} \left(1 + \hat{\beta}\right)}{2(1 + \sigma \alpha^2)} \left[1 + \sqrt{1 - 4\hat{\beta} \frac{1 + \sigma \alpha^2}{(1 + \hat{\beta})}}\right]. \quad (9.B.8) \]
Chapter 10.
Competitive equilibrium models

10.1. Strategies for pricing risky claims

In an economy with complete markets, state-date prices equal intertemporal marginal rates of substitution times conditional probabilities evaluated at an equilibrium allocation. Complete markets assure that intertemporal rates of substitution are equated across all consumers, making it possible to speak unambiguously of the intertemporal rate of substitution and also allowing us to synthesize a representative agent. In a pure endowment economy that specifies the preferences of a single representative consumer, like one studied by Robert E. Lucas, Jr. (1978), it is trivial to compute the intertemporal marginal rates of substitution and therefore equilibrium state-date prices. These prices can then be used to price risky claims that consist of bundles of the basic state-date commodities.

This pricing strategy can be extended beyond endowment economies to representative household economies that have endogenous state variables including productive capital stocks and household capital stocks. Household capital stocks can be used to represent non-separabilities over time in the household’s preferences. Brock (1982XXX) showed that Lucas’s method of pricing risky claims could be extended to handle settings with such endogenous state variables by proceeding as follows:

1. Solve a planning problem to compute optimal allocations.
2. Compute the shadow prices of state-date contingent consumption goods as intertemporal marginal rates of substitution evaluated at the optimal allocation times conditional probabilities. Take the shadow prices to be the state-date prices.
3. Represent a security as a stochastic process of pay outs, that is, as a sequence of measurable functions of the economy’s history of shocks.
4. Price a security by multiplying the state-date payouts by the state-date prices computed in step (2), then sum over time and across states.

This and the following two chapters implement this pricing strategy in a class of linear-quadratic economies. To begin, this chapter studies economies without
a preference for robustness. Chapters 11 and 12 then add a preference for robustness and study how asset prices are thereby affected.

Chapter 12 relies heavily on the four step strategy (1)-(4) for pricing assets. Here and in chapter 11, we lay out alternative decentralizations of our planning problem. As we shall see, our four step strategy is powerful partly because it allows us to price assets using an economical state vector \( x_t \), namely the state vector of the representative household. However, to express the idea that the household is a price taker in a competitive equilibrium, we have to augment the state \( x_t \) with additional components \( X_t \) that are comparable in dimension to \( x_t \); \( X_t \) becomes a part of the state vector in the household’s problem that the household takes as exogenous and in terms of which prices of Arrow securities are expressed.\(^1\) In a competitive equilibrium, we impose \( X_t = x_t \), but only after the household has optimized while taking \( X_t \) as beyond his control. Setting \( X_t = x_t \) after optimization makes the representative household representative. After this is done, we can cast asset pricing formulas solely in terms of the state \( x_t \) in the planning problem.

### 10.2. Types of competitive equilibria

We now describe three types of competitive equilibria in a class of economic environments that are congenial to the optimal linear regulator.\(^2\) The three types of competitive equilibrium share common specifications of information, preferences, and technologies, but they have different market structures. They are (1) an “Arrow-Debreu equilibrium” with trades at time 0 in a complete set of state-contingent dated commodities; (2) an “equilibrium with Arrow securities” that has a sequence of complete markets in current period commodities and one-period ahead state-contingent claims; and (3) a “partial equilibrium” model in which supply is determined by a competitive representative firm that acts as a price taker, and in which prices lie along a system of demand equations perturbed by shocks. The allocations (quantities) in all three types of competitive equilibria all solve a common planning problem. The three types of competitive equilibria are thus different ways of supporting the same Pareto optimal allocation. The particular price systems and trading opportunities that support the optimal allocation differ across the three types of equilibria. For applications, it is useful to know how to transform one type of equilibrium into another.

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1. In related contexts, this idea was used by Lucas and Prescott (1971) and Prescott and Mehra (1982XXX).
2. Hansen (1987) and Hansen and Sargent (200X) have studied such economies.
10.3. Information, Preferences, and Technology

10.3.1. Information

An exogenous information vector \( z_t \) is governed by

\[
z_{t+1} = A_{zz} z_t + C_z \epsilon_{t+1}, \tag{10.3.1} \]

where \( \{ \epsilon_t \} \) is an i.i.d. Gaussian vector with mean 0 and covariance matrix \( I \), and the eigenvalues of \( \tilde{A}_{zz} \equiv \sqrt{\beta} A_{zz} \) are bounded by unity in modulus. The vector \( z_t \) determines a time \( t \) preference shock \( b_t \) and a time \( t \) endowment shock \( d_t \) via

\[
d_t = U_d z_t \quad \text{and} \quad b_t = U_b z_t. \tag{10.3.2} \]

10.3.2. Preferences

A representative household has preferences ordered by

\[
- \frac{1}{2} E \left( \sum_{t=0}^{\infty} \beta^t \left( |s_t - b_t|^2 + \ell_t^2 \right) \ igg| J_0 \right), \tag{10.3.3} \]

where \( J_0 \) is defined below, \( \ell_t \) is a scalar process\(^3\) that constrains a vector \( g_t \) of intermediate activities (designed to capture generalized adjustment costs), and \( s_t \) is a vector of household services produced at time \( t \) via the household technology

\[
s_t = \Lambda h_{t-1} + \Pi c_t \quad \text{and} \quad h_t = \Delta h_{t-1} + \Theta h_t. \tag{10.3.4} \]

In (10.3.4), \( h_t \) is a vector of stocks of household durable goods at \( t \), \( c_t \) is a vector of consumption flows, and \( \Lambda, \Pi, \Delta_h, \Theta_h \) are matrices.

\(^3\) Sometimes we interpret \( \ell_t \) as labor input.
10.3.3. Technology

There is a constant returns to scale production technology

\[ \Phi c_t + \Phi i_t + \Phi g g_t = \Gamma k_{t-1} + d_t \]
\[ k_t = \Delta k_{t-1} + \Theta k i_t, \]  

(10.3.5)  

where \( k_t \) is a vector of capital goods used in production, \( i_t \) is a vector of investment goods, \( \Delta k \) is a matrix, and \( g_t \) is constrained by

\[ g_t \cdot g_t \leq \ell_t^2. \]

This fact allows us to compute an equilibrium allocation from the planning problem before computing the prices that support it.

10.3.4. Planning problem

The planning problem is to maximize (10.3.3) over choices of adapted processes for \( \{s_t, c_t, i_t, g_t, h_t, k_t, h_{t-1}\}_{t=0}^{\infty} \) subject to (10.3.1), (10.3.2), (10.3.4), and (10.3.5) with given initial conditions for \((z_0, h_{-1}, k_{-1})\). The planning problem takes the form of an optimal linear regulator. Let

\[ x_t = \begin{bmatrix} h_{t-1} \\ k_{t-1} \\ z_t \end{bmatrix}. \]

We view the first two components of the state vector to be endogenous and the third component to be exogenous. When the matrix \( \Phi \equiv [\Phi_c \ \Phi_g] \) is nonsingular, the control vector \( u_t \) can be chosen to be investment \( i_t \) because in this case

\[ \begin{bmatrix} c_t \\ g_t \end{bmatrix} = \Phi^{-1} (\Gamma k_{t-1} + U d z_t - \Phi i_t). \]  

(10.3.6)  

Using this relation, the constraints (10.3.4) and (10.3.5) can be rewritten

\[ x_{t+1} = Ax_t + Bu_t + C \epsilon_{t+1} \]  

(10.3.7)  

for appropriately chosen matrices \( A, B, C \). The matrix \( A \) is block triangular and the bottom row block of \( B \) is zero as required for the discounted stochastic

\[ ^4 \text{Under the constant returns to scale interpretation, } d_t \text{ is taken as an additional input available in fixed supply.} \]

\[ ^5 \text{The matrix } \Phi \text{ can be rendered nonsingular by augmenting the control vector to include some of the components of consumption or the labor-using intermediate activities.} \]
linear regulator problem. Moreover, using (10.3.6) and (10.3.4), the time \( t \) terms \(|s_t - b_t|^2\) and \(|g_t|^2\) in the objective function (10.3.3) of the planner can both be expressed as quadratic forms in the control \( i_t \) and the augmented state \( x_t \).

The planner’s optimal decision rule is \( u_t = -Fx_t \). Under this rule, the state evolves according to

\[
x_{t+1} = A^o x_t + Cc_{t+1},
\]

where \( A^o = A - BF \).

10.3.5. Imposing stability

In permanent income economies, optimality does not necessarily imply stability of the state vector process. For example, the economy of chapter 9 has a single consumption good, a single capital good, and no labor-using intermediate activities. The counterpart to equation (10.3.6) is

\[
c_t = \Gamma k_{t-1} + U_d z_t - i_t.
\]

The chapter 9 model constrains the subjective discount factor to be the reciprocal of the physical return to capital: \( \beta = \frac{1}{\Gamma + \Delta} \). Without imposing stability as an additional constraint, the optimal solution does not stabilize the capital stock sequence because the sequence of capital stocks diverges to minus infinity at a rate that is not dominated by \( \frac{1}{\sqrt{\beta}} \). We want to impose stability because solutions that require \( x_t \) not to explode at a rate exceeding \( \frac{1}{\sqrt{\beta}} \) are much better approximations to models that impose debt limits or various non-negativity constraints. Therefore, we impose stability as an additional constraint, with the consequence that the solution to the resulting infinite-horizon control problem is equal to the limit of the solutions to a sequence of corresponding finite-horizon problems, each of which has a zero restriction imposed on the terminal capital stock.
10.4. Arrow-Debreu equilibrium

10.4.1. The price system at time 0

An Arrow-Debreu equilibrium has complete markets at time 0 in claims to state-contingent dated commodities. We follow Harrison and Kreps (1979) and Hansen and Sargent (XXXX) in scaling the Arrow-Debreu prices in such a way that values can be represented as inner products of two stochastic processes, prices and quantities. The appropriately scaled prices are the ordinary Arrow-Debreu state-date prices divided by probabilities times discount factors. Using these scaled Arrow-Debreu prices makes present values (i.e., asset prices) become expected present values of geometric sums of quadratic forms (prices times quantities). These present values are easy to compute by solving Sylvester equations (see chapter 3).

Thus, we use a price system with components \[
\{p^0_{0t}, p^0_{ct}, p^0_{dtt}, p^0_{rt}\}_{t=0}^{\infty},\]
each element of which resides in a space \(L^2_0\) defined by

\[
L^2_0 = \{y_t\}_{t=0}^{\infty} : y_t \text{ is a random variable in } J_t \text{ for } t \geq 0, \text{ and } E \left[ \sum_{t=0}^{\infty} \beta^t y^2_t \mid J_0 \right] < +\infty .
\]

That ‘\(y_t\) is in \(J_t\)’ means that \(y_t\) can be expressed as a measurable function of \(J_t = [\epsilon^t, x_0]\), where \(J_0 = [x_0]\). The square summability requirement, \(E[\sum_{t=0}^{\infty} \beta^t y^2_t \mid J_0] < \infty\), imposes that \(y_t\) not grow too fast in absolute value.

Our price system \([p_{k0}, \{p^0_{ct}, p^0_{dtt}, p^0_{rt}\}_{t=0}^{\infty}]\) contains the following prices: \(p_{k0}\) is a vector that prices the initial capital stock \(k^{-1}\); \(p^0_{ct}\) is an \(n_c \times 1\) stochastic process that prices the consumption process \(c_t\); \(p^0_{dtt}\) is a scalar stochastic process that prices \(\ell_t\); \(p^0_{rt}\) is a vector stochastic process that prices the process \(\{d_t\}\); \(p^0_{dtt}\) is an \(n_k \times 1\) vector stochastic process that prices new investment goods; and \(p^0_{rt}\) is an \(n_k \times 1\) vector stochastic process of capital rental rates. A time \(t\) component of the price system is a random vector that is a function of

\[
J_t = [\epsilon^t, x_0] .
\]

The price system is a sequence of vector-valued measurable functions of the time \(t\) histories \(J_t\).

Both prices and quantities are stochastic processes. We require the stochastic processes for both prices and quantities to reside in \(L^2_0\). By virtue of a Cauchy-Schwartz inequality, this makes the conditional inner products to be used in the budget constraints and objective functions below well defined and
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finite in equilibrium. Later it will be convenient to obtain recursive representations for both prices and quantities.

We now describe the choice problems faced by households and two types of firms within a competitive equilibrium in which all trades occur at time 0. The household and firms act as price takers. The allocations chosen by the household and the firms must be “realizable” in the sense that time \( t \) decisions depend only on information available at time \( t \), i.e., they must reside in \( L^2_0 \).

10.4.2. Households

We let \( E \) denote the mathematical expectation evaluated with respect to the joint probability distribution of \([\epsilon, x_0]\). We also let \( E_t \) denote \( E(\cdot | J_t) \). The household chooses stochastic processes for \( \{c_t, s_t, h_t, \ell_t\}_{t=0}^{\infty} \), each element of which is in \( L^2_0 \), to maximize

\[
-\frac{1}{2} E_0 \sum_{t=0}^{\infty} \beta^t \left[ (s_t - b_t) \cdot (s_t - b_t) + \ell_t^2 \right]
\]  

(10.4.1) \( ^{\text{of1}} \) *]

subject to

\[
E \sum_{t=0}^{\infty} \beta^t p^0_{ct} \cdot c_t \mid J_0 = E \sum_{t=0}^{\infty} \beta^t \left( p^0_{ct} \ell_t + p^0_{dt} \cdot d_t \right) \mid J_0 + p_{k0} \cdot k_{-1} \]  

(10.4.2) \( ^{\text{of2}} \) *

\[
s_t = \Lambda h_{t-1} + \Pi c_t \]  

(10.4.3) \( ^{\text{of3}} \) *

\[
h_t = \Delta h_{t-1} + \Theta h c_t, \quad h_{-1}, k_{-1} \text{ given.} \]  

(10.4.4) \( ^{\text{of4}} \) *

10.4.3. Firms of type I

A firm of type I rents capital and labor, and buys the realization of the endowment process \( d_t \). It uses these inputs to produce consumption goods and investment goods, which it sells. The firm of type I chooses stochastic processes for \( \{c_t, i_t, k_t, \ell_t, g_t, d_t\} \), each element of which is in \( L^2_0 \), to maximize

\[
E_0 \sum_{t=0}^{\infty} \beta^t \left( p^0_{ct} \cdot c_t + p^0_{it} \cdot i_t - p^0_{kt} \cdot k_{t-1} - p^0_{\ell t} \cdot \ell_t - p^0_{dt} \cdot d_t \right) \]  

(10.4.5) \( ^{\text{of5}} \) *

subject to

\[
\Phi c_t + \Phi g_t + \Phi i_t = \Gamma k_{t-1} + d_t \]  

(10.4.6) \( ^{\text{of6}} \) *

\[
-\ell_t^2 + g_t \cdot g_t = 0. \]  

(10.4.7) \( ^{\text{of7}} \) *
10.4.4. Firms of type II

A firm of type II is in the business of purchasing investment goods and renting capital to firms of type I. As a price taker, a firm of type II faces the vector $v_0$ and the stochastic processes $\{p^0_{rt}, p^0_{it}\}$. The firm chooses $k_{-1}$ and stochastic processes for $\{k_t, i_t\}_{t=0}^{\infty}$ to maximize

$$E \left[ \sum_{t=0}^{\infty} \beta^t (p^0_{rt} \cdot k_{t-1} - p^0_{it} \cdot i_t) \mid J_0 \right] - p_{k0} \cdot k_{-1}$$

subject to

$$k_t = \Delta_k k_{t-1} + \Theta k_{i_t}.$$ 

10.4.5. Competitive equilibrium

We adopt a typical definition of a competitive equilibrium with all trades being made at time 0.

**Definition 10.4.1.** A competitive equilibrium is a price system $[p_{k0}, \{p^0_{ct}, p^0_{lt}, p^0_{dt}\}, p^0_{rt}, p^0_{it}]_{t=0}^{\infty}$ and an allocation $\{c_t, i_t, k_t, h_t, d_t, g_t\}_{t=0}^{\infty}$ that satisfy the following conditions:

a. Each component of the price system and the allocation resides in the space $L^2_0$.

b. Given the price system and given $h_{-1}, k_{-1}$, the stochastic process $\{c_t, s_t, \ell_t, h_t\}_{t=0}^{\infty}$ solves the consumer’s problem.

c. Given the price system, the stochastic process $\{c_t, i_t, k_t, \ell_t, d_t, g_t\}$ solves the problem of the firm of type I.

d. Given the price system, the vector $k_{-1}$ and the stochastic process $\{k_t, i_t\}_{t=0}^{\infty}$ solve the problem of the firm of type II.
10.4.6. Equilibrium computation

Our strategy for computing an equilibrium is first to solve the planning problem for equilibrium quantities, then compute shadow prices that can be transformed into equilibrium prices.

The optimal linear regulator can be used to solve a planning problem. The optimal law of motion for the state $x_t$ and the value function for the planning problem contain the information needed to compute the competitive equilibrium prices.\(^6\)

Let $V(x) = -x'Px - p$ be the optimal value of the planning problem starting from initial state $x$. The Bellman equation for the planning problem is

$$-x'Px - p = \max_{c,t,g} \{-0.5[(s-b) \cdot (s-b) + g \cdot g] + \beta E (-x''Px* - p)\} \quad (10.4.10)$$

subject to the linear constraints

$$\Phi_{c,c} + \Phi_{g,g} + \Phi_{i,i} = \Gamma_k + d$$
$$k^* = \Delta_{k}k + \Theta_{k,i}$$
$$h^* = \Delta_{h}h + \Theta_{h,c}$$
$$s = \Lambda h + \Pi c$$
$$z^* = A_{22}z + C_{2}\epsilon$$
$$b = U_b z$$
$$d = U_d z,$$

where $^*$ denotes a next period value. The time-invariant character of the planning problem makes the optimal policy functions or decision rules time invariant. The time $t$ state vector is $x_t' = (h_{t-1}', k_{t-1}', z_t')$. The time $t$ decision rules are linear in the state vector $x_t$. We denote these rules $c_t = S_c x_t, g_t = S_g x_t, h_t = S_h x_t, i_t = S_i x_t, k_t = S_k x_t, s_t = S_s x_t$.

Similarly, the law of motion for the state vector is linear:

$$x_{t+1} = A^o x_t + C \epsilon_{t+1} \quad (10.4.12)$$

where

$$A^o \equiv \begin{bmatrix} A_{11}^o & A_{12}^o \\ 0 & A_{22}^o \end{bmatrix}, C \equiv \begin{bmatrix} 0 \\ C_2 \end{bmatrix}. \quad (10.4.13)$$

The partitioning of the $A^o$ and $C$ matrices is according to the endogenous state vector $(h_{t-1}', k_{t-1}')'$ and the exogenous state vector $z_t$. The zero restriction on the

\(^6\) See Hansen and Sargent (2XXXX) for a complete account.
(2.1) partition of \( A^o \) reflects the fact that the exogenous state vector at time \( t+1 \) does not depend on the endogenous state vector at time \( t \). The zero restriction on the first rows in the partition of \( C \) reflects the fact that the endogenous state vector at time \( t+1 \) is predetermined (i.e., depends only on time \( t \) information).

The contingency plans for \( h_t \) and \( k_t \) are embedded in the part of (10.4.13) that determines the endogenous state vector \( [h'_t \; k'_t]' \) as a function of \( x_t \). In particular,

\[
\begin{bmatrix}
S_h \\
S_k
\end{bmatrix}
= \begin{bmatrix}
A_{11}^o & A_{12}^o
\end{bmatrix}.
\tag{10.4.14}
\]

Notice that the planner’s decision rules are recursive in the sense that the time \( t \) decision depends on the state vector at time \( t \), which in turn depends on the state vector at time \( t-1 \).

The eigenvalues of \( A^o \) determine the growth rates of the state vector \( \{x_t\} \). Since \( A^o \) is block triangular, the set of eigenvalues of \( A^o \) is the union of the set of eigenvalues of \( A_{11}^o \) and the set of eigenvalues of \( A_{22}^o \). We refer to the first set of eigenvalues as the \textit{endogenous eigenvalues} because \( A_{11}^o \) is determined by the solution to the planning problem. These eigenvalues must have absolute values strictly less than \( 1/\sqrt{\beta} \) to satisfy the requirement that the elements of \( \{x_t\} \) be in \( L_0^2 \). We refer to the second set of eigenvalues as the \textit{exogenous eigenvalues} because the matrix \( A_{22}^o \) is specified exogenously. By assumption, the eigenvalues of \( A_{22}^o \) have absolute values that are less than or equal to \( 1/\sqrt{\beta} \).

10.4.7. Shadow Prices

Equilibrium prices can be computed by first computing shadow prices, then appropriately reinterpreting them as prices. Formulas for shadow prices corresponding to the elements of the price system \([p^0, \{p^0_{ct}, p^0_{dt}, p^0_{ct}, p^0_{rt}\}_{t=0}^\infty]\) can be extracted from \( A^o \) and the matrix \( P \) in the quadratic form in the value function. Evaluating these shadow prices at the equilibrium allocation recovers the prices themselves.

The time \( t \) component of these shadow prices are linear functions of \( x_t \). In particular, the vector of shadow prices is given by\(^7\)

\[ p^S_{ct} = M_c x_t \]

where

\[ M_c = \Theta^h_h M_h + \Pi^t M_s. \tag{10.4.15} \]

\(^7\) Quantities \( M_j x_t \) emerge from derivatives of the planner’s value function and are measured in units of marginal utility.
Here $M_h x_t$ is the shadow price of consumer durables and $M_s x_t$ is the shadow price of household services. These shadow prices satisfy

$$M_h x_t = E \left[ \sum_{t=1}^{\infty} \beta^t (\Delta h_t)^{t-1} \Lambda' M_s x_{t+\tau} | J_t \right]$$  \hspace{1cm} (10.4.16)  \hspace{1cm} \text{["shadowh "]}

$$M_s x_t = (s_t - b_t),$$  \hspace{1cm} (10.4.17)  \hspace{1cm} \text{["shadows "]}

where the mathematical expectation is evaluated with respect to the model (10.4.12). Hansen and Sargent (XXXX) show that these other components of the shadow price system have representations

\begin{align*}
p^S_{ct} &= M_c x_t \\
p^S_{it} &= M_i x_t \\
p^S_{rt} &= \Gamma' M_d x_t \\
p^S_{dt} &= M_d x_t \\
p^S_{kt} &= (\Gamma'M_d + \Delta' M_k)x_t,
\end{align*}

where $p^S_{kt}$ is the shadow price of $k_{t-1}$. Hansen and Sargent (XXXX) give formulas for the various $M_j$ matrices.

10.4.8. Recursive representation of time 0 prices

The price system consists of sequences of measurable functions of the histories $J_t = [e^t, x_0]$. Fortunately, it is easy to obtain a recursive representation of these functions by introducing an additional state vector that is designed to keep track of the histories $J_t$. In particular, define the new state vector $X_t$ with components $H_{t-1}, K_{t-1}$ that have the same dimensions as $h_{t-1}, k_{t-1}$, respectively:

$$X_t = \begin{bmatrix} H_{t-1} \\ K_{t-1} \\ z_t \end{bmatrix},$$

where $z_t$ is also a component of $x_t$. Impose a given initial condition

$$X_0 = \begin{bmatrix} H_{-1} \\ K_{-1} \\ z_0 \end{bmatrix} = \begin{bmatrix} h_{-1} \\ k_{-1} \\ z_0 \end{bmatrix}.$$ 

Take the law of motion to be

$$X_{t+1} = A^o X_t + C\epsilon_{t+1}$$  \hspace{1cm} (10.4.18a)  \hspace{1cm} \text{["bigX;a "]}
where $A^o$ is the same matrix that appears in the representation (10.3.8) for the evolution of $x_t$ under the planner’s optimal control. Then we can represent the shadow price system as

\begin{align*}
  p_{ct}^S &= M_cX_t \\
  p_{lt}^S &= M_lX_t \\
  p_{rt}^S &= \Gamma' M_d X_t \\
  p_{st}^S &= M_d X_t \\
  p_{kt}^S &= (\Gamma'M_d + \Delta'_k M_k) X_t.
\end{align*}

(10.4.18b) ["bigX;b"]

What is the point of this “big $X$” representation for prices? First, note that by setting $X_0 = x_0$, we assure that (10.4.18) reproduces the planner’s shadow prices. But by expressing these prices in terms of $X_t$ rather than $x_t$, we have made these shadow prices depend only on a state variable that is beyond the control of the individual, and whose role is to account for the history $J_t = [\epsilon^t, x_0]$. In a competitive equilibrium, we want households and firms to influence the evolution of $h_t$ and $k_t$, the endogenous components of $x_t$, but still to be price takers. Therefore, in expressing the choices facing households and firms in a competitive equilibrium, we use a different state vector $X_t$ to provide a recursive representation of prices.

The shadow prices have the units of time 0 marginal utilities of the representative agent. We can choose a numeraire to express prices in terms of one of the consumption goods. In particular, denote the time $t$ marginal utility of the first consumption good $e_{1u_{c,t}}$ and assume that $e_{1u_{c,t}} \neq 0$ with probability one for all $t$. This assumption makes the first consumption good at time $t$ a legitimate numeraire. We choose to express the price system at time 0 in units of the first consumption good. Therefore, we set,

\begin{align*}
  p_{ct}^0 &= \frac{p_{ct}^S}{e_{1u_{c,0}}} \\
  p_{lt}^0 &= \frac{p_{lt}^S}{e_{1u_{c,0}}} \\
  p_{rt}^0 &= \frac{p_{rt}^S}{e_{1u_{c,0}}} \\
  p_{dt}^0 &= \frac{p_{dt}^S}{e_{1u_{c,0}}} \\
  p_{kt}^0 &= \frac{p_{kt}^S}{e_{1u_{c,0}}}.
\end{align*}

In their concept of a recursive competitive equilibrium, Prescott and Mehra (1980) distinguished between the market wide level of capital $k$ and the level chosen by an individual $k$ in order to represent price-taking behavior.
and so on. More generally, for $t \geq \tau$ where $\tau \geq 0$, we could define a time $\tau$ price system

$$p_{ct}^\tau = \frac{p_{ct}}{e_1 u_{c,\tau}}, \quad p_{lt}^\tau = \frac{p_{lt}}{e_1 u_{c,\tau}}$$

and so on. To convert the tail of the time 0 price system for $t \geq 1$ to the time 1 price system we can use

$$p_{ct}^1 = \frac{e_1 u_{c,0}}{e_1 u_{c,1}},$$

and so on.

### 10.5. Asset pricing in a nutshell

It is a significant practical convenience that we can dispense with $X_t$ as a state variable when we actually compute asset prices. After we have computed the equilibrium quantities and prices, for the purpose of computing asset prices, it is sufficient to express both streams of payouts and prices as functions of the state vector $x_t$ that appears in the planning problem. For example, an endowment shock is a linear function of the exogenous component $z_t$, while the endogenous component $h_{t-1}$ tracks movements in the intertemporal rate of substitution that drives the prices. Let $\{y_t\}_{t=0}^{\infty}$ be a stochastic process of ‘dividends’ or claims on the vector of consumption goods with representation $y_t = S_c x_t$. In units of the first time $t$ consumption good, let $a_{yt}$ denote the price at time $t$ of a claim on the tail of the dividend process $\{y_s\}_{s=t}^{\infty}$. The asset price $a_{yt}$ can be represented as

\begin{align*}
x_{t+1} &= A^o + C_{t+1} \quad (10.5.1a) \quad \text{["nutshell1;a"]} \\
y_t &= S_y x_t \quad (10.5.1b) \quad \text{["nutshell1;b"]} \\
p_{ct} &= (e_1 M_c x_t)^{-1} M_c x_t \quad (10.5.1c) \quad \text{["nutshell1;c"]} \\
a_{yt} &= E_t \sum_{t=0}^{\infty} \beta^t p_{ct+j} \cdot y_{t+j} \quad (10.5.1d) \quad \text{["nutshell1;d"]}
\end{align*}

Equation (10.5.1d) can be evaluated easily by solving a Sylvester equation.

---

9 This time $\tau$ price system would prevail if we were to reopen markets at time $\tau$, subject to appropriate initial conditions being inherited from earlier trading at time 0 prices.
Such formulas are in the spirit of Brock’s (1982) extension of Lucas’s asset pricing formulas. The single state vector \( x_t \) tracks both the state-date prices and the dividend process.

While a representation for prices that, like (10.5.1a), (10.5.1c), is cast in terms of the state vector \( x_t \) is most useful for pricing assets, to express the problem that faces the household in a competitive equilibrium we need to use the alternative representation of prices (10.4.18a), (10.4.18b). As we turn next to the problems of the agents in a competitive equilibrium with sequential trading, we again have recourse to the ‘big \( X \)’ representation (10.4.18).

**10.6. Sequential markets and Arrow securities**

We noted in section 10.4.8 that the time 0 prices for an Arrow-Debreu time 0 competitive equilibrium have a recursive representation based on (10.4.18a), (10.4.18b). This fact makes it easy to construct equilibrium prices for trading in a sequence of one-period markets, where each period there are trades in spot markets as well as a complete set of one-period state-contingent claims to wealth. Following Arrow (1954XXX), we can use (10.4.18) to form competitive equilibrium prices in a setting with sequential trading of one-period state-contingent claims. In that setting, the decision problems of households and firms are recursive. The equilibrium allocation will be the same as with time 0 trading.

**10.6.1. Arrow securities**

To indicate how to move from our equilibrium with time 0 trading toward an equilibrium with sequential trading, represent the consumer’s budget constraint (10.4.2) as

\[
E_0 \sum_{t=1}^{\infty} \beta^t \left( p_{c_t}^0 c_t - p_{t-1}^0 \ell_t - p_{t-1}^0 d_t \right) + p_{c_0}^0 c_0 = p_{\ell_0}^0 \ell_0 + p_{d_0}^0 d_0 + p_k^0 k_{-1}.
\]

Express the expected discounted sum on the left side as

\[
E_0 \beta \left( \frac{e_{1u,c,1}}{e_{1u,c,0}} \right) \sum_{t=1}^{\infty} \beta^{t-1} \left( p_{c_t}^1 c_t - p_{t-1}^1 \ell_t - p_{t-1}^1 d_t \right) = E_0 \beta \left( \frac{e_{1u,c,1}}{e_{1u,c,0}} \right) a_1 (X_1) = \int q (X_1|X_0) a_1 (X_1) dX_1,
\]

where \( a_1 = a_1 (X_1) \) measures wealth at time 1 in units of the time 1 consumption good, and the one-step ahead pricing kernel \( q (X_1|X_0) = \beta_{\frac{e_{1u,c,1}}{e_{1u,c,0}}} f (X_1|X_0) \). Here
$f(X_1|X_0)$ is the transition density of $X$ defined by (10.4.18a). Use (10.6.1) to express the budget constraint as

$$\int q(X_1|X_0) \, a_1(X_1) \, dX_1 + p_{\ell_0} \cdot c_0 = p_{\ell_0} \ell_0 + p_{d_0} \cdot d_0 + a_0(X_0) \quad (10.6.2) \quad \text{["arrow3"]}$$

where $a_0(X_0) \equiv p_{k_0} \cdot k_{-1}$. More generally, take prices without superscripts to be denominated in units of time $t$ consumption of the first good and write

$$\int q(X_{t+1}|X_t) \, a_{t+1}(X_{t+1}) \, dX_{t+1} + p_{c_t} \cdot c_t = p_{c_t} \ell_t + p_{d_t} \cdot d_t + a_t(X_t) \quad (10.6.3) \quad \text{["arrow4"]}$$

where

$$q(X_{t+1}|X_t) = \beta \frac{e_{u_c,t+1}}{e_{u_c,t}} f(X_{t+1}|X_t) \quad (10.6.4) \quad \text{["arrow6"]}$$

is the kernel for pricing claims of the first consumption good at time $t+1$ in terms of time $t$ consumption of the first good.

The spot prices are given by the appropriate time $t$ components of our original time 0 prices. Together with the pricing kernel defined as (10.6.4), they allow us to support the solution of the planning problem by a competitive equilibrium with sequential markets. Within that equilibrium, the problem of the household is recursive and the problems of both types of firms are static, as we now proceed to show.

### 10.6.1.1. The household’s problem in the sequential equilibrium

The household’s Bellman equation is

$$W(a_t, h_{t-1}, X_t) = \max_{c_t, h_t, a(X_{t+1})} \left\{ - (|s_t - b_t|^2 + \ell_t^2) + \beta \int W(a(X_{t+1}), h_t, X_{t+1}) \, f(X_{t+1}|X_t) \, dX_{t+1} \right\} \quad (10.6.5) \quad \text{["arrow10"]}$$

where the maximization is subject to

$$s_t = \Delta h_t + \Pi c_t \quad (10.6.6a) \quad \text{["arrow11;a"]}$$

$$h_t = \Delta h_{t-1} + \Theta_n c_t \quad (10.6.6b) \quad \text{["arrow11;b"]}$$

$$X_{t+1} = A^n X_t + C \epsilon_{t+1} \quad (10.6.6c) \quad \text{["arrow11;c"]}$$

$$[p_{c_t} \quad p_{d_t} \quad p_{\ell_t}] = M X_t \quad (10.6.6d) \quad \text{["arrow11;d"]}$$

$$b_t = S_b X_t \quad (10.6.6e) \quad \text{["arrow11;e"]}$$

$$\int a(X_{t+1}) q(X_{t+1}|X_t) \, dX_{t+1} = a_t + p_{c_t} \ell_t + p_{d_t} \cdot d_t - p_{c_t} \cdot c_t, \quad (10.6.6f) \quad \text{["arrow11;f"]}$$
and where the matrix $M$ consists of submatrices $M_j$ that pertain to the prices indicated and $b_t = S_b X_t \equiv U_b z_t$. The optimal policy functions express $c_t, \ell_t,$ and $a(X_{t+1})$ each as functions of $(a_t, h_{t-1}, X_t)$.

### 10.6.1.2. A type I firm

The problem of a firm of type I is static:

$$\max_{c_t, i_t, \ell_t, a_t} \left( p_{ct} \cdot c_t + p_{it} \cdot i_t - p_{rt} \cdot k_{t-1} - p_{dt} \cdot d_t - p_{\ell t} \ell_t \right)$$

subject to the technology (10.4.6), (10.4.7).

### 10.6.2. A type II firm

The problem of a firm of type II is also static. Hansen and Sargent (200XXX) show that in the Arrow-Debreu equilibrium with trading at time 0, the value of the initial capital stock is $p_{k0} \cdot k_{t-1}$ where $p_{k0} = (e_{1u,0})^{-1} M_k X_0$; they provide a formula for the matrix $M_k$. Let $p_{kt} = p_k(X_t) = (e_{1u,c,t})^{-1} M_k X_t$ be the corresponding price of the vector $k_{t-1}$ if markets were to be reopened at time $t$, as described above. For our sequential market structure, we posit that in period $t$ firms can trade claims on the capital vector at time $t+1$ contingent on the aggregate state being $X_{t+1}$. The price vector of such claims in terms of the first consumption good at time $t$ is given by the kernel

$$q_k(X_{t+1}|X_t) = \beta \left( \frac{e_{1u,c,t+1}}{e_{1u,c,t}} \right) p_k(X_{t+1}) f(X_{t+1}|X_t).$$

The firm of type II faces $p_{it}, p_{rt}, q_k(X_{t+1}|X_t)$ as a price taker and at time $t$ solves the static problem:

$$p_{kt} \cdot k_{t-1} = \max_{i_t, \ell_t} \left( p_{rt} \cdot k_{t-1} - p_{it} \cdot i_t + k_t \cdot \int q_k(X_{t+1}|X_t) dX_{t+1} \right)$$

where the maximization is subject to

$$k_t = \Delta_k k_{t-1} + \Theta_k i_t.$$

The solution is a decision rule expressing $i_t$ as a function of $(k_{t-1}, X_t)$. 
10.6.3. Equilibrium

In a recursive competitive equilibrium, the household takes the law of motion for \( X_t \) as given. However, the household and the firms choose elements \( h_s, k_s \) of the state \( x_{s+1} \) that correspond to the elements \( H_s, K_s \) of \( X_{s+1} \). A recursive competitive equilibrium requires that \( X_t = x_t \) for \( t \geq 1 \), starting from \( x_0 = X_0 \), which means that that the laws of motion chosen by firms and the household must be consistent with the law of motion (10.4.18) that inspires the household’s decisions and that \( v_t \cdot k_{t-1} = a(X_t) \).

It can be verified that the quantities that solve the planning problem are equilibrium quantities at the candidate prices described above. See Hansen and Sargent (20XXX) for some of the details.

10.7. Partial equilibrium interpretation

Another decentralization of the planning problem makes contact with partial equilibrium models in the style of Lucas and Prescott (1971), Rosen, Murphy, and Scheinkman (1994), Rosen and Topel (199??), and Sargent (1987). These models focus on a representative firm that acts as a price taker. The industry as a whole faces a stochastically shifting linear demand schedule.

Within the environment of this chapter, consider a representative firm that chooses stochastic processes \( \{c_t, g_t\} \) to maximize

\[
E_0 \sum_{t=0}^{\infty} \beta^t \left\{ p_t \cdot c_t - g_t \cdot g_t \right\} \tag{10.7.1} \]

subject to the constraints

\[
k_t = \Delta_k k_{t-1} + \Theta_k i_t \tag{10.7.2a} \]
\[
\Phi_c c_t + \Phi_g g_t + \Phi_i i_t = \Gamma k_{t-1} + d_t \tag{10.7.2b} \]
\[
X_{t+1} = A X_t + C \epsilon_{t+1} \tag{10.7.2c} \]
\[
d_t = U d X_t \tag{10.7.2d} \]
\[
p_t = M c X_t. \tag{10.7.2e} \]

Here (10.7.2e), (10.7.2c) are used to represent the demand curve. The state \( X_t \) is defined as above. The Bellman equation is

\[
V(k_{t-1}, X_t) = \max_{c_t, k_t, i_t, g_t} \left\{ p_t \cdot c_t - g_t \cdot g_t + \beta EV(k_t, X_{t+1}) \right\} \tag{10.7.3} \]
where the maximization is subject to \((10.7.2)\). The optimal decision rule expresses \(c_t, q_t\) as functions of \(k_t, X_t\), so that the firm chooses to make \(k_t\) follow a law that can be expressed as

\[
k_t = k(k_{t-1}, X_t) = k(k_{t-1}, H_{t-1}, K_{t-1}, z_t). \tag{10.7.4}
\]

Embedded in \((10.7.2c)\) is the firm’s perceived law of motion for \(K_t\), namely,

\[
K_t = K(X_t) = K(H_{t-1}, K_{t-1}, z_t). \tag{10.7.5}
\]

A competitive equilibrium requires that \(k_s \equiv K_s\) for all \(s\) and all \(z_s\), or

\[
k(k_{t-1}, H_{t-1}, K_{t-1}, z_t) = K(H_{t-1}, K_{t-1}, z_t). \tag{10.7.6}
\]

The left side of \((10.7.6)\) is the actual law of motion for \(K\) that emerges from optimization (this is the content of the function \(k(\cdot)\)) and equilibrium (the condition that \(k = K\), which makes the representative firm representative). The right side of \((10.7.6)\) is the representative firm’s perceived law of motion for \(K\). Thus, \((10.7.6)\) imposes equality between the ‘perceived’ law of motion for \(K\) and the ‘actual’ law of motion implied by those perceptions. Equality between these two laws imposes rational expectations while respecting the price taking behavior of the firm.\(^{10}\)

\section*{10.8. Concluding remarks}

This chapter has set forth a class of linear quadratic economies and described three standard types of equilibria for them. We have made the standard rational expectations assumption that a planner and the agents all trust their common model. In the next chapter we shall alter that assumption by instilling in both agents and a planner the same degree of preference for robustness of decision rules with respect to deviations of the actual data generation mechanism from their common approximating model. We shall aim to do this in a way that renders the worst case shocks chosen by the robust planner consistent with ones chosen by the robust consumer in the recursive competitive equilibrium with Arrow securities. Thus, the next chapter will contain modifications of the Bellman equation \((10.4.10), (10.4.11)\) for the planner, the Bellman equation \((10.6.5),\)

\(^{10}\) See Evans and Honkapohja (2001) for extensive exploitation of the definition of a rational expectations equilibrium as the fixed point of a mapping from a perceived to an actual law of motion.
(10.6.6) for the household trading Arrow securities, and the Bellman equation (10.6.8), (10.6.9) of the capital-purchasing firm. With a preference for robustness, the first two of these will be replaced by Bellman equations for two-player, zero-sum games. Decentralizing the robust planning problem will then require checking that the choices of both the maximizing and the minimizing players for the planning problem and the household within a competitive equilibrium, respectively, are mutually consistent.
Chapter 11.
Competitive equilibrium under robustness

This chapter modifies the situation of the representative household in the equilibrium models of chapter 10 by adding concern for model specification. Equilibrium quantities solve a robust planning problem. Competitive equilibrium prices can be computed from shadow prices for the robust planning problem.

This chapter parallels that of chapter 10, except that Bellman equations for the appropriate two-player zero-sum games replace those for the maximizing agents in chapter 10. The role of the minimizing agent is to help the maximizing explore the fragility of his decision rule with respect to various hard-to-detect perturbations of his approximating model. To decentralize the allocation that solves the planning problem, we specify two-player games for the for the representative household in the decentralized economy and verify that at the equilibrium prices both the allocation and the worst case model chosen by the household are aligned with their counterparts that are chosen by the planner. Chapter 12 then uses these results to apply and interpret some simple asset pricing formulas that emerge when agents have a preference for robustness.

11.1. Two robust planning problems

We describe two interpretations of the decentralization with sequential trading of Arrow securities. One of these has the household solve a robust control problem in which he imagines that a malevolent agent distorts innovations to the process driving endowment and preference shocks. The other is an ex post Bayesian interpretation along the lines of chapter 6. It endows the consumer with a belief about the law of motion that is distorted relative to his approximating model in a way that lets him attain a robust rule by solving an ordinary Bellman equation without a concern for robustness.
11.1.1. A direct robust planning problem

A planning problem that recovers the competitive equilibrium allocation takes the form of a robust linear regulator problem. Paralleling chapter 10, the value function and the robust law of motion for the state $x_t$ contain the information needed to compute the competitive equilibrium prices.\(^1\) Let $V(x) = -x'Px - p$ be the value of the robust planning problem starting from initial state $x$. The Bellman equation is

$$-x'Px - p = \min_w \max_{c,i,g} \left\{ -0.5 \left[ (s - b) \cdot (s - b) + g \cdot g \right] + \beta \theta w'w + \beta E \left( -x''Px'' - p \right) \right\}$$ (11.1.1) \(["rod13 "]\)

where the extremization is subject to the linear constraints

$$\Phi_c c + \Phi_g g + \Phi_i i = \Gamma k + d \quad (11.1.2a) \quad ["rod14;a "]$$
$$k^* = \Delta_k k + \Theta_k i \quad (11.1.2b) \quad ["rod14;b "]$$
$$h^* = \Delta_h h + \Theta_h c \quad (11.1.2c) \quad ["rod14;c "]$$
$$s = \Lambda h + \Pi c \quad (11.1.2d) \quad ["rod14;d "]$$
$$z^* = A_{2z} z + C_{2} (\epsilon + w) \quad (11.1.2e) \quad ["rod14;e "]$$
$$b = U_b z \quad (11.1.2f) \quad ["rod14;f "]$$
$$d = U_d z, \quad (11.1.2g) \quad ["rod14;g "]$$

where $^*$ denotes a next period value. Problem (11.1.1), (11.1.2) differs from (10.4.10), (10.4.11) in: (1) the addition of the distortion $Cw$ in the law of motion for $z$, (2) the appearance of $\beta \theta w'w$ in the continuation value function in (11.1.2), and (3) the minimization over $w$. As usual, $\theta > 0$ is a robustness parameter.

A Markov perfect equilibrium of the two-player zero-sum game (11.1.1), (11.1.2) is a pair of decision rules $u = -F(\theta)x, w = K(\theta)x$. The equilibrium determines two laws of motion for the state, namely,

$$x_{t+1} = A^o x_t + C\epsilon_{t+1} \quad (11.1.3) \quad ["nasset3 "]$$

and

$$x_{t+1} = (A^o + CK(\theta)) x_t + C\epsilon_{t+1}, \quad (11.1.4) \quad ["nasset4 "]$$

where $A^o = A - BF(\theta)$. Equation (11.1.3) is the approximating model under the robust rule, while (11.1.4) is the distorted worst case model associated with

\(^1\) See Hansen and Sargent (2XXX) for a complete account.
\( \theta \). In chapter 12, we show that either of these models can be used to price assets by appropriately adjusting the stochastic discount factor.

### 11.1.2. An ex post Bayesian planning problem

We can apply an idea of chapter 6 to confront the planner with a distorted law of motion for \( z \) that will inspire him to choose a robust decision rule. In particular, we can augment the state variable \( x \) by a vector \( X \) of the same dimension. The Bellman equation for the ex post Bayesian problem is

\[
-x'PX - p = \max_{c,i,g} \left\{ -0.5 \left[ (s - b) \cdot (s - b) + g \cdot g \right] + \beta E \left( -x'^*PX'^* - p \right) \right\} \tag{11.1.5} \]

subject to the linear constraints formed by (11.1.2) and the following additional exogenous law of motion for \( w \):

\[
X^* = A^*X + C\epsilon \tag{11.1.6a} \]
\[w = KX. \tag{11.1.6b} \]

In particular, notice how through equation (11.1.2e), the \( w \) determined by equation (11.1.6) feeds back into the \( z \) process that governs the shocks impinging on the consumer’s preference and endowment shock processes, \((b, d)\). Where \( u_t \) is the control, the planner chooses a decision rule \( u = -\tilde{F} \begin{bmatrix} x \\ X \end{bmatrix} \). Chapter 6 shows how, after equating \( X = x \), this decision rule satisfies

\[
-\tilde{F} \begin{bmatrix} x \\ x \end{bmatrix} = -Fx
\]

where \( u = -Fx \) solves the robust planning problem (11.1.1), (11.1.2).
11.1.3. Remarks on practicality

The direct way of solving the robust planning problem is obviously the more useful one computationally. The ex post Bayesian method is a useful reinterpretation of that solution. In particular, we’ll see that the ex post Bayesian interpretation is useful for asset pricing.

11.2. Two asset pricing strategies

Under a preference for robustness, there are counterparts to the strategy described in section 10.5 that succeed in expressing asset prices in terms of the state vector $x_t$ of the planner. In particular, under robustness, the allocation and shadow prices associated with a robust planning problem can be used to express asset prices in terms of $x_t$. We shall use such a pricing strategy in chapter 12.

There are actually two distinct strategies for getting shadow prices that can be transformed into prices that support the robust plan as a competitive equilibrium. The first uses the robust planning problem (11.1.1), (11.1.2), and the second uses the ex post Bayesian planning problem (11.1.5), (11.1.6).

11.2.1. Pricing from the robust planning problem

The following method was used by Hansen, Sargent, and Tallarini (1999).

1. Solve the robust planning problem.
2. Obtain representations for the planner’s shadow prices, based on marginal utilities of consumption.
3. Use the appropriate shadow prices to price assets as conditional expectations of inner products of (scaled) state-date prices, computing the conditional expectation by taking the distorted law of motion cast in terms of little $x$ that emerges from the robust planning problem and the corresponding sequence of information $J_t$.

This method leads to a representation for asset prices correspoding to (10.5.1) of the following form:

\[
x_{t+1} = \hat{A}^o x_t + C \epsilon_{t+1} \tag{11.2.1a} \]
\[
y_t = S y x_t \tag{11.2.1b} \]
\[
p_{cs} = (c_1 M_c x_t)^{-1} \hat{M}_c x_s \tag{11.2.1c} \]
where $\hat{E}$ is the expectation evaluated with respect to $\hat{A}^o = A^o + CK$, the transition matrix for the worst-case transition law under the robust decision rule $F$, where $A^o = A - BF$; and $\hat{M}$ also incorporates the worst case transition law through the substitution of $\hat{E}$ for $E$ in (10.4.16), (10.4.15).

11.2.2. Pricing from the ex post Bayesian planning problem

Although chapter 12 will use pricing formulas based on (11.2.1), an alternative strategy could be based on the following four step procedure:

1. Solve a robust planning problem.

2. Obtain a representation of the worst case shock process in terms of the new state variable $X_t$ as on page 212.

3. Solve the ordinary (i.e., non-robust) planning problem with the distorted law of motion for the augmented state $[x_t, X_t]$ as in (11.1.5), (11.1.6).

4. Use either of the complete-market decentralizations presented in chapter 10 for our economies without robustness and price assets using the standard (non-robust) asset pricing formulas.

Each of these pricing strategies shares with (10.5.1) the features that we represent prices and pay outs in terms of the planner’s state vector $x_t$.

11.3. Two decentralizations with Arrow securities

With sequential trading in Arrow securities, there are two decentralizations that support the solution of a robust planning problem as competitive equilibria. The two decentralizations differ in how they construe the problem of the representative household. (Being static, the firms’ problems are the same as in chapter 10.) One decentralization makes the household into an ex post Bayesian who confronts an ordinary (i.e., non-robust) problem with a law of motion for the state that is distorted relative to the approximating model. A second decentralization confronts a robust household with laws of motion for the augmented state $(x, X)$ and has the household solve a zero-sum two-player game in which it chooses distortions to the innovations in that composite law of motion. After solving for the decision rule for each type of problem and requiring that
the representative household be representative by imposing $X = x$, it turns out that both of these decentralizations support the allocation that solves the robust planning problem.

11.3.1. Ex Post Bayesian decentralization

11.3.1.1. The household’s problem under the ‘Bayesian’ interpretation

Let $u_t = -Fx_t$ be the planner’s decision rule in a robust control problem, and let $w_{t+1} = Kx_t$ be the associated worst case shock that emerges from the Markov perfect equilibrium of the robust planning problem. Define $A^o = A - BF$ to be the state transition matrix for the autonomous law of motion for $x_t$ under the approximating model and the robust rule. Let $\tilde{A}^o = A^o + CK$ be the state transition matrix under the robust rule and the distorted model from the Markov perfect equilibrium of the robust planning problem.

We can use $\tilde{A}^o = A^o + CK$ to define a transition law for $X_{t+1}$ in an equilibrium with sequential trading of Arrow securities. In this equilibrium, the household’s Bellman equation that corresponds to (10.6.6) becomes

\[
W(a_t, h_{t-1}, X_t) = \max_{c_t, \ell_t} \left\{ -\left(|s_t - b_t|^2 + \ell_t^2\right) + \beta \int W(a(X_{t+1}), h_t, X_{t+1}) f(X_{t+1} | X_t) dX_{t+1} \right\}
\]

where the maximization is subject to

\[
\begin{align*}
    s_t &= \Lambda h_t + \Pi c_t \\
    h_t &= \Delta_h h_{t-1} + \Theta_h c_t \\
    X_{t+1} &= (A^o + CK) X_t + C\epsilon_{t+1} \\
    [p_{ct} & \ p_{ct} & \ p_{ct}] = MX_t \\
    b_t &= S_b X_t \\
    \int a(X_{t+1})q(X_{t+1} | X_t) dX_{t+1} &= a_t + p_{ct} \ell_t + p_{ct} d_t - p_{ct} \cdot c_t,
\end{align*}
\]

and where the matrix $M$ consists of submatrices $M_j$ that pertain to the prices indicated. The household’s optimal policy functions express $c_t$ and $a(X_{t+1})$ each as functions of $(a_t, h_{t-1}, X_t)$. 


11.4. Robust representative household with Arrow securities

11.4.0.1. The household’s problem in the sequential equilibrium

Again let \( A^o = A - BF \) be the transition matrix for the autonomous law of motion for \( x_t \) that emerges from the robust planning problem for a given \( \theta \). As before, use \( A^o \) to define a law of motion for a process \( X_t \) that encodes the history of shocks \( J_t \). Consider the following robust household’s Bellman equation that we obtain by modifying (10.6.5), (10.6.6):

\[
W(a_t, h_{t-1}, X_t) = \min_{w_{t+1}} \max_{c_t, \ell_t, a(X_{t+1})} \left\{ - \left( |s_t - b_t|^2 + \ell_t^2 \right) + \beta \int W(a(X_{t+1}), h_t, X_{t+1}) f(X_{t+1}|X_t) dX_{t+1} + \beta \theta w_{t+1}' w_{t+1} \right\}
\]

(11.4.1) ["arrow10R "]

where the maximization is subject to

\[
\begin{align*}
  s_t &= \Lambda h_t + \Pi c_t \\
  h_t &= \Delta h_{t-1} + \Theta h_{t-1} \\
  X_{t+1} &= A^o X_t + C (\epsilon_{t+1} + w_{t+1}) \\
  M X_t &= [p_{ct} \ p_{lt} \ p_{dt}] \\
  b_t &= S_b X_t \\
  \int a(X_{t+1}) q(X_{t+1}|X_t) dX_{t+1} &= a_t + p_{lt} \ell_t + p_{dt} \cdot d_t - p_{ct} \cdot c_t,
\end{align*}
\]

(11.4.2a) ["arrow11R;a "]

(11.4.2b) ["arrow11R;b "]

(11.4.2c) ["arrow11R;c "]

(11.4.2d) ["arrow11R;d "]

(11.4.2e) ["arrow11R;e "]

(11.4.2f) ["arrow11R;f "]

and where the matrix \( M \) consists of submatrices \( M_j \) that pertain to the prices indicated. The Markov perfect equilibrium policy functions express \( w_{t+1} \) and \( c_t, \ell_t, a(X_{t+1}) \) each as functions of \( (a_t, h_{t-1}, X_t) \). In particular, \( w_{t+1} = w(a_t, h_{t-1}, X_t) \).

Notice how through equations (11.4.2c), (11.4.2d) the minimizing player distorts the transition probabilities. Among the equilibrium conditions is now one that requires

\[
w_{t+1} = w(a_t, H_{t-1}, X_t) = K X_t.
\]
11.5. Concluding remarks

Having assured ourselves that prices are well defined and that the standard decentralization with sequential trading of one-period Arrow securities makes sense, in chapter 12 we shall compute asset prices under a preference for robustness and interpret them. By carefully distinguishing between the approximating model and the worst case model chosen by the robust planner, we shall obtain a multiplicative decomposition of the appropriate stochastic discount factor into the ordinary intertemporal marginal rate of substitution and an adjustment that reflects agents’ fears of model misspecification. We shall interpret this decomposition as partitioning risk premia into premia for bearing ordinary risk times another kind of premium that compensates for bearing ignorance of model specification or fear of model misspecification.
Chapter 12.
Asset pricing

12.1. Introduction

This chapter explores how a preference for robustness affects prices of risky securities. Without a preference for robustness, we can represent the price of a claim to a random future payoff as a conditional expectation of the inner product of a stochastic discount factor and the random future payoff. The conditional expectation is evaluated using the representative agent’s model.\footnote{Without concerns about model misspecification, an agent can discard the adjective ‘approximating’.} A very similar representation for asset prices prevails in a model where the representative agent has a preference for robustness, but now the conditional expectation is to be evaluated with respect to the representative agent’s worst case model, a model that that depends on the parameter $\theta$ that calibrates his preference for robustness. However, because the approximating model and the worst case model put positive probabilities on the same events, there exists another representation for assets prices that evaluates the conditional expectation with respect to the approximating model. Rather than adjusting the probability distribution relative to the approximating model, this representation instead adjusts the stochastic discount factor to reflect the preference for robustness. In particular, if asset prices are to be represented in terms of conditional expectations under the approximating model, then we have to multiply the ordinary stochastic discount factor without a preference for robustness by the likelihood ratio, or Radon-Nikodym derivative, of the endogenous distorted model relative to the approximating model. That likelihood ratio, the expected value of which is the entropy measure that we used in chapter 2 to measure the proximity of models, also governs the detection statistics of chapter 8.

After reviewing asset pricing formulas in a standard model without a preference for robustness, this chapter modifies those formulas to express a representative agent’s preference for robustness. By way of examples, we study asset pricing in the permanent income economy of chapter 9 and a partial equilibrium occupational choice model of Sherwin Rosen.
12.2. Approximating and distorted models

Chapters 10 and 11 describe planning problems and competitive equilibria for a class of linear-quadratic models. The consumption smoothing model of chapter 9 and the occupational choice model of section 12.6 are special cases. The environment of chapter 10 is arranged so that without a preference for robustness, the planning problem fits into the optimal linear regulator problem. Chapter 11 then uses a robust linear regulator to create a model in which the representative household has a preference for robustness indexed by parameter $\theta > 0$. Equilibrium representations for prices and quantities can be determined from the solution of the robust linear regulator.

Chapter 10 describes matrices that portray the preferences, technology, and information structure of the economy. Those matrices can be assembled into matrices that define the robust linear regulator for a planning problem. The solution of the planning problem determines competitive equilibrium prices and quantities. Associated with the robust planning problem is the Bellman equation

$$-x'Px - p = \max_u \min_w \{ r(x,u) + \theta \beta w'w + \beta E(-x'^*Px'^* - p) \} \tag{12.2.1}$$

where the extremization is subject to

$$x^* = Ax + Bu + C(\epsilon + w), \tag{12.2.2}$$

where $\epsilon \sim \mathcal{N}(0, I)$ and $\theta \in [\theta, +\infty]$. A Markov perfect equilibrium of this two-player zero-sum game is a pair of decision rules $u = -F(\theta)x, \ w = K(\theta)x$. The equilibrium determines two laws of motion for the state

$$x_{t+1} = A^o x_t + C\epsilon_{t+1} \tag{12.2.3}$$

and

$$x_{t+1} = (A^o + CK(\theta))x_t + C\epsilon_{t+1}, \tag{12.2.4}$$

where $A^o = A - BF(\theta)$. Equation (11.1.3) is the approximating model under the robust rule, while (12.2.4) is the distorted worst case model associated with $\theta$.

Where there is no preference for robustness, $\theta = +\infty$. Chapter 10 describes a class of economies whose equilibria can be presented in the form (12.2.4) together with selector matrices that determine equilibrium prices and quantities as functions of the state $x_t$. In particular, quantities $Q_t$ and scaled state-contingent prices $p_t$ are linear functions of the state:

$$Q_t = S_Q x_t \tag{12.2.5a}$$

$$p_t = p_Q x_t. \tag{12.2.5b}$$
We shall soon describe what we mean by ‘scaled prices’.

To get equilibria under a preference for robustness, we simply set $\theta < +\infty$ in (12.2.1). Formulas for equilibrium prices and quantities from chapter 10 (i.e., the $S_0, M_0$ in (12.2.5)) apply directly. Associated with an equilibrium under a preference for robustness are the approximating transition law (12.2.3) and the distorted transition law (12.2.4) for the state $x_t$, as well as auxiliary equations for prices and quantities of the form (12.2.5).

The approximating and distorted equilibrium laws of motion (12.2.3) and (12.2.4) induce Gaussian transition densities

\[
\begin{align*}
  f (x_{t+1}|x_t) &\sim \mathcal{N}(A^o x_t, C C') \quad (12.2.6a) \quad [\text{nasset6; a }]
  \\
  \hat{f} (x_{t+1}|x_t) &\sim \mathcal{N}((A^o + CK) x_t, C C') , \quad (12.2.6b) \quad [\text{nasset6; b }]
\end{align*}
\]

where we use (\^) to denote a probability associated with the distorted model (12.2.4). These transition densities induce joint densities $f^{(t)}(x^t)$ on histories $x^t = [x_t, x_{t-1}, \ldots, x_0]$ via

\[
  f^{(t)}(x^t) = f (x_t|x_{t-1}) f (x_{t-1}|x_{t-2}) \ldots f (x_1|x_0) f (x_0),
\]

and similarly for $\hat{f}^{(t)}(x^t)$. Let $f_t(x_t|x_0)$ denote the $t$–step transition densities

\[
\begin{align*}
  f_t (x_t|x_0) &\sim \mathcal{N}(A^{ot} x_0, V_t) \quad (12.2.7a) \quad [\text{nasset8; a }]
  \\
  \hat{f}_t (x_t|x_0) &\sim \mathcal{N}((A^o + CK)^t x_0, \hat{V}_t) , \quad (12.2.7b) \quad [\text{nasset8; b }]
\end{align*}
\]

where $V_t$ satisfies the recursion $V_t = A^{ot} V_{t-1} A^o + C C'$ initialized from $V_1 = C C'$, and $\hat{V}_t$ satisfies the recursion $\hat{V}_t = (A^o + CK)^t \hat{V}_{t-1}(A_o + CK) + C C'$ initialized from $\hat{V}_1 = C C'$.

---

2 The formulation on page 333 allows a broader set of perturbations of a Gaussian approximating model by letting the minimizing agent choose an arbitrary density. Under such a formulation, the minimizing agent would still choose a Gaussian transition density with the same conditional mean as (12.2.6b) but with conditional covariance $C C' = C(I - \theta^{-1} C' P C)^{-1} C'$. 
### 12.3. Asset pricing without robustness

In section 10.5, we explained how claims on risky streams of returns can be represented as the inner product of a price process and a payout process, where both the price and payout process are expressed as functions of the planner’s state vector \( x_t \). Thus, although it portraying the household’s problem in a recursive competitive equilibrium we need to distinguish between the individual household’s \( x_t \) and its ‘market wide’ counterpart \( X_t \) that drives prices, for the purpose of computing asset prices we can actually exclude \( X_t \) from the state vector and simply use \( x_t \) as the state vector. Accordingly, in the remainder of this chapter, we express prices in terms of \( x_t \) and histories \( x_t \).

When \( \theta = +\infty \), there is no discrepancy between the distorted and worst case models and the following standard representative agent asset pricing theory applies. Let \( c_t \) denote a vector of time-\( t \) consumption goods. The price of a unit vector of consumption goods in period \( t \) contingent on the history \( x_t \) is

\[
q_t(x_t|x_0) = \beta^t \frac{u'(c_t(x_t))}{e_1 \cdot u'(c_0(x_0))} f_t(x_t|x_0),
\]

(12.3.1) ["nasset90"]

where \( c_t(x_t) \) is a possibly history-dependent state-contingent consumption process, \( u'(c) \) is the vector of marginal utilities of consumption, and \( e_1 \) is a selector vector that pulls off the first consumption good, the time-zero value of which we take as numeraire. To make (12.3.1) well defined, we assume that \( e_1 \cdot u'(c_0(x_0)) \neq 0 \) with probability one. If we assume that the consumption allocation is not history-dependent, so that \( c_t(x_t) = c(x_t) \) as is true in the models that occupy us, then we can use the \( t \)-step pricing kernel

\[
q_t(x_t|x_0) = \beta^t \frac{u'(c(x_t))}{e_1 \cdot u'(c(x_0))} f_t(x_t|x_0).
\]

(12.3.2) ["nasset9"]

Let an asset entitle its owner to \( \{y(x_t)\}_{t=0}^{\infty} \), a stream of a vector of consumption goods whose state-contingent price is given by (12.3.2). The time-0 price of the asset is

\[
a_0 = \sum_{t=0}^{\infty} \int_{x_t} q_t(x_t|x_0) \cdot y(x_t) \, dx_t
\]

3 The household in a competitive economy would face prices that are the same functions of \( X_t \) and \( X' \).

4 We denote by \( u'(c_t) \) the vector of marginal utilities of the consumption vector \( c_t \). In our model, \( u'(c_t) = M_c x_t \).
or
\[ a_0 = \sum_{t=0}^{\infty} \int_{x_t} \beta^t \frac{u'(c(x_t))}{e_1 \cdot u'(c(x_0))} y(x_t) f_t(x_t|x_0) \, dx_t. \]  \hspace{0.5cm} (12.3.3) \hspace{0.5cm} ["nasset10"]

We can represent (12.3.3) as
\[ a_0 = E_0 \sum_{t=0}^{\infty} \beta^t u'(c(x_t)) \cdot y(x_t). \]  \hspace{0.5cm} (12.3.4) \hspace{0.5cm} ["nasset11"]

In linear-quadratic general equilibrium models, \( u'(c(x_t)) \) and \( y(x_t) \) are both linear functions of the state. This means that the price of an asset is the conditional expectation of a geometric sum of a quadratic form, as portrayed in (12.3.4). Equation (12.3.4) implies a Sylvester equation (see page 75). Thus, let
\[ p_c(x_t) = \frac{u'(c(x_t))}{e_1 \cdot u'(c(x_0))}. \]

Then the asset price can be represented
\[ a_0 = E_0 \sum_{t=0}^{\infty} \beta^t p_c(x_t) \cdot y(x_t). \]  \hspace{0.5cm} (12.3.5) \hspace{0.5cm} ["nasset12"]

We can regard \( p_c \) as a scaled Arrow-Debreu price: it equals the Debreu state price divided by \( \beta^t \) times a conditional probability. Scaling the price system in this way facilitates computation of asset prices as conditional expectations of an inner product of state prices and pay outs. Often \( \beta^t p_c(x_t) \) is called a \( t \)-period stochastic discount factor. Below we shall also denote it as \( m_{0,t} \equiv \beta^t p_c(x_t) \), so that (12.3.5) becomes
\[ a_0 = E_0 \sum_{t=0}^{\infty} m_{0,t} \cdot y(x_t). \]

Hansen and Sargent (200??) provide a complete treatment of asset pricing within linear-quadratic general equilibrium models. They show that: (1) equilibrium scaled Arrow-Debreu prices and quantities have representations (12.2.5); (2) the matrix \( S_Q \) is embedded in \( F, A, B \) from the optimal linear regulator problem; and (3) the matrices \( M_p \) that pin down the scaled Arrow-Debreu prices are embedded in the matrix \( P \) in the value function \(-x'Px - p\). Thus, in such models
\[ p_c(x_t) = M_c x_t / e_1 M_c x_0. \]  \hspace{0.5cm} (12.3.6) \hspace{0.5cm} ["sasset1"]

See chapter 10 for a formula for \( M_c \) and more details.
12.4. Asset pricing with robustness

We activate a preference for robustness by setting $\theta < +\infty$, which causes the transition densities $(12.2.6a), (12.2.6b)$ under the approximating and distorted models to disagree. In addition, the formulas for $S_Q$ and $M_Q$ in (12.2.5) respond to the setting for $\theta$, via the dependence of $S_Q$ on $F(\theta)$ and the dependence of $M_Q$ on the $P$ that solves the Bellman equation (12.2.1). (We give an example in section 12.6.)

The price system that supports a competitive equilibrium can be represented in the forms (12.3.1) and (12.3.2), with the distorted densities $\hat{f}(t)$ and $\hat{f}_t$ replacing the corresponding densities for the approximating model in (12.3.1) and (12.3.2). Thus, with a preference for robustness, the time 0 price of the asset corresponding to (12.3.3) is

$$a_0 = \sum_{t=0}^{\infty} \int_{x_t} \beta^t p_c(y) \cdot y_t \cdot \hat{f}_t(x_t | x_0) \, dx_t.$$  \hspace{1cm} (12.4.1)  

We can represent (12.4.1) as

$$a_0 = \hat{E}_0 \sum_{t=0}^{\infty} \beta^t p_c(x_t) \cdot y_t$$  \hspace{1cm} (12.4.2)  

where $\hat{E}$ denotes mathematical expectation using the distorted model (12.2.4), and $u'(c(x_t))$ must be computed using the $M_Q$ in representation (12.2.5b) associated with $\theta$.

12.4.1. Adjustment of stochastic discount factor

Formula (12.4.2) represents the asset price in terms of the distorted measure that the planner uses to evaluate future utilities in the Bellman equation (12.2.1). To compute asset prices using this formula, we must solve a Sylvester equation using transition matrix $A + CK(\theta)$ from equation (12.2.4) to reflect that we are evaluating the expectation using the distorted transition law. We can also evaluate asset prices by computing expectations under the approximating model, but this requires that we adjust the stochastic discount factor to make the asset price satisfy (12.4.2). Thus, we can represent (12.4.2) as

$$a_0 = \sum_{t=0}^{\infty} \int_{x_t} \beta^t p_c(x_t) \left( \frac{\hat{f}_t(x_t | x_0)}{f_t(x_t | x_0)} \right) \cdot y_t \cdot f_t(x_t | x_0) \, dx_t.$$  \hspace{1cm} (12.4.3)  

["masset10 "]

["masset11 "]

["masset1 "]

["masset1 "]
or

\[ a_0 = E_0 \sum_{t=0}^{\infty} \beta^t p_c(x_t) \left( \frac{\hat{f}_t(x_t|x_0)}{f_t(x_t|x_0)} \right) \cdot y(x_t), \tag{12.4.4} \]

where the absence of a \( (\cdot) \) from \( E \) denotes that the expectation is evaluated with respect to the approximating model (12.2.3).\(^5\)

In summary, with a preference for robustness, if we want to evaluate asset prices under the approximating model, we have to adjust the ordinary \( t \)-period stochastic discount factor \( m_{0,t} = \beta^t p_c(x_t) \) for a concern about model misspecification and use the modified stochastic discount factor:

\[ m_{0,t} \left( \frac{\hat{f}_t(x_t|x_0)}{f_t(x_t|x_0)} \right). \]

Such a multiplicative adjustment to the stochastic discount factor \( m_{0,t} \) carries over to nonlinear models. For our linear-quadratic-Gaussian setting, the likelihood ratio is

\[ L_t = \frac{\hat{f}_t(x_t|x_0)}{f_t(x_t|x_0)} = \exp \left[ \sum_{s=1}^{t} \{ \epsilon_s w_s - .5w_s'w_s \} \right]. \]

### 12.4.2. Reopening markets

This section describes how to extend our asset pricing formulas to allow us to price 'tail assets' that are traded at time \( t \) and that pay off vectors of consumption payoffs \( \{ y_\tau \}_{\tau=t}^{\infty} \) for \( t > 0 \). We want the price to be stated in time \( t \) units of the numeraire good.

Letting the \( t \)-step discount factor at time 0 be \( m_{0,t} = \beta^t p_c(x_t) \), (12.4.2) can be portrayed as

\[ a_0 = \hat{E}_0 \sum_{t=0}^{\infty} m_{0,t} \cdot y_t \tag{12.4.5} \]

where \( m_{0,t} \) is a vector of time-0 stochastic discount factors for pricing a vector of time-\( t \) payoffs. Define \( m_{t,\tau} \) as the vector of corresponding time-\( t \) stochastic discount factors for discounting time-\( \tau \geq t \) payoffs:\(^6\)

\[ m_{t,\tau} = \beta^{\tau-t} p_c(x_\tau) / e_1 p_c(x_t). \tag{12.4.6} \]

\(^5\) Notice the appearance of the same likelihood ratio in (12.4.4) used to define entropy in chapters 2 and 17 and to describe detection error probabilities in chapter 8.

\(^6\) We assume that \( e_1 p_c(x_t) \neq 0 \) with probability 1.
Then in time \( t \) units of the numeraire consumption good, the vector of payoffs \( \{ y_\tau \}_{\tau=0}^\infty \) is

\[
a_t = \hat{E} \sum_{\tau=0}^\infty m_{t,\tau} y_\tau.
\]

Equation (12.4.7) is equivalent with

\[
a_t = E_t \sum_{\tau=0}^\infty \left( m_{t,\tau} m_{t,\tau}^u \right) \cdot y_\tau,
\]

where the appropriate multiplicative adjustment \( m_{t,\tau}^u \) to the stochastic discount factor is the likelihood ratio

\[
m_{t,\tau}^u = \frac{\hat{f}_t (x_\tau | x_t)}{f_t (x_\tau | x_t)} = e^{\sum_{s=t}^{\tau} \{ \epsilon_s w_s - 0.5 w_s' w_s \}}.
\]

**12.5. Pricing single period payoffs**

For the purpose of using the permanent income model of chapter 9 to shed light on the implications of a preference for robustness for the equity premium, let consumption be a scalar process and \( y_{t+1} \) a scalar random payoff at time \( t + 1 \). Without a preference for robustness, the price at time \( t \) of a time \( t + 1 \) payout is

\[
a_t = E_t m_{t,t+1} y_{t+1}.
\]

Applying the definition of a conditional covariance to (12.5.1) and using the Cauchy-Schwartz inequality implies

\[
\left( \frac{a_t}{E_t m_{t,t+1}} \right) \geq E_t y_{t+1} - \left( \frac{\sigma_t (m_{t,t+1})}{E_t m_{t,t+1}} \right) \sigma_t (y_{t+1}).
\]

The bound is attained by payoffs on the efficient frontier. The left side is the price of the risky asset relative to the price \( E_t m_{t,t+1} \) of a risk-free asset that pays off 1 for sure next period. The term \( \left( \frac{\sigma_t (m_{t,t+1})}{E_t m_{t,t+1}} \right) \) is the ‘market price of risk’: it tells the rate at which the price \( a_t \) deteriorates with increases in the conditional standard deviation of the pay out \( y_{t+1} \).
Various studies used versions of (12.5.1) to estimate the market price of risk from data on \((a_t, y_{t+1})\) alone without restricting or imposing any theory on \(m_{t,t+1}\). For post WWII quarterly data, estimates of the market price of risk hover around .25. Hansen, XXX, and XXX’s characterization of the equity premium puzzle is that .25 is much higher than would be implied by many theories that explicitly link \(m_{t,t+1}\) to aggregate consumption, for example, the theory \(m_{t,t+1} = \beta u'(c_{t+1})/u'(c_t)\) where \(u()\) is a power utility function with power \(\gamma\). That specification makes \(m_{t,t+1} = \beta \left(\frac{c_{t+1}}{c_t}\right)^\gamma\). But aggregate consumption is a smooth series, so that the growth rate of consumption has a standard deviation so small that unless \(\gamma\) is implausibly large, the market price of risk implied by this theory of the stochastic discount factor \(m_{t,t+1}\) remains far below the observed value of .25. Similarly, the permanent income model of chapter 9 that sets \(m_{t,t+1} = M_x x_{t+1}/M_x x_t\) also implies too low a value of the market price of risk, again because the volatility of consumption growth is too small.

Under a preference for robustness, we have

\[
a_t = E_t \left( m_{t,t+1} m_{t,t+1}^u \right) y_{t+1} \tag{12.5.3} \]

where from (12.4.9)

\[
m_{t,t+1}^u = \exp \left[ \epsilon_{t+1}' w_{t+1} - .5 w_{t+1}' w_{t+1} \right]. \tag{12.5.4}
\]

By construction, \(E_t m_{t,t+1}^u = 1\). Hansen, Sargent, and Tallarini computed that \(E_t (m_{t,t+1}^u)^2 = \exp(w_{t+1}' w_{t+1})\) so that

\[
\sigma_t \left( m_{t,t+1}^u \right) = \sqrt{\exp \left( w_{t+1}' w_{t+1} - 1 \right)} \approx |w_{t+1}' w_{t+1}|. \tag{12.5.5}
\]

HST refer to \(\sigma_t (m_{t,t+1})\) as the one-period market price of Knightian uncertainty. Similarly, the \(\tau - t\) period market price of Knightian uncertainty is the conditional standard deviation of \(m_{t,t}^u\) defined by (12.4.9). A preference for robustness can boost the market price of risk by increasing these objects.

This is how you put in
12.5.1. Calibrated market prices of Knightian uncertainty

HST computed one-period market prices of risk for a calibrated version of the permanent income model described in chapter 9. In particular, they proceeded as follows:
1. Setting $\sigma = 0$ and $\beta R = 1$, HST used the method of maximum likelihood to estimate the remaining free parameters of chapter 9's permanent income model.

2. HST used those maximum likelihood parameter estimates as the approximating model of the endowment processes $d_t^*, \hat{d}_t$ for a representative agent whose continuation values they use to price risky assets. Thus, HST took a particular stand on how the representative agent created his approximating model, something that our robust control theory is silent about.

3. To study the effects of a preference for robustness on asset prices while leaving the consumption–investment allocation $(c_t, i_t)$ intact, HST lowered $\sigma$ below zero, but adjusted the discount factor according to the relation $\beta = \hat{\beta}(\sigma)$ given by equation (9.3.19), which defines a locus of $(\beta, \sigma)$ pairs that freeze $\{c_t, i_t\}$. For each $(\beta, \sigma)$ thereby selected, HST calculate market prices of Knightian uncertainty and the detection error probabilities associated with distinguishing the approximating model from the worst case model associated with $\sigma$. Figure 9.6.3 in chapter 9 reports those detection error probabilities as a function of $\sigma$. We are interested in the relation between the detection error probabilities and the $j$-period market prices of Knightian uncertainty.

4. For one and four period horizons, Figures 12.5.1 and 12.5.2 report the calculated market prices of Knightian uncertainty plotted against the detection error probabilities. These graphs have two salient features. First, there appear to be approximately linear relationships between the detection error probabilities and the market prices of Knightian uncertainty. In a continuous time, diffusion specification, Anderson, Hansen, and Sargent (20XXX) establish an exact linear such relationship. Second, the market price of Knightian uncertainty is substantial even for values of the detection error probability sufficiently high that it seems plausible to seek robustness against models so close to the approximating model. Thus, a detection error probability of .3 leads to a one-period market price of uncertainty of about .15, which can explain about half of the observed equity premium.

In chapter 14, we shall return to the relationship between detection error probabilities and the market price of Knightian uncertainty in a version of a permanent income model in which the representative agent must use a Kalman filter because he does not observe the state variables that drive our two-factor endowment process.
12.6. A model of occupational choice and pay

Aloyisius Siow (1984) and Sherwin Rosen (1995) and have used pure time-to-build structures to represent price and quantity cycles in markets for occupations under rational expectations. In their models, prospective new entrants into an occupation respond to their forecasts of the present value of wages that will begin accruing only after a period of schooling. We want to study how in equilibrium those forecasts and workers’ decisions would behave under a concern for model misspecification.

Siow and Rosen used partial equilibrium models cast in terms of dynamic supply and demand curves. To analyze how a concern for model misspecification affects demand and supply, we first find the representative agent whose preferences induce the demand curve and the technology that generates the supply curve. It is straightforward to cast Rosen’s model within the class of general equilibrium models of Chapter 10. Then the methods of section 12.2 and 12.4 can be used to construct a version of the model in which the representative agent has a concern about model misspecification indexed by \( \theta \in [\theta, \infty) \).

12.6.1. A one-occupation model

For concreteness, let the occupation be called engineering. Rosen (1995)’s model determines the stock of engineers \( N_t \); the number of new entrants into engineering school, \( n_t \); and the wage \( W_t \) of engineers. It takes \( k \) periods of schooling to become an engineer. We’ll set \( k = 4 \) in our example. Rosen’s model consists of the following equations: first, an inverse demand curve for engineers

\[
W_t = \eta_d - \alpha_d N_t + u_{dt}, \quad \alpha_d > 0; \tag{12.6.1} \]

second, a description of the education process as a time-to-build structure

\[
N_{t+k} = \delta N_{t+k-1} + n_t, \quad 0 < \delta < 1; \tag{12.6.2} \]

third, a definition of the expected present value of each new engineering student

\[
v_t = \beta^k E_t \sum_{j=0}^{\infty} (\beta \delta)^j W_{t+k+j}; \tag{12.6.3} \]

and fourth, a supply curve of new students as a function of \( v_t \)

\[
n_t = \eta_s + \alpha_s v_t + u_{st}, \quad \alpha_s > 0. \tag{12.6.4} \]

\[ \]

\[ \]

7 The MATLAB program \texttt{school2.m} in \texttt{c:/projects:hansar} computes the Rosen model.
Here \( u_t = [u_{st} \ u_{dt}]' \) is a stochastic process of labor demand and supply shocks. Under a potentially distorted model indexed by \( w_{t+1} \), the shocks \( u_{st}, u_{dt} \) are given by

\[
\begin{aligned}
\ u_{st} &= U_s z_t \\
\ u_{dt} &= U_{d1} z_t
\end{aligned}
\] (12.6.5) ["rinfo1"]

where

\[
\begin{aligned}
\ z_{t+1} &= A_{22} z_t + C_2 (\epsilon_{t+1} + w_{t+1}) \\
\end{aligned}
\] (12.6.6) ["rinfo2"]

the eigenvalues of \( A_{22} \) are bounded in modulus by \( \frac{1}{\sqrt{\beta}} \). \( U_s, U_{d1} \) are selector vectors, \( \epsilon_{t+1} \) is an i.i.d. vector stochastic process with mean zero and covariance matrix \( I \), and \( w_{t+1} \) is a vector of perturbations to the conditional means of the innovations to the approximating model. As usual, the approximating model assumes that \( w_{t+1} \equiv 0 \). Specification (12.6.5)–(12.6.6) allows the demand and supply shocks to be serially correlated.

We use the following:

**Definition 12.6.1.** A rational expectations equilibrium without a preference for robustness is a stochastic process \( \{W_t, N_t, v_t, n_t\}_{t=0}^{\infty} \) satisfying (12.6.1), (12.6.2), (12.6.3), (12.6.4), (12.6.5) and (12.6.6), \( w_{t+1} \equiv 0 \), the stability condition \( E_0 \sum_{t=0}^{\infty} \beta^t N_t^2 < +\infty \), and the initial conditions for \( N_{-1}, n_{-s}, s = 1, \ldots, -k + 1 \).

**12.6.2. Equilibrium with no preference for robustness**

In the model without a preference for robustness, \( E_t \) in (12.6.3) is the mathematical expectation evaluated with respect to the distribution under the approximating \( (w_{t+1} \equiv 0) \) model. With a preference for robustness, the mathematical expectation under the distorted model \( \hat{E}_t \) replaces \( E_t \) in (12.6.3). But the distorted model is endogenous. Discovering it requires knowing the common preferences of the malevolent agent and the representative agent. To put a preference for robustness into Rosen’s model, it is necessary first to map the model without a preference for robustness into the equilibrium framework of chapter 10. This will identify the preferences of a representative agent that the malevolent agent also uses to formulate perturbations that promote robustness.

In terms of the class of general equilibrium models of chapter 10, we represent Rosen’s model by sweeping the time-to-build structure into the household technology and the demand for engineers into the preference specification, while putting the supply of new engineers into the technology for producing goods.
Here is how. Take the household technology to be

\[
s_t = [\alpha_d 0 0 0] \begin{bmatrix} N_t \\ n_{t-1} \\ n_{t-2} \\ n_{t-3} \end{bmatrix} + 0n_t
\]

\[
\begin{bmatrix} N_{t+1} \\ n_t \\ n_{t-1} \\ n_{t-2} \end{bmatrix} = \begin{bmatrix} \delta_N & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} N_t \\ n_{t-1} \\ n_{t-2} \\ n_{t-3} \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} n_t.
\]

In the notation of chapter 10, these equations can be represented as

\[
s_t = \Lambda h_{t-1} + \Pi c_t
\]

\[
h_t = \Delta h_{t-1} + \Theta h c_t,
\]

where we have set \( n_t \) in Rosen’s model to \( c_t \) and \( [N_t \ n_{t-1} \ n_{t-2} \ n_{t-3}]' \) to \( h_{t-1} \) in the model of chapter 10. To complete the representation of (12.6.1), we set the preference shock \( b_t = \eta_d + \eta dt \).

We represent the supply of entering students by using the technology side of the model. In particular, we assume

\[
\begin{bmatrix} 1 \\ 0 \end{bmatrix} c_t + \begin{bmatrix} -1 \\ \alpha_s^{-1} \end{bmatrix} i_t + \begin{bmatrix} 0 \\ -1 \end{bmatrix} g_t = \begin{bmatrix} 0 \\ 0 \end{bmatrix} k_{t-1} + \begin{bmatrix} u_{st} \\ 0 \end{bmatrix}.
\]

This equation matches the representation of technology

\[
\Phi_c c_t + \Phi_i i_t + \Phi_g g_t = \Gamma k_{t-1} + d_t
\]

in chapter 10. Associated with this model is a representative agent who has preferences over \( c_t \) paths that are ordered by

\[
-E_0 \sum_{t=0}^{\infty} \beta^t \{ .5 (s_t - b_t) \cdot (s_t - b_t) + .5 g_t \cdot g_t \},
\]

where the mathematical expectation is taken with respect to the approximating model. Hansen and Sargent use the shadow prices from a planning problem.

\[\text{\footnote{In the language of Hansen and Sargent (200?), this preference representation is not canonical, meaning that it must be transformed to a canonical representation in order to get convenient representations of dynamic demand functions.}}\]
to construct a competitive equilibrium, as described briefly in chapter 10. The shadow prices $M^c_t, M^c_t, M^b_t$ for $s_t, c_t, h_t$, respectively, satisfy

$$M^s_t = b_t - s_t \quad \text{(12.6.9a)}$$

$$M^h_t = E_t \sum_{\tau=1}^{\infty} \beta^\tau (\Delta_h^t)^{\tau-1} \Lambda'M^s_t \quad \text{(12.6.9b)}$$

$$M^c_t = \Theta'_h M^h_t + \Pi'M^s_t. \quad \text{(12.6.9c)}$$

Since $\Pi = 0$ for the present example, we have

$$M^c_t = \Theta'_h E_t \sum_{\tau=1}^{\infty} \beta^\tau (\Delta_h^t)^{\tau-1} \Lambda'M^s_t. \quad \text{(12.6.10)}$$

It can be verified that the wage $w_t$ in Rosen’s model matches the shadow price $M^s_t$ and that the present value $v_t$ matches $\alpha_d^{-1}M_{ct}$ (compare (12.6.3) with (12.6.10)). Where $x_t$ is the state, $^9$ Hansen and Sargent show that $M^c_t = M_c x_t$ and $M^s_t = M_s x_t$ and give formulas for the matrices $M_c, M_s$. We can use these objects to compute the equilibrium values of $w_t = M_h x_t, v_t = M_c x_t$ in Rosen’s model. The solutions for the quantities can be determined from the representation for the equilibrium in the state space form

$$x_{t+1} = A^o x_t + C e_{t+1}.$$

The next section computes examples of equilibria of the model both without and without a preference for robustness. Appendix A solves the model by hand and describes some its analytical features.

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$^9$ The state $x_t$ equals $[N_t \ n_{t-1} \ n_{t-2} \ 1 \ z_{st} \ z_{dt}]'$, where $z_s$ is the supply shock and $z_d$ is the demand shock. The presence of $z_s$ and $z_d$ means that we can accommodate demand and supply shocks that are first-order autoregressive processes.
12.6.3. Example

A version of the model with a preference for robustness replaces (12.6.3) by

\[ v_t = \beta^k \hat{E}_t \sum_{j=0}^{\infty} (\beta \delta_N)^j W_{t+k+j}, \]  

(12.6.11) ["rosen3d"]

where \( \hat{E}_t \) is the mathematical expectation with respect to the distorted model. Representation (12.4.8) above implies that an equivalent representation of \( v_t \) in the model with a preference for robustness is

\[ v_t = \beta^k E_t \sum_{j=0}^{\infty} (\beta \delta_N)^j m_{t,t+k+j}^u W_{t+k+j}, \]  

(12.6.12) ["rosen30d"]

where \( m_{t,t}^u \) is the Radon-Nikodym derivative defined in (12.4.9). In (12.6.12), the expectation is evaluated under the approximating model. Equation (12.6.12) shows how a preference for robustness puts an adjustment for model uncertainty into \( v_t \). That adjustment gets reflected in the behavior of \( N_t, n_t, W_t \) in ways that the following example illustrates.

For alternative versions of the same model without a concern robustness (the solid lines) and with a concern for robustness with \(-\theta^{-1} = -0.5\) (the dotted lines), Fig. 12.6.1 shows impulse responses to an i.i.d. supply shock where the inverse demand shock is also i.i.d. Both of these impulse responses are evaluated under the approximating model.\(^{10}\) We set the covariance matrices of the two shocks to be \( I \) and the remaining parameter values at \( \delta_N = 0.95, \alpha_s = 1, \alpha_d = 0.1, \eta_s = 10, \eta_d = 30, \beta = 1/1.05.\(^{11}\)

The effects of a preference for robustness operate through the forecasting equation (12.6.11). The bottom left panel of Fig. 12.6.1 shows that under a preference for robustness, the initial adverse effect on \( v_t \) of a supply shock is greater in absolute value (more negative) than when there is no concern for robustness. The top right panel shows how, because of its more adverse implications for \( v_t \), the supply shock causes a lower entry rates under a preference for robustness.

\(^{10}\) Thus, for the robust version of the model the agents inside the model are basing their decisions on the distorted model, but we are assuming that the data are actually generated by the approximating model.

\(^{11}\) See appendix A for a description of the role of the ratio \( \alpha_s \alpha_d \) of the slopes inverse demand and supply function in influencing the solution, and for under the i.i.d. specification an inverse demand shock has no persistent effects on any variable in the model.
This means that under the approximating model, the wage actually declines less in response to a supply shock under a preference for robustness (see the top left panel). The top left panel shows wages declining less while the bottom left panel shows the expected present value declining more under a preference for robustness. This discrepancy reflects the pessimistic forecasts that emanate from the worker’s use of the distorted model to form $\hat{E}_t$. Wages decline less under a preference for robustness because the lower entry rate induced by the pessimistic forecast $v_t$ causes the actual stock of engineers $N_t$ to increase less under a preference for robustness (see the bottom right panel).\textsuperscript{12}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{impulse_responses.png}
\caption{Impulse responses to supply shock without a preference for robustness (solid lines) and with a preference for robustness with $\sigma = -.5$.}
\end{figure}

\textsuperscript{12} Appendix A gives analytical expressions that help provide more intuition about the shapes of the impulse response functions and the relations among them.
A. Solving Rosen’s model by hand

Using methods described by Sargent (1987), we can solve Rosen’s model by hand and thereby discover a reduced description of the state. Substituting equations (12.6.1), (12.6.3), and (12.6.4) into (12.6.2) and rearranging yields

\[
\left(1 + \beta \delta^2 + \alpha_s \alpha_s \beta^k - \delta N L - \beta \delta_N L^{-1}\right) N_{t+k} = E_t \left[(1 - \beta \delta_N L^{-1}) (\eta_s + u_{st}) + \alpha_s \beta^k (\eta_d + u_{d,t+k})\right],
\]

where \(L\) is the backward shift operator. Notice the appearance in the characteristic polynomial of \(\alpha_s \alpha_s = \frac{\alpha_s}{\alpha_d}\), the ratio of the slope of the inverse demand schedule to the slope of the inverse supply schedule. The polynomial in \(L\) on the left side evidently can be factored as \(f_0 f(\beta L^{-1}) f(L)\) where \(f(L) = (1 - \psi L)\) and \(|\psi| < 1\). Then the stabilizing solution of (12.A.1) is

\[
N_{t+k} = \psi N_{t+k-1} + E_t \left\{ \left(\frac{f_0^{-1}}{1 - \psi \beta L^{-1}}\right) \left[(1 - \beta \delta_N L^{-1}) (\eta_s + u_{st}) + \alpha_s \beta^k (\eta_d + u_{d,t+k})\right]\right\}.
\]

(12.A.2)

It follows from (12.A.2) that \(N_{t+k-1}\) is a complete description of the endogenous part of the state vector at the beginning of time \(t\). We could have guessed this from (12.6.2) because \(N_{t+k-1}\) is independent of decisions or shocks that occur before time \(t\).

When \(u_{st}, u_{dt}\) are i.i.d., (12.A.2) simplifies to

\[
N_{t+k} = \psi N_{t+k-1} + \eta_s + \alpha_s \beta^k + u_{st}.
\]

(12.A.3)

In the i.i.d. case, it follows from (12.6.2) and (12.A.3) that the decision rule for \(n_t\) is

\[
n_t = (\psi - \delta_N) N_{t+k-1} + \eta_s + \alpha_s \beta^k + u_{st}.
\]

(12.A.4)
Part III

Robust filtering
Chapter 13.  
A dual filtering problem

13.1. Filtering

The Kalman filter, a recursive method for estimating a hidden state vector, assumes that a decision maker knows the statistical model linking the hidden state to observables. If the decision maker regards the statistical model as only approximating a true data generating mechanism that he cannot specify, he may want estimators of the hidden state that are robust to model misspecification. This chapter describes such estimators. In light the remark attributed to Fellner about how probability slanting depends on the ‘prize’ (see page 25), it is not surprising that a robust filter depends partly on the decision maker’s criterion function. In this chapter we assume that the decision maker cares about a weighted sum of current and past errors in estimating the state. An alternative but equivalent way of thinking about this criterion is that at some initial date, the decision maker must commit himself to a particular estimation rule, and that after many periods have passed, the decision maker will be evaluated according to that weighted sum of state estimation errors.

As we shall see, we are naturally led to this way of specifying the decision maker’s criterion by pursuing duality arguments that correspond to ones encountered in chapter 4. Although for some economic problems, this criterion is plausible for some economic problems, for others it is not, as we shall argue in chapter 14. But we adopt it now because it of how it illuminates the duality of robust control and filtering.
13.1.1. Warning about recycling of notation

As is not unusual in presentations of filtering and control, we have recycled some notation. For example, we use the matrices $A$ and $K$ to denote different objects in the robust filtering and control problems.

13.2. Robust filtering and duality

The duality described in chapter 4 doubles the usefulness of the optimal linear regulator problem because to each control problem there corresponds a filtering problem, and vice versa. It is natural to suspect that there is also be a filtering problem that is dual to the robust linear regulator problem studied in chapters 6 and 7. There is.

13.2.1. Duality of ordinary filtering and control

We begin by recalling the structure of the duality presented in chapter 4. Then we'll state a corresponding duality result under a preference for robustness.

Let $\tilde{z}_t = C\lambda_t + D\mu_t$. Consider the linear regulator problem

$$-\lambda_0^t \Sigma \lambda_0 = \max_{\{\mu_t\}} \sum_{t=0}^{\infty} -\tilde{z}_t^t \tilde{z}_t$$

(13.2.1)

where the maximization is subject to an initial condition for $\lambda_0$ and the transition law

$$\lambda_{t+1} = A'\lambda_t + G'\mu_t.$$  

(13.2.2)

Here $\lambda_t$ is the state vector and $\mu_t$ is the control vector.

Chapter 4 displayed a filtering problem that is dual to this regulator problem and interpreted the $\lambda_t$'s and $\mu_t$'s of the control problem as Lagrange multipliers associated with the filtering problem. For $t \geq 0$, the filtering problem has a state vector $x_{-t}$ and an observation vector $y_{-t}$ that satisfy

$$x_{-t} = Ax_{-t-1} + C\epsilon_{-t}$$  

(13.2.3a)

$$y_{-t} = Gx_{-t-1} + D\epsilon_{-t}$$  

(13.2.3b)

where $\epsilon_{-t}$ is an i.i.d. Gaussian vector with mean zero and covariance matrix $I$.

Consider a recursive estimator of the hidden state:

$$\hat{x}_{-t} = A\hat{x}_{-t-1} + K(y_{-t} - G\hat{x}_{-t-1}),$$  

(13.2.4)
where $K$ is the Kalman gain matrix. The error in reconstructing the state at $t$ is

$$e_{-t} = x_{-t} - \hat{x}_{-t}. \quad (13.2.5)$$

The decision maker wants to estimate a linear combination $Hx_{-t}$ of the state at each $t$ and so poses the minimization problem

$$\min_K E \lim_{T \to \infty} T^{-1} \sum_{t=0}^T z'_{-t} z_{-t} \quad (13.2.6)$$

or

$$\min_K \text{trace}(H'HH) \quad (13.2.7)$$

subject to (13.2.3), (13.2.4), (13.2.5) and where $\Sigma = E(e_{-t}e'_{-t})$. For $H'HH$ of full rank, the minimized value of trace($H'HH\Sigma$) is independent of $H$. Define the operators

$$K(\Sigma) = (CD' + A\Sigma G') (DD' + G\Sigma G')^{-1} \quad (13.2.8)$$

$$T^*(\Sigma) = (A - K(\Sigma)G) \Sigma (A - K(\Sigma)G)' + (C - K(\Sigma)D) (C - K(\Sigma)D)' \quad (13.2.9)$$

The minimized value of $\Sigma$ solves the Riccati equation

$$\Sigma = T^*(\Sigma) \quad (13.2.10)$$

and the associated minimizing $K$ satisfies

$$K = K(\Sigma). \quad (13.2.10)$$
13.2.2. A robust linear regulator

Let \( \theta \in [\bar{\theta}, +\infty] \) be our robustness parameter and consider a robust linear regulator corresponding to (13.2.1), (13.2.2):

\[
-\lambda_0' \Sigma \lambda_0 = \max_{\{\mu_t\}} \min_{\{\phi_{t+1}\}} \sum_{t=0}^{\infty} \{-\tilde{z}'_t \tilde{z}_t + \theta \beta \phi_{t+1}' \phi_{t+1}\} 
\]  
(13.2.11)

where the maximization is subject to

\[
\lambda_{t+1} = A' \lambda_t + G' \mu_t + H' \phi_{t+1} \tag{13.2.12}
\]

and an initial condition \( \lambda_0 \). The solution of this robust control problem is a pair of decision rules

\[
\mu_t = -K' \lambda_t \tag{13.2.13a}
\]

\[
\phi_{t+1} = K'' \lambda_t. \tag{13.2.13b}
\]

We seek the robust filtering problem that corresponds to this robust linear regulator problem.

13.2.3. A dual robust filtering problem

The main purpose of this chapter is to find and interpret the filtering problem that is dual to (13.2.11), (13.2.12). The robust filtering problem surrounds the approximating model (13.2.3) with a set of perturbed models of the form

\[
x_{-t} = Ax_{-t-1} + C (\epsilon_{-t} + w_{-t}) \tag{13.2.14a}
\]

\[
y_{-t} = Gx_{-t-1} + D (\epsilon_{-t} + w_{-t}) \tag{13.2.14b}
\]

where \( \epsilon_{-t} \) is another i.i.d. Gaussian vector with mean zero and covariance matrix \( I \) and \( w_{-t} \) is a vector of measurable functions of \([y^{-t}, x^{-t}]\). The \( w_{-t} \) process represents specification errors that can feed back on the histories of the unobserved state and the observed variables. The decision maker constructs a robust filter by solving the following two-player zero-sum multiplier game:

\[
\text{trace } (H' H \Sigma) = \max_{\{w_{-t}\}} \min K \lim_{T \to \infty} T^{-1} \sum_{t=0}^{T} (z_{-t}' z_{-t} - \theta w_{-t}' w_{-t}) \tag{13.2.15}
\]

The extremizing \( \Sigma \) is the fixed point of iterations on \( T^* \circ D^* \) where \( T^* \) is the operator (13.2.10) associated with iterations on the Riccati equation associated with the ordinary Kalman filter, and the distortion operator \( D^* \) is defined as

\[
D^* (\Sigma) = \Sigma + \theta^{-1} \Sigma H' (\theta I - H \Sigma H')^{-1} H' \Sigma. \tag{13.2.16}
\]
The robust Kalman filter $K$ then satisfies

$$K = \mathcal{K} \circ \mathcal{D}^*(\Sigma)$$

where $\Sigma = T^* \circ \mathcal{D}^*(\Sigma)$. The maximizing $w_{-t}$ sequence has the recursive representation

$$w_{-t} = -\theta^{-1} [I + \theta^{-1} (C - KD)' \Sigma^{-1} (C - KD)]^{-1} (C - KD)' \Sigma^{-1} (A - KG) e_{-t-1},$$

which shows how the worst case mean distortions $w$ feed back on both $x_{-t-1}$ and $\hat{x}_{-t-1}$. This formula shows how the evil agent exploits his information advantage over the decision maker, who cannot observe $x_{-t-1}$.

### 13.3. The robust filtering problem

We now substantiate these claims about duality. We’ll actually seek a slightly more general result. Because chapters 6 and 7 both study discounted optimal linear regulators, we shall generalize (13.2.11), (13.2.12) to make it into a discounted optimal linear regulator, and we’ll seek a filtering problem that is dual to that discounted problem.

As we saw in chapter 4, though the filtering problem is typically applied in stochastic contexts, because the mathematics merely manipulates moment matrices, we can present an entirely nonstochastic derivations of a robust Kalman filter. Thus, we shall appeal a version of certainty equivalence to drop the random process $\epsilon_{-t}$ from the state space system and also shall drop mathematical expectations from the prediction error criterion that the decision maker seeks to minimize. Then we let $w_{-t} \equiv 0$ in the decision maker’s approximating model and allow for specification errors to be of the form:\footnote{See the discussion in section 4.2 for why we make time recede into the past with increases in $t$.}

$$x_{-t} = Ax_{-t-1} + Cw_{-t} \quad (13.3.1a)$$
$$y_{-t} = Gx_{-t-1} + Dw_{-t}. \quad (13.3.1b)$$

Here $t \geq 0$. Let $y^{-t}$ denote the history of $y$ up to $-t$. Let $\hat{E}[\cdot | y^{-t-1}]$ denote a filtered value conditioned on the history of $y$ up to time $-t-1$. We seek filtered values $\hat{x}_{-t} = \hat{E}[x_{-t} | y^{-t}], \hat{y}_{-t} = \hat{E}[y_{-t} | y^{-t-1}]$, where $\hat{E}(\cdot)$ is a distorted expectations operator. We restrict the filter to be time-invariant.

We shall eventually construct a robust filter by solving a non-stochastic zero-sum two-player game in which an evil prediction-error-maximizing agent
chooses a sequence of shocks \(\{w_{-t}\}\) to maximize a prediction error criterion. For now, we take as given the choice of the prediction-error-minimizing agent, and focus on the decision of an evil maximizing agent. To set the problem facing the evil agent, we form an observer system (see Kwakernaak and Sivan (1972)).

Emulating the measurement equation, we require the estimator of \(y\) to take the form:

\[
\hat{y}_{-t} = G \hat{x}_{-t-1}. \tag{13.3.2}
\]

It follows that the prediction error for \(y_{-t}\) is:

\[
y_{-t} - \hat{y}_{-t} = G (x_{-t-1} - \hat{x}_{-t-1}) + D w_{-t}.
\]

We consider updating schemes for \(\hat{x}\) that are parameterized by a fixed gain matrix \(K\). Such updating schemes are optimal for infinite horizon filtering problems. The forecast-error-minimizing agent is supposed to have chosen \(K\). The updating rule takes the form:

\[
\hat{x}_{-t} = A \hat{x}_{-t-1} + K (y_{-t} - \hat{y}_{-t}) \tag{13.3.3}
\]

or

\[
\hat{x}_{-t} = A \hat{x}_{-t-1} + K (y_{-t} - G \hat{x}_{-t-1}). \tag{13.3.4}
\]

Subtracting (13.3.3) from (13.3.1a) gives

\[
x_{-t} - \hat{x}_{-t} = (A - KG) (x_{-t-1} - \hat{x}_{-t-1}) + (C - KD) w_{-t}. \tag{13.3.5}
\]

Define the state reconstruction error:

\[
e_{-t} = x_{-t} - \hat{x}_{-t}. \tag{13.3.6}
\]

Then (13.3.5) can be expressed

\[
e_{-t} = (A - KG) e_{-t-1} + (C - KD) w_{-t}. \tag{13.3.7}
\]

The filter \(K\) is to be designed to minimize a quadratic form in the following linear combination of the forecast errors in the state:

\[
z_{-t} = He_{-t}. \tag{13.3.8}
\]

The criterion of the multiplier form of a two-person game for a robust filter is

\[
.5 \sum_{t=0}^{\infty} \beta^t (z'_{-t} z_{-t} - \theta w'_{-t} w_{-t}), \quad \beta \in (0, 1) \tag{13.3.9}
\]
Notice how $\beta^t$ weights forecast errors from the more recent past more heavily. The agent composing the filter aims to minimize this criterion by choosing $K$, while the evil agent aims to maximize it by choosing $w_{-t}$’s subject to (13.3.7) and (13.3.8). Here $\theta > 0$ is a penalty on the $w'_{-t} w_{-t}$ sequence. Given $K$, the maximized value of (13.3.9) is given by a value function

$$0.5 e'_0 \Sigma^{-1} e_0$$ (13.3.10)

where $\Sigma$ satisfies a Riccati equation for a dual problem to be shown below and the maximizing $w_{-t}$ sequence can be represented in the recursive form $w_{-t} = O e_{-t-1}$ where $e_{-t}$ evolves according to (13.3.7). Given $K$, we can compute $O$ by solving the linear regulator problem associated with (13.3.9), (13.3.7), (13.3.8). We give another algorithm for computing the $w_{-t}$ in the next section.

13.3.1. The evil agent’s problem

We now focus on the evil agent’s problem. As in chapter 7, we can use the optimal value function that emerges from this problem as a criterion function that the minimizing agent then uses to devise a robust $K$. Thus, in posing the problem of the evil agent in chapter 7, we frequently took the control law $u = -Fx$ as fixed and let the evil agent respond. We now take $K$ as fixed and study the problem of maximizing (13.3.9) by choice of $\{w_t\}$. We form the conjugate problem associated with choosing the $w_{-t}$’s to maximize (13.3.9). Let $\beta^t \lambda'_t$ denote the vector of Lagrange multipliers on (13.3.7), let $\beta^t \phi'_t$ be the vector of multipliers on (13.3.8), and form a Lagrangian. Among the first-order conditions for the problem of maximizing the Lagrangian with respect to $\{w_{-t}, e_{-t}\}_{t=0}^\infty$ and minimizing it with respect to $\{\lambda_t, \phi_t\}_{t=0}^\infty$ are:

$$w_{-t} : \quad w_{-t} = -\frac{1}{\theta} (C - K D)' \lambda_t$$ (13.3.11a)
$$z_{-t} : \quad z_{-t} = -\phi_t$$ (13.3.11b)
$$e_{-t-1} : \quad \beta \lambda_{t+1} = (A - K G)' \lambda_t + \beta H' \phi_{t+1}$$ (13.3.11c)
$$e_{-0} : \quad \lambda_0 = H' \phi_0$$ (13.3.11d)

We can use (13.3.11a) to get a convenient formula for the distortion $w_{-t}$. First, note that $\lambda_t$ is the vector of shadow prices of $e_{-t}$, so that $\lambda_t = \Sigma^{-1} e_{-t}$ where $\Sigma$ appears in the value function (13.3.10). This equation and (13.3.11a) imply

$$w_{-t} = -\theta^{-1} (C - K D)' \Sigma^{-1} e_{-t}.$$ (13.3.12)
To compute the feedback rule for the worst-case shock $w_{-t}$, substitute (13.3.7) into (13.3.12) and solve for $w_{-t}$ to get

$$w_{-t} = -\theta^{-1} \left[ I + \theta^{-1} (C - KD)' \Sigma^{-1} (C - KD) \right]^{-1} (C - KD)' \Sigma^{-1} (A - KG) e_{-t-1}$$

(13.3.13)

XXXXX Pierre: the above formula is new – it meets your criticism, but differs from your suggested formula. Below, we shall show how both $\Sigma$ and $K$ can be computed by solving the linear regulator (13.2.11), (13.2.12).

13.3.2. The dual problem

Use (13.3.11a) and (13.3.11b) to write

$$w'_{-t} w_{-t} = \frac{1}{\theta^2} \lambda_t' (C - KD) (C - KD)' \lambda_t$$

$$z'_{-t} z_{-t} = \phi_t' \phi_t.$$

Substituting these into (13.3.9) gives the dual criterion

$$\frac{1}{2\theta} \sum_{t=0}^{\infty} \beta^t \left\{ -\lambda_t' (C - KD) (C - KD)' \lambda_t + \theta \phi_t' \phi_t \right\}.$$  (13.3.14)

The dual problem is to minimize (13.3.14) by choice of $\{\phi_t\}_{t=0}^{\infty}$, subject to (13.3.11c) and (13.3.11d). For convenience, rewrite (13.3.11c), (13.3.11d) as

$$\lambda_{t+1} = \beta^{-1} (A - KG)' \lambda_t + H' \phi_{t+1}$$  (13.3.15a)

$$\lambda_0 = H' \phi_0.$$  (13.3.15b)

This is a discounted linear regulator problem with state $\lambda_t$ and control $\phi_{t+1}$. The optimized value of the objective functions of the original and dual problems are equal.

We can reinterpret the dual problem in terms of a time-domain version of the multiplier problem associated with (7.2.3), (7.2.4):

$$\min_{\{w_t\}} \sum_{t=0}^{\infty} \beta^t \left\{ -x_t' (H_0 - JF)' (H_0 - JF) x_t + \theta w_t' w_t \right\}$$

(13.3.16)

subject to

$$x_{t+1} = (A_o - BF) x_t + C w_{t+1}$$  (13.3.17a)

$$x_0 = C w_0.$$  (13.3.17b)

Notice that the dual filtering problem (13.3.14), (13.3.15a), (13.3.15b) corresponds to (13.3.16), (13.3.16), (13.3.17) with the settings in Table 13.3.1. This means that all of the computational methods that apply to the control problem can be used to solve the filtering problem, as we describe in the following section.
A dual filtering problem

Table 13.3.1: Duality of filter and control

<table>
<thead>
<tr>
<th>Filter</th>
<th>Σ</th>
<th>φ₀</th>
<th>λₜ</th>
<th>A'/β</th>
<th>G'/β</th>
<th>H'</th>
<th>C'</th>
<th>K'</th>
<th>D'</th>
</tr>
</thead>
<tbody>
<tr>
<td>Control</td>
<td>P</td>
<td>w₀</td>
<td>xₜ</td>
<td>A₀</td>
<td>B</td>
<td>C</td>
<td>H₀</td>
<td>F</td>
<td>J</td>
</tr>
</tbody>
</table>

13.3.3. Computing $K$

This section formulates a two-person zero-sum game that tells us how to compute a robust filter $K$. From chapters 6 and 7 we know the form of the analogous game for the control problem, so we can use the links in Table 13.3.1 to figure out the game that corresponds to the dual to the filtering problem.

The two-player zero-sum game associated with the robust control problem is

$$\max_{\{u_t\}} \min_{\{w_t\}} \sum_{t=0}^{\infty} \beta^t \{-z_t'z_t + \theta w_t'w_t\} \tag{13.3.18}$$

subject to

$$z_t = H_0 x_t + Ju_t \tag{13.3.19a}$$

$$x_{t+1} = A_o x_t + Bu_t + Cw_{t+1} \tag{13.3.19b}$$

$$x_0 = Cw_0. \tag{13.3.19c}$$

The $u_t$ component of the solution is a time-invariant feedback rule

$$u_t = -Fx_t, \tag{13.3.20}$$

where formulas for $F$ are given in chapters 2, 6, and 7.

Using Table 13.3.1, it follows that the two-player zero-sum game for the dual filtering problem is

$$\max_{\{\nu_t\}} \min_{\{\phi_t\}} \sum_{t=0}^{\infty} \beta^t \{-\tilde{z}_t'\tilde{z}_t + \theta \phi'_t \phi_t\} \tag{13.3.21}$$

subject to

$$\tilde{z}_t = C'\lambda_t + D'\mu_t \tag{13.3.22a}$$

$$\lambda_{t+1} = \beta^{-1} A'\lambda_t + \beta^{-1} G' \mu_t + H' \phi_{t+1} \tag{13.3.22b}$$

$$\lambda_0 = H' \phi_0. \tag{13.3.22c}$$

This problem (13.3.21), (13.3.22) can also be formulated as an optimal linear regulator. Equilibrium choices of $\mu_t, \phi_{t+1}$ have representations of the form (13.2.13) where $K', K'_φ$ can be calculated with formulas analogous to those used to solve the corresponding control problem (13.3.18), (13.3.19).
13.3.4. Another way to compute the worst-case shock

Formula (13.3.13) shows how to compute the worst-case shock associated with the robust filter. An alternative way to compute it is to use the dual problem to compute $K$, then to formulate the primal problem, say with the following convention for our time index $t$:

$$
\max_{\{w_t\}} \sum_{t=0}^{\infty} \beta^{-t} (e'_t H' H e_t - \theta w'_t w_t) 
$$

subject to

$$
e_t = (A - K G) e_{t-1} + (C - K D) w_t
$$

given an initial $e_{-1}$. Note how the discounting of the past in problem (13.3.9), (13.3.7), (13.3.8) corresponds to anti-discounting the future in (13.3.23a) (because $|\beta| < 1$). The feedback rule for the worst case shock is

$$
w_t = O e_{t-1},
$$

where $O$ equals the usual feedback matrix\footnote{Namely, $-F$ in $u_t = -Fx_t$ in the linear regulator in chapters 2 and 6.} for the optimal linear regulator associated with (13.3.23).

Having found $O$ either from formula (13.3.13) or (13.3.24), we return to our original convention for the index $t$ and use (13.3.2) to represent the robust filter recursively as

\begin{align*}
\hat{y}_{-t} &= G \hat{x}_{-t-1} \\
\hat{x}_{-t} &= A \hat{x}_{-t-1} + K (y_{-t} - \hat{y}_{-t}).
\end{align*}

The associated worst-case law of motion for the state and the observed variables is

\begin{align*}
w_{-t} &= O (x_{-t-1} - \hat{x}_{-t-1}) \\
x_{-t} &= A x_{-t-1} + C w_{-t} \\
y_{-t} &= G x_{-t-1} + D w_{-t}.
\end{align*}
13.4. Robustifying Muth’s filter

As an example, we set $\beta = 1$ and consider Muth’s (1960) problem of estimating the position of a random walk disturbed by measurement error. We assume the same approximating model that Muth used:

\begin{align}
  x_{t+1} &= x_t + \alpha \hat{e}_{1,t+1} \quad \text{(13.4.1a)} \\
  y_{t+1} &= x_t + \hat{e}_{2,t+1} \quad \text{(13.4.1b)}
\end{align}

where $\alpha$ is the signal to noise ratio. The process $\hat{e}_{t+1} = [\hat{e}_{1,t+1} \hat{e}_{2,t+1}]'$ is an i.i.d. Gaussian process with mean zero and identity covariance matrix. The state $x_t$ is to be estimated from current and past values of $y_t$. We consider the filter

\begin{equation}
  \hat{x}_{t+1} = \hat{x}_t + K (y_{t+1} - \hat{x}_t) \quad \text{(13.4.2)}
\end{equation}

where $\hat{x}_{t+1}$ is the estimate of the state using the history of $y_s$ through $t+1$. We want $K$ to be robust to misspecification of (13.4.1).

To attain robustness, we consider a family of perturbed models:

\begin{align}
  x_{t+1} &= x_t + \alpha (e_{1,t+1} + w_{1,t+1}) \quad \text{(13.4.3a)} \\
  y_{t+1} &= x_t + e_{2,t+1} + w_{2,t+1} \quad \text{(13.4.3b)}
\end{align}

where $e_{t+1} = [e_{1,t+1} e_{2,t+1}]'$ is another i.i.d. Gaussian process with mean zero and identity covariance matrix; and $[w_{1,t+1}, w_{2,t+1}]$ are distortions to the conditional means of the two shocks $\hat{e}_{t+1}$ in (13.4.1). Subtracting (13.4.2) from (13.4.3a) and using (13.4.3b) gives

\begin{equation}
  e_{t+1} = (1 - K) e_t + \alpha e_{1,t+1} - K e_{2,t+1} + \alpha w_{1,t+1} - K w_{2,t+1} \quad \text{(13.4.4)}
\end{equation}

where $e_t \equiv x_t - \hat{x}_t$. Because of the form of the solution of the linear regulator problem for the worst case errors, we can represent the worst case mean distortions as

\begin{align}
  w_{1,t+1} &= -N_1 e_t \\
  w_{2,t+1} &= -N_2 e_t, \quad \text{(13.4.5)}
\end{align}

where $N_1, N_2$ are submatrices of $F^*$ in (13.3.24). Please notice that $N_1, N_2$ are functions of $\theta$ and $K$ in the linear regulator problem (13.3.23).

For arbitrary $K$ and fixed $w_{1,t+1} = -N_1 e_t, w_{2,t+1} = -N_2 e_t$, the error in reconstructing the state when the worst case model associated with $N_1, N_2$ prevails is

\begin{equation}
  e_{t+1} = (1 - K) e_t + \alpha N_1 e_t + KN_2 e_t + \alpha e_{1,t+1} - K e_{2,t+1} \quad \text{(13.4.6)}
\end{equation}
or

\[ e_{t+1} = \chi e_t + \alpha e_{1,t+1} - Ke_{2,t+1}, \]  

(13.4.7)

where

\[ \chi = 1 - K - \alpha N_1 + KN_2. \]  

(13.4.8)

Equation (13.4.7) gives the law of motion of the error \( e_t \) in reconstructing the state for filter \( K \) when the conditional means of the shocks are feeding back on \( e_t \) via \( N_1, N_2 \). Denote the variance of \( e_t \) by \( \text{var}_e(K; N_1, N_2) \). From (13.4.7) it follows directly that

\[ \text{var}_e(K; N_1, N_2) = \frac{\alpha^2 + K^2}{1 - \chi^2}. \]  

(13.4.9)

We also have the decomposition of the spectral density of \( \text{var}_e(K; N_1, N_2) \):

\[ S_e(\omega; K, N_1, N_2) = g_1(\omega) g_1(-\omega) + g_2(\omega) g_2(-\omega) \]  

(13.4.10)

where \( g_1(\omega) = \frac{\alpha}{1 - \chi \exp(-i\omega)} \) and \( g_2(\omega) = \frac{K}{1 - \chi \exp(-i\omega)} \).

Consider \( \text{var}_e \) as a function of \( K \). Let \( \hat{K}(\theta) \) be the robust filter associated with \( \theta \). When \( N_1, N_2 \) correspond to the worst case distortions for a given \( \theta \), we know that \( \text{var}_e(K; N_1, N_2) \) is minimized at \( \hat{K}(\theta) \).

### 13.4.1. Ordinary Kalman filter

Let \( K^* = \hat{K}(+\infty) \) denote the standard Kalman filter. If \( \theta = +\infty \), then \( N_1 = N_2 = 0 \) and the variance of \( e_t \) simplifies to:

\[ \text{var}_e(K; 0, 0) = \frac{\alpha^2 + K^2}{1 - (1 - K)^2} = \frac{\alpha^2 + K^2}{2K - K^2}. \]  

(13.4.11)

Minimizing (13.4.11) with respect to \( K \) gives a formula for \( K \) that agrees with that produced by the ordinary Kalman filter: \( K^* = \frac{\sqrt{\alpha^2 + 4\alpha^2 - \alpha^2}}{2} \). When \( \alpha = 1 \), this equals \( \frac{\alpha}{2} \), the golden ratio.
Figure 13.4.1: Variance of \( e_t(K; N_1, N_2) \) as function of \( K \) for \( N_1 \) and \( N_2 \) evaluated at \( \theta = 10^8 \). Here \( K^* \approx \hat{K}(\theta) \), both denoted by the asterisk. The two curves are for two values of the signal-noise ratio \( \alpha = 1 \) and \( \alpha = 1.78 \).

Figure 13.4.2: Variance of \( e_t(K; N_1(\theta), N_2(\theta)) \) as function of \( K \) for \( N_1 \) and \( N_2 \) evaluated at \( \theta = 7 \). Here \( K^* < \hat{K}(\theta) \) (where \( \hat{K} \) is denoted by the x and \( K^* \) by the small vertical line on the curve \( \text{var}_e(K) \)). The two curves are for two values of the signal-noise ratio \( \alpha = 1, 1.78 \).

13.4.2. Illustrations

For fixed \( \theta < \infty \), we can determine a \( \hat{K}(\theta) \) by solving the two player game (13.3.21), (13.3.22). We can also find the associated feedback rules for the
conditional means of the shocks $N_1(\theta), N_2(\theta)$ associated with $\hat{K}(\theta)$ from the optimal linear regulator (13.3.23).

In Fig. 13.4.1, we have fixed $\theta = 10^8$ and derived the associated $\hat{K}, N_1, N_2$ (all three are functions of $\theta$) and have plotted $\Delta(K; N_1, N_2)$, the variance of $e_1(K)$, as a function of $K$. It has a minimum at $\hat{K}(\theta)$. We have also put $K^*$ =
Figure 13.4.5: Frequency decomposition of the reconstruction error variance \( \text{var}_r(K; N_1, N_2) \) for \( \theta = 7 \) for \( \hat{K}(\theta) \) and \( K^* \), \( \alpha = 1.78 \). The solid curve is for \( \hat{K} \), the dotted one for \( K^* \).

Figure 13.4.6: The robust Kalman gain \( \hat{K}(\theta) \) as a function of \( \log(\theta) \) and \( \alpha \).

\( \hat{K}(+\infty) \) and \( \hat{K}(\theta) \) on the graph. For this large value of \( \theta \), \( K^* \) is indistinguishable from \( \hat{K}(\theta) \).

Fig. 13.4.2 contains the same information as Fig. 13.4.1, except for the value of \( \theta = 7 \). Now \( \hat{K}(7) > K^* = K(\infty) \), though the state reconstruction error variances \( \text{var}_r \), associated with them are close.

Fig. 13.4.3 displays the frequency decomposition of \( \Delta(K^*; 0, 0) \). This is the frequency decomposition of the variance of \( e_t \) under the assumption of no specification error, using the optimal \( K^* \) (the ordinary Kalman filter) with \( \alpha = 1 \).
Fig. 13.4.7: The robust Kalman gain \( \hat{K}(\theta) \) as a function of \( \log(\theta) \), given \( \alpha = 1.78 \).

Fig. 13.4.4 displays the frequency decomposition of \( \Delta(K; N_1(7), N_2(7)) \) for two values of \( K: K^* \) and \( \hat{K}(7) \). Here 7 is the value of \( \theta \). Thus, the dotted line is the frequency decomposition of \( \Delta(K^*; N_1(7), N_2(7)) \) while the solid line is the frequency decomposition of \( \Delta(\hat{K}(7); N_1(7), N_2(7)) \). Fig. 13.4.4 is for \( \alpha = 1 \), while Fig. 13.4.5 if for \( \alpha = 1.78 \).

Fig. 13.4.3 shows that the ordinary Kalman filter \( K^* \) is most vulnerable to low frequency components of \( e_t \), which can be induced by having the worst case conditional means feed back positively on \( e_t \). Fig. 13.4.4 shows how the worst case conditional means associated with \( \theta = 7 \) pump up the low frequencies of \( e_t \), and how the robust \( \hat{K}(7) \) filter achieves a lower variance \( \Delta(\hat{K}; N_1, N_2) \) by flattening the spectrum, accepting higher variance at higher frequencies in exchange for lower variance at the low frequencies where the worst case conditional means operate the strongest.

Fig. 13.4.6 and Fig. 13.4.7 show the robust Kalman gain \( \hat{K} \) as functions of \( \log(\theta) \) and \( \alpha \). These figures show how increasing the preference for a robust filter (i.e., decreasing \( \theta \)) raises the Kalman gain.
13.5. Discounting and the direction of time

This chapter has been partly motivated by mechanical questions associated with duality. From the duality of ordinary (non-robust) filtering and control described in chapter 4, we should expect there to exist a filtering problem that is dual to the robust discounted optimal linear regulator problem analyzed in chapters 6 and 7. By reverse engineering, this chapter has found that robust filtering problem.

We ask the reader to notice the timing incorporated in the criterion function that the robust filter minimizes: a geometrically discounted sum of current and past forecast errors. We discover this criterion by mechanically pursuing the implications of duality. However, that criterion is not appropriate in many economic models, in particular, those that tell us and the agents in them to care about current and future returns. In devising a robust filter, such agents should limit their attention to forecast errors of the current and future variables that influence payoffs. We study the formulation of this chapter partly because of the light it throws on duality and also because it is widely used in the control literature. In chapter 14, we shall describe a different robust filtering problem with a payoff function that is exclusively forward looking. Nevertheless, it is possible to imagine economic contexts in which the timing convention of the filter of this chapter can be defended, for example, where the agent who filters must commit himself in advance to a filtering rule and then be judged on the average forecasting behavior only after much time has elapsed.

Without a preference for robustness, the dual filtering problem (the Kalman filter) described in chapter 4 has been widely used in macroeconomics. However, it is questionable whether the robust filter that engineered in this chapter should be so widely used in economics because of the peculiar backward-looking objective function that it imputes to the decision maker.\(^3\) In dynamic economic problems, it is more natural to expect filtering problems to arise in conjunction with maximization of forward-looking criteria like (6.2.5). Here the decision maker is indifferent to past estimation errors, but cares about current and future ones. We shall derive a robust filter for that case in chapter 14. It will differ from the one in this chapter because the distortions induced by the error-variance-maximizing evil agent depend on the decision maker’s criterion function. Under

\(^3\) The robust filter derived in this chapter seems applicable in situations where the decision maker must commit himself to a filtering rule at the beginning of time, then submit his forecast errors for evaluation after a long time has passed.
objective functions like (6.2.5), the ordinary Kalman filter will turn out to be robust to misspecification.
Chapter 14.

Estimation and decision

_In commerce bygones are forever bygones and we are always starting clear at each moment, judging the value of things with a view to future utility. Industry is essentially prospective not retrospective._
— William Stanley Jevons, 1871

14.1. Introduction

This chapter combines and extends ideas about control and filtering from chapters 6, 4, and 13. We study a setting where a decision maker does not observe parts of the state that help forecast variables that he cares about. We formulate a joint control and prediction problem and show how it can be represented recursively. The problem separates neatly into an ordinary Kalman filtering problem and an ordinary robust decision problem with appropriately adjusted state and disturbance vectors. In getting this two-part representation, the Kalman filter emerges as the solution of a minimum statistical discrepancy problem like that studied in chapter 4.

This chapter describes how the ordinary Kalman filter of chapter 4 is robust with respect to the purely ‘forward looking’ criteria that often appear in economic models. By way of contrast, the filter of chapter 13 is robust with respect to a ‘backward-looking’ criterion. The filter of chapter 13 is often recommended in the control literature (see Whittle (1990, 1996) and Başar and Bernhard (1996)). But because the decision makers in economists’ models typically have forward-looking objective functions like those studied in this chapter, the sense of robustness that this chapter ascribes to the ordinary Kalman filter seems particularly relevant for economists.
14.2. Economic Setting

A decision maker wants to maximize
\[ E_0 \sum_{t=0}^{\infty} \beta^t r(f_t, y_t, u_t), \quad 0 < \beta < 1 \] (14.2.1a)
where \( E_0 \) is the mathematical expectation conditioned on information known at 0 and the one-period return function is
\[ r(f, y, u) = -[f' \ y'] R \begin{bmatrix} f \\ y \end{bmatrix} - u'Q u - u' W \begin{bmatrix} f \\ y \end{bmatrix}. \] (14.2.1b)

Here \( f, y \) are elements of a state vector and \( u \) is a control vector for a system with the state-space representation
\[
\begin{bmatrix} f_{t+1} \\ y_{t+1} \\ z_{t+1} \\ \end{bmatrix} = \begin{bmatrix} A_{ff} & A_{fy} & 0 \\ A_{uf} & A_{yy} & A_{yz} \\ A_{zf} & A_{zy} & A_{zz} \end{bmatrix} \begin{bmatrix} f_t \\ y_t \\ z_t \end{bmatrix} + \begin{bmatrix} B_f \\ B_y \\ B_z \end{bmatrix} u_t + \begin{bmatrix} 0 \\ 0 \\ C_y \end{bmatrix} (\epsilon_{t+1} + w_{t+1}) \] (14.2.2)
where \( \epsilon_{t+1} + w_{t+1} \) is a composite shock process to be described in detail in the next subsection and where \( C_y C_y' \) is nonsingular. Throughout this chapter we maintain:

**Assumption 1**: Current and past values of \((f, y)\) are in the information set of the decision-maker.

Notice that \( f, y, u \), but not \( z \), appear in the current period return function. The vector \( z_t \) consists of information variables that help forecast variables that appear in the objective function. In some of our economic examples \( B_z = 0 \), though that is not necessary. We assume that \( z \) is not observed and that it is related to an observable estimate \( \hat{z}_t \) by
\[ z_t = \hat{z}_t + G_z (\epsilon_{zt} + w_{zt}). \] (14.2.3)
where \( G_z \) satisfies \( G_z G_z' = \Sigma \) and \( \Sigma \) is the asymptotic covariance matrix of errors in reconstructing the state from a Kalman filter to be described soon. Define
\[ \hat{x}' = [f' \ y' \ \hat{z}']. \]
We shall eventually show how \( \hat{z} \) follows the law of motion
\[ \hat{z}_{t+1} = A_z \hat{x}_t + B_z u_t + \hat{C}_z \epsilon_{t+1} \] (14.2.4)
where \( A_z = [A_{zf} \ A_{zy} \ A_{zz}] \); \( \hat{C}_z \) is a matrix determined below by the same Kalman filtering problem that also determines \( G_z \), and \( \epsilon_{t+1} \) is a function of the innovation in \( y_{t+1} \).
14.2.1. Shocks and filtered processes

The random version of the model assumes that $\epsilon_x$ and $\epsilon_z$ are i.i.d. Gaussian disturbances with mean vectors zero and identity covariance matrices. All calculations of robust rules and distortions in this chapter can be done while setting $\epsilon_x, \epsilon_z$ to be zero, as the result of an applicable certainty equivalence principle. So we set $\epsilon_x, \epsilon_z$ to be zero until further notice. The vectors $w_x$ and $w_z$ are distortions to the conditional means of $\epsilon_x$ and $\epsilon_z$, respectively.

14.3. A two-step procedure

Problem (14.2.1), (14.2.2), (14.2.3) impels the decision maker to choose $u_t$ in light of his estimate of the hidden part of the state $z_t$. The main finding of this chapter is that a robust decision rule for (14.2.1) can be constructed by solving the estimation and control problems separately, and that the estimation part of the problem is solved by an ordinary Kalman filter. That the decision maker must act on the basis of an estimate $\hat{z}_t$ of $z_t$ opens additional sources of misspecification to consider in designing a robust decision rule. The Kalman filter allow us to construct an innovations representation for $z$ that identifies two orthogonal components of the shock processes that impinge on $z$ under the approximating model. Robust decision rules are attained by contemplating the effects of distortions to the conditional means of both of those components. Misspecified dynamics are accommodated by allowing those distortions to feed back on the history of the state.

The principal outcomes of our analysis can be summarized by saying that robust decision rules for model (14.2.1), (14.2.2), (14.2.3) can be computed sequentially in two steps:

1. The first step is to find the appropriate $G_z$ and $\Sigma = G_zG_z'$ associated with (14.2.3). To do this, we solve an ordinary Kalman filter problem. Let $\Sigma$ be the positive semi-definite matrix that solves the associated algebraic

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1 However, we must appeal to the presence of the Gaussian terms to justify the detection error probability calculations of chapter 8.

2 This filter can be computed by using the Matlab command $[K, S] = kfilter(A_{22}, A_{y2}, C_yC_y', C_zC_z', C_yC_z')$, where the last three matrices are the covariance matrices for the state noise, the measurement noise, and the cross-covariance matrix between state and measurement noise.
Riccati equation
\[ \Sigma = A_{zz} \Sigma A'_{zz} + C_z C'_z \]
\[ - \left[ A_{zz} \Sigma A'_{yz} + C_z C'_y \right] \left[ A_{yz} \Sigma A'_{yz} + C_y C'_y \right]^{-1} \left[ A_{yz} \Sigma A'_{zz} + C_y C'_z \right]. \]  \hspace{1cm} (14.3.1)

Then compute \( \tilde{C}_z \) as the Cholesky factor of \( \Sigma \):
\[ \Sigma = \tilde{C}_z \tilde{C}'_z. \]

Set \( G_z \) equal to \( \tilde{C}_z \). Define
\[ \Lambda = \begin{bmatrix} A_{yz} \Sigma A'_{yz} + C_y C'_y & A_{yz} \Sigma A'_{zz} + C_y C'_z \\ A_{zz} \Sigma A'_{yz} + C_z C'_y & A_{zz} \Sigma A'_{zz} + C_z C'_z \end{bmatrix}. \]  \hspace{1cm} (14.3.2)

Factor \( \Lambda \) according to
\[ \Lambda = \begin{bmatrix} \hat{C}_y & 0 \\ \hat{C}_z & \hat{C}_z \end{bmatrix} \begin{bmatrix} \hat{C}_y & 0 \\ \hat{C}_z & \hat{C}_z \end{bmatrix}', \]  \hspace{1cm} (14.3.3)

where \( \hat{C}_y \) is the Cholesky factor of \( \Lambda_{11} \), \( \hat{C}_z \) is the Cholesky factor of \( \Lambda_{22} - \Lambda_{21} \Lambda_{11}^{-1} \Lambda_{12} \), and \( \hat{C} = K\hat{C}_y \) where \( K = \Lambda_{21} \Lambda_{11}^{-1} \) is the Kalman gain. By construction, \( \hat{C}_y, \hat{C}_z \) are nonsingular. In summary, the inputs to the first step are \([A_{zz}, A_{yz}, C_y, C_z]\) and the outputs are \([\hat{C}_y, \hat{C}_z, \hat{C}_z]\). Below we shall interpret these outputs and also \( \Sigma, \Lambda \).

2. Define the filtered state vector \( \hat{x}' = [f' \ y' \ \hat{z}']' \). Let \(^*\) superscripts denote next period values. Compute the decision rule for \( u \) that solves the following zero-sum, two-person game:
\[ -\hat{x}' V \hat{x} = \max_u \min_{\hat{w}, \tilde{w}} \rho(f, y, u) - \beta \hat{x}'' V^* \hat{x}^* + \beta \theta (\hat{w}' \hat{w} + \tilde{w}' \tilde{w}) \]  \hspace{1cm} (14.3.4a)

where the extremization is subject to:
\[ f^* = A_f \hat{x} + B_f u \]
\[ y^* = A_y \hat{x} + B_y u + \hat{C}_y \hat{w} \]
\[ \tilde{z}^* = A_z \hat{x} + B_z u + \hat{C}_z \tilde{w} + \tilde{C}_z \tilde{w}, \]  \hspace{1cm} (14.3.4b)

where \( A_h = [A_{hf} \ A_{hy} \ A_{hz}] \) for \( h = f, y, z \). Note the roles of \([\hat{C}_y, \hat{C}_z, \hat{C}_z]\) and the two new shocks \( \hat{w}, \tilde{w} \) that are the controls of the minimizing agent.
There are several remarkable features of this algorithm. First, (14.3.1) is the Riccati equation associated with the ordinary (i.e., non-robust) Kalman filter for a state-space model with system matrices $A_{zz}, C_z, A_{yz}, C_y, C_z C'_y$. Second, (14.3.4) defines an ordinary linear quadratic robust decision problem with state $\tilde{x}$. It can be solved using one of the methods from chapter 3. Next, the third equation of (14.3.4) is interpretable as an extension of a standard innovations representation for $z_t$. An ordinary innovations representation would be of the form (14.2.4), to which the third equation of (14.3.4) adds the additional shock distortion $\tilde{C}_z \tilde{w}$. This extra mean distortion identifies additional directions of misspecifications against which the decision maker wants robustness, which he does with the assistance of the minimizing agent.

The main purpose of this chapter is to defend and interpret this two step procedure. We do so by formulating a zero-sum, two-person game in terms of the original system (14.2.1), (14.2.2), (14.2.3) and showing how it can be represented as (14.3.4).

14.4. A game to get a robust filter and control

We begin by formulating a game directly in terms of problem (14.2.1), (14.2.2), (14.2.3). We start with a deterministic version, so we set $\epsilon_x$ and $\epsilon_z$ to zero. We can produce a robust decision rule by formulating a zero-sum, two-player game recursively via the following Bellman equation:

$$-\tilde{x}' V \tilde{x} = \max_{u, w_x, w_z} \min_{x, y} r(f, y, u) + \beta \theta (w'_x w_x + w'_z w_z) - \beta x'^* V x^*$$  \hspace{1cm} (14.4.1)

where the extremization is subject to

$$\begin{bmatrix} f^* \\ y^* \\ z^* \end{bmatrix} = \begin{bmatrix} A_{ff} & A_{fy} & 0 \\ A_{fy} & A_{yy} & A_{yz} \\ A_{zf} & A_{zy} & A_{zz} \end{bmatrix} \begin{bmatrix} f \\ y \\ z \end{bmatrix} + \begin{bmatrix} B_f \\ B_y \\ B_z \end{bmatrix} u + \begin{bmatrix} 0 \\ C_y \\ C_z \end{bmatrix} w_x$$  \hspace{1cm} (14.4.2)

and

$$z = \tilde{z} + G_z w_z,$$  \hspace{1cm} (14.4.3)

where again $x' = (f' \ y' \ z')$ and $\tilde{x}' = (f' \ y' \ \tilde{z}')$. Given $G_z$, we shall seek the fixed point $V = V^*$ of the mapping from $V^*$ to $V$ defined by (14.4.1). For now, we take $G_z$ as given, and focus on the two-period problem on the right.

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3 See Hansen and Sargent (2004, chapter 8) for an extensive discussion of innovations representations and some of their applications in economics.
side of (14.4.1). (Later we’ll describe how $G_z$ is determined as part of this game from the fixed point of an equation for updating $\hat{z}$.) The separability in $u$ and $(w_x, w_z)$ of the problem on the right side of (14.4.1) lets us solve it in two steps by forming an inner problem that minimizes over $(w_x, w_z)$ and an outer problem that maximizes over $u$.

14.4.1. Inner problem

The inner problem is:

$$\min_{w_x, w_z} (w_x' w_x + w_z' w_z)$$  \hspace{1cm} (14.4.4)

subject to (14.4.3) and the first two rows of (14.4.2), namely,

$$y^* = A_y f + A_{yy} y + A_{yz} z + B_y u + C_y w_x$$  \hspace{1cm} (14.4.5a)

$$z^* = A_z f + A_{zy} y + A_{zz} z + B_z u + C_z w_x.$$  \hspace{1cm} (14.4.5b)

This problem takes next period’s state $(f^*, y^*, z^*)$ as given, which in random settings corresponds to conditioning on these variables. Note that although $f^*$, $y^*$ will actually be observed next period, $z^*$ won’t, so that for the ‘outer problem’ we’ll have to take into account that we have conditioned on $z^*$.

We can ignore the equation for $f^*$ in (14.4.5) because $f^*$ is not altered by either $z$ or $w_z$. Use (14.4.5c) to eliminate $w_z$ and rewrite the objective:

$$\min_{z, w_x} \left[ (z - \hat{z})' \Sigma^{-1} (z - \hat{z}) + w_x' w_x \right].$$  \hspace{1cm} (14.4.6)

Put Lagrange multipliers $2\lambda_y$ and $2\lambda_z$ on (14.4.5a) and (14.4.5b), respectively, and obtain the first-order conditions:

$$z - \hat{z} = \Sigma \left( A_y' \lambda_y + A_{zz}' \lambda_z \right)$$  \hspace{1cm} (14.4.7a)

$$w_x = C_y' \lambda_y + C_z' \lambda_z.$$  \hspace{1cm} (14.4.7b)

Define $A_y' = [A_{fy}' \ A_{yy}' \ A_{yz}']$ and $A_z' = [A_{fz}' \ A_{yz}' \ A_{zz}']$. A direct calculation applying the definition of $\hat{x}$ yields

$$y^* - A_y \hat{x} - B_y u = C_y' w_x + A_{yz} (z - \hat{z})$$  \hspace{1cm} (14.4.8a)

$$z^* - A_z \hat{x} - B_z u = C_z' w_x + A_{zz} (z - \hat{z}).$$  \hspace{1cm} (14.4.8b)

---

4 The inner problem is evidently a version of the Kalman filtering problem as it was posed in chapter 4.
Substituting for $w_x$ and $z - \hat{z}$ from the first order conditions (14.4.7) gives
\[
\begin{bmatrix}
y^* - A_y\hat{x} - B_yu \\
z^* - A_z\hat{x} - B_zu
\end{bmatrix}
= \Lambda \begin{bmatrix}
\lambda_y \\
\lambda_z
\end{bmatrix}
\tag{14.4.9}
\]
where
\[
\Lambda = \begin{bmatrix}
A_{yz}\Sigma A_{yz} + C_yC_y' & A_{yz}\Sigma A_{zz} + C_yC_z' \\
A_{zz}\Sigma A_{yz} + C_zC_y' & A_{zz}\Sigma A_{zz} + C_zC_z'
\end{bmatrix}.
\tag{14.4.10}
\]
Solving (14.4.9) for $\lambda_y, \lambda_z$ gives
\[
\begin{bmatrix}
\lambda_y \\
\lambda_z
\end{bmatrix} = \Lambda^{-1} \begin{bmatrix}
y^* - A_y\hat{x} - B_yu \\
z^* - A_z\hat{x} - B_zu
\end{bmatrix}.
\tag{14.4.11}
\]
Finally, substitute (14.4.11) into the first-order conditions (14.4.7) to solve for $(w_x, z - \hat{z})$, and use the result to evaluate $w_x'w_x + (z - \hat{z})'\Sigma^{-1}(z - \hat{z})$:
\[
\begin{align*}
\min_{w_x, z} & \left(w_x'w_x + (z - \hat{z})'\Sigma^{-1}(z - \hat{z})\right) \\
&= \begin{pmatrix}
y^* - A_y\hat{x} - B_yu \\
z^* - A_z\hat{x} - B_zu
\end{pmatrix}' \Lambda^{-1} \begin{pmatrix}
y^* - A_y\hat{x} - B_yu \\
z^* - A_z\hat{x} - B_zu
\end{pmatrix}.
\tag{14.4.12}
\end{align*}
\]
Here, ‘ent’ stands for entropy, a measure of discrepancy between $(y^*, z^*)$ and $(A_y\hat{x} + B_yu, A_z\hat{x} + B_zu)$.

14.4.2. Outer problem

Using the solution of the inner problem, and remembering that we conditioned on $y^*, z^*, u$, we can represent the outer problem as:
\[
-x'V\hat{x} = \max_u \min_{y^*, z^*} r\left(f, y, u\right) - \beta x'^*V'^*x^* + \beta \theta \text{ent}(y^*, z^*, u|x^*)
\tag{14.4.13}
\]
subject to
\[
f^* = A_{ff}f + A_{fy}y + B_{fu}.
\]
We need not carry along the transition laws for $y^*, z^*$ because they are embedded by construction in ent$(y^*, z^*, u|x^*)$. Given $G_z$, (14.4.13) defines a mapping from $V^*$ to $V$, the fixed point of which is the $V^*$ that induces the robust decision rule. When $V^* = V$, the robust decision rule for $u$ maximizes the right side of (14.4.13) Where $V^* = V$.

We can simplify problem (14.4.13) by obtaining an alternative representation for entropy. We do this in the following subsections, and describe how $G_z$ is determined by a Kalman filter.
14.4.3. Representing entropy

For the purpose of simplifying the outer problem, we use the following result.

**Theorem 14.4.1.** Entropy \( \text{ent}(y^*, z^*, u|x) \) defined by (14.4.12) can be represented as

\[
\text{ent} \left( y^*, z^*, u|x \right) = \tilde{w}' \tilde{w} + \tilde{w}' \tilde{w}
\]

(14.4.14)

where \( \tilde{w}, \tilde{w} \) are shocks related to \( y^*, z^*, u \) via the ‘innovations representation’

\[
y^* = A_y \hat{x} + B_y u + \tilde{C}_y \hat{w}
\]

\[
z^* = A_z \hat{x} + B_z u + \tilde{C}_z \hat{w} + \hat{C}_z \hat{w}
\]

where \( \tilde{C}_y \) is a Cholesky factor of \( \Lambda_{11} \), \( \tilde{C}_z \) is a Cholesky factor of \( \Lambda_{22} - \Lambda_{21} \Lambda_{11}^{-1} \Lambda_{12} \), and \( \tilde{C}_z = K \tilde{C}_y \) where \( K = \Lambda_{21} \Lambda_{11}^{-1} \) is the Kalman gain.

**Proof.** Let

\[
L = \begin{bmatrix} I & 0 \\ -K & I \end{bmatrix}
\]

where \( K = \Lambda_{21} \Lambda_{11}^{-1} \). Straightforward calculations show\(^5\)

\[
LAL' = \begin{bmatrix} \Lambda_{11} & 0 \\ 0 & \Lambda_{22} - \Lambda_{21} \Lambda_{11}^{-1} \Lambda_{12} \end{bmatrix} = \begin{bmatrix} \tilde{C}_y \tilde{C}_y' \\ 0 & \tilde{C}_z \tilde{C}_z' \end{bmatrix}
\]

(14.4.15)

where \( \tilde{C}_y \) is a Cholesky factor of \( \Lambda_{11} \) and \( \tilde{C}_z \) is a Cholesky factor of \( \Lambda_{22} - \Lambda_{21} \Lambda_{11}^{-1} \Lambda_{12} \). Now compute

\[
L \begin{bmatrix} y^* - A_y \hat{x} - B_y u \\ z^* - A_z \hat{x} - B_z u \end{bmatrix} = \begin{bmatrix} y^* - A_y \hat{x} - B_y u \\ z^* - A_z \hat{x} - B_z u - K (y^* - A_y \hat{x} - B_y u) \end{bmatrix}
\]

(14.4.16)

Define the mean distortions to innovations \( \hat{w}, \hat{w} \) by\(^6\)

\[
y^* - A_y \hat{x} - B_y u = \tilde{C}_y \hat{w}
\]

(14.4.17a)

\[
z^* - A_z \hat{x} - B_z u = \tilde{C}_z \hat{w} + \hat{C}_z \hat{w}.
\]

(14.4.17b)

---

5 It can be verified that \( \Lambda = \begin{bmatrix} \tilde{C}_y & 0 \\ \tilde{C}_z & \tilde{C}_y \end{bmatrix} \begin{bmatrix} \tilde{C}_y & 0 \\ \tilde{C}_z & \tilde{C}_z \end{bmatrix}' \).

6 Had we explicitly carried along randomness, the distorted innovations representation would be

\[
y^* - A_y \hat{x} - B_y u = \tilde{C}_y (\hat{w} + \tilde{\epsilon})
\]

\[
z^* - A_z \hat{x} - B_z u = \tilde{C}_z (\hat{w} + \tilde{\epsilon}) + \tilde{C}_z (\hat{w} + \tilde{\epsilon}).
\]

where \( \begin{bmatrix} \tilde{\epsilon} \\ \tilde{\epsilon} \end{bmatrix} \) is an i.i.d. Gaussian vector with mean zero and identity covariance matrix.
Here $\hat{C}_v\hat{w}$ is the mean distortion to the innovation in $y^*$. We want the zero-mean random term $\tilde{\epsilon}_z$ to which $\hat{C}_z\tilde{w}$ is the mean distortion to be the part of the innovation in $z^*$ that is orthogonal to the innovation in $y^*$, a condition that we will attain soon by setting $\hat{C}_z$ appropriately. Using these definitions in (14.4.16) shows that

$$z^* - A_z\hat{x} - B_zu - K(y^* - A_y\hat{x} - B_yu) = \hat{C}_z\tilde{w} + (\hat{C}_z - K\hat{C}_y)\tilde{w} = \hat{C}_z\tilde{w}$$

provided that we set $\hat{C}_z = K\hat{C}_y$, a condition that makes $\tilde{\epsilon}$ and $\hat{\tilde{\epsilon}}$ orthogonal.\(^7\)

Collecting these results, we have

$$\mathbf{L}\begin{bmatrix} y^* - A_y\hat{x} - B_yu \\ z^* - A_z\hat{x} - B_zu \end{bmatrix} = \begin{bmatrix} \hat{C}_y & 0 \\ 0 & \hat{C}_z \end{bmatrix} \begin{bmatrix} \tilde{w} \\ \hat{C}_z \tilde{w} \end{bmatrix}$$

Using (14.4.15) and (14.4.18), we have

$$\text{ent}(y^*, z^*, u|\hat{x}) = \begin{bmatrix} y^* - A_y\hat{x} - B_yu \\ z^* - A_z\hat{x} - B_zu \end{bmatrix}'L'(L\Lambda L')^{-1}L \begin{bmatrix} y^* - A_y\hat{x} - B_yu \\ z^* - A_z\hat{x} - B_zu \end{bmatrix}$$

$$= \begin{bmatrix} \tilde{w}' \\ \hat{C}_z \tilde{w} \end{bmatrix}' \begin{bmatrix} \hat{C}_y & 0 \\ 0 & \hat{C}_z \end{bmatrix}^{-1} \begin{bmatrix} \hat{C}_y & 0 \\ 0 & \hat{C}_z \end{bmatrix} \begin{bmatrix} \tilde{w}' \\ \hat{C}_z \tilde{w} \end{bmatrix}$$

$$= \tilde{w}'\tilde{w} + \hat{w}'\hat{w}.$$
14.4.3.1. Entropy updating and determination of $G_z$

Equation (14.4.17b) can be represented as

$$z^* = 	ilde{z}^* + \tilde{C}_z \tilde{w}, \quad (14.4.20)$$

where $\tilde{z}^*$ can be regarded as the ordinary Kalman filter updating of the unobserved state, namely, $\tilde{z}^* = A_z \tilde{x} + B_z u + \tilde{C}_z \tilde{w}$. Formula (14.4.20) suggests a way to update $G_z$ in (14.2.3). Because $y^*$ is observed next period, the estimate $z^*$ can be conditioned on it. That eliminates $\tilde{w}$ – the mean distortion to the innovation in $y^*$ – as a component of the distortion of the estimate of $z^*$ and leaves only the contribution to entropy associated with $\tilde{w}$ – the mean distortion to the part of the innovation in $z^*$ that is orthogonal to the innovation in $y^*$. Thus we define:

$$\text{ent}^* (z^* | \tilde{z}^*) = \tilde{w}' \tilde{w} = (z^* - \tilde{z}^*)' \left( \Sigma^* \right)^{-1} (z^* - \tilde{z}^*)$$

where $\tilde{z}^*$ is defined by (14.3.4b) and

$$\Sigma^* = \tilde{C}_z \left( \tilde{C}_z \right)' .$$

or

$$\Sigma^* = \Lambda_{22} - \Lambda_{21} \Lambda_{11}^{-1} \Lambda_{12}$$

or

$$\Sigma^* = A_{zz} \Sigma A_{zz}' + C_z C_z'$$

$$- \left[ A_{zz} \Sigma A_{yz}' + C_z C_y' \right] \left[ A_{yz} \Sigma A_{yz}' + C_y C_y' \right]^{-1} \left[ A_{yz} \Sigma A_{zz}' + C_y C_z' \right].$$

(14.4.21)

This is the typical Riccati equation associated with the Kalman filter. At the next iteration, $G_z$ is set equal to $\tilde{C}_z$, the Cholesky factor of $\Sigma^*$. On the assumption that an infinite history of observations is available, the appropriate thing to do is to iterate to convergence on (14.4.21) and to set $G_z \equiv \tilde{C}_z$ as the Cholesky factor of the fixed point $\Sigma^* = \Sigma$. Notice that the value function matrices $V$ and $V^*$ from (14.4.13) do not appear in (14.4.21). Therefore, filtering separates from control.
14.5. Alternative representation of the game

We can use the formulas for entropy from Theorem 14.4.1 to obtain a more convenient representation of the outer game, one that takes the form of an ordinary robust decision problem with an observed state. Thus, we can use the representation of entropy (14.4.14) and the innovations representation (14.4.17) to reformulate game (14.4.13) as:

\[-\dot{x}' V \dot{x} = \max_{u} \min_{\hat{w}, \tilde{w}} r(f, y, u) - \beta x^* V^* x^* + \beta \theta (\hat{w}' \hat{w} + \tilde{w}' \tilde{w})\]  \hspace{1cm} (14.5.1a)

subject to:

\[f^* = A_f \dot{x} + B_f u\]
\[y^* = A_y \dot{x} + B_y u + \hat{C}_y \hat{w}\]
\[z^* = A_z \dot{x} + B_z u + \hat{C}_z \hat{w} + \tilde{C}_z \tilde{w}.\]  \hspace{1cm} (14.5.1b)

Notice the gap between \(z^*\) and \(\dot{z}^* = A_z \dot{x} + B_z u + \hat{C}_z \hat{w}\) that appears in the last transition equation. The solution of problem (14.5.1b) induces a worst-case estimate of \(z^*\) that is distinct from \(\dot{z}^*\); it also defines a mapping from \(V^*\) to \(V\).

With a change of notation, we can also represent this game as

\[-\dot{x}' V \dot{x} = \max_{u} \min_{\hat{w}, \tilde{w}} r(f, y, u) - \beta \dot{x}' V^* \dot{x}^* + \beta \theta (\hat{w}' \hat{w} + \tilde{w}' \tilde{w})\]  \hspace{1cm} (14.5.2a)

subject to:

\[f^* = A_f \dot{x} + B_f u\]
\[y^* = A_y \dot{x} + B_y u + \hat{C}_y \hat{w}\]
\[\dot{z}^* = A_z \dot{x} + B_z u + \hat{C}_z \hat{w} + \tilde{C}_z \tilde{w}.\]  \hspace{1cm} (14.5.2b)

The difference in the two representations is simply in the notation used for the choice of next period’s element of the state \(z^*\). In the notation of (14.5.2), \(\dot{z}^*\) no longer denotes \(A_z \dot{x} + B_z u + \hat{C}_z \hat{w}\). The virtue of the notation in (14.5.2) is that it makes evident how (14.5.2) is an ordinary robust decision problem with \(\dot{x}\) treated as an observed state.

In choosing \(\hat{w}, \tilde{w}\), the decision maker chooses the worst-case estimates of \(y^*, z^*\) against which to plan.\(^8\) We note that for fixed \(\hat{C}_y, \hat{C}_z, \tilde{C}_z\), (14.5.2) is an

\[^8\] The approximating law of motion in the corresponding stochastic system is

\[\dot{z}^* = A_z \dot{x} + B_z u + \hat{C}_z (\hat{\epsilon} + \hat{w}),\]

where \(\hat{\epsilon}\) is an i.i.d. error with mean zero and identity covariance matrix.
ordinary deterministic linear quadratic robust decision problem, with observed state $\tilde{x}$ and shock process $\begin{bmatrix} \tilde{w} \\ \tilde{w} \end{bmatrix}$.

The presence of the mean distortion $\tilde{C}_z \tilde{w}$ in (14.5.2b) explicitly recognizes the gap between $z$ and the decision maker’s estimate $\tilde{z}$. Without a concern for robustness, an appeal to a certainty equivalence result would allow ignoring (i.e., setting to zero) the random term corresponding to $\tilde{C}_z \tilde{w}$. That implies that after using the Kalman filter to construct the filtered process $\tilde{z}$, one could proceed as though $\tilde{z}$ itself were an observed process. But under a preference for robustness, the fact that $\tilde{z}$ emerged from filtering should be taken into account in considering model misspecification. The term $\tilde{C}_z$ is essential in telling us how to explore misspecifications associated with the filtering process. We say more about this issue in subsection 14.5.2.

14.5.1. Summary

Because a fixed point $\Sigma$ of (14.4.21) is independent of $V$, we can proceed sequentially:

1. Solve an ordinary Kalman filtering problem by iterating to convergence on (14.4.21). Form $\tilde{C}_z, \tilde{C}_y, \tilde{C}_z$ for representation (14.5.2b).

2. Iterate to convergence on the mapping from $V^*$ to $V$ defined by (14.5.2).

These are the two steps that we promised at the beginning of this chapter.

14.5.2. A comparison model

Asset pricing models of Veronesi (????) and David (????) and (MORE REFERENCES) let payoffs reflect hidden states, thereby confronting decision makers with filtering problems. Several commentators$^9$ have interpreted those models as being isomorphic with alternative models that start with a sufficiently rich stochastic process for payoffs, with no hidden states. In particular, without a preference for robustness, the implications of hidden state models are identical to those from a model that simply takes the innovation representation$^{10}$ for dividends as the stochastic process for dividends in the first place, ignoring the origin of that process in a filtering problem. From this standpoint, the only

$^9$ In oral remarks at seminars, John Cochrane, Darrell Duffie, John Heaton, and Kenneth Singleton have all expressed this view.

$^{10}$ See Hansen and Sargent (200XXX, chapter 8).
defense of the hidden state, filtering setup is that it provides a possibly parsi-
monious parameterization of a rich stochastic process for dividends. However
things are different in models in which decision makers have concerns about
robustness. Under a concern for robustness, filtering contributes an additional
avenue of deception $\tilde{C}_z \tilde{w}$. The distortion $\tilde{C}_z \tilde{w}$ injects an additional kinds of
ambiguity because the innovations process comes from a filtering problem.\footnote{This insight is exploited in the context of a linear-quadratic asset pricing model by Hansen, Sargent, and Wang (2002) and in a nonlinear model by Cagetti, Hansen, Sargent, and Williams (2002).}

We can highlight how the filtering problem interacts with the robust con-
trol problem by considering the following alternative model that would arise by
simply using the ordinary innovations representation to describe the $\tilde{z}$ process,
then ignoring that it arose from a filtering problem as would be appropriate in
a model without a preference for robustness. This model takes as its starting
point the ordinary innovations representation for $y$ as a given process. The
decision maker ignores the fact that this process is itself the outcome of a filtering
problem and so allow the evil agent to manipulate not $\tilde{w}$ but only the innovation
in the observable process $y$. This leads to a game that can be characterized in
terms of the value function recursion

$$-\tilde{x}'V \tilde{x} = \max_u \min_{\tilde{w}} r(f, y, u) - \beta \tilde{x}'V^* \tilde{x}^* + \beta \theta \tilde{w}' \tilde{w} \tag{14.5.3a}$$

subject to:

$$f^* = A_f \tilde{x} + B_f u \tag{14.5.3b}$$
$$y^* = A_y \tilde{x} + B_y u + \tilde{C}_y \tilde{w} \tag{14.5.3c}$$
$$\tilde{z}^* = A_z \tilde{x} + B_z u + \tilde{C}_z \tilde{w}. \tag{14.5.3d}$$

This is another ordinary robust linear quadratic decision problem, one in which
the law of motion for $y^*$ is immediately replaced by its innovation representation.
Thus we have absorbed the hidden state structure into a Wold representation for
$y^*$ with one shock process $\tilde{w}$ (or in the stochastic version $\tilde{e} + \tilde{w}$). This limits the
ability of the evil agent to deceive the maximizing agent via the filtering prob-
lem. Without a concern for robustness the comparison model leads to the same
decision rule as the basic model of this chapter, but with a concern for robust-
ness, the evil agent in general manipulates $\tilde{w}$ and thereby promotes robustness
along dimensions that the comparison model (14.5.3) ignores by forgetting that
the process for $\tilde{z}$ originates in the solution of a filtering problem.
14.6. Stochastic version

The stochastic version of (14.5.2) is formed by adding random shocks $\hat{\epsilon}$ and $\hat{\bar{\epsilon}}$ to the transition law and taking expectations appropriately in (14.5.2a):

$$-\bar{x}'V\bar{x} - \alpha = \max_{\bar{w},\bar{w}'} \min_{\bar{w},\bar{w}'} \left( \bar{r}(f, y, u) - \beta E\bar{x}'V\bar{x}^* + \beta \theta (\bar{w}'\bar{w} + \bar{w}'\bar{w}) - \beta \alpha \right)$$

subject to:

$$f^* = A_f \bar{x} + B_f u$$
$$y^* = A_y \bar{x} + B_y u + \dot{C}_y (\bar{\epsilon} + \bar{\bar{\epsilon}})$$
$$\bar{z}^* = A_z \bar{x} + B_z u + \ddot{C}_z (\bar{\epsilon} + \bar{\bar{\epsilon}}).$$

Here $\epsilon$ and $\bar{\epsilon}$ are both i.i.d. random sequences with means zero and identity covariance matrices; the mathematical expectation in (14.6.1a) is taken with respect to these disturbances.

The matrices $V, G_z$ and the decision rules from (14.5.2) for $u$ and the mean distortions $\bar{w}, \bar{\bar{w}}$ also solve (14.6.1). The only additional parameter of (14.6.1) is the constant $\alpha$ in the value function.

14.7. Simulations

For simulating, it is useful to represent a comprehensive system that includes $[f' \ y' \ z']'$ and $\bar{z}$ within a state space form. For this purpose, define the prediction

$$\ddot{y}_{t+1} = A_y f_t + A_{yy} y_t + A_{yz} \bar{z}_t + B_y u_t.$$  

Subtracting this equation from the equation for $y_{t+1}$ in (14.2.2) gives

$$y_{t+1} - \ddot{y}_{t+1} = A_{yz} (z_t - \bar{z}_t) + C_y \epsilon_{t+1}.$$  

We can write the equation for $\bar{z}_{t+1}$ in (14.5.2b) as

$$\bar{z}_{t+1} = A_{zz} \bar{z}_t + K (y_{t+1} - \ddot{y}_{t+1})$$  

or

$$\bar{z}_{t+1} = A_{zz} \bar{z}_t + K \left[ A_{yz} (z_t - \bar{z}_t) + C_y \epsilon_{t+1} \right].$$  

Then we can write the system under the robust control law as

$$f_{t+1} = A_{ff} f_t + A_{fy} y_t - B_f F \begin{bmatrix} f_t \\ y_t \\ \bar{z}_t \end{bmatrix}$$

(14.7.1a)
\[ y_{t+1} = A_{yf}f_t + A_{yy}y_t + A_{yz}z_t - B_{y}F_{f} \begin{bmatrix} f_t \\ y_t \\ z_t \end{bmatrix} + C_{y}\epsilon_{xt+1} \] (14.7.1b)

\[ z_{t+1} = A_{zz}z_t + C_{z}\epsilon_{xt+1} \] (14.7.1c)

\[ \dot{z}_{t+1} = A_{zz}\dot{z}_t + KA_{yz} (z_t - \dot{z}_t) + KC_{y}\epsilon_{xt+1}. \] (14.7.1d)

Let \( F = [F_f \quad F_y \quad F_z] \). Then rewrite (14.7.1) in the state-space form:

\[
\begin{bmatrix}
  f_{t+1} \\
  y_{t+1} \\
  z_{t+1}
\end{bmatrix} =
\begin{bmatrix}
  A_{ff} - B_{f}F_f & A_{fy} - B_{f}F_y & 0 & -B_{f}F_z \\
  A_{yf} - B_{y}F_f & A_{yy} - B_{y}F_y & A_{yz} & \quad -B_{y}F_z \\
  0 & 0 & A_{zz} & 0
\end{bmatrix}
\begin{bmatrix}
  f_t \\
  y_t \\
  z_t
\end{bmatrix}
\]

\[
+ \begin{bmatrix}
  0 \\
  C_y \\
  C_z
\end{bmatrix} \epsilon_{xt+1}.
\] (14.7.2)

System (14.7.2) describes the joint behavior of the state and the filtered piece of the state \( \dot{z} \) under the joint robust control and filter.

### 14.8. Asset pricing in a permanent income model

In chapter 12, we used a model of Hansen, Sargent, and Tallarini’s (HST, 1999) to describe some of the implications of a preference for robustness for asset pricing. Hansen, Sargent, and Wang (HSW, 2002) modified HST’s permanent income model by withholding knowledge of the two separate components of the endowment process posited by HST, thereby impelling the representative to base consumption-saving decisions on filtered estimates of the state of those two components. HSW used their model to study the effects of filtering on market prices of risk. HSW’s representative agent faces a problem that falls within the setting of this chapter.

We briefly summarize HST’s model, which was described in detail in chapters 9 and 12. A planner values a scalar process \( s \) of consumption services according to

\[ V_0 = -\sum_{t=0}^{\infty} \beta^t (s_t - b_t)^2 \] (14.8.1)

where the service \( s \) is produced by the scalar consumption process \( c \) via the household technology

\[
\begin{align*}
  s_t &= (1 + \lambda) c_t - \lambda h_{t-1} \\
  h_t &= \delta h_{t-1} + (1 - \delta) c_t
\end{align*}
\] (14.8.2)
where \( \lambda \geq 0 \) and \( \delta_h \in (0,1) \), \( b \) is an exogenous preference shock process, and \( h \) is a scalar stock of household habits. HST set \( b \) to a constant. There is a linear technology for converting an exogenous scalar endowment \( d_t \) into consumption or capital:

\[
c_t + k_t = Rk_{t-1} + d_t
\]

(14.8.3)

where \( k_t, d_t \) are the capital stock and the exogenous stochastic endowment at time \( t \), respectively. HST showed that \( R \) is the gross return on the risk free asset, made constant by the technology.

**Figure 14.8.1:** Market price of Knightian uncertainty for four-period securities \( \sigma_t(m_{t,t+4})^u \) as function of detection error probability in HST (*) no filtering model and HSW (○) filtering model.

HST assumed that the agent and the planner observe histories of each component of the following two-component model for the endowment:

\[
d_{t+1} = \mu_d + d_{t+1}^1 + d_{t+1}^2
\]

\[
d_{t+1}^1 = g_1 d_t + g_2 d_{t-1} + c_1 \epsilon_{t+1}^1
\]

\[
d_{t+1}^2 = a_1 d_t^2 + a_2 d_{t-1}^2 + \epsilon_{t+1}^2,
\]
where \( \epsilon_{t+1} = \begin{bmatrix} \epsilon^1_{t+1} \\ \epsilon^2_{t+1} \end{bmatrix} \) is an i.i.d. 2 \( \times \) 1 Gaussian disturbance vector with mean zero and identity covariance matrix.

While HST assumed that the planner observes current and lagged values of both components \( d^1_t \) at time \( t \), HSW assumed instead that the planner sees values only current and lagged values of the sum \( d_t \) at time \( t \).

The law of motion for the state of the model can be written

\[
\begin{bmatrix}
    h_t \\
    k_t \\
    d^1_t \\
    d^1_{t+1}
\end{bmatrix}
= \begin{bmatrix}
    \delta_h (1 - \delta_h) \gamma & 0 & 0 & 0 & (1 - \delta_h) & 0 & 0 \\
    0 & \delta_k & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 1 & 0 & 0 & 0 \\
    0 & 0 & a_2 & \mu_d (1 - a_1 - a_2) & a_1 & g_1 - a_1 & g_2 - a_2 \\
    0 & 0 & 0 & 1 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & g_1 & g_2 & 0 \\
    0 & 0 & 0 & 0 & 1 & 0 & 0 \\
\end{bmatrix}
\begin{bmatrix}
    h_{t-1} \\
    k_{t-1} \\
    d^1_{t-1}
\end{bmatrix}
\begin{bmatrix}
    \epsilon^1_{t+1} \\
    \epsilon^2_{t+1}
\end{bmatrix}
+ \begin{bmatrix}
    (1 - \delta_h) \\
    0 \\
    0 \\
    0 \\
    0 \\
    0 \\
    0 \\
\end{bmatrix}
\begin{bmatrix}
    1 \\
    y_t + c_1 \\
    c_2 \\
    0 \\
    0 \\
\end{bmatrix}
\begin{bmatrix}
    h_t \\
    k_t \\
    d^1_t
\end{bmatrix}
\begin{bmatrix}
    \epsilon^1_{t+1} \\
    \epsilon^2_{t+1}
\end{bmatrix}.
\]

Notice that this in the form (14.2.2) where \( B_y = 0 \), \( B_z = 0 \), and

\[
f_t = \begin{bmatrix} h_{t-1} \\ k_{t-1} \\ d^1_{t-1} \end{bmatrix}, \quad y_t = d_t, \quad z_t = \begin{bmatrix} d^1_t \\ d^1_{t-1} \end{bmatrix}.
\]

For HST, the planner knows current and lagged values of \( f_t, y_t \) and \( z_t \), when \( i_t \) is chosen. HSW instead assume that current and lagged values of \( f_t, y_t \) are in the planner’s information set when \( i_t \) is to be chosen, but that \( z_t \) is never observed. The planner bases his decisions on an estimate of \( z_t \) from the history of \( y_t, f_t \). This makes the Bellman equation for the robust planner take the form of (14.6.1).
14.8.1. Asset pricing with robustness and filtering

As already noted, although game (14.6.1) takes account of the fact that part of the state is estimated, in the end it still takes the form of the robust linear regulator without filtering of the kind analyzed in chapters 2 and 6. It follows that the asset pricing theory from chapter 12 applies directly. HSW used this theory to construct multi-period versions of market prices of risk for their model and compared them with HST’s model.

Fig. 14.8.1 shows four-period market prices of risk for the HST and HSW models, each expressed as functions of the detection probability.\(^\text{12}\) We computed the detection probability for the HSW model by applying the methods described in chapter 8 to representation (14.6.1).\(^\text{13}\) As with the figures on page 237, we have chosen to plot the market price of risk against the detection probability rather against the robustness parameter \(\theta\). The reason for this choice is that Fig. 14.8.1 reveals the existence of a relationship between detection probabilities and the market price of risk that seems to prevail across the two models. In Anderson, Hansen, and Sargent (2002), we have analytically established such a model independent relationship for a class of continuous time specifications. Heuristically, the key to the deducing the relationship is that the same object, namely \(w'w\) for the distorted model, governs both the detection probabilities and the market prices of risk.\(^\text{14}\)

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\(^\text{12}\) HSW also report market prices of risk for shorter horizons. See HSW for an explanation of why the four period market price of risk is the one we choose to compare with the HST model.

\(^\text{13}\) See HSW (2002, appendix A) for details.

\(^\text{14}\) Fig. 14.8.1 conceals that for the same value of the robustness parameter \(\theta\), the HST models and HSW models imply different market prices of risk.
Part IV

More applications
15.1. Introduction

This chapter and the next describe equilibria in which there are multiple decision makers who share a common approximating model but are all concerned about model misspecification. Imputing a common approximating model to all of the agents is a way to stay as close as possible to rational expectations. We make this modelling choice because we desire to preserve as much as possible of the structure and empirical power of rational expectations. Because they have different objective functions, the the context-specific worst case models of different decision makers will in general differ.

In the present chapter, we study two player dynamic games. We adapt the concept of Markov perfect equilibrium to incorporate preferences for robustness to model misspecification. Here the timing protocol is that both players choose sequentially and simultaneously in each period. In chapter 16, we study a different timing protocol in which a Stackelberg leader or Ramsey planner chooses once and for all at time 0, while Stackelberg followers or members of a competitive fringe choose sequentially.

15.2. Markov perfect equilibria with robustness

There are two agents each of whose decisions affect the motion of a common state vector that impinges on both the return functions of both agents. The environment is one in which, without preferences for robustness, a Markov perfect equilibrium can be computed by working backwards appropriately on pairs of standard Bellman functions and the associated equations that express decision rules as functions of continuation value functions. We modify the Markov perfect equilibrium concept by imputing concerns about robustness to both decision makers. The equilibrium concept insists that the two decision makers share a common approximating model, which from the point of view of each agent incorporates the robust decision rule used by the other agent.

The two agents, \( i = 1, 2 \) share a common approximating model

\[
x_{t+1} = Ax_t + B_1u_{1t} + B_2u_{2t} + C\epsilon_{t+1}
\]  

(15.2.1)
where $u_{it}$ is a control vector chosen by agent $i$ as a function of the state $x_t$, and $\epsilon_{t+1}$ is an i.i.d. Gaussian random vector with mean zero and identity covariance matrix. Agent $i$ conceives of model misspecification by thinking that the actual data generating mechanism can be represented as a member of a set of perturbations to (15.2.1) of the form
\begin{equation}
    x_{t+1} = Ax_t + B_1 u_{1t} + B_2 u_{2t} + C (\epsilon_{t+1} + w_{it+1})
\end{equation}

where $w_{it+1}$ represents misspecified dynamic components that depend on the history of $x_s$ up to time $t$. Agent $i$ wants to maximize
\begin{equation}
    E_0 \sum_{t=0}^{\infty} \beta^t r_i (x_t, u_{it})
\end{equation}

where $\beta \in (0,1)$ and $r_i (x_t, u_{it}) = - [x_t' R_i x_t + u_{it}' Q_i u_{it} + 2u_{it}' H_i x_t]$.

We appeal to the version of certainty equivalence cited on page 20 to allow us to drop the $\epsilon_{t+1}$ term from (15.2.2) and the conditional expectation $E$ from (15.2.3) and proceed to solve nonstochastic versions of our problems.

We define a Nash equilibrium with robust decision makers and a common approximating model. In equilibrium, player $i$ selects a robust decision rule of the form
\begin{equation}
    u_{it} = - F_{it} x_t.
\end{equation}

Though we will eventually find a time invariant rule $F_i$, to accommodate backward induction we allow a time-varying rule. The set of laws of motion confronting agent $i$ has the form
\begin{equation}
    x_{t+1} = (A - B_{-i} F_{-it}) x_t + B_i u_{it} + C w_{it+1}
\end{equation}

where a subscript $-i$ refers to the other player. Notice that (15.2.5) incorporates the robust rule $F_{-it}$ of the other player and that each player has his own distortion process $w_{it}$. Player $i$ solves a multiplier control problem with multiplier $\theta_i$.

**Definition 15.2.1.** A Markov perfect equilibrium with robustness consists of pairs of value functions $V_i$, decision rules $u_i = -F_i x_i$, and rules for worst case shocks $w_i = K_i x_i$ such that the decision rules for $u_i, w_i$ attain $V_i(x)$ and the value functions $V_i$ satisfy the Bellman equations
\begin{equation}
    V_i (x) = \max_{u_i} \min_{w_i} \{ r_i (x, u_i) + \beta \theta_i w_i' w_i + \beta V_i (x^*) \} \quad (15.2.6)
\end{equation}
where * denotes next period’s value and the extremization is subject to

\[ x^* = (A - BF_{-i})x + B_iu_i + Cw_i. \]  

(15.2.7)

The value functions assume the form

\[ V_i(x) = -x'P_ix, \]

where \( P_i = T \circ D_i \) is a fixed point defined in terms of the composition of modified versions of two familiar operators:

\[
T_i (P_i) = Q_i + \beta(A - B_{-i}F_{-i})'P_i(A - B_{-i}F_{-i}) \\
- \left( \beta(A - B_{-i}F_{-i})'P_iB_i + H_i \right) \left( R_i + \beta B_i'P_iB_i \right)^{-1} \\
\times (\beta B_i'P_i(A - B_{-i}F_{-i}) + H_i) \\
D_i (P_i) = P_i + \theta_i^{-1}P_iC \left( I - \theta_i^{-1}CP_iC \right)^{-1}C'P_i.
\]

(15.2.8)

(15.2.9)

The \( T \) operator is associated with the maximization part of the problem on the right side of (15.2.7), while the \( D \) operator is associated with the minimization part.

### 15.2.1. Computational algorithm: iterating on stacked Bellman equations

Define the iterations

\[
F_{it} = (R_i + \beta B_i'D_i(P_{it+1})B_i)_{-i}^{-1} \left( \beta B_i'P_{it+1}(A - B_{-i}F_{-i}) + H \right) \quad (15.2.10)
\]

\[
P_{it} = T_i \circ D_i (P_{it+1}). \quad (15.2.11)
\]

We propose to use iterations on these operators to find fixed points \( F_i, P_i, i = 1, 2 \) that satisfy

\[
F_i = (R_i + \beta B_i'D_i(P_i)B_i)_{-i}^{-1} \left( \beta B_i'P_i(A - B_{-i}F_{-i}) + H \right) \quad (15.2.12)
\]

\[
P_i = T_i \circ D_i (P_i). \quad (15.2.13)
\]

Suppose that \( u_i \) is \( k_i \times n \). Given \( P_{it+1}, P_{2t+1}, \) equations (15.2.10) for \( i = 1, 2 \) form \((k_1 + k_2) \times n\) linear equations in the same number of variables, namely, \( F_{it}, F_{2t} \). To compute an equilibrium, start with zero terminal value matrices \( P_{1T}, P_{2T} \), then solve (15.2.10) for \( F_{1T}, F_{2T} \), then iterate backwards on (15.2.10),(15.2.11) until, hopefully, the \( F_{it}, P_{it} \) sequences converge. If they converge, we say that there is an asymptotically time invariant equilibrium law of motion.
When both players use time invariant robust rules, the approximating model becomes

\[ x_{t+1} = A^*x_t + C\epsilon_{t+1} \tag{15.2.14} \]

where \( A^* = A - B_1F_1 - B_2F_2 \) and where we have reactivated the Gaussian disturbance. The two agents share this approximating model but in general have different worst case models. The worst case model for agent \( i \) is

\[
\begin{align*}
x_{t+1} &= A^*x_t + C(\epsilon_t + w_{it+1}) \\
w_{it+1} &= K_i x_t.
\end{align*}
\]

where

\[
K_i = \theta^{-1} (I - \theta_i^{-1}C'P_tC)^{-1} C'P_tA^*.
\tag{15.2.15}
\]

Thus the worst case model of player \( i \) is

\[
x_{t+1} = (A^* + CK_i)x_t + C\epsilon_{t+1}.
\tag{15.2.16}
\]

Each player can be regarded as solving an ordinary control problem using its own twisted law of motion (15.2.16).
Chapter 16.
Robustness in forward looking models

16.1. Introduction

Here and in chapter 18 we study Ramsey plans under a preference for robustness. A benevolent government acts as a Stackelberg leader with respect to a competitive private sector, modeled in terms of a representative agent, that acts as a follower. At time 0, the government chooses a sequence of actions, taking into account how the private sector’s decisions at each date will respond to its forecasts of future government actions. The government’s policy instruments appear as ‘forcing variables’ in the private sector’s Euler equations. Those Euler equations thus describe how private decisions depend on the sequence of government actions. When the agents composing the private sector also have a preference for robustness, some of those Euler equations describe the motion of the private agents’ worst case shocks.

Without preferences for robustness, it is known that these Stackelberg or Ramsey problems can be solved by forming a Lagrangian in which a sequence of multipliers adheres to a sequence of private agents’ Euler equations. The private sector’s Euler equations are ‘implementability constraints’ that require the government’s decision at time $t$ to confirm forecasts that had informed private agents’ earlier decisions. The Lagrange multipliers on the implementability constraints make the government’s actions depend on the history of the economy and allow a recursive representation of the history-dependent decision rule for the government that solves the Stackelberg problem.

This chapter formulates an equilibrium in which both the leader and the follower are concerned about model misspecification. As a natural counterpart or extension of rational expectations, we assume that private agents and the government share a common approximating model. When both types of agent have a preference for robustness, the approximating model for one agent must include a description of the robust decision rules of the other types of agent, and of how they respond to other decision makers’ actions. Though they share a common approximating model, because their preferences may differ, the different types of agent may not share the same worst-case model. In order completely to describe the common approximating model of the two types of agents, we require an adequate description of the dynamics of their possibly different worst case shocks.
The remainder of this chapter is organized as follows. Section 16.2 states a Stackelberg problem in which decision makers fear model misspecification and therefore want robustness. Section 16.3 describes how to solve the robust Stackelberg problem by properly rearranging and reinterpreting some state variables and some Lagrange multipliers after having solved a robust linear regulator. As an example, section 16.5 describes a dynamic model of a monopolist facing a competitive fringe. Section 16.6 describes a monetary policy example of Woodford. Section 16.7 shows how to compute a robust Ramsey rule when some of the state variables are latent, requiring that agents and the government filter. Section 16.8 computes an example for Woodford’s model, showing a sense in which the robust rule for setting the interest rate is less aggressive than the rule formed without a concern for model misspecification. Section 16.9 concludes. Appendix A briefly describes how the invariant subspace methods of chapter 3 can also be used to compute robust Ramsey plans. Appendix B studies the Riccati equation that solves the robust Ramsey problem. Appendix C describes the connection of our work to a Bellman equation that Marcet and Marimon (1999) have used to solve problems with implementability constraints like ours. Appendix D describes how to decentralize... Change or move.

16.1.1. Related literature

¹ More generally, Hurwicz (1951) had advocated zero-sum games as a way of making decisions when a decision could not specify a unique model.
² Chapter 3 describes efficient computational algorithms for such models.
models where part of the state is unknown and must be filtered. DeJong, In-
gram, and Whiteman (1996), Otrok (In press), and others study the Bayesian
estimation of forward looking models. They summarize the econometrician’s
doubts about parameter values with a prior distribution, meanwhile attributing
no doubts about parameter values to the private agents in their models. Mark
Giannoni (2000) studies robustness in a forward looking macro model. He mod-
el the policy maker as knowing all parameters except two, for which he knows
only bounds. The policy maker then computes the min – max policy rules. Kasa
designs simple (not history dependent) robust policy rules for a forward looking
monetary model. Christiano and Gust (1999) study robustness from the view-
point of the determinacy and stability of rules under nearby parameters. They
adopt a perspective of robust control theorists like Ba¸sar and Bernhard (1995)
and Zhou, Doyle, and Glover (1996), who are interested in finding rules that
stabilize a system under the largest set of departures from a reference model.

16.2. The robust Stackelberg problem

This section defines a robust Stackelberg problem where the Stackelberg leader
is concerned about model misspecification. In macroeconomic problems, the
Stackelberg leader is often a government and the Stackelberg follower is a rep-
resentative agent within a private sector. In section 16.5, we present an applica-
tion with an interpretation of the two players as a monopolist and a competitive
fringe.

Let $z_t$ be an $n_z \times 1$ vector of natural state variables, $x_t$ an $n_x \times 1$ vector
of endogenous variables free to jump at $t$, and $U_t$ a vector of the leader’s in-
struments. The $z_t$ vector is inherited from the past. The model determines the
‘jump variables’ $x_t$ at time $t$. Included in $x_t$ are prices and quantities that ad-
just to clear markets at time $t$. Let $y_t = \begin{bmatrix} z_t \\ x_t \end{bmatrix}$. Define the Stackelberg leader’s
one-period loss function$^3$

$$r(y, u) = y'Qy + u'Ru.$$  \hspace{1cm} (16.2.1) \hspace{1cm} ["target "]

$^3$ The problem assumes that there are no cross products between states and con-
trols in the return function. A simple transformation converts a problem whose return
function has cross products into an equivalent problem that has no cross products. See
Chapter 3.
The leader wants to maximize
\[ -\sum_{t=0}^{\infty} \beta^t r(y_t, U_t). \] (16.2.2) ["new1"]

The leader makes policy in light of a set of models indexed by a vector of specification errors \( W_{t+1} \) around its approximating model:
\[
\begin{bmatrix}
I \\
 G_{21} \\
 G_{22}
\end{bmatrix}
\begin{bmatrix}
z_{t+1} \\
x_{t+1}
\end{bmatrix}
= \begin{bmatrix}
\hat{A}_{11} & \hat{A}_{12} \\
\hat{A}_{21} & \hat{A}_{22}
\end{bmatrix}
\begin{bmatrix}
z_t \\
x_t
\end{bmatrix}
+ \hat{B}U_t + \hat{C}W_{t+1}. \] (16.2.3) ["new2"]

We assume that the matrix on the left is invertible, so that
\[
\begin{bmatrix}
z_{t+1} \\
x_{t+1}
\end{bmatrix}
= \begin{bmatrix}
A_{11} & A_{12} \\
A_{21} & A_{22}
\end{bmatrix}
\begin{bmatrix}
z_t \\
x_t
\end{bmatrix}
+ BU_t + CW_{t+1}. \] (16.2.4) ["new3"]

or
\[ y_{t+1} = Ay_t + BU_t + CW_{t+1}. \] (16.2.5) ["new30"]

The followers’ behavior is summarized by the second block of equations of (16.2.3) or (16.2.4). These typically include the first-order conditions of private agent’s optimization problem (i.e., their Euler equations). These equations summarize the forward looking aspect of the followers’ behavior. The particular structure of these equations and the variables composing \( x_t \) depend on the followers’ optimization problems, and in particular, whether we impute a concern about robustness to them. As we shall see later, if we impute a motive for robustness to the followers, then it is necessary to include \( w_{t+1} \), the specification errors of the followers, among the variables in \( x_t \). In sections 16.5 and 16.6, we’ll display some concrete examples.

Returning to (16.2.3) or (16.2.4), the vector \( W_{t+1} \) of unknown specification errors can feed back, possibly nonlinearly, on the history \( y' \), which lets the \( W_{t+1} \) sequence represent misspecified dynamics. The leader regards its approximating model (which has \( W_{t+1} = 0 \)) as a good approximation to the unknown true model in the sense that the unknown \( W_{t+1} \) sequence satisfies
\[
\sum_{t=0}^{\infty} \beta^{t+1} W'_{t+1} W_{t+1} \leq \eta_0 \] (16.2.6) ["new7"]

\[ ^4 \text{We have assumed that the matrix on the left of (16.2.3) is invertible for ease of presentation. However, by appropriately using the invariant subspace methods described in Chapter 3 and appendix A, it is straightforward to adapt the computational method when this assumption is violated.} \]

\[ ^5 \text{If } C_{t+1} \text{ were added to the right side of (16.2.5), we would take the expectation of (16.2.6).} \]
where \( \eta_0 > 0 \).

The certainty equivalence principle stated on page 20 allows us to work with non stochastic approximating and distorted models. We would attain the same decision rule if we were to replace \( x_{t+1} \) with the forecast \( E_t x_{t+1} \) and to add a shock process \( \epsilon_{t+1} \) to the right side of (16.2.3) or \( C \epsilon_{t+1} \) to the right side of (16.2.4), where \( \epsilon_{t+1} \) is an i.i.d. random vector with mean of zero and identity covariance matrix.

Let \( X^t \) denote the history of any variable \( X \) from 0 to \( t \). Kydland and Prescott (1980), Miller and Salmon (1982, 1985), Hansen, Epple, and Roberds (1985), Pearlman, Currie and Levine (1986), Sargent (1987), Pearlman (1992) and others have studied non-robust (i.e., \( \eta_0 = 0 \)) versions of the following problem:

**Definition 16.2.1.** For \( \eta > 0 \), the constraint version of the Stackelberg problem is to extremize (16.2.2) by finding a sequence of decision rules expressing \( U_t \) and \( W_{t+1} \) as sequences of functions mapping the time \( t \) history of the state \( z^t \) into the time \( t \) decision. The leader chooses these decision rules at time 0 and commits to them evermore.

**Definition 16.2.2.** When \( \eta_0 > 0 \), the decision rule for \( U_t \) that solves the Stackelberg problem is called a robust Stackelberg plan or robust Ramsey plan.

Note that the decision rules are designed to depend on the history of the true state \( z_t \) and not on the history of the jump variable \( x_t \). For a non-robust version of the problem, the forementioned authors show that the optimal rule is history-dependent, meaning that \( U_t, W_{t+1} \) depend not only on \( z_t \) but also on lags of it. The history dependence comes from two sources: (a) the leader’s ability to commit to a sequence of rules at time 0,\(^6\) and (b) the forward-looking behavior of the followers that is embedded in the second block of equations in (16.2.3) or (16.2.4).

Fortunately, there is a recursive way of expressing this history dependence by having decisions \( U_t, W_{t+1} \) depend linearly only on the current value \( z_t \) and on a new component of the state vector, \( \mu_{xt} \). The component \( \mu_{xt} \) is a vector of Lagrange multipliers on the last \( n_z \) equations of (16.2.3) or (16.2.4). Part of the solution of the problem in Definition 16.2.2 is then a law of motion expressing \( \mu_{xt+1} \) as a linear function of \( (z_t, \mu_{xt}) \). The history dependence of the leader’s plan is expressed in the dynamics of \( \mu_{xt} \). These multipliers track past leader

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\(^6\) The leader would make different choices were it to choose sequentially, that is, were it to set \( U_t \) at time \( t \) rather than at time 0.
promises about current and future settings of \( U \). At time 0, if there are no past promises to honor, it is appropriate for the leader to initialize the multipliers to zero (this maximizes its criterion function). The multipliers take non-zero values thereafter, reflecting the subsequent costs to the leader of adhering to its commitments.

### 16.2.1. Multiplier version of the robust Stackelberg problem

In chapter 6 and 7, we showed that it is usually more convenient to solve a multiplier game rather than a constraint game. Accordingly, we use:

**Definition 16.2.3.** The multiplier version of the robust Stackelberg problem is the zero-sum two-player game:

\[
\max_{\{U_t\}_{t=0}^{\infty}} \min_{\{W_{t+1}\}_{t=0}^{\infty}} -\sum_{t=0}^{\infty} \beta^t \{ r(y_t, U_t) - \beta \Theta W_{t+1}' W_{t+1} \} \tag{16.2.7} \]

where the extremization is subject to (16.2.5) and \( \Theta < \Theta < \infty \).

### 16.3. Solving the robust Stackelberg problem

This section describes a three step algorithm for solving a multiplier version of the robust Stackelberg problem.

#### 16.3.1. Step 1: solve a robust linear regulator

Step 1 temporarily disregards the forward looking aspect of the problem (step 3 will take account of that) and notes that superficially the multiplier version of the robust Stackelberg problem (16.2.7), (16.2.5) has the form of a robust linear regulator problem. Mechanically, we can solve this artificial robust linear regulator by noting that associated with problem (16.2.7) is the Bellman equation

\[
v(y) = \max_u \min_W \{ -r(y, u) + \beta \Theta W' W + \beta v(y^*) \}, \tag{16.3.1} \]

where \( y^* \) denotes next period’s value of the state and the extremization is subject to the transition law \( y^* = Ay + Bu + CW \). The solution has the form \( v(y) = -y' P y \), where \( P \) is a fixed point of the operator \( T \circ D \) defined in chapters 2 and 6, namely,

\[
T(P) = Q + \beta A' PA - \beta^2 A' PB (R + \beta B' PB)^{-1} B' PA \tag{16.3.2} \]

\[
D(P) = P + \Theta^{-1} PC (I - \Theta^{-1} C' PC)^{-1} C' P. \tag{16.3.3} \]


Thus, the Bellman equation (16.3.1) leads to the Riccati equation
\[ P = T \circ D(P) . \] (16.3.4) ["bell3 "]

Here the \( T \) operator emerges from the maximization over \( U \) on the right side of (16.3.1), while the \( D \) operator emerges from the minimization over \( W \). The extremizing decision rules are given by \( U_t = -F_1 y_t \) where
\[ F_1 = \beta (R + \beta B' D(P) B)^{-1} B' D(P) A. \] (16.3.5) ["robrulen1"]

and \( W_{t+1} = -F_2 y_t \) where
\[ F_2 = -\Theta^{-1} (I - \Theta^{-1} C' P C)^{-1} C' P (A - BF). \] (16.3.6) ["robrule2n1 "]

(See page 22.) The next steps recognize how the solution of the Riccati equation
\[ P = T \circ D \] encodes objects that solve the robust Stackelberg problem. That will tell us how to manipulate the decision rules for \( U_t \) and \( W_{t+1} \) (linear functions identified by the vectors (16.3.5) and (16.3.6) ) to get the solution of the robust Stackelberg problem.

16.3.2. Step 2: use the stabilizing properties of shadow price \( P y_t \)

At this point we use \( P \) to describe how shadow prices on the transition law relate to the artificial state vector \( y_t = [z'_{t'}. x'_{t'}]' \) (we say ‘artificial’ because \( x_t \) is a vector of jump variables.) Recall the Lagrangian methods used in chapters 3 and 6. Thus, another way to solve the multiplier version of the robust Stackelberg problem (16.2.7), (16.2.5) is to form the Lagrangian:
\[ \mathcal{L} = -\sum_{t=0}^{\infty} \beta^t [y_t Q y_t + U_t' R U_t + 2\beta \mu_{t+1}' (A y_t + B U_t + C W_{t+1} - y_{t+1}) - \beta \Theta W_{t+1}' W_{t+1}] . \] (16.3.7) ["olrp3 "]

We want to maximize (16.3.7) with respect to sequences for \( U_t \) and \( y_{t+1} \) and minimize it with respect to a sequence for \( W_{t+1} \). The first-order conditions with respect to \( U_t, y_t, W_{t+1} \), respectively, are:
\[ 0 = RU_t + \beta B' \mu_{t+1} \] (16.3.8a) ["foc1;a "]
\[ \mu_t = Q y_t + \beta A' \mu_{t+1} \] (16.3.8b) ["foc1;b "]
\[ 0 = \beta \Theta W_{t+1} - \beta C' \mu_{t+1} \] (16.3.8c) ["foc1;c "]

Solving (16.3.8a) and (16.3.8c) for \( U_t \) and \( W_{t+1} \) and substituting into (16.2.5) gives
\[ y_{t+1} = A y_t - \beta (BR^{-1}B' - \beta^{-1} \Theta^{-1} CC') \mu_{t+1} . \] (16.3.9) ["olrp4 "]
Write (16.3.9) as
\[ y_{t+1} = A y_t - \beta \tilde{B} \tilde{R}^{-1} \tilde{B}' \mu_{t+1}. \] (16.3.10) \[\text{"olrp6\tiny"]\]

We can represent the system formed by (16.3.10) and (16.3.8b) as
\[
\begin{bmatrix} I & \beta \tilde{B} \tilde{R}^{-1} \tilde{B}' \\ 0 & \beta A' \end{bmatrix} \begin{bmatrix} y_{t+1} \\ \mu_{t+1} \end{bmatrix} = \begin{bmatrix} A & 0 \\ -Q & I \end{bmatrix} \begin{bmatrix} y_t \\ \mu_t \end{bmatrix}. \] (16.3.11) \[\text{"olrp7\tiny"]\]
or
\[
L^* \begin{bmatrix} y_{t+1} \\ \mu_{t+1} \end{bmatrix} = N \begin{bmatrix} y_t \\ \mu_t \end{bmatrix}. \] (16.3.12) \[\text{"olrp8\tiny"]\]

We want to find a ‘stabilizing’ solution of (16.3.12), i.e., one that satisfies
\[
\sum_{t=0}^{\infty} \beta^t y_t y_t' < +\infty.
\]

The stabilizing solution is attained by setting \( \mu_0 = Py_0 \), where \( P \) solves the matrix Riccati equation \( P = T \circ D(P) \). The solution for \( \mu_0 \) replicates itself over time in the sense that
\[ \mu_t = Py_t. \] (16.3.13) \[\text{"king4\tiny"]\]

16.3.3. Key insight

In a typical robust linear regulator problem, \( y_0 \) is a state vector inherited from the past; the multiplier \( \mu_0 \) jumps at \( t = 0 \) to satisfy \( \mu_0 = Py_0 \). See chapters 3 and 6. But in the Stackelberg problem, pertinent components of both \( y_0 \) and \( \mu_0 \) must adjust to satisfy \( \mu_0 = Py_0 \), as shown in step 3.

16.3.4. Step 3: convert implementation multipliers into state variables

Partition \( \mu_t \) conformably with the partition of \( y_t \) into \( \begin{bmatrix} z_t & x_t \end{bmatrix}' \):

\[ \mu_t = \begin{bmatrix} \mu_{zt} \\ \mu_{xt} \end{bmatrix}. \]

\[7\] This argument just adapts one in Pearlman (1992). The Lagrangian associated with the robust Stackelberg problem remains (16.3.7). Then the logic of section 16.3.2 implies that the stabilizing solution must satisfy (16.3.13). It is only in how we impose (16.3.13) that the solution diverges from that for the linear regulator.
For the robust Stackelberg problem, only the first \( n_z \) elements of \( y_t = [z_t \quad \mu_{xt}]' \) are predetermined while the remaining components are free. And while the first \( n_z \) elements of \( \mu_t \) are free to jump at \( t \), the remaining components are not. The third step completes the solution of the robust Stackelberg problem by taking note of these facts. We convert the last \( n_x \) Lagrange multipliers \( \mu_{xt} \) into state variables by using the following procedure after we have performed the key step of computing the \( P \) that solves the Riccati equation \( P = T \circ D(P) \).

Write the last \( n_x \) equations of (16.3.13) as

\[
\mu_{xt} = P_{21}z_t + P_{22}x_t. \tag{16.3.14} \]

The vector \( \mu_{xt} \) becomes part of the state at \( t \), while \( x_t \) is free to jump at \( t \). Therefore, solve (16.3.13) for \( x_t \) in terms of \( (z_t, \mu_{xt}) \):

\[
x_t = -P_{22}^{-1}P_{21}z_t + P_{22}^{-1}\mu_{xt}. \tag{16.3.15} \]

Then we can write

\[
y_t = \begin{bmatrix} I \\ -P_{22}^{-1}P_{21} \end{bmatrix} \begin{bmatrix} z_t \\ \mu_{xt} \end{bmatrix}. \tag{16.3.16} \]

and from (16.3.14)

\[
\mu_{xt} = [P_{21} \quad P_{22}]y_t. \tag{16.3.17} \]

With these modifications, the key formulas (6.11.2) and (16.3.4) from the optimal linear regulator for \( F \) and \( P \), respectively, continue to apply. Using (16.3.16), the solutions for the control and worst case shock are

\[
\begin{bmatrix} U_t \\ W_{t+1} \end{bmatrix} = \begin{bmatrix} -F_1 \\ -F_2 \end{bmatrix} \begin{bmatrix} I \\ -P_{22}^{-1}P_{21} \end{bmatrix} \begin{bmatrix} z_t \\ \mu_{xt} \end{bmatrix}. \tag{16.3.18} \]

Using the law of motion for \( y_{t+1} \) together with (16.3.16) and (16.3.17) allows us to represent our solution recursively as

\[
\begin{bmatrix} z_{t+1} \\ \mu_{x,t+1} \end{bmatrix} = \begin{bmatrix} I \\ P_{21} \end{bmatrix} \begin{bmatrix} A - BF_1 - CF_2 \end{bmatrix} \begin{bmatrix} I \\ -P_{22}^{-1}P_{21} \end{bmatrix} \begin{bmatrix} z_t \\ \mu_{xt} \end{bmatrix}. \tag{16.3.19a} \]

\[
x_t = [-P_{22}^{-1}P_{22} \quad P_{22}^{-1}] \begin{bmatrix} z_t \\ \mu_{xt} \end{bmatrix}. \tag{16.3.19b} \]

When the random shock \( \epsilon_{t+1} \) is present, we must add

\[
\begin{bmatrix} I \\ P_{21} \end{bmatrix} C\epsilon_{t+1}. \tag{16.3.20} \]
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to the right side of (16.3.19). Equation (16.3.19a) is the worst-case law of motion for \( z_t \). To get the law of motion under the approximating model and the robust Stackelberg or Ramsey plan, we replace \( (A - BF_1 - CF_2) \) with \( A - BF_1 \) in (16.3.19a). By doing so, we set the worst-case shock \( W_{t+1} \) to zero. Then we have the following description of the approximating model under the robust Stackelberg plan:

\[
\begin{bmatrix}
  z_{t+1} \\
  \mu_{x,t+1}
\end{bmatrix} = \begin{bmatrix}
  I & 0 \\
  P_{21} & P_{22}
\end{bmatrix} \begin{bmatrix}
  I \\
  -P_{22}^{-1} P_{21}
\end{bmatrix} \begin{bmatrix}
  z_t \\
  \mu_{xt}
\end{bmatrix}
\]  

(16.3.21a)  ["king11;a "]

\[
x_t = \begin{bmatrix}
  -P_{22}^{-1} P_{21} & P_{22}^{-1}
\end{bmatrix} \begin{bmatrix}
  z_t \\
  \mu_{xt}
\end{bmatrix}
\]  

(16.3.21b)  ["king11;b "]

Again, in the random case we must add (16.3.20) to the right side of (16.3.21). The difference equation (16.3.21a) is to be initialized from the given value of \( z_0 \) and the value \( \mu_{0,z} = 0 \). The latter setting reflects that at time 0 there are no past promises to keep.

In summary, we solve the robust Stackelberg problem by formulating a particular optimal linear regulator, solving the associated matrix Riccati equation (16.3.4) for \( P \), computing \( F_1, F_2 \), and then partitioning \( P \) to obtain representation (16.3.21).

16.3.5. Alternative representation of decision rule

For some purposes, it is useful to eliminate the implementation multipliers \( \mu_{xt} \) and to express the decision rule for \( U_t \) as a function of \( z_t, z_{t-1} \) and \( U_{t-1} \). This can be accomplished as follows.\(^8\) First represent (16.3.21a) compactly as

\[
\begin{bmatrix}
  z_{t+1} \\
  \mu_{x,t+1}
\end{bmatrix} = \begin{bmatrix}
  m_{11} & m_{12} \\
  m_{21} & m_{22}
\end{bmatrix} \begin{bmatrix}
  z_t \\
  \mu_{xt}
\end{bmatrix}
\]  

(16.3.22)  ["vonzer1 "]

and write the feedback rule for \( U_t, W_{t+1} \) (16.3.18) as

\[
U_t = f_{11} z_t + f_{12} \mu_{xt} \]  

(16.3.23)  ["vonzer2 "]

\[
W_{t+1} = f_{21} z_t + f_{22} \mu_{xt}. \]  

(16.3.24)  ["vonzer22"]

Then where \( f_{12}^{-1} \) denotes the generalized inverse of \( f_{12} \), (16.3.23) implies \( \mu_{x,t} = f_{12}^{-1} (U_t - f_{11} z_t) \). Equate the right side of this expression to the right side of the second line of (16.3.22) lagged once and rearrange by using (16.3.23) lagged once to eliminate \( \mu_{x,t-1} \) to get

\[
U_t = f_{12} m_{22} f_{12}^{-1} U_{t-1} + f_{11} z_t + f_{12} (m_{21} - m_{22} f_{12}^{-1} f_{11}) z_{t-1}
\]  

(16.3.25a)  ["vonzer3;a "]

\(^8\) Peter Von Zur Muehlen suggested this representation to us.
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or

\[ U_t = \rho U_{t-1} + \alpha_0 z_t + \alpha_1 z_{t-1} \]  

(16.3.25b)  

for \( t \geq 1 \). For \( t = 0 \), the initialization \( \mu_{x,0} = 0 \) implies that

\[ U_0 = f_{11}Z_0. \]  

(16.3.25c)  

Similarly, the worst case shock can be represented as

\[ W_{t+1} = f_{22}m_{22}f_{12}^{-1}U_{t-1} + f_{21}z_t + f_{22}(m_{21} - m_{22}f_{12}^{-1}f_{11})Z_{t-1}. \]  

(16.3.26)  

By making the leader’s control or ‘instrument’ feed back on itself, the form of (16.3.25) potentially allows ‘instrument-smoothing’ to emerge as an optimal rule under commitment. This insight partly motivated Woodford to use his model as a tool to interpret empirical evidence about interest rate smoothing in the U.S. We discuss this further in section 16.6.

By following the approaches of Kydland and Prescott (1980) and Marcet and Marimon (2000), appendix C describes a closely related Bellman equation that can be used to compute a robust Ramsey plan.

16.4. Incorporating robustness for the followers

So far we have concentrated on getting a robust rule for the leader, taking as given the Euler equations that characterize the followers’ behavior. In this section, we point out that by including the appropriate Euler equations for the followers among the implementability constraints, we can impute a concern for robustness to the followers as well as to the leader. For a representative follower example, we shall index the concern for robustness among the followers by a multiplier \( \theta \) that can but need not equal the robustness parameter \( \Theta \) of the leader.
16.4.1. Strategy

To apply the preceding results to a problem in which the Stackelberg leader and the Stackelberg followers both want robust decision rules, we have to include Euler equations for the follower that incorporate a concern about robustness. To formulate these implementability constraints, we can rely on findings from chapter 6 that allow us to represent the solution of the followers’ problem as the stabilizing solution of a system of Euler equations that are the first-order conditions for extremizing with respect to both a ‘natural control’ $u_t$ and a pseudo-control $w_{t+1}$, the worst case shocks, in a robust linear regulator. The Bellman-Isaacs condition from chapter 6 allows us to characterize the solution of the robust control problem in this way. Then to formulate the robust Stackelberg problem, we regard the first-order conditions of the competitive firm, including those for choosing the follower’s worst-case shock process, as among the implementability conditions for the monopolist. This leads to an equilibrium of the game between the leader and the follower in which each understands the decision rules of the other, and in which the leader takes into account how the follower’s decisions respond to its own. To know how the follower responds, the leader has to keep track of how the worst case shocks of the follower responds to the leader’s decisions. This impels us to include the worst case shock process of the followers in the state vector for the leader.

By following this recipe, we construct an equilibrium in which leaders and followers share a common approximating model. However, the difference in their preferences can lead them to slant their worst case models in different directions away from their common approximating model, as the two types of agent use their own worst-case analyses to investigate the fragility of alternative rules to possible misspecification of that common approximating model. In the next section, we illustrate our equilibrium concept with an example.

16.5. A monopolist with a competitive fringe

As an example, this section studies an industry with a large firm that acts as a Stackelberg leader with respect to a competitive fringe. The industry produces a single nonstorable homogeneous good. One large firm called the monopolist produces $Q_t$ and a representative firm in a competitive fringe produces $q_t$. We use $q_t$ to denote the quantity chosen by the individual competitive firm and $\bar{q}_t$ to denote the equilibrium quantity. In equilibrium, $q_t = \bar{q}_t$, but it is necessary to distinguish between $q_t$ and $\bar{q}_t$ in posing the optimum problem of the representative competitive firm. The representative firm in the competitive fringe takes
\( Q_t, \bar{Q}_t \) as exogenous and chooses sequentially. In light of the responses of the representative firm in the competitive fringe, the monopolist commits to a policy at time 0, taking into account its ability to manipulate the price sequence and the worst case beliefs of the representative competitive firm through its quantity choices. Subject to the competitive fringe’s best response, the monopolist views itself as choosing \( q_{t+1} \) and \( Q_{t+1} \) for \( t \geq 0 \), as well as the representative competitive firm’s worst-case shock process \( w_{t+1} \) for \( t \geq 0 \).

Costs of production are
\[
C_t = eQ_t + \frac{5gQ_t^2}{2} + 5c(Q_{t+1} - Q_t)^2
\]
for the monopolist and
\[
\sigma_t = dq_t + \frac{5hq_t^2}{2} + 5c(q_{t+1} - q_t)^2
\]
for the representative competitive firm, where \( d > 0, e > 0, c > 0, g > 0, h > 0 \) are cost parameters. There is a linear inverse demand curve
\[
p_t = A_0 - A_1 (Q_t + \bar{Q}_t) + v_t,
\]
where \( A_0, A_1 \) are both positive and \( v_t \) is a disturbance to demand governed by
\[
v_{t+1} = \rho v_t + C_v \tilde{\epsilon}_{t+1}
\]
and where \( |\rho| < 1 \) and \( \tilde{\epsilon}_{t+1} \) is an i.i.d. sequence of random variables with mean zero and variance 1. The monopolist and the representative competitive firm share equation (16.5.2) as their approximating model for the demand shock. The monopolist and the representative competitive firm both want decision rules that are robust to alternative specifications of the process for the demand shock. Because the monopolist and the representative firm in the competitive fringe potentially have different worst case models of the demand shock, we distinguish between them by letting \( v_t \) denote the process perceived by the representative firm, and \( V_t \) the process perceived by the monopolist. For the representative competitive firm, the alternative models of the demand shock have the form
\[
v_{t+1} = \rho v_t + C_v (\epsilon_{t+1} + w_{t+1}).
\]
For the monopolist, they have the form
\[
V_{t+1} = \rho V_t + C_v (\tilde{\epsilon}_{t+1} + W_{t+1}).
\]
It is appropriate to set initial conditions so that \( V_0 = v_0 \). Here \( w_{t+1}, W_{t+1} \) are specification errors for the representative competitive firm and the monopolist, respectively, and \( \epsilon_{t+1}, \tilde{\epsilon}_{t+1} \) are other i.i.d. random processes with mean zero and variance 1. The distortions \( w_{t+1}, W_{t+1} \) can feed back on the history of the state of the market, namely, \((\bar{Q}, Q, v, V)\). The distortions \( w_{t+1} \) and \( W_{t+1} \) will typically differ because the monopolist and the representative competitive firm have different objectives.
16.5.1. The competitive fringe

The representative competitive firm regards \{Q_t, \overline{q}_t\}_{t=0}^\infty as given stochastic processes and chooses an output plan \{q_t\}_{t=0}^\infty and shock distortion process \{w_{t+1}\}_{t=0}^\infty to extremize

\[
E_0 \sum_{t=0}^\infty \beta^t \{p_t q_t - \sigma_t + \beta\theta w_{t+1}^2\}, \quad \beta \in (0, 1)
\]  \hfill (16.5.5) ["oli3"]

subject to \(q_0\) given, where \(E_t\) is the mathematical expectation based on time \(t\) information evaluated with respect to a distorted model that includes (16.5.3). Here \(\theta\) is the robustness parameter of the representative firm in the competitive fringe, which could differ from \(\Theta\), the robustness parameter of the monopolist. Let \(u_t = q_{t+1} - q_t\). We take \((u_t, w_{t+1})\) as the representative competitive firm’s composite control vector at \(t\). First order-conditions for extremizing (16.5.5) with respect to \(u_t, w_{t+1}\) are

\[
u_t = E_t \beta u_{t+1} - c^{-1}\beta h q_{t+1} + c^{-1}\beta E_t (p_{t+1} - d)
\]

\[
w_{t+1} = -\frac{1}{2\theta} C_v q_{t+1} + \beta \rho E_t w_{t+2}
\]  \hfill (16.5.6) ["oli4"]

for \(t \geq 0\).

We can appeal to the certainty equivalence principle stated on page 20 to justify working with a non-stochastic version of (16.5.6) that we form by dropping the expectation operator and the random terms \(\hat{\epsilon}_{t+1}\) and \(\epsilon_{t+1}\) from (16.5.2) and (16.5.3).\(^9\) Shift (16.5.1) forward one period, set \(q_t = \overline{q}_t\) for all \(t \geq 0\), and substitute for \(p_{t+1}\) in (16.5.6) to get

\[
u_t = \beta u_{t+1} - c^{-1}\beta h \overline{q}_{t+1} + c^{-1}\beta (A_0 - d) - c^{-1}\beta A_1 \overline{q}_{t+1} - c^{-1}\beta A_1 Q_{t+1} + c^{-1}\beta v_{t+1}
\]

\[
w_{t+1} = -\frac{1}{2\theta} C_v q_{t+1} + \beta \rho w_{t+2}.
\]  \hfill (16.5.7) ["oli5"]

Equation (16.5.7) combines the Euler equations of the representative firm in the competitive fringe with market clearing.\(^10\) Note that \(v\), and not \(V\), appears in the

---

\(^9\) We use a method that Sargent (1979) and Townsend (1983) used to compute a rational expectations equilibrium. The key step is to eliminate price and output by setting \(q_t = \overline{q}_t\) and substituting from the inverse demand curve and the production function into the firm’s first-order conditions to get a difference equation in capital.

\(^10\) As shown in Sargent (1979) in the case without robustness, (16.5.7) is also the Euler equation for a fictitious planner who takes \(Q_t\) as exogenous and who chooses...
first equation of (16.5.7). This reflects how the representative competitive firm’s forecasts influence its decisions, a fact that the monopolist will acknowledge when he designs his policy.

16.5.2. The monopolist’s problem

The monopolist views the sequence of Euler equations-cum-market clearing conditions (16.5.7) as implementability constraints. We can represent the constraints impinging on the monopolist, including (16.5.7), in terms of the transition law:

\[
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
A_0 - d & 1 & 0 & -A_1 & -A_1 - h & c & 0 & u_{t+1} \\
0 & 0 & 0 & 0 & -\frac{1}{2\rho}C_v & 0 & \beta \rho & w_{t+2}
\end{bmatrix}
\]

(16.5.8) [“oli6 “]

\[
= \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \rho & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \rho & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0
\end{bmatrix}
\begin{bmatrix}
1 \\
v_t \\
V_t \\
Q_t \\
R_t \\
u_t \\
w_{t+1}
\end{bmatrix}
+ \begin{bmatrix}
0 \\
0 \\
0 \\
1 \\
0 \\
0 \\
0
\end{bmatrix}U_t + \begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0
\end{bmatrix} \begin{bmatrix}
C_t \\
W_{t+1}
\end{bmatrix}
\]

(16.5.9) [“oli6a “]

where \(U_t = Q_{t+1} - Q_t\) is the control of the monopolist. The last row portrays (16.5.7). Represent (16.5.8) as

\[
y_{t+1} = Ay_t + BU_t + CW_{t+1}.
\]

Although we have included \((u_t, w_{t+1})\) as components of the ‘state’ \(y_t\) in the monopolist’s transition law (16.5.9), \((u_t, w_{t+1})\) are actually ‘jump’ variables that correspond to \(x_t\) in section 16.3. The analysis in section 16.3 implies that the solution of the monopolist’s problem is encoded in the Riccati equation associated with a robust linear regulator that takes (16.5.9) as the transition law.

a sequence for \(\{q_{t+1}\}_{t=0}^{\infty}\) to maximize the discounted sum of consumer and producer surplus. Given stable sequences \(\{Q_t, v_t\}\), we could solve (16.5.7) and \(u_t = \theta_{t+1} - \theta_t\) to express the competitive fringe’s output sequence as a function of the monopolist’s output sequence.
To match the setup of section 16.3, we partition $y_t$ as $y_t' = [z_t' \ x_t']$ where $z_t' = [1 \ v_t \ V_t \ Q_t \ \bar{\pi}_t]$, $x_t' = [u_t \ w_{t+1}]$, and let $\mu_{xt} = \begin{bmatrix} \mu_{ut} \\ \mu_{wt} \end{bmatrix}$ be the vector of multipliers associated with the Euler equations for $(u_t, w_{t+1})$. The monopolist’s problem can be expressed

$$
\max_{\{U_t\}} \min_{\{W_{t+1}\}} \sum_{t=0}^{\infty} \beta^t \left\{ p_t Q_t - C_t + \beta \Theta W_{t+1} W_{t+1} \right\}
$$

or

$$
\max_{\{U_t\}} \min_{\{W_{t+1}\}} \sum_{t=0}^{\infty} \beta^t \left\{ (A_0 - A_1 (\bar{\pi}_t + Q_t) + V_t) Q_t - eQ_t - .5gQ_t^2 - .5eU_t^2 + \beta \Theta W_{t+1}^2 \right\}
$$

subject to (16.5.9). Notice that the monopolist’s perceived demand shock appears in (16.5.10). The monopolist’s problem can be written

$$
\max_{\{U_t\}} \min_{\{W_{t+1}\}} \sum_{t=0}^{\infty} \beta^t \left\{ y_t' Q y_t + U_t' R U_t - \beta \Theta W_{t+1}^2 \right\}
$$

subject to (16.5.9) where

$$
Q = \begin{bmatrix}
0 & 0 & 0 & \frac{\Delta_0 - e}{2} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\Delta_0 - e & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

and $R = \frac{c}{2}$. The results of section 16.3 apply.
16.5.3. Representation of the monopolist’s decision rule

Recall that $z_t = [1 \ v_t \ V_t \ Q_t \ \bar{u}_t]'$ and $x_t = [u_t \ w_{t+1}]'$. The monopolist’s decision rule is given by equations (16.3.23), (16.3.24), which become

$$
\begin{bmatrix}
U_t \\
W_{t+1}
\end{bmatrix} = o_1 \begin{bmatrix}
v_t \\
V_t \\
Q_t \\
\bar{u}_t
\end{bmatrix} + o_2 \begin{bmatrix}
\mu_{at} \\
\mu_{wt}
\end{bmatrix}.
\tag{16.5.12}
$$

Equation (16.3.15) describes the decisions of the representative competitive firm; (16.3.15) becomes

$$
\begin{bmatrix}
u_t \\
w_{t+1}
\end{bmatrix} = n_1 \begin{bmatrix}
v_t \\
V_t \\
Q_t \\
\bar{u}_t
\end{bmatrix} + n_2 \begin{bmatrix}
\mu_{at} \\
\mu_{wt}
\end{bmatrix}.
\tag{16.5.13}
$$

Here $n_1, n_2, o_1, o_2$ are matrices to be defined by matching the formulas from section 16.3. In addition, (16.3.22) gives the law of motion of $[z_t \ x_t]'$ with the Stackelberg plan under the approximating model.

16.5.4. Interpretation

The approximating model incorporates the robust decision rules for both types of firm, but does not add $C_v W_{t+1}$ to the right side of (16.3.22). However, $C_v w_{t+1}$ is by construction incorporated in the appropriate equation for $v_{t+1}$. The absence of $C_v W_{t+1}$ from the right side of (16.3.22) reflects that under the approximating model, concern about robustness is just in the head of the monopolist firm, meaning that it affects its decision rule, but not the actual course of the demand shock process $V_t$. However, $v_t$ tracks the worst case shock process of the competitive firm, which helps the monopolist to determine the effects of his decisions on the decisions of the competitive firms.

To simulate the random version of the model, we add $[0 \ C_v \ C_v \ 0 \ 0 \ 0]' \epsilon_{t+1}$ to the right side of (16.5.8) and the appropriate counterpart to the right side of (16.3.22). Note that the innovation $\epsilon_{t+1}$ impinges on both $V_t$ and $v_t$ under the approximating model.
16.5.5. Numerical example

For parameter settings $A_0, A_1, \rho, C_v, c, d, e, g, h, \beta = 100, 1, .8, 2, 1, 20, 2, .2, .95, 10$, and $\theta = \Theta = 100$, Fig. 16.5.1, Fig. 16.5.2, Fig. 16.5.3, Fig. 16.5.4, Fig. 16.5.5, Fig. 16.5.6 display features of a simulation of the approximating model under the robust rule, i.e., we use $M$, not $M_w$.

Figures 16.5.1, 16.5.2, 16.5.3, and 16.5.4 show impulse responses to the demand innovation $\epsilon_t$ while figures 16.5.5 and 16.5.6 show aspects of a simulated sample path using a Gaussian i.i.d. $\epsilon_t$ with mean zero and unit variance. The simulation was initialized with all variables except $\mu_w, \mu_u$ set to the nonstochastic steady state of (16.3.22); we set $\mu_w, \mu_u$ to zero. The impulse responses show that a demand innovation pushes the implementation multipliers $\mu_w, \mu_u$ down, and leads the monopolist to expand output while the representative competitive firm contracts output in subsequent periods. The response of price to a demand shock innovation is to rise on impact but then to decrease in subsequent periods in response to the increase in total supply $q+Q$ engineered by the monopolist. Note that both of the worst case shocks $W$ and $w$ fall in response to an innovation in demand. This tells us that both types of firms’ decision rules are most fragile in the direction of overestimating demand.

Note in figure 16.5.6 how starting from 0 the implementation multipliers $\mu_u$ and $\mu_w$ head from zero toward the vicinities of their steady state values, which are negative. The negative values of the multipliers $\mu_u, \mu_w$ reflect the cost to the monopolist of adhering to its plan. The time inconsistency of the monopolist’s plan is reflected in the incentive the monopolist would have to reset the multipliers to zero in any period and thereby reinitialize its plan (see Hansen, Epple, and Roberds (1985)). Fig. 16.5.5 illustrates that the monopolist is acting to smooth total output $Q+q$, and that it does so by inducing a negative contemporaneous covariance between its own output and the price. Fig. 16.5.7 shows the worst case shocks $W$ and $w$ and their difference. Notice that $W$ and $w$ are both negative, so that the both types of firm are protecting themselves against lower demand than given by the approximating model. Fig. 16.5.8 shows sample paths of $V, v$ and their difference. Since we are simulating the approximating model under the robust rules for the monopolist and the competitive fringe, $V$ is the actual demand shock process, while $v$ is the worst-case demand shock process that the representative firm in the competitive fringe uses to construct a robust rule, and that the monopolist manipulates, subject to the implementability constraint given by the Euler equation that is the second block of (16.5.7). The worst case shock reflects state-dependent pessimism about the state of demand.
16.6. A ‘new synthesis’ model

To illustrate a robust Ramsey plan, we adopt the economic setting of Woodford (1998). Woodford’s model is one of a class of ‘new synthesis’ macroeconomic models (see Goodfriend and King (1997), Rotemberg and Woodford...
Robustness in forward looking models

Figure 16.5.3: Impulse response of $v$, $V$, and $V-v$ to demand shock innovation $\epsilon$.

Figure 16.5.4: Impulse response of $W$ and $w$ to demand shock innovation $\epsilon$.

(1997), Clarida, Gali, and Gertler (1999)) that add nominal price stickiness to a real business cycles model, thereby putting nominal prices into the model and creating an avenue by which monetary policy can influence output.
The heart of Woodford’s model is two log-linear approximations to Euler equations that describe decisions of households and firms. Households’ behavior is summarized by a standard Euler equation for consumption. Log linearizing this equation and using the production function delivers what Woodford calls the
Monopolistically competitive firms set nominal prices according to timing protocols that imply first order conditions that generate what Woodford calls a ‘Phillips curve.’ We begin our analysis from these two log-linearized

\[ W, w, W - w \]

\[ V, v, V - v \]

Figure 16.5.7: Sample path of $W, w, W - w$.

Figure 16.5.8: Sample path of $V, v, V - v$.

‘IS curve’. Monopolistically competitive firms set nominal prices according to timing protocols that imply first order conditions that generate what Woodford calls a ‘Phillips curve.’ We begin our analysis from these two log-linearized

\[ W, w, W - w \]

\[ V, v, V - v \]

Figure 16.5.7: Sample path of $W, w, W - w$.

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\[ V, v, V - v \]

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\[ V, v, V - v \]

Figure 16.5.7: Sample path of $W, w, W - w$.

Figure 16.5.8: Sample path of $V, v, V - v$.

‘IS curve’. Monopolistically competitive firms set nominal prices according to timing protocols that imply first order conditions that generate what Woodford calls a ‘Phillips curve.’ We begin our analysis from these two log-linearized

\[ W, w, W - w \]

\[ V, v, V - v \]

Figure 16.5.7: Sample path of $W, w, W - w$.

Figure 16.5.8: Sample path of $V, v, V - v$.

‘IS curve’. Monopolistically competitive firms set nominal prices according to timing protocols that imply first order conditions that generate what Woodford calls a ‘Phillips curve.’ We begin our analysis from these two log-linearized

\[ W, w, W - w \]

\[ V, v, V - v \]

Figure 16.5.7: Sample path of $W, w, W - w$.

Figure 16.5.8: Sample path of $V, v, V - v$.

‘IS curve’. Monopolistically competitive firms set nominal prices according to timing protocols that imply first order conditions that generate what Woodford calls a ‘Phillips curve.’ We begin our analysis from these two log-linearized

\[ W, w, W - w \]

\[ V, v, V - v \]
Euler equations. The expectational Phillips curve and the dynamic IS curve, respectively, are:

\[
\pi_t = \kappa \gamma_t + \beta \tilde{E}_t \pi_{t+1} + c_1 \epsilon_{1t} \tag{16.6.1} \]

\[
\gamma_t = \tilde{E}_t \gamma_{t+1} - \sigma^{-1} \left[ (r_t - r^*_{t}) - \tilde{E}_t \pi_{t+1} \right] + c_2 \epsilon_{2t} \tag{16.6.2}
\]

where \(\gamma_t = o_t - o^*_t\), the deviation of the log of output from the log of potential output, \(r_t\) is the nominal short term interest rate (the monetary authority’s instrument), and \(r^*_t\) is the natural rate of interest. We have added constants \(c_1, c_2\) times two i.i.d. unit variance shock processes \(\epsilon_{1t}, \epsilon_{2t}\) to Woodford’s two equations.\(^{12}\) Here \(\tilde{E}_t(\cdot)\) is a (possibly distorted) conditional expectation operator. Woodford assumes rational expectations, so for him there is no distortion. We will find a class of distorted expectations operators indexed by \(\theta\).

Woodford assumes that the government seeks to maximize

\[
- \sum_{t=0}^{\infty} \beta^t l_t \tag{16.6.3} \]

where the loss function \(L\) is taken to be

\[
l_t = \pi^2_t + \lambda_\gamma (\gamma_t - \gamma^*)^2 + \lambda_r (r_t - r^*)^2. \]

Following Woodford, we regard (16.6.3) as the log-linear-quadratic approximation of the welfare of a representative household. Both firms and households use the criterion (16.6.3) to formulate their robust pure forecasting problems.\(^{13}\) Woodford shows that the natural rate of interest obeys

\[
r^*_t = \sigma \tilde{E}_t \left[ (o^*_{t+1} - o_t^*) - (g_{t+1} - g_t) \right]. \tag{16.6.4}
\]

Rather than using (16.6.4) directly, Woodford adopted the specification that \(r^*_t\) is simply a stationary univariate exogenous process, about which we shall soon say more.\(^{14}\) Then the arrival of information can be described by

\[
S_{t+1} = A_{11} S_t + C_1 (w_{t+1} + \epsilon_{t+1}) \tag{16.6.5a} \]

\[
I_t = e S_t \tag{16.6.5b}
\]

\(^{12}\) Woodford's equations have no such errors. He assumed an exact model.

\(^{13}\) See Whittle (1996) and chapter 6.

\(^{14}\) An important feature of this specification is that \(r^*_t\) be a stationary process to validate Woodford’s procedure of using linear approximations around a nonstochastic steady state.
where $I = \begin{bmatrix} r^*_n & \epsilon_{1t} & \epsilon_{2t} & \gamma^* & r^* \end{bmatrix}'$, $e$ is a $5 \times 4$ matrix whose last three columns have a one on the diagonal and zeros in other entries and the first column is $[\gamma^* \ r^* \ 0 \ 0 \ 0]$, $\epsilon_{t+1}$ is an i.i.d. Gaussian sequence of shocks to the state vector $S_t$, and $w_{t+1}$ is a vector of model specification errors. As above, we shall appeal to a certainty equivalence principle that is induced by the linear-quadratic structure to allow drop the $\eta_{t+1}$ term while computing the optimal rules. The approximating model assumes that $w_{t+1} = 0$.

To put Woodford’s model into the framework of section 16.3, define $z_t = S_t, U_t = r_t, x_t = [\pi_t \ \gamma_t]'$, $C = \begin{bmatrix} C_1 \\ 0 \end{bmatrix}$. Then the model (16.6.1), (16.6.2) (16.6.5) and the associated robust control problem fit within the framework of section 16.3.

16.6.1. Specification of $r^*_t$

For the purposes of preparing the way for issues to be discussed in section 16.7, we say more about the specification of the natural rate $r^*_t$. Woodford’s specification that the natural rate of interest $r^*_t$ is a first order autoregressive process is compatible with the following law of motion for $(o^*_t, g_t)$:

$$
\begin{align*}
o^*_{t+1} &= 2o^*_t - o^*_{t-1} + c_o \epsilon_{o,t+1} \\
g_{t+1} - o^*_{t+1} &= a_g + s_g,t+1 \\
s_{g,t+1} &= \rho s_g,t + c_g \epsilon_{g,t+1}, \quad |\rho| < 1,
\end{align*}
$$

where $[\epsilon_{o,t} \ \epsilon_{g,t}]$ is a vector white noise orthogonal to the $(\epsilon_1, \epsilon_2)$ process. Remember that $o^*_t$ represents the log of potential output and $g_t$ the log of aggregate demand. The system (16.6.6) says that the growth rate of $o^*_t$ takes a random walk. This allows ‘trend breaks’ in potential output. Equations (16.6.6b), (16.6.6c) allow $g_t$ to deviate from $o^*_t$ by a constant plus a stable first-order autoregressive process. Equations (16.6.4) and (16.6.6) imply

$$
r^*_t = \sigma (1 - \rho) s_{yt} = c_g (1 - \rho L) s_{g,t},
$$

where $L$ is the lag operator. Equation (16.6.7) is Woodford’s specification, which we adopt in the numerical calculation in section 16.8.
16.7. Filtering

This section analyzes the combined filtering and control of forward-looking models that was studied without a concern for model misspecification by Pearlman, Currie, and Levine (1986), Pearlman (1992), and Svensson and Woodford (1999). Drawing heavily on results from chapter 14, we briefly describe how the calculations in section 16.3 can be adapted when agents and the government have incentives to estimate a latent variable, like potential GDP. Designing a robust Ramsey rule now requires deriving a robust filter and adjusting the control problem to admit additional possible model distortions when filtering is done with a possibly misspecified model. As a laboratory, we use a version of Woodford’s model, suitably adjusted to make potential GDP a latent variable.

16.7.1. Specification with observed state

For concreteness, we adopt the particular specification (16.6.6). To represent this specification with matrix notation, let

\[ s_{ut} = \begin{bmatrix} o_{t-1}^{o} & o_{t-1}^{o} & s_{gt} \end{bmatrix} \]  \hspace{1cm} (16.7.1)  

Soon we shall assume that \( s_{ut} \) is not observed and that it must be estimated by both the public and the government. But for now let \( s_{ut} \) be observed at \( t \). Let \( s_{ot} \) be a vector of information variables whose histories up to \( t \) are observed at time \( t \). Assume that

\[ \begin{bmatrix} s_{u,t+1} \\ s_{o,t+1} \end{bmatrix} = \begin{bmatrix} A_{uu} & A_{uo} \\ A_{ou} & A_{oo} \end{bmatrix} \begin{bmatrix} s_{ut} \\ s_{ot} \end{bmatrix} + \begin{bmatrix} C_u \\ C_o \end{bmatrix} \eta_{t+1}, \]  \hspace{1cm} (16.7.2)  

where \( \eta_{t} \) is an i.i.d. vector process with mean zero and unit covariance. Let \( s_{o,t} = [\epsilon_{1t} \epsilon_{2t} 1]' \). Under the assumption that both \( s_{ut}, s_{ot} \) are observed at \( t \), specification (16.7.2) is consistent with system (16.6.5a), so that the Ramsey plan can be computed as in section 16.3.
16.7.2. Latent state variables

The solution procedure can be altered to handle the case when $s_{ut}$ is not observed. As an example, we now assume that potential output is observed by neither the government nor the private sector. Everyone treats potential output as a hidden variable that must be estimated as a function of the history of observed variables. Consistent with this assumption, we adjust the model to be

$$\pi_t = \kappa \gamma_t + \beta \tilde{E}_t \pi_{t+1} + c_1 \varepsilon_{1t}$$  \hspace{1cm} (16.7.3) \hspace{1cm} ["wood23 "]

$$\gamma_t = \tilde{E}_t \gamma_{t+1} - \sigma^{-1} \left( (r_t - r_t^n) - \tilde{E}_t \pi_{t+1} \right) + c_2 \varepsilon_{2t}$$  \hspace{1cm} (16.7.4) \hspace{1cm} ["wood24 "]

where $\gamma_t = o_t - \tilde{o}_t^n$ and for any variable $a$, $\tilde{a}_{t+1}$ represents an estimate of $a$ using observations up to time $t + 1$. Now the natural rate of interest satisfies

$$r_t^n = \sigma \tilde{E}_t \left[ (\tilde{o}_{t+1}^n - \tilde{o}_t^n) - (g_{t+1} - g_t) \right].$$  \hspace{1cm} (16.7.5) \hspace{1cm} ["rn4 "]

Equation (16.7.5) assumes that $g_{t+1}$ is observed but that $o_{t+1}^n$ is not at $t + 1$, so that a robust estimator $\tilde{o}_t^n$ must be constructed.

The government and private agents now face a robust filtering problem on top of their robust decision problems. Results of chapter 14 can be applied to show that filtering and ‘control’ aspects of the problem separate so that the following two step procedure is appropriate.

**Step 1.** Solve an ordinary Kalman filtering problem for the system (16.7.2) to attain the corresponding ‘innovations representation’

$$\begin{bmatrix} s_{u, t+1} \\ s_{o, t+1} \end{bmatrix} = \begin{bmatrix} A_{uu} & A_{uo} \\ A_{ou} & A_{oo} \end{bmatrix} \begin{bmatrix} s_{ut} \\ s_{ot} \end{bmatrix} + \begin{bmatrix} \tilde{C}_u & \tilde{C}_o \\ 0 & \tilde{C}_o \end{bmatrix} \begin{bmatrix} \tilde{\eta}_{t+1} \\ \tilde{\eta}_{t+1} \end{bmatrix}$$  \hspace{1cm} (16.7.6) \hspace{1cm} ["fi12 "]

where $\begin{bmatrix} \tilde{\eta} \\ \tilde{\eta} \end{bmatrix}$ is an i.i.d. vector disturbance with mean zero and identity covariance matrix, and $\tilde{C}_u, \tilde{C}_u, \tilde{C}_o$ are matrices determined by a Kalman filter, as described below.

**Step 2.** Using the following evolution for information

$$\begin{bmatrix} \hat{s}_{u, t+1} \\ \hat{s}_{o, t+1} \end{bmatrix} = \begin{bmatrix} A_{uu} & A_{uo} \\ A_{ou} & A_{oo} \end{bmatrix} \begin{bmatrix} \hat{s}_{ut} \\ \hat{s}_{ot} \end{bmatrix} + \begin{bmatrix} \tilde{C}_u & \tilde{C}_o \\ 0 & \tilde{C}_o \end{bmatrix} \begin{bmatrix} \tilde{w}_{t+1} \\ \tilde{w}_{t+1} \end{bmatrix},$$  \hspace{1cm} (16.7.7) \hspace{1cm} ["fi13 "]

solve the problem using the method described in section 16.3. System (16.7.7) is obtained from (16.7.6) by replacing the stochastic shocks $\tilde{\eta}, \tilde{\eta}$ with distortions to the conditional means, $\tilde{w}, \tilde{w}$.

We next briefly describe the filtering problem in enough detail to implement the procedure.

---

15 See Anderson and Moore (1979) for a study of innovations representations.
16.7.3. Filtering details

System (16.7.6) or (16.7.7) is obtained from (16.7.2) by an application of the standard Kalman filter. In particular, the steady-state covariance for the error in reconstructing the state is given by $\Sigma$, the positive semi-definite matrix that solves the Riccati equation

$$
\Sigma = A_{uu} \Sigma A'_{uu} + C_u C'_u
- [A_{uu} \Sigma A'_{ou} + C_u C'_o] [A_{ou} \Sigma A'_{ou} + C_o C'_o]^{-1} [A_{ou} \Sigma A'_{uu} + C_o C'_u].
$$

(16.7.8) \[\text{"riccatif "}\]

Define the innovation covariance matrix

$$
\Lambda = E \begin{bmatrix} s_{u,t+1} - \hat{s}_{u,t+1} \\ s_{o,t+1} - \hat{s}_{o,t+1} \\ \end{bmatrix} \begin{bmatrix} s_{u,t+1} - \hat{s}_{u,t+1} \\ s_{o,t+1} - \hat{s}_{o,t+1} \\ \end{bmatrix}',
$$

which can be computed from (16.7.2):

$$
\Lambda = \begin{bmatrix} A_{ou} \Sigma A'_{ou} + C_o C'_o & A_{ou} \Sigma A'_{uu} + C_o C'_u \\ A_{uu} \Sigma A'_{ou} + C_u C'_o & A_{uu} \Sigma A'_{uu} + C_u C'_u \\ \end{bmatrix}.
$$

(16.7.9) \[\text{"lambdadef "}\]

Obtain $\hat{C}_u, \hat{C}_o, \hat{C}_u$ from the Cholesky factorization:

$$
\Lambda = \begin{bmatrix} \hat{C}_u & \hat{C}_o \\ 0 & \hat{C}_o \\ \end{bmatrix} \begin{bmatrix} \hat{C}_u & \hat{C}_o \\ 0 & \hat{C}_o \\ \end{bmatrix}'.
$$

(16.7.10) \[\text{"chol "}\]

These are the objects we use to create the innovations representation (16.7.6).

In (16.7.6), $\hat{C}_u \hat{\eta}_{t+1} \equiv a_{o,t+1}$ is the innovation in $s_{o,t+1}$, the error in predicting $s_{o,t+1}$ linearly from its own past. The term $\hat{C}_u \hat{\eta}_{t+1} \equiv K a_{o,t+1}$ is the projection of the unobserved $s_{u,t+1}$ on the innovation $a_{o,t+1}$, where $K = \hat{C}_u \hat{C}_o^{-1}$ is the stationary Kalman gain. The formula for updating estimates $\hat{s}_{ut}$ of the hidden state is

$$
\hat{s}_{u,t+1} = A_{ou} \hat{s}_{ot} + A_{uo} s_{ot} + \hat{C}_u \hat{\eta}_{t+1}.
$$

(16.7.11) \[\text{"newerror2 "}\]

The reconstruction error associated with $\hat{s}_{u,t+1}$ is

$$
s_{u,t+1} - \hat{s}_{u,t+1} = \hat{C}_u \hat{\eta}_{t+1}.
$$

(16.7.12) \[\text{"newerr "}\]

Combining (16.7.11) and (16.7.12) leads us to use (16.7.6) as the representation for information at time $t$.

---

16 See Anderson and Moore (1979).
16.7.4. Interpretation

A concern for robustness leaves the basic form of the Kalman filter intact, but nevertheless contributes a new term to the decision problem: namely, the $\tilde{\eta}_{t+1}$. This term represents an additional avenue for model misspecification that originates in concern about robustness of the filter. See Whittle (1996), Başar and Bernhard (1995, chapters 5 and 7), and Hansen, Sargent, and Wang (2000) for related discussions of the separation of combined filtering into a two-step process that is afforded by the linear-quadratic structure.

![Figure 16.7.1: Impulse responses of $r_t$ to $\eta_t$ (the innovation to $r^n_t$). Solid line is $\theta = +\infty$, dotted line is $\theta = 22.75$.](image)

16.8. Numerical example

For the version of Woodford’s model where the state is observed, we have computed ordinary ($\theta = +\infty$) and robust ($\theta = 22.75$) rules for Woodford’s model. Following Woodford, we assume the following parameter values: $(\beta, \sigma, \kappa, \lambda, \lambda_r) = (.99, .157, .024, .047, .233)$. Woodford assumed a first-order a.r. process for $r^n_t$ with a.r. coefficient .35 and standard deviation of $r^n_t$ of 3.72. Woodford had indirect evidence about the serial correlation of $r^n_t$ and studied the consequences of

\[\text{Note to Sargent: computed using managegv7.m, task24gv4.m in projects/robust/woodford.}\]
Figure 16.7.2: Impulse responses of conditional mean of worst case shocks $w_t$ to $\eta_t$. Solid line is $\theta = +\infty$, dotted line is $\theta = 22.75$.

Figure 16.7.3: Responses of implementation multipliers $\mu_{s1}$ (on IS curve) and $\mu_{s2}$ (on Phillips curve) to $\eta_t$ (innovation to $r^n_t$). Solid line is $\theta = +\infty$, dotted line is $\theta = 22.75$.

varying it. In our example, we set $\rho = .8$ to magnify the effects of robustness in
particular directions. By trial and error, we discovered that the value $\theta = 22.75$ is close to the ‘breakdown point’, so that it gives the maximum effects of robustness for the particular parameter values we, following Woodford, have chosen for the approximating model. Also, following Woodford, we assume that the IS and Phillips curves are exact, so that $\epsilon_{1t} = \epsilon_{2t} = 0$ in (16.6.1), (16.6.2). Let $\mu_{s1,t}, \mu_{s2,t}$ denote the multipliers on the implementability constraints (16.6.1), (16.6.2), respectively.

Here we report impulse responses under the approximating model and the robust decision rules. Figures 16.7.1, 16.7.2, and 16.7.3 report equilibrium impulse response functions for $\theta = +\infty$ (solid line) and $\theta = 22.75$ (dotted line). In figure 16.7.1, the impulse response of $r_t$ to innovations in $r^*_t$ shows that on impact, the government’s instrument $r_t$ responds somewhat less aggressively under the robust (dotted) rule than under $\theta = +\infty$ rule. Note that the response function under the robust rule eventually crosses that under the $\theta = +\infty$ rule, indicating that the government draws out its response to the shock longer under the robust rule, even though it responds less immediately. Evidently, this response mirrors the altered response of the implementability multiplier on the IS curve, shown in the top panel of Fig. 16.7.3. Qualitatively, such a result is potentially interesting in light of a literature that often finds that empirical Taylor rules are less aggressive than optimal rules computed from various structural models.\(^{18}\) However, quantitatively, the effect is not very large in our example. Before claiming that a concern for model misspecification can help account for the apparently conservative nature of empirical Taylor rules, we would want to carry out an analysis with a richer model having more stochastic components and more state variables.

Figures 16.7.2 and 16.7.3 display the impulse response functions of the worst case shocks and the multipliers to an innovation $\eta_{t+1}$ in the natural rate.

\(^{18}\) For analyses of this question in the context of backward-looking models, see comments by Sargent (1999) and Stock (1999) as well as the paper by Onatksi and Stock (2000). They show that with backward looking models, a concern for model misspecification typically makes the monetary authority respond more aggressively than it does without concerns for robustness.
16.9. Concluding remarks

This chapter has generalized standard methods for solving Ramsey problems in linear-quadratic forward looking models to include a common concern for model misspecification to both the government and private agents. The government and private agents share an approximating model that describes the shocks and other exogenous variables hitting the economy. We add one parameter to the standard rational expectations setup, the penalty parameter $\theta$ that indexes the size of a set of models near the approximating model with respect to which policy rules and private agents’ forecasts are to be robust. We compute the Ramsey rule by forming an optimal linear regulator problem and carefully exchanging the roles of the forward looking model’s artificial state variables and the Lagrange multipliers on their laws of motion. Mechanically, robustness is achieved simply by adding another control to the regulator problem, a distortion to the conditional mean of the disturbances that is chosen by a fictitious evil agent. Anderson, Hansen, and Sargent (2000), Hansen, Sargent, and Wang (2000), and chapter 8 indicate how a Bayesian model detection problem can be used to calibrate the parameter $\theta$. We intend the preference for robustness as measured by $\theta$ not to be a primitive, but to be context-specific, as determined by model detection error probabilities for the approximating model and, say, the endogenous worst-case model (see, e.g., Hansen, Sargent, and Wang (2000)).

Using similar methods, it would be possible to generalize our results to relax our assumption that private agents and the government share a common approximating model and a preference for robustness (i.e., $\theta$). In particular, one could construct a ‘two-$\theta$’ model, reflecting different concerns about model uncertainty on the part of the private sector and the government. Such a generalization would move part way toward the specification used by the papers of Giannoni (1999), Otrok (in press) and others cited in section 16.1.

A. Invariant subspace method

Let $L = L^* \beta^{-5}$ and transform the system (16.3.12) to

$$
L \begin{bmatrix} y_t^{*+1} \\ \mu_t^{*+1} \end{bmatrix} = N \begin{bmatrix} y_t^* \\ \mu_t^* \end{bmatrix},
$$

(16.A.1) ["symplec2"]

where $y_t^* = \beta^{t/2} y_t$, $\mu_t^* = \mu_t \beta^{t/2}$. Now $\lambda L - N$ is a symplectic pencil, so that the generalized eigenvalues of $L, N$ occur in reciprocal pairs: if $\lambda_i$ is an eigenvalue, then so is $\lambda_i^{-1}$.

We can use Evan Anderson’s Matlab program schurg.m to find a stabilizing solution of system (16.A.1). The program computes the ordered real generalized Schur decomposition of the matrix pencil. Thus, schurg.m computes matrices $L, N, V$ such that $L$ is upper triangular,
\( \bar{N} \) is upper block triangular, and \( V \) is the matrix of right Schur vectors such that for some orthogonal matrix \( W \) the following hold:

\[
\begin{align*}
WLV &= \bar{L} \\
WNV &= \bar{N}.
\end{align*}
\]  

Let the stable eigenvalues (those less than 1) appear first. Then the stabilizing solution is

\[
\mu_t^* = Py_t^*
\]  

where

\[
P = V_{21}V_{11}^{-1},
\]

\( V_{21} \) is the lower left block of \( V \), and \( V_{11} \) is the upper left block.

If \( L \) is nonsingular, we can represent the solution of the system as

\[
\begin{bmatrix}
y_{t+1}^* \\
\mu_{t+1}^*
\end{bmatrix} = L^{-1}N \begin{bmatrix}
I \\
P
\end{bmatrix} y_t^*.
\]  

The solution is to be initiated from (16.A.3). We can use the first half and then the second half of the rows of this representation to deduce the following recursive solutions for \( y_{t+1}^* \) and \( \mu_{t+1}^* \):

\[
\begin{align*}
y_{t+1}^* &= A_0^* y_t^* \\
\mu_{t+1}^* &= \psi^* y_t^*
\end{align*}
\]  

Now express this solution in terms of the original variables:

\[
\begin{align*}
y_{t+1} &= A_0 y_t \\
\mu_{t+1} &= \psi y_t,
\end{align*}
\]  

The solution method in the text assumes that \( L \) is nonsingular and well conditioned. If it is not, the following method proposed by Evan Anderson will work. We want to solve for a solution of the form

\[
y_{t+1}^* = A_0^* y_t^*.
\]

Note that with (16.A.3),

\[
L \left[ I; P \right] y_{t+1}^* = N \left[ I; P \right] y_t^*
\]

The solution \( A_0^* \) will then satisfy

\[
L \left[ I; P \right] A_0^* = N \left[ I; P \right].
\]

Thus \( A_0^* \) can be computed via the Matlab command

\[
A_0^* = (L \ast [I; P]) \setminus (N \ast [I; P]).
\]
where \( A_o = A_o^\star \beta^{-5} \), \( \psi = \psi^\star \beta^{-5} \). We also have the representation

\[ \mu_t = P y_t \quad \text{(16.A.7)} \]  

The matrix \( A_o = A - \tilde{B} F \), where \( F \) is the matrix for the optimal decision rule.

### B. The Riccati equation

#### 16.B.1. The Riccati equation

The stabilizing \( P \) obeys a Riccati equation coming from the Bellman equation. Substituting \( \mu_t = P y_t \) into (16.3.10) and (16.3.8b) gives

\[ (I + \beta \tilde{B} \tilde{R}^{-1} \tilde{B}' P) y_{t+1} = A y_t \quad \text{(16.B.1a)} \]  
\[ \beta A' P y_{t+1} = -Q y_t + P y_t. \quad \text{(16.B.1b)} \]

A matrix inversion identity implies

\[ (I + \beta \tilde{B} \tilde{R}^{-1} \tilde{B}' P)^{-1} = I - \beta \tilde{B} \left( \tilde{R} + \beta \tilde{B}' P \tilde{B} \right)^{-1} \tilde{B}' P. \quad \text{(16.B.2)} \]

Solving (16.B.1a) for \( y_{t+1} \) gives

\[ y_{t+1} = (A - \tilde{B} F) y_t \quad \text{(16.B.3)} \]

where

\[ F = \beta \left( \tilde{R} + \beta \tilde{B}' P \tilde{B} \right)^{-1} \tilde{B}' P A. \quad \text{(16.B.4)} \]

Pre multiplying (16.B.3) by \( \beta A' P \) gives

\[ \beta A' P y_{t+1} = \beta \left( A' P A - A' P \tilde{B} F \right) y_t. \quad \text{(16.B.5)} \]

For the right side of (16.B.5) to agree with the right side of (16.B.1b) for any initial value of \( y_0 \) requires that

\[ P = Q + \beta A' P A - \beta^2 A' P \tilde{B} \left( \tilde{R} + \beta \tilde{B}' P \tilde{B} \right)^{-1} \tilde{B}' P A. \quad \text{(16.B.6)} \]

Equation (16.B.6) is the algebraic matrix Riccati equation associated with the ordinary linear regulator for the system \( A, \tilde{B}, Q, \tilde{R} \).
C. Another Bellman equation

We briefly indicate the connection of the preceding formulation to that of Kydland and Prescott (1980) and Marcet and Marimon (2000). For a class of problems with structures close to ours, they construct a Bellman equation in a state vector defined as \((z, \mu_x)\): these are the ‘natural’ state variables and the vector of multipliers on the laws of motion for the ‘jump’ variables \(x_t\). We show how to modify that Bellman equation to include a concern about model misspecification.

Let \(\mu_x\) denote the sub vector of multipliers attaching to the implementability constraints that summarize the Euler equations of the private sector. Then the Lagrangian for the optimum problem (16.C.7) can be written

\[
L = -\sum_{t=0}^{\infty} \beta^t \left\{ \left[ \begin{matrix} z_t \\ x_t \end{matrix} \right]' Q \left[ \begin{matrix} z_t \\ x_t \end{matrix} \right] + U_t' RU_t - \beta \theta w_{t+1}' w_{t+1} + \beta \mu_x'(A_{21} z_t + B_{2} U_t + C_{2} w_{t+1} - x_{t+1}) \right\}.
\]

This Lagrangian is to be ‘extremized’ (i.e., maximized or minimized, as appropriate) with respect to sequences \(\{z_t, x_t, \mu_x, w_{t+1}\}\) subject to \(\lambda_0 = 0\) and the transition law

\[
z_{t+1} = A_{11} z_t + A_{12} x_t + B_{1} U_t + C_{1} w_{t+1}.
\]

Equation (16.C.1) can be rewritten

\[
L = -\sum_{t=0}^{\infty} \beta^t \left\{ \left[ \begin{matrix} z_t \\ x_t \end{matrix} \right]' Q \left[ \begin{matrix} z_t \\ x_t \end{matrix} \right] + U_t' RU_t - \beta \theta w_{t+1}' w_{t+1} + \left( \beta \mu_x'(A_{21} z_t + B_{2} U_t + C_{2} w_{t+1} - x_{t+1}) \right) \right\},
\]

which is to be extremized with respect to the same constraints (16.C.2). Define the one-period return function

\[
-\tilde{r}(z, \mu_x, x, \mu_x^*, w) = \left[ \begin{matrix} z \\ x \end{matrix} \right]' Q \left[ \begin{matrix} z \\ x \end{matrix} \right] + u' R u - \theta u' w + (\beta \mu_x'^* A_{21} z + B_{2} U + C_{2} w),
\]

where * superscripts denote one-period ahead values. Let \(v(z, \mu_x)\) be the optimum value of the problem starting with augmented state \((z, \mu_x)\). Problem (16.C.3) is recursive and has the following Bellman equation:

\[
v(z, \mu_x) = \max_{\{u, x\}} \min_{\{w, \mu_x^*\}} \left\{ r(z, \mu_x, x, \mu_x^*, w) + \beta v(z^*, \mu_x^*) \right\}
\]

where the extremization is subject to

\[
z^* = A_{11} z + A_{12} x + B_{1} u + C_{1} w.
\]

The Bellman equation (16.C.4), (16.C.5) is a version of the recursive saddle problem described by Kydland and Prescott (1980) and Marcet and Marimon (2000). We have added
a concern for robustness via the extra minimization with respect to the shock distortion \( w \). In related contexts, Marcet and Marimon stress that while such problems are not recursive in the natural state variables \( z \) alone, they becomes recursive when the multipliers \( \mu_x \) are included.

Although one could solve our problem by iterating to convergence on (16.C.4), (16.C.5), it is more convenient for us to use the method described in section 16.3 that solves the Riccati equation (16.3.4) and its associated Bellman equation.

### D. Decentralization of partial equilibrium

For a partial equilibrium model with adjustment costs, this appendix studies a recursive competitive equilibrium in which the representative firm has a preference for robust decisions. We show that the standard trick of computing an equilibrium by solving the fictitious planning problem of maximizing a discounted sum of consumer plus producer surplus goes extends to a setting where the firm wants robustness. In this case, the planning problem becomes a robust planning problem in which the planner extremizes over decision, model distortion pairs.

Consider an adaptation for robustness of Sargent’s (1987, chapter XVI) version of Lucas and Prescott’s model of investment under uncertainty. Demand for a single good is governed by an inverse demand function

\[
q_t = p_t - \ell(q_{t+1})
\]

where \( \ell \) is a linear function. The representative firm solves the two-player zero-sum game

\[
\min_{\{w_{t+1}\}} \max_{\{q_{t+1}\}} \sum_{t=0}^{\infty} \beta^t \left\{ p_t q_t - \sigma(q_t, q_{t+1}) + \beta^2 w_{t+1}^2 \right\}
\]

where \( \beta \) is a discount factor. An equilibrium of the representative agent’s two-player zero-sum game is a pair of decision rules

\[
q_{t+1} = \phi_q(q_t, w_t, \pi_t)
\]

\[
w_{t+1} = \phi_w(q_t, v_t, \pi_t)
\]

The representative agent’s extremization problem reduces a mapping from \( \ell \) in (16.3.4) to \( (\phi_q, \phi_w) \). When the representative firm perceives the law of motion for \( \pi_t \) to be (16.3.4), it acts to make the actual law of motion to be \( \theta_{t+1} = \phi_q(\pi_t, v_t, \pi_t) \). A competitive equilibrium under robustness is a fixed point of the mapping from \( \ell (\pi_t, v_t) \) to \( \phi_q(\pi_t, v_t, \pi_t) \). That is, for the representative firm to be representative, it must be true that \( \ell \) satisfies

\[
\phi_q(\pi_t, v_t, \pi_t) = \ell(q_t, v_t)
\]
Fortunately, by extending lines of argument of Lucas and Prescott (1972) and Sargent (1987), it is not necessary to attack this fixed point problem directly. In particular, we can compute $\ell_q$ and an associated $\ell_w$ directly by solving a fictitious robust planning problem. The fictitious planning problem is

$$\min_{\{w_{t+1}\}} \max_{\{\pi_{t+1}\}} \sum_{t=0}^{\infty} \beta^t \{ S(\pi_t, v_t) - \sigma(\pi_t, \pi_{t+1}) + \beta \theta w'_{t+1} w_{t+1} \}$$  \hspace{1cm} (16.D.7) \hspace{1cm} [*lpr6 *]

where $S(\pi, v)$ is consumer surplus defined as

$$S(\pi, v) = \int_{0}^{\pi} (A_0 - A_1 x + v) \, dx = A_0 \pi - \frac{A_1}{2} \pi^2 + \pi v.$$

The state of the market is $\pi_t, v_t$. A solution of this two-player zero-sum game is

$$\pi_{t+1} = \ell_q(\pi_t, v_t) \hspace{1cm} (16.D.8a) \hspace{1cm} [*partial1;a *]$$

$$w_{t+1} = \ell_w(\pi_t, v_t) \hspace{1cm} (16.D.8b) \hspace{1cm} [*partial1;b *]$$

It turns out that $\ell_j(\pi, v) = \phi_j(\pi, v, \pi)$ for $j = q, w$. This assertion can be proved by extending the proof in Sargent (1987, ch. XIV). The proof strategy is to obtain the Euler equations for extremizing (16.D.4), then to use the demand curve (16.D.1) to eliminate price, rearrange, and note that these Euler equations-cum equilibrium conditions match the Euler equations for extremizing the fictitious planning criterion (16.D.7).
Chapter 17.
Non-linear models

17.1. Introduction

This chapter discusses a preference for robustness in settings that extend beyond the linear-quadratic examples we have concentrated on up to now. We permit both the return function and the transition law to be of general functional forms. We describe the evolution of the state as a Markov process. We show how a preference for robustness to model misspecification can be expressed by altering the conditional expectation operator in the Bellman equation. That new operator is connected with the risk-sensitive control theory of chapter 6.

17.1.1. The $\mathcal{R}$ operator for LQ problems

Chapters 6 and 7 showed how for a linear quadratic problem, the key to computing a robust decision rule is to iterate to convergence on a composite operator $T \circ \mathcal{D}$ in place of the ordinary operator $T$ defined by the right side of the Riccati equation for the matrix $P$ in the value function $-x'Px$. Thus, for the non-stochastic linear-quadratic case studied in chapter 6, a Bellman equation that induces a robust rule is

$$-x'Px = \max_u [r(x, u) - \beta y'\mathcal{D}(P)y]$$

where $r(x, u) = -x'Qx - u'Ru$, the maximization is subject to $y = A_ox + Bu$, and

$$\mathcal{D}(P) = P + \theta^{-1}PC(I - \theta^{-1}C'PC)^{-1}C'P.$$  

Using the $\mathcal{D}$ operator in the preceding Bellman equation rewards robust decision rules. The operator $\mathcal{D}$ verifies the following equality:

$$J \equiv -x'A'\mathcal{D}(P)Ax = \min_w \left[ \theta w'w - (Ax + Cw)' P (Ax + Cw) \right]. \quad (17.1.1)$$

The problem on the right is to minimize $\theta w'w + y'Py$ subject to the approximating model $y = Ax + Cw$, where $y'Py$ is the continuation value function of next period’s state $y$.

Chapter 7 interprets the operator $\mathcal{D}$ in terms of risk-sensitivity where the law of motion is $y = A_ox + Bu + C\epsilon$ and $\epsilon$ is a Gaussian vector with mean zero.
Non-linear models

and identity covariance matrix. Define $\alpha = 2\theta$. The Bellman equation for the value function $V(x)$ for a risk-sensitive control problem is

$$V(x) = \max_u \{ r(x, u) + \beta R(V(y))(x) \}$$  \hspace{1cm} (17.1.2)

where

$$R(V(y))(x) = -\alpha \log E \exp \left( \frac{-V(y)}{\alpha} \right) \bigg| x. $$  \hspace{1cm} (17.1.3)

For the quadratic continuation value function $V(y) = -y'Py - p$ and the fixed control law $u = -Fx$, we have

$$R(V(y))(x) = -x'A' \mathcal{D}(P)Ax - p - \theta^{-1} \log \det (I - \theta^{-1}C'PC).$$

Notice the appearance of $\mathcal{D}$ on the right side. It encapsulates a ‘twisting’ of probabilities induced by risk-sensitive preferences.

In the linear-quadratic case studied in chapters 6 and 7, the robust decision rule associated with a given $\theta > 0$ also attains the value function $V(x)$ that solves the risk-sensitive Bellman equation (17.1.2). That implies that robustness to model misspecification can be achieved by iterating on the Bellman equation while replacing the expectations operator $E$ with the distorted operator $R$.

This chapter shows how these ideas extend beyond linear-quadratic problems. By using $R$ as defined in the first line of (17.1.3) to replace the conditional expectations operator in the Bellman equation, decision rules that are robust to model misspecification can be computed as easily as can the ordinary nonrobust optimal rules.

---

1 In earlier chapters, we typically used $\theta$ as the multiplier on $w'w$ which itself equaled two times entropy. We use $\alpha$ as the corresponding multiplier on entropy.
17.1.2. Markov perturbations

In this chapter, an approximating model is a controlled Markov chain. To express a concern about model misspecification, we suppose that a decision maker believes that an unknown member of a set of unspecified nearby models generates the data. We form the set of models by perturbing the Markov transition density of an approximating model. This way of proceeding lets us avail ourselves of formalizations from large deviation theory for Markov processes (e.g. see Dupuis and Ellis, 1997).

Thus, the decision maker’s approximating model is a Markov process with state \( x \in X \) and time-invariant transition density \( \pi(x', x) \), where \( x \) denotes the state today and \( x' \) the state tomorrow. We form a perturbed model by multiplying \( \pi \) by a function \( w(x', x) > 0 \), then rescaling to make the resulting object a transition density:

\[
\pi^w(x', x) = w(x', x) \frac{\pi(x', x)}{\int w(x', x) \pi(x', x) \, dx'}.
\]  

(17.1.4)

The likelihood ratio, or Radon-Nikodym derivative of \( \pi^w \) with respect to \( \pi \), is evidently

\[
\frac{w(x', x)}{\int w(x', x) \pi(x', x) \, dx'}.
\]  

(17.1.5)

From \( w(x', x) > 0 \), it follows that \( \pi^w \) puts positive probability on the same events as does \( \pi \) (i.e., \( \pi^w \) is said to be absolutely continuous with respect to \( \pi \)). This means that statistically \( \pi^w \) can be difficult to distinguish from \( \pi \) using a small number of observations.

It is sometimes convenient to describe a Markov process with a conditional expectations operator \( \mathcal{T} \) defined as follows. For any test function \( \phi \) belonging to the class \( \Phi \) of bounded continuous functions, let

\[
\mathcal{T} \left( \phi \right) (x) = E \left[ \phi \left( x_{t+1} \right) \mid x_t = x \right].
\]  

(17.1.6)

For a rich enough set of functions in \( \Phi \), the conditional expectations operator \( \mathcal{T} \) characterizes the transition density \( \pi \). The expectations operator associated with the distorted model (17.1.4) can be expressed

\[
\mathcal{T}^w \left( \phi \right) = \frac{\mathcal{T} \left( w\phi \right)}{\mathcal{T} \left( w \right)}.
\]

We let \( E^w(\cdot|x) \) denote the mathematical expectation with respect to the distorted model.
17.1.3. Relative entropy

To embody the idea that the approximating model is good, we want a convenient way to measure the discrepancy of another model from the approximating model. We measure discrepancy by ‘relative entropy,’ defined as the expected value of the log-likelihood ratio conditional on $x$, where the conditional expectation is evaluated with respect to the density associated with the twisted model $\pi^w$. As in chapter 2, we define relative entropy for a candidate model indexed by $w$ as

$$I(w)(x) \equiv E^w \left[ \log \frac{\pi^w(x', x)}{\pi(x', x)} \right]$$

(17.1.7)

Relative entropy is not a metric because it treats the approximating model $\pi$ and the alternative model $\pi^w$ asymmetrically. This asymmetry emerges because the expectation is evaluated with respect to the ‘twisted’ distribution $\pi^w$. Relative entropy is prominent in both information theory and large deviation theory and satisfies several attractive properties: \( I(w) \) nonnegative, but $I(w) = 0$ if $w$ is constant. Substituting for $T^w$ in (17.1.7) gives:

$$I(w) = \frac{T[w \log (w)]}{T(w)} - \log [T(w)]$$

(17.1.8)

To describe a preference for robustness, we shall use relative entropy $I(w)$ to measure how far other models (indexed by $w$) are from the approximating model. A robust decision maker pays special attention to models with small relative entropies because they are difficult to distinguish empirically from the approximating model.

---

2 For readers of Dupuis and Ellis (1997, Chapter 1, Section 4), think of the transition density associated with $T$ as Dupuis and Ellis’s $\theta$; and think of $\frac{w(z)}{T(z)(y)}$ as Dupuis-Ellis’s Radon-Nikodym derivative $\frac{d\gamma}{d\theta}$. For Dupuis and Ellis, relative entropy is $\int \log \left( \frac{d\gamma}{d\theta} \right) d\gamma$.


4 For Markov specifications with stochastically singular transitions, $\frac{w(x', x)}{T^w(x', x)}$ may be one even when $w$ is not constant. For these systems, we have in effect over parameterized the perturbations, although in a harmless way.
17.2. Value function for robustness

Given a current-period reward function $U(x)$ and a known Markov process, a value function $W(x)$ for a discounted infinite horizon solves the functional equation

$$W(x) = U(x) + \beta T(W)(x)$$  \hspace{1cm} (17.2.1)

where $\beta \in (0, 1)$ is a discount factor. To represent a concern about model misspecification while preserving recursivity, we shall generalize (17.2.1) by replacing the conditional expectations operator $T$ with an alternative transformation $R$ of the continuation value function. The operator $R$ distorts the conditional expectation operator $T$ with a single parameter $\alpha > 0$ and is defined as

$$R(W) = -\alpha \log \left( T \left[ \exp \left( \frac{-W}{\alpha} \right) \right] \right). \hspace{1cm} (17.2.2)$$

The parameter $\alpha = 2\theta$ is restricted to be nonnegative; as it diverges to $\infty$, $R$ becomes the conditional expectation operator $T$. The so-called risk sensitivity parameter is $\sigma = -2\alpha^{-1} = -\theta^{-1}$. In the absence of discounting, replacing $T$ with $R$ in (17.2.1) delivers the risk sensitive evaluation used in control theory.

To express a preference for robustness, we propose to iterate on the following recursion:

$$V(x) = U(x) + \beta R(W)(x). \hspace{1cm} (17.2.3)$$

Here $W(y)$ is a continuation value function and $V(x)$ is a current value function.

To help motivate (17.2.3), we now describe an inequality that bounds how much the conditional expectation of a continuation value function deteriorates across different probability specifications. Assume $\alpha > 0$ and consider the following problem:

**Problem A:**

$$\inf_{w > 0} J(w) \hspace{1cm} (17.2.4a)$$

where

$$J(w) \equiv \alpha I(w) + T^w(W). \hspace{1cm} (17.2.4b)$$

---

5 See Whittle (1990, 1996). As a formulation of recursive utility in the style of Epstein and Zin (1989), Weil (1993) used $R$ to make risk adjustments in a value function recursion that is not additively separable, in contrast to (17.2.1). While there exists a transformation of the value function that has a recursion that is additively separable for Weil’s formulation, the corresponding risk adjustment is different.

6 Note how (17.2.4b) generalizes (17.1.1).
The first term on the right of (17.2.4b) is a weighted entropy measure and the second is the expectation of the continuation value function using the twisted probability model indexed by \( w \), i.e., the expectation is evaluated with respect to the approximating model. The second term is the expectation of next period’s value function when the current period’s beliefs are as indexed by \( w \). The objective is to find a worst-value model \( w \), where the departures \( w \) from the approximating model are penalized at a utility-price \( \alpha \) applied to their relative entropy. Increasing the absolute magnitude of \( \alpha \) increases the penalty for deviating from the approximating model.

**Theorem 17.2.1.** Assume that \( T \) can be evaluated at \( \exp \left( -\frac{W}{\alpha} \right) \). For any constant \( k > 0 \), a solution to Problem A is:

\[
w^* = k \exp \left( -\frac{W}{\alpha} \right),
\]

which attains the minimized value

\[
J(w^*) = R(W)
\]

where

\[
R(W) = -\alpha \log \left( T \left[ \exp \left( -\frac{W}{\alpha} \right) \right] \right)
\]

The solution \( w^* \) is not unique (any \( k > 0 \) works), but the minimized value of the objective is unique and so is the associated probability law.

**Proof.**\(^7\) To verify that \( w^* \) is the solution, write:

\[
I(w) = I^* \left( \frac{w}{w^*} \right) + \frac{T \left( w \log w^* \right)}{T(w)} - \log T(w^*)
\]

where

\[
I^* (w) = \frac{T^* \left( w \log w \right)}{T^* (w)} - \log T^* (w)
\]

and

\[
T^* \phi = \frac{T \left( w^* \phi \right)}{T \left( w^* \right)}.
\]

\(^7\) This proof emulates the proof of Proposition 1.4.2 in Dupuis and Ellis (1997).
Notice that $I^*$ is itself interpretable as a measure of relative entropy and hence $I^*(w/w^*) \geq 0$. Thus the criterion $J$ satisfies the inequality:

$$J(w) = \alpha \left[ I^*(w/w^*) + \frac{T(w \log w^*)}{T(w)} - \log T(w^*) \right] + T^w(W) \geq \alpha \left[ \frac{T(w \log w^*)}{T(w)} - \log T(w^*) \right] + T^w(W)$$

$$= -\alpha \log T \left[ \exp \left( -\frac{W}{\alpha} \right) \right] = J(w^*).$$

Equation (17.2.4b) implies an inequality in terms of robust evaluations of value functions.

**Corollary 17.2.1.** The conditional expectation of the value function $W$ evaluated under $T^w$ satisfies the bound

$$T^w(W) \geq R(W) - \alpha I(w). \quad (17.2.5)$$

**Proof.** This follows immediately from $J(w^*) = R(W)$ and the definition of $J(w)$. □

The first term on the right depends on $\alpha$, but not on the alternative model parameterized by $w$. The second term is $-\alpha$ times entropy. Thus, inequality (17.2.5) justifies interpreting $\alpha$ as a type of utility price of robustness.

The robust value function $W$ solves the functional equation:

$$W(x) = \inf_w \{ U(x) + \beta [\alpha I(w) + T^w(W)(x)] \}. \quad (17.2.6)$$

This can also be expressed as

$$W(x) = U(x) + \beta R(W)(x). \quad (17.2.7)$$

These equations display how the continuation value function is adjusted for fear of possible model misspecification. **Elaborate**
17.2.1. Gaussian example

Theorem 17.2.1 shows that the distorted transition measure obeys
\[
\pi^w(x', x) \propto \pi(x', x) \exp \left( \frac{-W(x')}{\alpha} \right). 
\] (17.2.8)

To link this result to the linear-quadratic-Gaussian setting of earlier chapters, assume that the continuation value function is \( W(x) = -x'Px - \rho \) and that \( \pi(x', x) \) is Gaussian with mean \( A^*x \) and conditional covariance matrix \( C'C \), so that \( x' \) can be represented as \( x' = \mu + C \epsilon \) where \( \mu = A^*x \) and \( \epsilon \) is a Gaussian random vector with mean zero and identity covariance matrix. Using the definition \( \sigma = -\theta^{-1} = -2\alpha^{-1} \) and the preceding assumption about the conditional distribution of \( x' \), (17.2.8) implies
\[
\pi^w(x', x) \propto \exp \left( \frac{\sigma W(x')}{2} \right) \exp \left( \frac{\epsilon' \epsilon}{2} \right) 
= \exp \left( -\sigma (\mu + C \epsilon)' P (\mu + C \epsilon) - \rho \sigma \right) \exp \left( \frac{\epsilon' \epsilon}{2} \right) 
\propto \exp \left( -\epsilon' (I + \sigma C'PC) \epsilon \right) - \epsilon' (I + \sigma C'PC) (I + \sigma C'PC)^{-1} \sigma C'P \mu
\]

The last line portrays \( x' \) under \( \pi^w \) as having a Gaussian distribution with shocks that have mean vector \( \tilde{\mu} \) and covariance matrix \( \tilde{\Sigma} \) defined by
\[
\tilde{\mu} = -\sigma (I + \sigma C'PC)^{-1} C'P \mu 
\tilde{\Sigma} = (I + \sigma C'PC)^{-1}. 
\] (17.2.9) (17.2.10)

It follows that under the distorted conditional distribution, the mean and covariance matrix for \( x' \) are \( A^*x + C \tilde{\mu} \) and \( C \tilde{\Sigma} C' \). Notice that \( -\sigma (I + \sigma C'PC)^{-1} C'P = \theta^{-1}(I - \theta^{-1}C'PC)^{-1}C'P \mu \), so that the formula for the distortion in the mean agrees with the distortion under the worst case model from chapters 2 and 6. In those chapters, we allowed the minimizing agent to distort only the mean, not the variance, of the conditional distribution for next period’s state. We have just shown, however, that when the minimizing agent is allowed to choose any distribution near the approximating transition density \( \pi \), and when the density is Gaussian under the approximating model, the minimizing agent will select a Gaussian distorted distribution, but will choose to distort both the mean and the covariance matrix of the shocks. Further, the formula for the mean distortion matches the one that prevails when the minimizing agent is allowed to alter only the mean vector.
17.3. Large deviation interpretation of $\mathcal{R}$

We have interpreted (17.2.7) in terms of a preference for robustness that is achieved by substituting the operator $\mathcal{R}$ for the conditional expectations operator $\mathcal{T}$ in a corresponding Bellman equation without a concern for robustness. In this section, we use ideas from the theory of large deviations to describe how the operator $\mathcal{R}(W(x'))(x)$ contains information about the left tail of the distribution of $W(x')$. Recall from (17.2.2) that $\mathcal{R}$ depends on $\alpha$, and collapses to $\mathcal{T}$ as $\alpha \nearrow +\infty$. We shall show that $\mathcal{R}$ contains more information about the left tail of $W(x')$ as $\alpha$ is decreased. We gather this interpretation from an exponential inequality that bounds the (conditional) tail probabilities of $W$. The role of $\mathcal{R}$ in these tail probability bounds allows us to express a form of enhanced risk aversion generated by having the decision-maker care about more than just the conditional mean of the continuation value.

The tail probability bound comes from the theory of large deviation approximations. It uses the inequality

$$1_{\{W:W \leq -r\}} \leq \exp \left[ \frac{-(W + r)}{\alpha} \right]$$

depicted in Figure 17.3.1. This inequality holds for any real number $r$ and any $\alpha > 0$. Let $z$ denote the state tomorrow. Then computing expectations conditioned on the current state vector $x$ yields:

$$\Pr\{W(x') \leq -r|x\} \leq E \left( \exp \left[ \frac{-W(x')}{\alpha} \right] \right) \exp \left( -\frac{r}{\alpha} \right),$$

or

$$\log \left[ \Pr\{W(x') \leq -r|x\} \right] \leq -\frac{1}{\alpha} \mathcal{R}(W)(x) - \frac{r}{\alpha}. \quad (17.3.1)$$

The first term on the right side of this inequality is independent of $r$ but depends on $\alpha$. We can express (17.3.1) as

$$\Pr\{W(x') \leq -r|x\} \leq \exp \left\{ -\alpha^{-1} \mathcal{R}(W)(x) \right\} \exp \left( -\frac{r}{\alpha} \right). \quad (17.3.2)$$

Notice what this implies as $r$ increases. Inequality (17.3.2) bounds the tail probability on the left by an exponential in $r$. The right side declines with increases in $r$ at rate $-\alpha^{-1}$; $\mathcal{R}$, a function of $\alpha$, influences the constant in the bound. Decreasing $\alpha$ increases the exponential rate at which the bound sends the tail probabilities to zero, thereby expressing how a lower $\alpha$ heightens concern about tail events. This tells us how using $\mathcal{R}$ to replace the mathematical expectation $\mathcal{T}$ in a typical Bellman equation enhances risk aversion.
Figure 17.3.1: Ingredients of large deviation bounds: \( \exp \left( \frac{- (W + r)}{\alpha} \right) \)
and \( 1_{\{W \leq -r\}} \) for \( \alpha = 1, r = 1 \) and \( \alpha = 2, r = 1 \). \textbf{redraw}

Figure 17.3.2: The function \( h^{-1} E(h(W)) \) for \( h(W) = \exp \left( \frac{-W}{\theta} \right) \),
\( 0 < \alpha < +\infty \).
Figure 17.3.2 shows how $R$ induces additional caution about continuation utilities $W$. In the figure, $E(W)$ is the expected utility of a gamble between two continuation utility levels $W_2, W_1$ with $W_2 > W_1$. Where $h(W)$ is a convex function, like $\exp(-W/\alpha)$ for $0 < \alpha < +\infty$, $h^{-1}E(h(W)) < E(W)$. 
18.1. Introduction

This chapter provides an example of an equilibrium under robustness in which a government and the public share a common preference for robustness. We present and modify Sargent and Velde’s linear quadratic version of Lucas and Stokey’s (1983) model of optimal taxation in an economy without capital.\footnote{We thank Cristobal Huneeus and Yongseok Shin for excellent help with the computations.} We want to replace Lucas and Stokey’s government and agent with ones that share a preference for robustness against a set of model misspecifications parameterized by a common parameter \( \theta \). We shall study how activating a preference for robustness alters a single parameter in Lucas and Stokey’s formula for consumption and labor supply. First we review the model without a preference for robustness.

18.1.1. Exogenous processes and information

Let \( x_t \) be an exogenous information vector. We shall use \( x_t \) to drive exogenous stochastic processes \( g_t, d_t, b_t, 0s_t \), representing, respectively, government expenditures, an endowment, a preference shock, and a stream of promised coupon payments owed by the government at the beginning of time 0:

\[
\begin{align*}
g_t &= S_g x_t, \\
d_t &= S_d x_t, \\
b_t &= S_b x_t, \\
0s_t &= 0S_x x_t.
\end{align*}
\]

Sargent and Velde made one of two alternative assumptions about the underlying stochastic process \( x_t \).

Assumption 1: The process \( x_t \) is an \( n \times 1 \) vector with given initial condition \( x_0 \) and is governed by

\[
x_{t+1} = Ax_t + C\hat{\epsilon}_{t+1}.
\]

Here \( \hat{\epsilon}_{t+1} \) is an i.i.d. random vector with mean 0 and identity covariance matrix, and \( A \) is a stable matrix.
**Assumption 2:** The process \( x_t \) is an \( n \) state Markov chain with transition probabilities arranged in the \( n \times n \) matrix \( \pi \) with \( \pi_{ij} = \text{Prob}(x_{t+1} = x_j | x_t = x_i) \).

Later, when we find robust rules, we shall use one of the following assumptions about a class of alternative specifications:

**Assumption 1':** The process \( x_t \) is an \( n \times 1 \) vector with given initial condition \( x_0 \) and is governed by

\[
x_{t+1} = Ax_t + C(\epsilon_{t+1} + w_{t+1}).
\]

(18.1.3a)

Here \( \epsilon_{t+1} \) is another i.i.d. random vector with mean 0 and identity covariance matrix, and \( w_{t+1} \) is a measurable function of \( x_s, s \leq t; \) \( w_{t+1} \) is a distortion to the conditional mean of \( \hat{\epsilon}_{t+1} \) in the approximating model (18.1.2). It satisfies

\[
E_0 \sum_{t=0}^{\infty} \beta^{t+1} w_{t+1} \cdot w_{t+1} \leq \eta_0.
\]

(18.1.3b)

**Assumption 2':** The process \( x_t \) is one of a continuum of \( n \) state Markov chains, indexed by matrices \( w \) with elements \( w_{i,j} > 0 \). The transition probabilities are arranged in the \( n \times n \) matrix \( \pi^w \) with \( \pi^w_{ij} = (w_{ij} \pi_{ij}) / \sum_k w_{i,k} \pi_{ik} \). Let \( I(w) \), be called conditional entropy in state \( i \), where

\[
I(w)_i = \sum_j \ln \left( \frac{w_{i,j}}{\sum_k w_{i,k} \pi_{ik}} \right) \left( \frac{w_{i,j} \pi_{ij}}{\sum_k w_{i,k} \pi_{ik}} \right).
\]

(18.1.4)

The constraint on misspecification is

\[
E_0 \sum_{t=0}^{\infty} \beta^{t+1} I(w)_{t+1} \leq \eta,
\]

(18.1.5)

where \( I(w)_{t+1} \) is the value of conditional entropy at \( t \) when in the \( i \)th state.

In (18.1.4), the term \( w_{ij} / \sum_k w_{i,k} \pi_{ik} \) is the Radon-Nikodym derivative of the distorted transition density \( \pi^w \) with respect to the approximating one \( \pi \). Inequality (18.1.5) is a constraint on the relative entropy of Markov chain \( \pi^w \) relative to Markov chain \( \pi \).

We begin by staying with Sargent and Velde’s assumptions 1 and 2, then later adopt assumptions 1' and 2'.

---

2 See Anderson, Hansen, and Sargent (2000) for a discussion of how to specify the set of alternative models when the approximating model is Markov.
18.1.2. Technology

There is a technology for converting one unit of labor $\ell_t$ into one unit of a single nonstorable consumption good. Feasible allocations satisfy:

$$c_t + g_t = d_t + \ell_t. \quad (18.1.6)$$

18.1.3. Households

Markets are complete. At time 0, a representative consumer faces a scaled Arrow-Debreu price system $\{p^0_t\}$ and a flat rate tax on labor $\{\tau_t\}$ and chooses consumption and labor supply to maximize:

$$-0.5E_0 \sum_{t=0}^{\infty} \beta^t \left[ (c_t - b_t)^2 + \ell^2_t \right]. \quad (18.1.7)$$

subject to the time 0 budget constraint

$$E_0 \sum_{t=0}^{\infty} \beta^t p^0_t \left[ d_t + (1 - \tau_t) \ell_t + o_s - c_t \right] = 0. \quad (18.1.8)$$

This states that the present value of consumption equals the present value of the endowment plus coupon payments on the initial government debt plus after-tax labor earnings. The scaled Arrow-Debreu price system is a stochastic process.\(^3\)

Later we shall change (18.1.7) to induce a preference for robustness to model misspecification on the part of both the household and the government.

---

\(^3\) The scaled Arrow-Debreu prices are ordinary state prices divided by probabilities and the time $t$ power of the discount factor, transformations that permit representing values as conditional expectations of scaled prices times quantities. See chapter 12 as well as Hansen (1987) and Hansen and Sargent (1999).
18.1.4. Government

The government’s time-0 budget constraint is

\[ E_0 \sum_{t=0}^{\infty} \beta^t p_t^0 [(g_t + s_t) - \tau_t \ell_t] = 0. \]  

(18.1.9)

Given the government expenditure process and the present value \( E_0 \sum_{t=0}^{\infty} \beta^t p_t^0 s_t \), a feasible tax process must satisfy (18.1.9).

18.1.5. Equilibrium

**Definition:** \( L_2^0 \) is the space of random variables \( y_t \) measurable with respect to \( x_t \) and such that \( E_0 \sum_{t=0}^{\infty} \beta^t y_t^2 < +\infty \).

**Definitions:** A **feasible allocation** is a stochastic process \( \{c_t, \ell_t\} \) that satisfies (18.1.6). A **tax system** is a scalar stochastic process \( \{\tau_t\} \). A **price system** is a stochastic process \( \{p^0_t\} \). The time \( t \) elements of each of these processes are assumed to be measurable with respect to \( x_t \), and to belong to \( L_2^0 \).

**Definition:** An **equilibrium** is a feasible allocation, a price system, and a tax system that have the following properties:

1. Given the tax and price systems, the allocation solves the household’s problem.
2. Given the price system, the allocation and the tax system satisfy the government’s budget constraint.

18.1.6. Properties

The first-order conditions for the household’s problem imply that the equilibrium price system satisfies \( p_t^0 = \mu (b_t - c_t) \), where \( \mu \) is a numeraire that we set at \( b_0 - c_0 \).

As in chapter 12, the preference specification permits the scaled Arrow-Debreu price \( p_t^0 \) to be expressed in terms of ratios of linear functions of the state:

\[ p_t^0 = M_p x_t / M_p x_0, \]

where \( M_p \) is a matrix defined so that \( M_p x_t = b_t - c_t \). The preference specification will make it possible to express government time \( t \) revenues as the ratio of a quadratic function of the state at \( t \) to a linear function of the state at 0. The forms of these prices and taxes and of the other objects in (18.1.7) reduce the technical problem to evaluating geometric sums of a quadratic form in the state.

For assumptions 1 and 2, Appendix A of Sargent and Velde (1999) shows how
to compute such sums, using standard formulas for expectations of geometric sums of a quadratic form.

18.1.7. Ramsey problem

There are many equilibria, indexed by tax systems. The Ramsey problem is to choose the tax system that delivers the equilibrium preferred by the representative household. The Ramsey problem assumes that at time 0 the government commits itself to the tax system, once and for all.

*Definition:* The *Ramsey problem* is to choose an equilibrium that maximizes the household’s welfare (18.1.7). The allocation that solves this problem is called the *Ramsey allocation*, and the associated tax system is called the *Ramsey plan*.

18.1.8. Solution strategy

In solving the Ramsey problem, the government chooses all of the objects in an equilibrium, subject to the constraint on the equilibrium imposed by its budget constraint. Following a long line of researchers starting with Frank Ramsey (1929), we shall solve this problem using a ‘first-order’ approach that involves the following steps. The steps incorporate the properties required by the definition of equilibrium.

1. Obtain the first-order conditions for the household’s problem and use them to express the tax system and the price system in terms of the allocation alone.
2. Substitute the expressions for the tax system and the price system obtained in step 1 into the government’s budget constraint to obtain a single intertemporal restriction on allocations.
3. Use Lagrangian methods to find the feasible allocation that maximizes the utility of the representative household subject to the restriction derived in step 2. The maximizer is the Ramsey allocation.
4. Use the expressions from step 1 to find the associated Ramsey equilibrium price and tax systems by evaluating them at the Ramsey allocation.
18.1.9. Computation with no preference for robustness

We now execute these four steps for the version of the model without a preference for robustness. The problem is set so that the mathematics of linear systems can support a solution.

**Step 1.** The household’s first order conditions imply

\[ p_t^0 = \frac{(b_t - c_t)}{(b_0 - c_0)} \]  \hspace{1cm} (18.1.10)

\[ \tau_t = 1 - \frac{\ell_t}{b_t - c_t} \]  \hspace{1cm} (18.1.11)

**Step 2.** Using (18.1.10) and (18.1.11) express (18.1.9) as

\[ E_0 \sum_{t=0}^{\infty} \beta^t \left[ (b_t - c_t)(g_t + \_0 s_t) - (b_t - c_t) \ell_t + \ell_t^2 \right] = 0 \]  \hspace{1cm} (18.1.12)

Equation (18.1.12) is often called the implementability constraint on the allocation.

**Step 3.** Consider the maximization problem associated with the Lagrangian:

\[ J = E_0 \sum_{t=0}^{\infty} \beta^t \left\{ -0.5 \left[ (c_t - b_t)^2 + \ell_t^2 \right] \right. \]

\[ \left. + \lambda_0 \left[ (b_t - c_t) \ell_t - \ell_t^2 - (b_t - c_t) (g_t + \_0 s_t) \right] \right. \]

\[ + \mu_{0t} \left[ d_t + \ell_t - c_t - g_t \right] \}

where \( \lambda_0 \) is the multiplier associated with the government’s budget constraint, and \( \mu_{0t} \) is the multiplier associated with the time \( t \) feasibility condition. Obtain the first-order conditions:

\[ c_t : - (c_t - b_t) + \lambda_0 \left[ -\ell_t + (g_t + \_0 s_t) \right] = \mu_{0t} \]  \hspace{1cm} (18.1.13a)

\[ \ell_t : \ell_t - \lambda_0 \left[ (b_t - c_t) - 2\ell_t \right] = \mu_{0t} \]  \hspace{1cm} (18.1.13b)

\[ d_t + \ell_t = c_t + g_t \]  \hspace{1cm} (18.1.13c)

We want to solve equations (18.1.13a), (18.1.13b), (18.1.13c) and the government’s budget constraint (18.1.9) for an allocation.
18.1.10. Key idea

Our solution strategy is to begin by taking $\lambda_0$ as given and to solve (18.1.13) for an allocation contingent on $\lambda_0$. Then we shall use (18.1.9) to solve for $\lambda_0$.

18.1.11. Execution

Using the feasibility constraint $c_t = d_t + \ell_t - g_t$, we can express (18.1.13a), (18.1.13b) as

$$\ell_t - \lambda_0 [(b_t - d_t - \ell_t + g_t) - 2\ell_t] = -(d_t + \ell_t - g_t - b_t) + \lambda_0 [-\ell_t + (g_t + \sigma s_t)]$$

or

$$\ell_t = \frac{1}{2} (b_t - d_t + g_t) - \frac{\lambda_0}{2 + 4\lambda_0} (b_t - d_t - \sigma s_t).$$

We also derive

$$c_t = \frac{1}{2} (b_t + d_t - g_t) - \frac{\lambda_0}{2 + 4\lambda_0} (b_t - d_t - \sigma s_t).$$

Define

$$\tilde{c}_t = (b_t + d_t - g_t) / 2 \quad (18.1.14a)$$
$$\tilde{\ell}_t = (b_t - d_t + g_t) / 2 \quad (18.1.14b)$$
$$m_t = (b_t - d_t - \sigma s_t) / 2 \quad (18.1.14c.)$$

We have:

$$\ell_t = \tilde{\ell}_t - \mu m_t \quad (18.1.15a)$$
$$c_t = \tilde{c}_t - \mu m_t \quad (18.1.15b)$$

where, for convenience, we define

$$\mu = \frac{\lambda_0}{1 + 2\lambda_0} \quad (18.1.16)$$

Using (18.1.15), the general term of (18.1.12) can be written as:

$$(b_t - \tilde{c}_t) (g_t + \sigma s_t) - (b_t - \tilde{c}_t) \tilde{\ell}_t + \tilde{\ell}_t^2$$
$$- \mu m_t \left[- (g_t + \sigma s_t) + \tilde{\ell}_t - (b_t - \tilde{c}_t) + 2\tilde{\ell}_t \right] + \mu^2 m_t^2$$
$$= (b_t - \tilde{c}_t) (g_t + \sigma s_t) - 2m_t^2 \mu + 2m_t^2 \mu^2.$$
where we used $2\tilde{\ell}_t = b_t - d_t + g_t$ and $\tilde{\ell}_t = b_t - \tilde{c}_t$ to reduce the bracketed factor in the second line.

This allows us to write (18.1.12) as:

$$a_0(x_0) (\mu^2 - \mu) + b_0(x_0) = 0 \quad (18.1.17)$$

where

$$a_0(x_0) = E_0 \sum_{t=0}^{\infty} \beta^t \frac{1}{2} (b_t - d_t - o_s t)^2$$

$$= E_0 \sum_{t=0}^{\infty} \beta^t x'_t \frac{1}{2} [S_b - S_d - oS_s]' [S_b - S_d - oS_s] x_t \quad (18.1.18)$$

and

$$b_0(x_0) = E_0 \sum_{t=0}^{\infty} \beta^t \left[ (b_t - \tilde{c}_t) (g_t + o_s t) - (b_t - \tilde{c}_t) \tilde{\ell}_t + \tilde{\ell}_t^2 \right] \quad (18.1.19)$$

$$= E_0 \sum_{t=0}^{\infty} \beta^t \frac{1}{2} (b_t - d_t + g_t) (g_t + o_s t)$$

$$= E_0 \sum_{t=0}^{\infty} \beta^t x'_t [S_b - S_d + S_S]' [S_g + oS_s] x_t \quad (18.1.20)$$

where the fact that $b_t - \tilde{c}_t = \tilde{\ell}_t$ was used. The 0 subscripts on the forms $a_0$ and $b_0$ denote their dependence on $oS_s$. The coefficients in the polynomial expression of (18.1.17) are functions of $x_0$ alone because, given the law of motion for the exogenous state $x_t$, the infinite sums can be computed using the algorithms described in Appendix A of Sargent and Velde (1999).

Notice that $b_0(x_0)$, when expressed by (18.1.19), is simply the infinite sum on the left side of (18.1.12) evaluated for the specific allocation $\{\tilde{c}_t, \tilde{\ell}_t\}$. That allocation solves the problem:

$$\max_{c, \ell} -0.5 \left[ (c - b_t)^2 + \ell^2 \right]$$

subject to $c + g_t = \ell + d_t$. In other words, $\{\tilde{c}_t, \tilde{\ell}_t\}$ is the allocation that would be chosen by a benevolent dictator not restricted to competitive equilibrium allocations, or the Ramsey allocation when the government can resort to lump-sum taxation. The term $b_0(x_0)$ is the present-value of the government stream spending commitments $\{g_t + o_s t\}$, evaluated at the prices corresponding to the $\{\tilde{c}_t, \tilde{\ell}_t\}$
allocation. If that present value is 0, distortionary taxation is not necessary, and 
\( \mu = 0 \) (that is, \( \lambda_0 = 0 \)) solves (18.1.17): the government’s budget constraint is 
not binding. One configuration for which \( b_0(x_0) = 0 \) is when \( g_t = -\alpha s_t \) for all

\( t \), but there are many others. Because markets are complete, the timing of the
government’s claims on the household does not matter. If the government were able to acquire such claims on the private sector in a non-distortionary way, it would be able to implement a first-best allocation.

When the net present value of the government’s commitments is positive, we must solve (18.1.17) for a \( \mu \) in \((0, 1/2)\), corresponding to \( \lambda_0 > 0 \). The polynomial 
\( a_0(x_0)\mu(1 - \mu) \) is bounded above by \( a_0(x_0)/4 \), which means that government commitments that are “too large” cannot be supported by a Ramsey plan. If 
\( b_0(x_0) < a_0(x_0)/4 \) there exists a unique solution \( \mu \) in \((0, 1/2)\) and a unique 
\( \lambda_0 > 0 \). The Ramsey allocation can then be computed as:

\[
c_t = \hat{c}_t - \mu m_t \\
= \frac{1}{2} ([S_b + S_d - S_g] - \mu [S_b - S_d - \alpha S_s]) x_t \quad (18.1.21a)
\]

\[
\ell_t = \hat{\ell}_t - \mu m_t \\
= \frac{1}{2} ([S_b - S_d + S_g] - \mu [S_b - S_d - \alpha S_s]) x_t \quad (18.1.21b)
\]

and the Ramsey plan as:

\[
\tau_t = 1 - \frac{\ell_t}{b_t - c_t} \\
= 1 - \frac{\hat{\ell}_t - \mu m_t}{b_t - \hat{c}_t + \mu m_t} \\
= \frac{2 \mu m_t}{\hat{\ell}_t + \mu m_t} \\
= \frac{2 \mu [S_b - S_d - \alpha S_s] x_t}{([S_b - S_d + S_g] + \mu [S_b - S_d - \alpha S_s]) x_t}. \quad (18.1.22)
\]

Expression (18.1.22) shows how the stochastic properties of the tax rate mirror those for government expenditures when the endowment and the preference shocks are constant.
18.2. Modifications for robustness

We now compute a Ramsey plan under a preference for robustness induced by either Assumption 1’ or 2’. Our equilibrium concept has the government and public share a common preference for robustness as parameterized by $\theta$. We let $\theta$ be a multiplier on the constraint on the specification error (18.1.3b) for Assumption 1’ or (18.1.5) for Assumption 2’. We take $\theta \in (\theta_0, +\infty]$ as a parameter. The value of $\theta$ is context specific because it depends on the government expenditure process.\footnote{See chapter 7 for a discussion of the lower limit on $\theta$.}

18.2.1. Main idea

Recall our basic solution strategy of (1) taking $\lambda_0$ as given, (2) solving (18.1.13) for an allocation contingent on $\lambda_0$, then (3) using (18.1.9) to solve for $\lambda_0$. We solved the Ramsey problem by searching for a $\lambda_0$ that solves (18.1.9). For computing a robust rule, steps (1) and (2) remain the same, and (3) is modified by the extra step of finding a distorted expectations operator with which to evaluate the conditional expectations operator in (18.1.9). We now describe how to compute this modified expectations operator.

18.2.2. Assumption 1’

We first adopt Assumption 1’. As with all analyses of robustness, we require the preferences of the maximizing agent, the minimization of which will induce a worst case probability law. Because the Ramsey problem will embed (18.1.14), (18.1.15) regardless of the law of motion, the minimization problem in this case becomes a pure prediction problem. Thus, we use (18.1.14), (18.1.15), (18.1.1), and (18.1.7) to define $H$ in the following representation of the robust pure prediction problem:

$$\min_w E_0 \sum_{t=0}^{\infty} \beta^t \left\{-x_t' H' H x_t + \beta \theta w_{t+1}' w_{t+1}\right\}$$

(18.2.1)

where the expectation and the minimization are both subject to

$$x_{t+1} = Ax_t + C (\epsilon_{t+1} + w_{t+1}) .$$

(18.2.2)

This is a discounted optimal linear regulator problem with optimal feedback rule

$$w_{t+1} = K x_t .$$

(18.2.3)
Substituting (18.2.3) into (18.2.2) gives the distorted law of motion
\[ x_{t+1} = \hat{A}x_t + C\epsilon_{t+1} \]  
(18.2.4)

where
\[ \hat{A} = A + CK. \]  
(18.2.5)

An alternative formula for $K$ can be found as follows. First, compute a matrix $V$ by iterating to convergence on
\[ S(V_{j+1}) = -H'H + \beta A'D(V_j)A \]
where $D$ is the operator
\[ D(V) = V + \sigma VC(I - \sigma C'VC)^{-1}C'V \]  
(18.2.6)

and $\sigma \equiv -\theta^{-1}$. Then compute $K$ from
\[ K = \sigma (I - \sigma C'VC)^{-1}C'V. \]  
(18.2.7)

Notice that $H$ depends on $\mu$, or $\lambda$, through (18.1.14), (18.1.15). This dependence will necessitate that we use an iterative method to compute $\lambda$, whereas Sargent and Velde could calculate it directly.

Here is a four step process to compute the Ramsey plan with a preference for robustness with given $\theta$:

**Step 1.** Guess a value of $\lambda$. Find the associated $\mu$ from (18.1.16).

**Step 2.** Compute $c_t, \ell_t$ from (18.1.14), (18.1.15). Find the associated $H$ for (18.2.1). Find the associated $O$ and $\hat{A}$.

**Step 3.** Using model (18.2.4), (18.2.5) to evaluate the expectation operator $\hat{E}$, evaluate $a_0(x_0)$ and $b_0(x_0)$ using the formulas
\[ a_0(x_0) = \hat{E}_0 \sum_{t=0}^{\infty} \beta^t x_t \frac{1}{2} [S_b - S_d - \rho S_s]' [S_b - S_d - \rho S_s] x_t \]  
(18.2.8)

and
\[ b_0(x_0) = \hat{E}_0 \sum_{t=0}^{\infty} \beta^t x_t' [S_b - S_d + \rho S_g]' [S_g + \rho S_s] x_t. \]  
(18.2.9)

**Step 4.** Check whether (18.1.17) is satisfied. Iterate on steps 1 through 4 to find a $\lambda$ that is a zero of (18.1.17).

**Step 5.** Having found such a $\lambda$, use (18.1.21) and (18.1.22) to compute the Ramsey allocation and Ramsey plan.
18.2.3. Assumption 2′

Here is the corresponding step by step plan for computing the Ramsey plan under a preference for robustness under the Markov assumption 2′. First again fix a $\lambda$ and use (18.1.14), (18.1.15), (18.1.1), and (18.1.7) to define $H$ in the following representation of the one-period return function:

$$u(x_t) = -x_t' H' H x_t.$$  \hspace{1cm} (18.2.10)

This is the same $H$ as found above. Then follow these steps.

**Step 1.** Compute the value function $V_i$ (a vector) by iterating to convergence on

$$V_i = u_i - \beta \theta \ln \left( \sum_j \exp \left( \frac{-V_j}{\theta} \right) \pi_{ij} \right).$$  \hspace{1cm} (18.2.11)

This is a simple modification of an iteration on a Bellman equation.

**Step 2.** Form

$$w^*_j = \exp \left( \frac{-V_j}{\theta} \right).$$  \hspace{1cm} (18.2.12)

**Step 3.** Form the distorted transition density

$$\pi_{w^*} = \frac{w^*_i \pi_{ij}}{\sum_k w^*_k \pi_{ik}}.$$  \hspace{1cm} (18.2.13)

**Step 4.** Use $\pi_{w^*}$ to evaluate $a_0(x_0), b_0(x_0)$, using formulas for evaluating geometric sums of a quadratic form for a Markov chain.\(^5\)

**Step 5.** Check whether (18.1.17) is satisfied. Iterate on steps 1 through 4 to find a $\lambda$ that is a zero of (18.1.17).

**Step 6.** Having found such a $\lambda$, use (18.1.21) and (18.1.22) to compute the Ramsey allocation and Ramsey plan.

18.3. Computations

We have activated a preference for robustness by setting values of $\theta < +\infty$. We want to study how the allocations, taxes, and prices change as we accentuate a preference for robustness (i.e., lower $\theta$) for both the 1’ and 2’ cases. In addition to these objects, we also want to calculate the value of government debt and interest rates. Along the Ramsey allocation, the value of government debt $B_t$ can be computed as

$$B_t = \frac{\hat{E}_t \sum_{j=0}^{\infty} \beta^j \left[ (b_{t+j} - c_{t+j}) \ell_{t+j} - \ell^2_{t+j} - (b_{t+j} - c_{t+j}) g_{t+j} \right]}{(b_t - c_t)},$$

which can evidently be expressed as a function of the time $t$ state $x_t$, in particular, a quadratic form in $x_t$ plus a constant divided by a linear form in $x_t$. The quantity $B_t$ can be regarded as the time $t$ value of government state contingent debt (Arrow securities) issued at $t-1$ and priced at $t$. We also compute the one period gross interest rate $R_t$ from

$$R_t^{-1} = \frac{\hat{E}_t \beta p_{t+1}^I}{p_t},$$

where $p_{t+1}^I = \frac{M_{x_{t+1}}}{M_{x_t}}$.

18.4. Simulations at approximating model

Under assumption 1’, for an approximating model $A, C$ and a given $\theta$, a robust Ramsey plan is associated with a distorted law of motion $\hat{A}, \hat{C}$. We are interested in comparing the robust Ramsey plan with the optimal (non-robust or $\theta = +\infty$ Ramsey plan) at the approximating law of motion. Therefore, we’ll simulate systems with different $\theta$’s at the approximating model (i.e., $A$ or $\pi$, not $\hat{A}$ or $\pi^w$). These simulations thus indicate outcomes when the approximating model for the government expenditure process is indeed correct, although both the government and private agents use robust decision analysis to guide their decisions. Similarly, for Assumption 2’, we shall simulate the robust Ramsey plan using the approximating Markov chain $\pi$ (and not the distorted chain $\pi^{w*}$) to generate government expenditures.
18.5. Example computations

18.5.1. Markov case

We specify a three state Markov chain with transition matrix

\[ \pi = \begin{bmatrix} .95 & .05 & 0 \\ 0 & .8 & .2 \\ 0 & 0 & 1 \end{bmatrix} , \]

and initial distribution \( \pi_0 = [1 \ 0 \ 0]' \), so that the system starts in state 1. Government expenditures in states 1, 2, 3 are [.5 .5 .25]. Thus, state 1 is designed to represent war, state 2 armistice, and state 3 peace. Peace is an absorbing state. Moving to peace requires spending at least one period in the state of armistice. We specify that \( \beta = .97 \) and \( b = 2.2 \).

For two values of \( \theta \), namely, 4 and 1, Fig. 18.5.1 and Fig. 18.5.2 show the Ramsey plans and allocations with (solid lines) and without (dotted lines) a preference for robustness. Recall that lowering \( \theta \) raises the preference for robustness. For both values of \( \theta \), we have computed the associated distorted Markov transition matrices. With \( \theta = 4 \) we have

\[ \pi^{w'} = \begin{bmatrix} .9836 & .0164 & 0 \\ 0 & .8581 & .1419 \\ 0 & 0 & 1 \end{bmatrix} \]

With \( \theta = 1 \) we have

\[ \pi^{w'} = \begin{bmatrix} .99975 & .00025 & 0 \\ 0 & .99393 & .00607 \\ 0 & 0 & 1 \end{bmatrix} \]

Notice how raising the preference for robustness (i.e., lowering \( \theta \)) shifts probability toward longer wars and a longer armistice.

Relative to Lucas and Stokey’s Ramsey plan, the robust Ramsey plan has taxes higher and consumption and labor supply both lower. Once peace occurs, taxes and government debt are both higher in the robust Ramsey plan. Notice that once peace breaks out, the worst case beliefs coincide with the approximating beliefs. Thus, the higher taxes and government debt during peace time in the robust plan reflect the government’s having to honor its past promises, entered into during earlier periods of war or armistice, when it was planning against the pessimistic beliefs of itself and the public.
Figure 18.5.1: Markov case, $\theta = 4$. Dotted line is for Lucas and Stokey’s Ramsey plan, the solid line is the robust Ramsey plan.

During war and peace, interest rates coincide for the Lucas and Stokey and robust plans, however, they are lower for the robust plan under the armistice. This reflects the more pessimistic expectation about the one-period rate of growth of consumption under the distorted Markov chain $\pi^w$ used to price assets.\footnote{Notice that during war the expected rate of change of consumption under $\pi$ and $\pi^w$ coincide.}

When $\theta$ is pushed down to 1, the government sets taxes so high that the value of its debt is negative at first. It is so pessimistic that it actually acquires claims on the public, which it uses to fund some of its expenditures during armistice.
18.5.2. Stochastic difference equation

Fig. 18.5.3 shows the optimal (Lucas Stokey) and robust Ramsey plan when government expenditures follow the first order autoregression:

\[ x_{t+1} = .0175 + .95x_t + C\epsilon_{t+1}, \]

where \( C = .013660 \). When \( \theta = .1 \), the associated worst case distorted law of motion for government expenditures is

\[ x_{t+1} = .0476 + .9580x_t + C\epsilon_{t+1} \]

where again \( C = .013660 \).\(^7\) The mean of \( x_t \) under \( A \) is .35 while under the distorted law it is 1.1324. Thus, pessimism translates into more persistence and

\(^7\) We have set \( \theta \) to a low value to accentuate the effects of robustness so that they show up well on the graphs.
a higher mean for government expenditures. We set the other parameters at \( \beta = 0.97 \) and \( b = 2.135 \).

Fig. 18.5.3 again shows taxes and government debt to be higher, and labor and consumption lower under the robust Ramsey plan. Interest rates are uniformly lower under the robust Ramsey plan, reflecting the pessimism about the rate of growth of consumption embedded in the distorted law of motion \( \hat{A} \).

**Figure 18.5.3:** Stochastic difference equation case, \( \theta = 0.1 \). Dotted line is for Lucas and Stokey’s Ramsey plan, the solid line is the robust Ramsey plan.
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