TIME PREFERENCE AND CAPITAL ASSET PRICING MODELS

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Received May 1983, final version received July 1984

Results of the theory of individual optimal consumption–investment choice under uncertainty are extended to a class of intertemporally dependent preferences for consumption streams. These results are then used to show that with intertemporally dependent preferences, which are more realistic than the separable time-additive preference structure, Merton's (1973) multi-beta intertemporal capital asset pricing model is still valid, but it can no longer be collapsed to Breeden's (1979) single consumption-beta model.

1. Introduction and discussion

This paper investigates the role played by the assumption of separable time-additive preference structures in the derivation of continuous-time intertemporal capital asset pricing models. The results come in the form of both good news and not so good news. The good news is that when the separable time-additive preferences are replaced by more realistic intertemporally dependent preference structures, Merton's (1971, 1973) intertemporal capital asset pricing model (ICAPM), with the $S + 2$ funds separation theorem and the multi-beta risk premia relations, is still valid. The not-so-good news is that the former can no longer be collapsed to Breeden's (1979) elegant single consumption-beta capital asset pricing model (CCAPM).

Although economists have made extensive use of separable time-additive preferences for lifetime consumption streams, gaining significant insights in the process, they have done so half-heartedly. The reason for the extensive use is the great simplification obtained in calculations, but this benefit comes at the high cost of the severe restrictions that the time-additive preference structure imposes on consumers' choice behavior. Separable time-additive preferences imply complete time independence of desirability of consumption levels in different periods. Alternatively viewed, time-additive utilities imply discounting satisfaction from future consumption at a discount rate that is independent of consumption levels in other periods.

*I wish to thank Doug Breeden, the referee, René Stulz, the Editor for the Journal, and Harl Ryder for their very helpful comments and suggestions. I am also grateful to Robert Merton for a very stimulating discussion.

Pronounced objections to the assumption of separable time-additive utilities are Hicks (1965, p. 261): 'It is highly objectionable to assume that successive consumptions are independent; the normal condition is that there is a strong complementarity between them.' Also Lucas (1978): 'The time-additive preference structure is a nuisance, and it has no rationale beyond tractability.' On the empirical front, recent studies by Eichenbaum, Hansen and Singleton (1982) and by others, seem to indicate that time additivity of preferences is inadequate in explaining equilibrium relations, and that non-additive structures are needed.

As early a writer as Fisher (1930) advocates that relative weights given to future as against present consumption should vary with the level of overall satisfaction derived from a whole consumption program. This motivated Koopmans (1960) and Koopmans, Diamond and Williamson (1964) to investigate the choice behavior assumptions that would give rise to such a time-preference structure. One of their results is that these behavioral assumptions imply a broad class of preference structures that are recursive and contain the time-additive preference as a special case. These recursive preference structures have been applied to the theory of optimal economic growth by, among others, Beals and Koopmans (1969), Iwai (1972), and Boyer (1975). Recently, Lucas and Stokey (1982) used Koopmans’ recursive preferences to study 'economies with heterogeneous agents – economies that do not seem analyzable in an interesting way under the limits imposed by the assumption of time-additive preferences'.

Motivated quite differently, Uzawa (1968) suggests a non-additive consumption preference structure which is similar in form to the separable time-additive except that it features an instantaneous discount rate that depends on the concurrent consumption rate. Uzawa's preference structure has been used in growth theory and in macroeconomic and trade theory by Kouri (1980), Obstfeld (1981) and others. It will be shown in section 2 that the Uzawa preference structure is the continuous-time limit of a special case of the Koopmans preference structures, one that is characterized by a linear Koopmans aggregator function. Other special cases include the separable time-additive and time-multiplicative structures. In addition to providing a unifying viewpoint, this observation will be useful in several ways. First, by demonstrating that the Uzawa structure is a special case of the Koopmans class we set the former on the sound economic base provided by the latter via its postulated behavioral foundations. Second, the discrete-time formulation of the Koopmans class of preferences will facilitate generalization of some well-known results of the individual's consumption-investment problem to the preferences under the linear Koopmans aggregator.

Hicks (1965) observes that the marginal rate of substitution between current and future consumption should depend on whether the increment in the latter is to be a sudden spurt, out of line with its neighbors, or is needed to fill a gap, to make up a deficiency.
We will derive the first-order conditions for consumer's optimum, proving in particular that given linear aggregator preferences and a Markov assumption for return and state variables, the optimal consumption and investment strategies are also Markov, i.e., they neither depend on the history of consumption nor on the history of any other variable. This in turn implies that despite the role played by future consumption streams in discounting instantaneous utilities in the Uzawa structure, consumption history is irrelevant for the optimal control strategies. Consequently, consumption history is not a state variable in the preference structures that are based on the linear Koopmans aggregator. This fact lies at the heart of our choice to demonstrate the results using the Uzawa preference structure, since it emphasizes that (in this paper) it is not state dependence that causes the incollapsability of Merton’s ICAPM to Breeden’s CCAPM, but that it is rather a consequence of the non-separability of the preference structure. Third, working in the discrete-time framework will provide intuitive appeal and simplicity of exposition. This program is carried out in section 3.

In section 4, we take the results of section 3 to their continuous-time limit, showing that the first-order portfolio optimality conditions are identical to those which arise in Merton’s ICAPM and in related works that assume separable time-additive preferences. However, the consumption first-order condition takes different forms depending on the preference structure. It no longer is the familiar intertemporal envelope condition which equates marginal utility of current consumption $\partial u/\partial c$ to marginal utility of wealth $\partial J/\partial W$. Instead, for the Uzawa utility it is

$$\frac{\partial u}{\partial c} - \rho(c) = \frac{\partial J}{\partial W},$$

where $\rho(c)$ is the (non-constant) consumption-dependent subjective discount rate, and the rest of the variables have the standard meaning.

These results imply that Merton’s ICAPM relations and also the general results of the Cox, Ingersoll and Ross (CIR) (1977) model of equilibrium prices are robust to relaxation of the assumption of a separable time-additive preference structure, while Breeden’s consumption based CCAPM is not.

This deserves some immediate discussion. It may come somewhat as a surprise that the assumption that agents optimally control their consumption programs is not necessary at all in deriving Merton’s ICAPM relations ($S + 2$ funds separation and the risk premia relations). In fact, even in an economy where agents are exogenously forced to adhere to consumption plans that are ad-hoc (non-optimal) functions of wealth and the state variables, but they are still free to optimize their expected utility of consumption streams using the investment controls only, Merton’s ICAPM relations still hold. (An example, realistic to a certain degree, may be a wartime economy with centrally rationed consumption.)
The derivation of Merton's ICAPM relations utilizes the investment first-order conditions only, but does not use the consumption first-order condition. This is the reason why Merton's results are immune to a change in the form of the consumption first-order conditions, which results from the non-additive Uzawa preference structure. In fact, this argument leads to the conclusion that Merton's ICAPM relations are invariant to any change in the preference structure. This is discussed in section 5. Similar observations also apply to some of the results of CIR (1977) and related models, i.e., those results that do not make explicit use of the consumption first-order condition or of specific separable time-additive utility functions.

On the other hand, the consumption first-order condition in the familiar envelope condition format plays a crucial role in collapsing Merton's multi-beta ICAPM into Breeden's (1979) single consumption-beta model. Consequently, when the consumption first-order condition takes a different form as a result of adopting a more realistic non-separable preference description, this model no longer follows.

The point made in this paper is that the separable time-additive preference structure assumption is a crucial condition for derivation of the continuous-time single consumption-beta CCAPM, and that more realistic non-separable preferences would not, in general, give rise to this model. This is somewhat analogous to the well-known observation that the Sharpe–Lintner single-period CAPM is predicated on either normality of assets returns or, again, on a restrictive preference structure, namely, quadratic utilities. To draw the analogy further – it may still be the case that the single consumption-beta model would turn out to be a good approximation to reality, just as the Sharpe–Lintner model is. But this is, of course, an empirical issue, and it should be decided by an empirical test.

Except for some interpretations which are not an essential part of the ICAPM relations.

Invariance of the CIR results under exogenously enforced consumption programs is also true, provided all programs are identical so as to maintain the representative consumer assumption.

Single-beta CAPM risk premia relations can be represented for general preference structures as shown, for example, by Hansen, Richard and Singleton (1982), but these involve the empirically difficult-to-observe marginal rates of substitution of optimal consumption and thus are not, in general, preference free. We are concerned here only with preference free risk premia relations at the continuous time limit such as Merton's ICAPM and Breeden's CCAPM.

Empirically, consumption controls are influenced by transaction costs (of applying the control). This may be the main reason that consumers purchase durables and do not purchase the service flows directly by renting. (One does not terminate a lease for an apartment when going on a week's vacation and renew it when returning.) To the extent that the transaction costs associated with applying the consumption controls are much higher than those associated with applying the investment controls, yet another distinction between the CCAPM and the ICAPM follows from the foregoing discussion which emphasizes the role of unconstrained optimal consumption control in derivation of the CCAPM. Note also that direct optimal control of the consumption program is a crucial condition in Grossman and Shiller's (1982) generalization of the consumption based CAPM. Hence, according to the foregoing discussion, it may be expected that their alternative derivation would not be immune to relaxation of the time-additive preference assumption. It can be shown that this is, in fact, the case.
2. Recursive preference structures

From a number of behavioral as well as technical postulates on consumption preferences, Koopmans (1960) and Koopmans, Diamond and Williamson (1964) derived a general class of utility functionals on infinite discrete-time consumption sequences. A utility function in this class satisfies a recursive relation

\[ U(c_0, c_h, c_{2h}, \ldots) = V(c_0, U(c_h, c_{2h}, \ldots)) , \quad (2) \]

where \( V(\cdot, \cdot) \) is called the Koopmans aggregator. Note that instead of the original unit time interval we choose to work with an arbitrary increment \( h \). This will facilitate handling both the discrete-time, \( h = 1 \), and the continuous-time, \( h \to 0 \), frameworks using the same formulation. Accordingly, \( c_{kh} \) is interpreted as consumption rate during \((kh, (k + 1)h)\).

**Definition.** \( V(\cdot, \cdot) \) will be called the **linear Koopmans aggregator with increment** \( h \), or simply the **linear aggregator**, if it is of the form

\[ V(c, U) = h \cdot u(c) + [\alpha(c)]^h \cdot U, \quad \alpha(\cdot) > 0. \quad (3) \]

The linear aggregator gives rise to the economically meaningful cases summarized in the following proposition. Subsequent results are easier to prove for the general linear aggregator directly, and consequently, will be true for all these cases.

**Proposition 1.** (a) For \( 0 < \alpha(\cdot) \leq \bar{\alpha} < 1 \), \( u(\cdot) \) increasing concave, the linear Koopmans aggregator defines the Uzawa (1968) utility functional for \( h \to 0 \), and defines for \( h = 1 \) what we will call, the Uzawa discrete-time functional. \( u(\cdot) \) and \( \alpha(\cdot) \) are interpreted as the single-period utility and subjective consumption-dependent discount rate, respectively.

(b) For \( \alpha(\cdot) \) a constant in \((0,1)\), the Uzawa utility reduces to the separable time-additive functional. For the discrete version let \( h = 1 \); for the continuous, let \( h \to 0 \).

(c) For \( u(\cdot) \equiv 0 \) and for \( \alpha(\cdot) \) increasing concave, the linear aggregator defines the time-multiplicative utility functional. Again, discrete and continuous-time versions have \( h = 1 \) and \( h \to 0 \), respectively. \( \alpha(\cdot) \) is interpreted then as the single-period utility.
Proof. Applying (2) recursively and using (3) results in

\[ U(c_0, c_h, c_{2h}, \ldots) = V(c_0, V(c_h, U(c_{2h}, c_{3h}, \ldots))) = h u(c_0) + \alpha(c_0)^h \left[ h u(c_h) + \alpha(c_h)^h U(c_{2h}, c_{3h}, \ldots) \right] = h u(c_0) + h \alpha(c_0)^h u(c_h) + \alpha(c_0)^h \alpha(c_h)^h U(c_{2h}, c_{3h}, \ldots). \]

which can be written

\[ U(c_0, c_h, c_{2h}, \ldots) = \sum_{n=0}^{\infty} u(c_{nh}) \cdot h \cdot \prod_{k=0}^{n-1} [\alpha(c_{kh})]^h. \tag{4} \]

Writing the product in (4) as exponent of the sum of \( h \cdot \log \alpha(c_{kh}) \) and letting \( h \to 0 \) results in

\[ U = \int_0^\infty u(c_t)e^{-\int_0^t \rho(c_r) dr} dt, \tag{5} \]

where \( \rho(c_r) = -\log \alpha(c_r) \). Eq. (5) is the Uzawa (1968) utility functional. [Uzawa writes \( \delta(u(c_r)) \) instead of \( \rho(c_r) \).] Taking \( h = 1 \) in (4) results in what we called the discrete-time Uzawa functional. Part (c) is proved analogously. Part (b) is trivial. Q.E.D.

Note that the discount factor \( \Delta_t = \int_0^t \rho(c_r) dr \) of the Uzawa functional depends on the segment of the consumption stream under evaluation, between evaluation (decision) time zero and time \( t \) of the instantaneous utility contribution \( u(c_t) \), which is being discounted. Assuming \( \rho' > 0 \), an increase in the consumption level in this segment from 0 to \( t \) makes the contribution to overall satisfaction from consumption at time \( t \) less important, and the opposite for a decrease. Thus, intertemporal dependence of satisfaction from consumption in different time periods is naturally introduced. This is to be contrasted with the restrictive special case \( \rho(c) = \rho \), a constant, i.e., the separable time-additive preference structure with complete intertemporal independence.

3. Consumption–investment with the linear aggregator

The consumption–investment problem of an individual with a separable time-additive preference structure has been studied in a discrete-time frame-
work by Levhari and Srinivasan (1969), Samuelson (1969) and by others, and in continuous-time by Merton (1969, 1971). The time-multiplicative preference structure has been studied by Meyer (1969) and by Pye (1973) in a continuous-time framework, and by Ingersoll (1982) in discrete-time. The purpose of the following theorem is to unify and generalize the results of these studies to the preference structure defined by the linear Koopmans aggregator.

We first extend the preference order to the set of uncertain prospects of consumption sequences, appending the Savage rationality axioms. All the aforementioned functionals become then von Neumann–Morgenstem utility functionals. We will use the framework and notation as in Merton (1982). It is assumed that the environment (e.g., technology) is described by a stochastic $m$-vector state process $X(t)$, and that there are $n$ risky assets whose returns over $(t, t + h)$ are described by a stochastic $n$-vector $Z(t + h)$ such that $(Z(t + h), X(t + h))$ is Markov with respect to $X(t)$. There is also a riskless asset whose return $R(t)$ over $(t, t + h)$ is known with certainty at time $t$.

The individual's problem is to maximize $E_0U(c_0, c_h, \ldots)$ subject to the wealth accumulation and to the state dynamics, where the maximization is over all possible consumption–investment strategies (including artificially randomized but excluding future anticipating strategies). $E_0$ is expectation given information at time 0. Define the derived utility of wealth as

$$J = \max E_0U(c_t, c_{t+h}, \ldots),$$

where $U$ is defined by the recursive linear aggregator (3), and $E_0$ is expectation given information at time $t$. We can now state the following:

**Theorem 1.** (a) In a world with a Markov environmental state–returns process, an individual with a preference order that is described by the linear aggregator will have a derived utility which is at most a function of current wealth $W(t)$, current state $X(t)$ and time $t$. The same is true of the optimal consumption and investment strategies, i.e., the optimal strategies are (non-randomized) Markov; they do not depend on histories.

(b) Assuming differentiability where necessary and interior solutions, the optimal strategies are characterized by the following necessary first-order conditions:

$$0 = E_t\left[\left(\frac{\partial J}{\partial W}\right)_{t+h} \cdot (Z_j(t+h) - R(t))\right], \quad j = 1, \ldots, n, \quad (7)$$

$$\frac{\partial u}{\partial c} + \left(\frac{\partial \log \alpha}{\partial c}\right)(J - hu) = \frac{\partial J}{\partial W}. \quad (8)$$

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$^7$Recently, Epstein (1983) provided a set of postulates that give rise to the discrete-time Uzawa utility functional under uncertainty.

$^8$Note that although in the Uzawa and in the multiplicative cases $J$ does not admit exactly the same interpretation as in the additive case, it nevertheless serves the same purpose.
Proof. A modest generalization of Theorem 6.3 in Ross (1970, p. 124) to the case of a control dependent discount factor would constitute a rigorous proof. However, we will give a more intuitive proof. Instead of working with the infinite horizon directly, we will truncate it at an arbitrary finite time $T$, prove the results, and then let the horizon recede back to infinity.

Truncation is done by letting $\alpha(c_T) = 0$. $u(c_T)$ is then interpreted as utility of bequest. [For the multiplicative case, let $\alpha(c_T) = 1, \tau > T; \alpha(c_T)$ is then bequest utility.] It is not difficult to see\(^9\) that the linearity of the aggregator allows writing a Bellman functional equation for $J$,

$$J_{t} \equiv \max E_{t} U(c_{t}, c_{t+h}, \ldots)$$

$$= \max E_{t} \left[ hu(c_{t}) + \alpha(c_{t})^h \max E_{t+h} U(c_{t+h}, \ldots) \right],$$

or

$$J_{t} = \max \left\{ hu(c_{t}) + \left[ \alpha(c_{t}) \right]^h E_{t} J_{t+h} \right\},$$

where the maximization is over the current decision variables. From this point on the proof of (a), (b) for the finite-horizon problem follows in the footsteps of Merton (1982) and is omitted. Note, though, that the dependence of derived utility and of optimal consumption and investment on current wealth, state and time and nothing else, follows from the optimization at time $T - h$ and from the recursive dynamic programming procedure. Observing that (a), (b) in the theorem are true for an arbitrary horizon $T$, we let $T \to \infty$ and the theorem is proved.\(^{10}\) Q.E.D.

The following is an explanation for the Markov property (history independence) of the optimal strategies. The Uzawa utility (4) can be written ($h = 1$ for simplicity)

$$U(c_{0}, c_{1}, \ldots, c_{t}, \ldots) = A + B \cdot U^*(c_{t}, c_{t+1}, \ldots),$$

where

$$A = \sum_{n=0}^{t-1} u(c_{n}) \prod_{k=0}^{n-1} \alpha(c_{k}),$$

$$B = \prod_{k=0}^{t-1} \alpha(c_{k}),$$

$$U^*(c_{t}, c_{t+1}, \ldots) = \sum_{n=t}^{\infty} u(c_{n}) \prod_{k=t}^{n-1} \alpha(c_{k}).$$

\(^9\)See also Brock (1980).

\(^{10}\)At most, time dependence may drop out of the optimal strategies, and this only if the state–return process is time-homogeneous. For sufficient conditions for existence of the limits see Bertsekas and Shreve (1978).
Following the dynamic programming procedure, for a given realization of history up to (and excluding) time $t$, the agent maximizes expected utility using $U$ on the LHS of (9). But for the given history, $A$ and $B$ are just known numbers, the latter being positive. It follows that the agent may as well base his decisions at time $t$ on $U^*$ instead of $U$, since both are equivalent von Neumann–Morgenstern utilities. But $U^*$ does not depend on the realized history up to $t$, hence neither do the optimal strategies.

Clearly, this argument applies only to the linear aggregator. (The argument of the multiplicative case is similar to that of the Uzawa case.) It would not hold for nonlinear aggregators or for general non-recursive preference structures. In these cases the optimal strategies would in general depend on realized histories of consumption as discussed in Fama (1970). The effect of consumption histories on CAPM's is discussed in section 5.

As mentioned in the introduction, while the investment first-order condition (7) is identical for the three cases of Proposition 1, the consumption first-order condition (8) takes a different form for each case. In particular, we have:

**Corollary.** For continuous-time Markov state–return processes and preferences defined by the linear aggregator, the first-order conditions for optimal consumption are

\[
\frac{\partial J}{\partial W} = \frac{\partial u}{\partial c} \frac{\partial J}{\partial c} \quad \text{(Uzawa),} \tag{10}
\]

\[
\frac{\partial J}{\partial W} = \frac{\partial u}{\partial c} \quad \text{(time-additive),} \tag{11}
\]

\[
\frac{\partial J}{\partial W} = \frac{\partial \log \alpha}{\partial c} J \quad \text{(time-multiplicative).} \tag{12}
\]

This follows by letting $h \to 0$ in (8), considering each case separately ($\rho = -\log \alpha$ for the Uzawa case). Note that in the corollary we need only assume that the return–state process is continuous-time Markov. This includes as special cases diffusion processes as well as Poisson jump processes. [For example, eq. (11) is in agreement with the envelope condition for a continuous-time Poisson process derived by Merton (1971, p. 397).]

Observe that (10) exhibits the dual effect of an increment to the instantaneous consumption rate in the Uzawa case. First, an instantaneous increase in the consumption rate adds to the instantaneous utility of consumption; this is the first term on the RHS of (10). Second, it simultaneously causes a decrease in satisfaction from subsequent consumption by increasing the subjective discount rate; this is the second term on the RHS. Eq. (10) then implies that at
the optimum, the net of these two effects equals the marginal utility of wealth. A similar interpretation applies to the multiplicative case.

4. Diffusion state–return processes and the CCAPM

As mentioned earlier, the investment first-order condition (7) for optimality takes the same form for all preferences under the linear Koopmans aggregator. Little more can be said about (7) without adding more structure. At this point we specialize the assumption of continuous-time Markov state–returns process and require that it be a diffusion described by the following system of Ito stochastic differential equations:

\[
\begin{align*}
\mathrm{d} \mathbf{x} &= \mathbf{\mu}_x(\mathbf{x}, t) \, \mathrm{d}t + \mathbf{\sigma}_x(\mathbf{x}, t) \, \mathrm{d}\mathbf{z}_x \\
\mathrm{d} \mathbf{a} &= \mathbf{\mu}_a(\mathbf{x}, t) \, \mathrm{d}t + \mathbf{\sigma}_a(\mathbf{x}, t) \, \mathrm{d}\mathbf{z}_a
\end{align*}
\] (13)

\[
\begin{align*}
\mathrm{d} \mathbf{a} &= \mathbf{\mu}_a(\mathbf{x}, t) \, \mathrm{d}t + \mathbf{\sigma}_a(\mathbf{x}, t) \, \mathrm{d}\mathbf{z}_a
\end{align*}
\] (14)

where \( \mathbf{x}(t) \) is the state process and where \( \mathrm{d} \mathbf{a}_j(t) \) is the infinitesimal return on asset \( j \) over \( (t, t + \mathrm{d}t) \), i.e., the discrete-time returns of section 3 transform:

\[
\begin{align*}
Z_j(t + h) &\to \mathrm{d} \mathbf{a}_j(t), \\
R(t) &\to r(t) \, \mathrm{d}t.
\end{align*}
\] (15)

This formulation allows interpretation of \( \mathrm{d} \mathbf{a}_j \) either as a return on capital asset \( j \) (\( \mathrm{d} \mathbf{a}_j = \mathrm{d} P_j / P_j \)) as in Merton (1973), or as return from a physical production process (\( \mathrm{d} \mathbf{a}_j = \mathrm{d} \eta_j / \eta_j \)) as in Cox, Ingersoll and Ross (1977). The instantaneous interest rate \( r(t) \) may be stochastic and endogenously determined in equilibrium. Also \( \mathbf{\sigma}_x, \mathbf{\sigma}_a \) are diagonal matrices (regular on the state space of \( \mathbf{x} \)) and \( \mathrm{d}\mathbf{z}_x, \mathrm{d}\mathbf{z}_a \) are correlated Wiener processes. Denote by \( \mathbf{w} \) the optimal portfolio weights, and the per-unit-time covariance matrices by

\[
\begin{align*}
\mathbf{V}_{aa} \cdot \mathrm{d}t &= \mathbf{E}_t \left[ \mathbf{\sigma}_a \, \mathrm{d}\mathbf{z}_a \, \mathrm{d}\mathbf{z}_a' \mathbf{\sigma}_a' \right], \\
\mathbf{V}_{ax} \cdot \mathrm{d}t &= \mathbf{E}_t \left[ \mathbf{\sigma}_a \, \mathrm{d}\mathbf{z}_a \, \mathrm{d}\mathbf{z}_x \mathbf{\sigma}_x' \right].
\end{align*}
\] (16)

We can now state the following proposition which complements the corollary of section 2.

**Proposition 3.** In a world with a diffusion state–returns process and preferences defined by the linear aggregator, the interior first-order condition for optimal investment is

\[
\mathbf{V}_{aa} \mathbf{w} W + \mathbf{V}_{ax} \left( J_x \mathbf{w} / J \mathbf{w} \mathbf{w} \right) = \left( -J \mathbf{w} / J \mathbf{w} \mathbf{w} \right) \left( \mathbf{\mu}_a - r \right).
\] (17)
Proof. Using (15) and denoting $h = \Delta t$, rewrite (7) as

$$0 = \mathbb{E}_t \left[ J_W(W(t + \Delta t), x(t + \Delta t), t + \Delta t) \cdot (\Delta a(t) - r(t) \Delta t) \right].$$

(18)

Applying Ito’s lemma to $J_w$ in (18) and letting $\Delta t \to 0$ proves the proposition. The details are given in the appendix. Q.E.D.

The implications of the foregoing results to continuous-time CAPM’s are presented in the following:

**Theorem 2.** The ICAPM (*S + 2 funds separation and the risk premium relations*) is valid for all preference structures defined by the linear aggregator, but, in general, it cannot be collapsed, for these preferences, to a single consumption-beta CCAPM unless the preference structure is time-additive.

*Proof.* The first part of the theorem follows from Proposition 2, which shows that the investment first-order condition (17) is identical for all preferences defined by the linear aggregator, and from the fact that (17) is all that is needed to derive the ICAPM. The second part will only be demonstrated for the Uzawa utility functional, since the proof for the time-multiplicative case is very similar. We first prove three lemmas:

**Lemma 1.** Let $c = c(W, x, t)$ and $w = w(W, x, t)$ be the optimal consumption and investment (portfolio) controls and $J = J(W, x, t)$ the derived utility of wealth for the Uzawa preference, then a necessary and sufficient condition for validity of the CCAPM is

$$G^x = c_w(J_{xw}/J_{ww}) - c_s = 0.$$  

(19)

*Proof.* Apply Ito’s lemma to $c$ and use (14) to get

$$\frac{1}{dt} \text{cov}(dc, da) = c_w W\mu_a^x + V_{ax}c_x.$$  

(20)

Multiply (17) by $c_s$, then add and subtract $V_{ax}c_x$ from the LHS of the resultant equation, and use (20) to get

$$\frac{1}{dt} \text{cov}(dc, da) + V_{ax}G^x = (-c_w(J_{xw}/J_{ww})) (\mu_a - \xi),$$  

(21)

which proves the proposition. (It is useful to think of a representative consumer.) Q.E.D.
Lemma 2. A necessary and sufficient condition for $G^* = 0$ is $H^* = c_w J_x - c_z J_w = 0$.

Proof. Take the partial derivatives of the consumption first-order condition (10) (the Uzawa case) w.r.t. $W$ and w.r.t. $x$. Multiply the former by $c_x$ and the latter by $c_w$, then subtract to get

$$J_{ww} G^x + \frac{d\rho}{dc} H^x = 0.$$  

Noting that $J_{ww} \neq 0$ and $d\rho/dc \neq 0$ for the Uzawa preference, the proposition is proved. Q.E.D.

Lemma 3. For the Uzawa (and for the time-additive) preference, $H^x$ is in general not identically zero.

Proof. In fact, there is no reason why $H^x$ should identically vanish, but let us demonstrate a concrete counter-example. The argument goes as follows. If for the special case $\rho(c) = \text{constant}$ (time-additive preferences), one could demonstrate that $H^x$ is not identically zero, then, when $\rho(\cdot)$ is perturbed continuously from being a constant to being a non-constant function of $c$, $H^x$ would still be non-zero for at least a small enough perturbation (and most likely for a large too). We will use an example due to Cox, Ingersoll and Ross (1978), in which $\rho(\cdot) = \text{constant}$, $u(c) = \log c$ and there is only one state variable $x$ (that is proportional to the interest rate). In this case Cox, Ingersoll and Ross show that the optimal consumption rule is linear in wealth $W$ but independent of the state variables $x$, and that the indirect utility for wealth $J$ depends non-trivially on the state variable. Formally, this implies $c_w \neq 0, c_x = 0, J_x \neq 0$, hence $H^x \neq 0$. Q.E.D.

To recapitulate, it was shown in Lemma 3 that $H^x$ is in general not identically zero, whence by Lemma 2 so is $G^x$. Using Lemma 1 now completes the proof of Theorem 2. Note that the validity of the CCAPM for the time-additive case results from $d\rho/dc = 0$ which implies $G^x = 0$ in (21). Q.E.D.

The following is an intuitive explanation for the incollapsability result. It is well known [e.g., Hansen, Richard and Singleton (1982) and references therein] that a CAPM risk premium relation that involves marginal utility of consumption can be written for any preference structure. Current consumption is a 'sufficient statistic' for marginal utility for the time-additive preference structure but not for non-separable preferences. Hence, a (current) consumption-beta CCAPM is valid in the former case but not in the latter.
5. When consumption histories are relevant

In this section we argue that when the optimal controls do depend upon consumption history, the multi-beta ICAPM holds, while again it cannot be collapsed to the single consumption-beta CAPM.

As noted earlier, if the preference order is not associated with a Koopmans aggregator, or even when it is, but the aggregator is not linear, then the optimal consumption–investment strategies are, in general, not Markov and do depend on consumption history. To avoid technical difficulties associated with infinite dimensions, let us focus on cases where a sufficient statistic for consumer $j$ consumption history up to time $t$ can be represented as a vector $y^k(t)$ of a finite and constant dimension.\footnote{This will suffice to make our point. Functional analytic techniques would be necessary to deal with infinite dimensional state processes.} We shall call $y^k(t)$ the personalized consumption–state vector. An example is Ryder and Heal’s (1973) analysis of optimal growth under certainty using a utility functional defined by

$$U = \int_0^\infty e^{-at}u[c(t), y(t)] \, dt,$$

where

$$y(t) = \beta e^{-\beta t} \int_{-\infty}^t e^{\beta \tau} c(\tau) \, d\tau$$

is the consumption–state variable, a scalar. To solve the individual consumption–investment problem the personalized consumption–state vector must be added to the environmental state vector $\mathbf{X}$ in order to make the system Markov. Let $Y'$ be the vector $(y^1, y^2, \ldots, y^K)$ of consumption–state vectors for all consumers. Let $Y$ be the vector one gets from $Y'$ after removing all redundancies, i.e., components that are real functions of other components, and let $N$ be the dimension of $Y$.

It is straightforward to see that the ICAPM holds in this set-up with an extended state vector $(\mathbf{X}, Y)$ of dimension $S + N$; separation will be into $S + 2 + N$ funds (as opposed to $S + 2$ for the linear Koopmans aggregator), and the risk premia relations will formally look the same.\footnote{If $N$ is small or of the order of $S$, then the consumption–state variables play a role similar to that of the state variables and the ICAPM implications to assets pricing will not be different from those in an economy without consumption–state variables. If $N$ is large – of the order of the number of consumers – then the differential demands for assets by consumers would be small and would tend to wash out, so that the macro effects on asset pricing would again be unchanged.} On the other hand, personalized consumption–state variables introduce state dependence into the utility functionals and, as Breeden (1979) notes, the CCAPM does not hold then.
6. Summary

We have shown that the assumption of a separable time-additive preference structure is crucial for the derivation of the single consumption-beta CAPM. When more general preference structures are used, like those non-time-additive and state-independent preferences that are defined by the linear Koopmans aggregator, then the consumption CAPM does not generally hold. It was also shown that although Merton's multi-beta CAPM and related models have been derived assuming a time-additive preference structure and optimality of the consumption policy, these assumptions are not necessary. In fact, only optimality of the investment policy is necessary. The multi-beta CAPM holds, then, under a wide class of intertemporally dependent preference structures.

Appendix: Proof of Proposition 2

Rewrite (18):

\[ 0 = E_t[(\Delta a - r\Delta t)(J_w + \Delta J_w)], \]

or

\[ E_t[\Delta a \Delta J_w + o(\Delta t)] = -J_w E_t[\Delta a - r\Delta t], \]

where by Itô's lemma

\[ \Delta J_w = J_{ww}\Delta W + \Delta \alpha' J_{wx} + O(\Delta t), \]

and where by Merton's (1971) wealth dynamics

\[ \Delta W = W\Delta z'\gamma_w + O(\Delta t). \]

Substitute for \( \Delta \alpha \) and \( \Delta a \) from (13) and (14), and for \( \Delta J_w \) and \( \Delta W \) in (A.1), use (16), divide by \( \Delta t \), let \( \Delta t \to 0 \) and (17) follows.

References


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