Sources of Entropy in Dynamic Representative Agent Models

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Understanding dynamic models

- Are dynamic features important for the disaster story?
- More generally, how does one discern critical features of modern dynamic models?
  - The size of equity premium is no longer an overidentifying restriction
### Market-adjusted excess returns

<table>
<thead>
<tr>
<th>Asset Class</th>
<th>Value</th>
<th>Momentum</th>
</tr>
</thead>
<tbody>
<tr>
<td>US stocks</td>
<td>4.3%</td>
<td>6.1%</td>
</tr>
<tr>
<td>UK stocks</td>
<td>2.7%</td>
<td>10.8%</td>
</tr>
<tr>
<td>Euro stocks</td>
<td>4.2%</td>
<td>10.9%</td>
</tr>
<tr>
<td>Jpn stocks</td>
<td>11.3%</td>
<td>4.2%</td>
</tr>
<tr>
<td>FX</td>
<td>4.9%</td>
<td>2.7%</td>
</tr>
<tr>
<td>Bonds</td>
<td>0.3%</td>
<td>0.3%</td>
</tr>
<tr>
<td>Commodities</td>
<td>6.4%</td>
<td>8.8%</td>
</tr>
</tbody>
</table>

- Annualized alphas relative to the MSCI world equity index in excess of the US Treasury Bill rate
- Source: Asness, Moskowitz, and Pedersen (2009)
Understanding dynamic models

- Are dynamic features important for the disaster story?
- More generally, how does one discern critical features of modern dynamic models?
  - The size of equity premium is no longer an overidentifying restriction
Are dynamic features important for the disaster story?

More generally, how does one discern critical features of modern dynamic models?

- The size of equity premium is no longer an overidentifying restriction
- The models are built up from different state variables
- Which pieces are most important quantitatively?

We start by thinking about how risk is priced in these models

- What is the source of the evident high entropy in the data?
- We use ACE to characterize this
AJ bound, non-i.i.d. case

- AJ bound

\[ L(m) \geq E(\log r^j - \log r^1) + L(q^1) \]

non-i.i.d. piece
AJ bound, non-i.i.d. case

- **AJ bound**
  \[
  L(m) \geq E(\log r^j - \log r^1) + L(q^1)
  \]
  non-i.i.d. piece

- **Conditional entropy:**
  \[
  L_t(m_{t+1}) = \log E_t m_{t+1} - E_t \log m_{t+1}
  \]

- **Average conditional entropy (ACE)**
  \[
  L(m) = EL_t(m_{t+1}) + L(E_t(m_{t+1})) = EL_t(m_{t+1}) + L(q^1)
  \]
  \[
  EL_t(m_{t+1}) \geq E(\log r^j - \log r^1)
  \]
Advantages of average conditional entropy (ACE)

- Transparent lower bound: expected excess return (in logs)

- Alternatively, ACE measures the highest risk premium in the economy

- Conditional entropy is easy to compute; to compute ACE evaluate conditional entropy at steady-state values

- ACE is comparable across different models with different state variables, preferences, etc.
Key models

- External habit

- Recursive preferences

- Heterogeneous preferences

- ....
A change in notation

- $\alpha$ is replaced by $1 - \alpha$

Example: CRRA preferences; $RA = 5$

- Old $\alpha = 5$

- New $\alpha = -4$
External habit

- Equations (Abel/Campbell-Cochrane/Chan-Kogan/Heaton)

\[ U_t = \sum_{j=0}^{\infty} \beta^j u(c_{t+j}, x_{t+j}), \]

\[ u(c_t, x_t) = \frac{(f(c_t, x_t)\alpha - 1)}{\alpha}. \]

- Habit is a function of past consumption: \( x_t = h(c_t^{t-1}) \), e.g., Abel: \( x_t = c_{t-1} \).

- Dependence on habit
  - Abel: \( f(c_t, x_t) = c_t / x_t \)
  - Campbell-Cochrane: \( f(c_t, x_t) = c_t - x_t \)

- Pricing kernel:

\[ m_{t+1} = \beta \frac{u_c(c_{t+1}, x_{t+1})}{u_c(c_t, x_t)} = \beta \left( \frac{f(c_{t+1}, x_{t+1})}{f(c_t, x_t)} \right)^{\alpha-1} \left( \frac{f_c(c_{t+1}, x_{t+1})}{f_c(c_t, x_t)} \right) \]
Example 1: 

- Preferences: \( f(c_t, x_t) = \frac{c_t}{x_t} \)
- Chan and Kogan have extended the habit formulation:

\[
\log x_{t+1} = (1 - \phi) \sum_{i=0}^{\infty} \phi^i \log c_{t-i} = \phi \log x_t + (1 - \phi) \log c_t
\]

- Relative (log) consumption

\[
\log s_t \equiv \log \left( \frac{c_t}{x_t} \right) = \phi \log s_{t-1} + \log g_t
\]

- Pricing kernel:

\[
\log m_{t+1} = \log \beta + (\alpha - 1) \log g_{t+1} - \alpha \log \left( \frac{x_{t+1}}{x_t} \right) = \log \beta + (\alpha - 1) \log g_{t+1} - \alpha (1 - \phi) \log s_t
\]
ACE: Abel-Chan-Kogan

- Pricing kernel:
  \[
  \log m_{t+1} = \log \beta + (\alpha - 1) \log g_{t+1} - \alpha (1 - \phi) \log s_t
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- Conditional entropy: 
  \[
  L_t(m_{t+1}) = \log E_t e^{\log m_{t+1}} - E_t \log m_{t+1}
  \]
  \[
  \log E_t e^{\log m_{t+1}} = \log \beta + k(\alpha - 1; \log g) - \alpha (1 - \phi) \log s_t (= - \log r^1)
  \]
  \[
  E_t \log m_{t+1} = \log \beta + (\alpha - 1) \kappa_1(\log g) - \alpha (1 - \phi) \log s_t
  \]
  \[
  L_t(m_{t+1}) = k(\alpha - 1; \log g) - (\alpha - 1) \kappa_1(\log g)
  \]

- ACE: 
  \[
  EL_t(m_{t+1}) = k(\alpha - 1; \log g) - (\alpha - 1) \kappa_1(\log g)
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- **Pricing kernel:**
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  \[
  \log E_t e^{\log m_{t+1}} = \log \beta + k(\alpha - 1; \log g) - \alpha (1 - \phi) \log s_t \left(=- \log r^1 \right)
  \]

  \[
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- **ACE:**
  \[
  EL_t(m_{t+1}) = k(\alpha - 1; \log g) - (\alpha - 1) \kappa_1 (\log g)
  \]

- It is exactly the same as in the CRRA case
Example 2: Campbell and Cochrane (1999)

- Preferences: $f(c_t, x_t) = c_t - x_t$
- Campbell and Cochrane specify (log) surplus consumption ratio directly:

\[
\log s_t = \log \left( \frac{c_t - x_t}{c_t} \right) \\
\log s_t = \phi (\log s_{t-1} - \log \bar{s}) + \lambda (\log s_{t-1}) (\log g_t - \kappa_1 (\log g)).
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- Compare to relative (log) consumption in Chan and Kogan

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- Pricing kernel:

\[
\log m_{t+1} = \log \beta + (\alpha - 1)\log g_{t+1} + (\alpha - 1)\log \left( s_{t+1}/s_t \right) \\
= \log \beta - (\alpha - 1)\lambda(\log s_t)\kappa_1(\log g) \\
+ (\alpha - 1)(1 + \lambda(\log s_t))\log g_{t+1} \\
+ (\alpha - 1)(\phi - 1)(\log s_t - \log \bar{s})
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Example 2: Campbell and Cochrane (1999)

Preferences: \( f(c_t, x_t) = c_t - x_t \)

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\log m_{t+1} = \log \beta + (\alpha - 1) \log g_{t+1} + (\alpha - 1) \log (s_{t+1}/s_t) \\
= \log \beta - (\alpha - 1)\lambda (\log s_t) \kappa_1 (\log g) \\
+ (\alpha - 1)(1 + \lambda (\log s_t)) \log g_{t+1} \\
+ (\alpha - 1)(\phi - 1)(\log s_t - \log \bar{s})
\]

Conditional entropy:

\[
L_t(m_{t+1}) = k((\alpha - 1)(1 + \lambda (\log s_t)); \log g) \\
- (\alpha - 1)(1 + \lambda (\log s_t)) \kappa_1 (\log g)
\]
Additional assumptions

- To compute ACE, we have to specify $\lambda$ and $\log g$
- Conditional volatility of the consumption surplus ratio

$$
\lambda(\log s_t) = \frac{1}{\sigma} \sqrt{\frac{1 - \phi - b/(1 - \alpha)}{1 - \alpha}} \sqrt{1 - 2(\log s_t - \log \bar{s})} - 1
$$

- In discrete time, there is an upper bound on $\log s_t$ to ensure positivity of $\lambda$
- In continuous time, this bound never binds so we will ignore it
- In Campbell and Cochrane, $b = 0$ to ensure a constant $\log r^1$
- Consumption growth is i.i.d.
  - Case 1. $\log g_{t+1} = w_{t+1}$, $w_{t+1} \sim \mathcal{N}(\mu, \sigma^2)$
  - Case 2. $\log g_{t+1} = w_{t+1} - z_{t+1}$, $z_{t+1} \mid j \sim \text{Gamma}(j, \theta^{-1})$, $\bar{j} = \omega$
Conditional entropy:

\[ L_t(m_{t+1}) = ((\alpha - 1)(\phi - 1) - b)/2 + b(\log s_t - \log \bar{s}) \]

ACE: \( EL_t(m_{t+1}) = ((\alpha - 1)(\phi - 1) - b)/2 \)
Conditional entropy:

\[ L_t(m_{t+1}) = \frac{((\alpha - 1)(\phi - 1) - b)}{2} + b(\log s_t - \log \bar{s}) \]

ACE: \( EL_t(m_{t+1}) = \frac{((\alpha - 1)(\phi - 1) - b)}{2} \)

All authors use \( \alpha = -1 \)

ACE for different calibrations (quarterly)
ACE: Campbell and Cochrane, Case 1

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ACE for different calibrations (quarterly)
- Campbell and Cochrane (1999): \( \phi = 0.97, b = 0 \);
  \[ EL_t(m_{t+1}) = 0.0300 \text{ (0.120 annual)} \]
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- Conditional entropy:
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  - Campbell and Cochrane (1999): \( φ = 0.97, b = 0; \)
    \( EL_t(m_{t+1}) = 0.0300 \) (0.120 annual)
  - Wachter (2006): \( φ = 0.97, b = 0.011; \)
    \( EL_t(m_{t+1}) = 0.0245 \) (0.098 annual)
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  - Wachter (2006): \( \phi = 0.97, b = 0.011; \)
    \( EL_t(m_{t+1}) = 0.0245 \) (0.098 annual)
  - Verdelhan (2009): \( \phi = 0.99, b = -0.011; \)
    \( EL_t(m_{t+1}) = 0.0155 \) (0.062 annual)
ACE: Campbell and Cochrane, Case 2

- Conditional entropy:

\[ L_t(m_{t+1}) = (\alpha - 1)(1 + \lambda(\log s_t))\omega \theta \]
\[ + ((1 + (\alpha - 1)(1 + \lambda(\log s_t))\theta)^{-1} - 1)\omega \]
\[ + ((\alpha - 1)(\phi - 1) - b)/2 + b(\log s_t - \log \bar{s}) \]

- ACE: use log-linearization around \( \log \bar{s} \)

\[ EL_t(m_{t+1}) = \omega d^2/(1 + d) + ((\alpha - 1)(\phi - 1) - b)/2 \]
\[ d = \frac{\theta}{\sigma} \sqrt{(\alpha - 1)(\phi - 1) - b} \]
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- Conditional entropy:

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\]

\[
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\]

- Calibration as above + vol of \( \log g \) + jump parameters:

- \( \sigma^2 = (0.035)^2/4 - \omega \theta^2 \)
- BNSU: \( \omega = 0.01/4, \theta = 0.15 \)
- BCM: \( \omega = 1.3864/4, \theta = 0.0229 \)
## ACE: Campbell and Cochrane, Case 2

<table>
<thead>
<tr>
<th>Calibration</th>
<th>ACE</th>
<th>ACE (case 1)</th>
<th>ACE jumps</th>
</tr>
</thead>
<tbody>
<tr>
<td>CC + BNSU</td>
<td>0.0341</td>
<td>0.0300</td>
<td>0.0041</td>
</tr>
<tr>
<td>W + BNSU</td>
<td>0.0281</td>
<td>0.0245</td>
<td>0.0036</td>
</tr>
<tr>
<td>V + BNSU</td>
<td>0.0181</td>
<td>0.0155</td>
<td>0.0026</td>
</tr>
<tr>
<td>CC + BCM</td>
<td>0.0883</td>
<td>0.0300</td>
<td>0.0583</td>
</tr>
<tr>
<td>W + BCM</td>
<td>0.0737</td>
<td>0.0245</td>
<td>0.0492</td>
</tr>
<tr>
<td>V + BCM</td>
<td>0.0487</td>
<td>0.0155</td>
<td>0.0332</td>
</tr>
</tbody>
</table>
Time dependence via external habit

- No time-dependence in consumption growth

- Nevertheless: habit with varying volatility may have a substantial impact on the entropy of the pricing kernel

- Could be relevant for option prices (Du, 2008)
Recursive preferences: traditional version

Equations (Kreps-Porteus/Epstein-Zin/Weil)

\[ U_t = \left[ (1 - \beta) c_t^\rho + \beta \mu_t(U_{t+1})^\rho \right]^{1/\rho} \]

\[ \mu_t(U_{t+1}) = \left( E_t U_{t+1}^\alpha \right)^{1/\alpha} \]

\[ IES = 1/(1 - \rho) \]

\[ CRRA = 1 - \alpha \]

\[ \alpha = \rho \Rightarrow \text{additive preferences} \]
Recursive preferences: pricing kernel

- Scale problem by $c_t \ (u_t = U_t/c_t, \ g_{t+1} = c_{t+1}/c_t)$

  $$u_t = [(1 - \beta) + \beta \mu_t (g_{t+1} u_{t+1})^\rho]^{1/\rho}$$

- Pricing kernel (mrs)

  $$m_{t+1} = \beta \left(\frac{c_{t+1}}{c_t}\right)^{\rho-1} \left(\frac{U_{t+1}}{\mu_t (U_{t+1})}\right)^{\alpha-\rho}$$

  $$= \beta g_{t+1}^{\rho-1} \left(\frac{g_{t+1} u_{t+1}}{\mu_t (g_{t+1} u_{t+1})}\right)^{\alpha-\rho}$$
Loglinear approximation

\[ \log u_t = \rho^{-1} \log \left[ (1 - \beta) + \beta \mu_t (g_{t+1} u_{t+1})^\rho \right] \]

\[ = \rho^{-1} \log \left[ (1 - \beta) + \beta e^{\rho \log \mu_t (g_{t+1} u_{t+1})} \right] \]

\[ \approx b_0 + b_1 \log \mu_t (g_{t+1} u_{t+1}) \]

Exact if \( \rho = 0 \): \( b_0 = 0, b_1 = \beta \)

Solve by guess and verify
Example 1: Bansal-Yaron

- Consumption growth

\[
\log g_t = g + \gamma(L)\nu_{t-1}^{1/2}w_{1t} \\
\nu_t = \nu + \nu(L)w_{2t} \\
(w_{1t}, w_{2t}) \sim \text{NID}(0, I)
\]

- Guess value function

\[
\log u_t = u + \omega_g(L)\nu_{t-1}^{1/2}w_{1t} + \omega_v(L)w_{2t}
\]

- Solution includes

\[
\omega_{g0} + \gamma_0 = \gamma(b_1) \equiv \sum_{i=0}^{\infty} b_1^i \gamma_i \\
\omega_{v0} = b_1 (\alpha/2) \gamma(b_1)^2 \nu(b_1)
\]
ACE: Bansal-Yaron

- **Pricing kernel**

\[
\log m_{t+1} = \log \beta + (\rho - 1)g - (\alpha - \rho)(\alpha/2)\omega_v^2
\]
\[
+ (\rho - 1)[\gamma(L)/L] + v_{t-1}^{1/2}w_1t - (\alpha - \rho)(\alpha/2)\gamma(b_1)^2v_t
\]
\[
+ [(\rho - 1)\gamma_0 + (\alpha - \rho)\gamma(b_1)]v_{t}^{1/2}w_{1t+1}
\]
\[
+ (\alpha - \rho)\omega_v^2w_{2t+1}
\]

- **Conditional entropy**

\[
L_t(m_{t+1}) = [(\rho - 1)\gamma_0 + (\alpha - \rho)\gamma(b_1)]^2v_t/2 + (\alpha - \rho)^2\omega_v^2/2
\]

- **ACE (Bansal, Kiku, Yaron, 2007; monthly)**

\[
0.0218 = 0.0065 + 0.0153
\]
\[
0.0026 = 0.0026 + 0.0000 \text{ if } \rho = \alpha
\]
Example 2: Wachter

- Consumption growth

\[ \log g_t = g + \sigma w_{1t} + z_t \]
\[ \lambda_t = (1 - \varphi)\lambda + \varphi \lambda_{t-1} + \sigma_\lambda w_{2t} \]
\[ (w_{1t}, w_{2t}) \sim \text{NID}(0, I) \]
\[ z_t|j \sim \mathcal{N}(j\theta, j\delta^2) \]
\[ j \geq 0 \text{ has jump intensity } \lambda_{t-1} \]

- Guess value function

\[ \log u_t = u + \omega_\lambda \lambda_t \]

- Solution includes

\[ \omega_\lambda = (1 - b_1 \varphi)^{-1} b_1 \left[ e^{\alpha \theta + (\alpha \delta)^2 / 2} - 1 \right] / \alpha \]
ACE: Wachter

- Pricing kernel

\[
\log m_{t+1} = \log \beta + (\rho - 1)x - (\alpha - \rho)(\alpha/2)[\sigma^2 + (\omega_\lambda \sigma_\lambda)^2]
\]
\[
- \lambda_t(e^{\alpha \theta + (\alpha \delta)^2/2 - 1})/\alpha
\]
\[
+ (\alpha - 1)(\sigma w_{1t+1} + z_{t+1}) + (\alpha - \rho)(\omega_\lambda \sigma_\lambda)w_{2t+1}
\]

- Conditional entropy (monthly)

\[
L_t(m_{t+1}) = (\alpha - 1)^2 \sigma^2/2 + (\alpha - \rho)^2(\omega_\lambda \sigma_\lambda)^2/2
\]
\[
+ \lambda_t \left\{ e^{(\alpha-1)\theta + (\alpha-1)^2 \delta^2/2 - 1} - (\alpha - 1)\theta \right\}
\]

- ACE (monthly)

\[
0.0100 = 0.0001 + 0.0087 + 0.0012
\]
\[
0.0013 = 0.0001 + 0.0000 + 0.0012 \text{ if } \rho = \alpha
\]
Time dependence via recursive preferences

- Little time-dependence in pricing kernel

- Nevertheless: interaction of (modest) dynamics in consumption growth and recursive preferences can have a substantial impact on the entropy of the pricing kernel

- Not clear it’s relevant to option prices, but it’s a route to magnify the impact of disasters on excess returns