An Equilibrium Guide to Designing Affine Pricing Models

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Abstract

The paper examines equilibrium models based on Epstein-Zin preferences in a framework in which exogenous state-variables follow affine jump diffusion processes. A main insight is that the equilibrium asset prices can be computed using a standard machinery of affine asset pricing theory by imposing parametric restrictions on market prices of risk, determined inside the model by preference and model parameters. An appealing characteristic of the general equilibrium setup is that the state variables have an intuitive and testable interpretation as driving the consumption and dividend dynamics. We present a detailed example where large shocks (jumps) in consumption volatility translate into negative jumps in equilibrium prices of the assets as agents demand a higher premium to compensate for higher risks. This endogenous ”leverage effect,” which is purely an equilibrium outcome in the economy, leads to significant premiums for out-of-the-money put options. Our model is thus able to produce an equilibrium ”volatility smirk” which realistically mimics that observed for index options.

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1 Introduction

A cornerstone of modern finance, no-arbitrage models are routinely applied to price basic securities such as stocks and bonds as well as derivative assets. No-arbitrage models place very few restrictions on the behavior of asset prices. Indeed, no-arbitrage models say (almost) nothing about the relationship between the assumed, objective probability law of the ”state-variables” in the model, and the arbitrage-induced ”risk neutral” measure used for pricing. This is convenient from the point of view of practitioners who wish to maintain an infinite number of degrees of freedom in adjusting no-arbitrage models to observed asset prices. It is inconvenient and non-informative to academics who wish to design asset pricing models to study the dynamics of financial markets to learn about such things as market efficiency, investors’ risk aversion, and the link between the macro economy and financial market prices.

In this paper we describe a framework for designing affine pricing models based on general equilibrium. We show that in designing a consumption based equilibrium model where the representative agent is endowed with Epstein-Zin preferences over intermediate consumption and wealth, we recover approximately standard affine dynamics for stocks and bonds. This is done under the assumption that the underlying state dynamics is an \( n \) dimensional affine jump diffusion process.

The main message of the paper is that we can proceed to price stocks, bonds and derivatives by using the standard machinery of affine no-arbitrage models, under the conditions that 1) we constrain the market prices of risk to be explicit functions of the preference parameters, and 2) that we choose state-variables that relate specific payoff streams to aggregate consumption. We provide explicit expressions for the market-risk-prices which depend on exogenous dynamics as well as preferences. We verify that risk prices of immediate Weiner consumption shocks have the standard linear form known from the no-arbitrage literature. The market risk prices of other sources of uncertainty are non-zero only if preferences do not degenerate into standard power utility of consumption (CRRA) case. Thus, the Epstein-Zin preference structure offers an important extension of the standard CRRA model in explaining non-zero risk premiums for factors which are not directly related to aggregate consumption.\(^1\)

While standard no-arbitrage models offer no guide to specifying a link between the objective and risk neutral measures for discontinuous jumps, our framework provides an explicit formula for connecting the two measures. Specifically, we show that risk neutral Poisson arrival intensities as well as the risk-neutral jump size distributions are obtained through simple scalar adjustments of the objective arrival intensities and jump size distributions. In an example application, we show that both the

\(^1\)This point has been emphasized elsewhere in context of general-equilibrium Epstein-Zin models, see e.g. Bansal & Yaron (2004) and Tauchen (2005).
jump arrival intensity and jump sizes are larger under the risk-neutral measure. The differences increase in the level of risk aversion for the representative agent.

In illustrating our approach, we present a detailed example model where aggregate consumption and dividends exhibit stochastic volatility. The volatility process, which we assume affects both dividend and consumption growth, follows a mean reverting process where shocks may be continuous (Wiener process), discontinuous (compound Poisson), or both. Our model generates a negative correlation between shocks to the volatility process and the equilibrium stock prices. This correlation approaches negative one when the jumps dominate the variation in the volatility, and is different under the objective and risk neutral distributions. No-arbitrage models to date have assumed that the negative volatility/stock price correlation is exogenously determined and identical under the two measures.

We study the equilibrium impact of volatility shocks on theoretical options prices under our model. Theoretical options prices are computed through the fourier inversion technique of Lewis (2001), adapted to our setting with random, equilibrium-determined interest rates. The model produces several interesting stylized facts about options. The implied volatilities computed with our model tend to mimic those observed empirically in that the implied volatility is U shaped, and with significantly higher prices for out-of-the-money puts. Low levels of risk aversion, conversely, produces a flatter volatility smile. This effect is not present in a model with CRRA utility, and this model tends to produce a reversed pattern in the implied volatility with relatively higher prices for ITM puts, than OTM puts. The large impact of volatility shocks on OTM put options is related to two facts. First, the equilibrium stock price process is heavily influenced by the the possibility of sudden increases (jumps) in economic uncertainty even under relatively modest levels of risk aversion. This generates large (negative) price jumps in the physical probability law of the stock price. Second the adjustment of the physical probability law into the risk-neutral law which makes convenient the computation of derivative prices, implies an increase in both volatility jump arrival intensity, as well as the average sizes of the jumps. These risk adjustments are only present under the full Epstein-Zin preference model, and no such adjustment takes place for CRRA utility as shocks in the volatility are not explicitly correlated with the immediate innovations into the consumption growth.

Our paper is connected to the extant literature in several ways. Bansal & Yaron (2004) introduce the idea of long run risks and show that persistence in state-variables coupled with an Epstein-Zin based equilibrium pricing kernel magnifies risk premiums relative to i.i.d. economies. Aase (2002) studies time-additive equilibrium with general utility under jump processes. Shaliastovich and Tauchen (2006) study equilibrium under subordinated Levy processes. Eraker (2006) examines a similar modelling environment in discrete time and studies example models for pricing stocks and bonds. Our paper generalizes most of the previous works, which are based on conditionally normal processes, to general affine processes. The advantage of continuous time ap-
approach in the current paper is that analytical tractability allows specific formulae for market prices of risks, risk neutral dynamics, etc. to be developed.

A number of papers have examined the implications of stochastic volatility and jump on options prices. Early examples include Hull and White (1987), Heston (1993), Bates (1996,2000), Bakshi, Cao and Chen (1997), and Duffie, Pan and Singleton (2000). Madan, Carr and Chang (1998) and Carr, Geman, Madan and Yor (2003) examine stochastic volatility models driven by subordinated Levy processes. Option pricing under recursive preferences has been studied by Liu, Pan and Wang (2005), Garcia, Luger and Renault (2003) and Benzoni, Colin-Dufresne and Goldstein (2005). Liu, Pan and Wang (2005) argue that Epstein-Zin preferences cannot explain the high valuations of OTM put options in their i.i.d economy and argue that the results are similar to results under CRRA utility. They conclude that a model in which investors do not know the true probability of a crash and exhibit uncertainty aversion is needed to explain high OTM put options. Benzoni, Colin-Dufresne and Goldstein (2005) show that Epstein-Zin preferences generate high valuations for OTM puts if the economy is not i.i.d, and expected consumption growth exhibits persistence along the lines of Bansal and Yaron (2004). Unlike these paper, our example application focuses on stochastic volatility as the driving force behind fat tailed return distributions and option premiums.

The remainder of this paper is organized as follows. Section 2 discusses the specification of exogenous state-variables, the Epstein-Zin utility specification, and derives the corresponding pricing kernel. In section 3 we discuss the pricing of assets with various payoffs, including dividend paying stocks, and equity options. In section 4 we present the example model and derive the equilibrium pricing kernel, stock price process and we present several stylized facts about equity options prices computed with this model. Section 5 concludes.

2 Model

We start with a discrete-time formulation of the real endowment economy where the investors preferences over the uncertain consumption stream \( C_t \) can be described by a recursive utility function of Epstein & Zin (1989) and Weil (1989):

\[
U_t = \left[ (1 - \delta)C_t^{1-\gamma} + \delta(E_tU_{t+1}^{1-\gamma})^\frac{\theta}{\gamma} \right]^{\frac{\theta}{\gamma}}.
\]  

(1)

The representative agent’s preferences are thus characterized by a subjective discount factor \( \delta \), the intertemporal elasticity of substitution (IES) \( \psi \) and the local risk aver-
sion coefficient $\gamma$. $E_t$ denotes the standard expectation operator conditional on the information available to the agent in period $t$, and for notational convenience we set
\[ \theta = \frac{1 - \gamma}{1 - \psi}. \]

Notably, when the risk aversion coefficient is equal to the reciprocal of the IES, (equivalently, $\theta = 1$), the preferences collapse to the familiar power utility case with risk aversion parameter $\gamma = \frac{1}{\psi}$. The novel and appealing characteristic of the generalized preferences is that they break the link between $\gamma$ and $\psi$ and allow to capture the difference in agent’s attitudes toward risk over time and across the states of the world.

In discrete time, the Epstein-Zin (EZ) preference structure leads to the following Euler equation
\[ E_t \left[ \delta^\theta \left( \frac{C_{t+1}}{C_t} \right)^{-\frac{\theta}{\psi}} R_{a,t+1}^{-(1-\theta)} R_{t,t+1} \right] = 1, \]
where $R_{a,t}$ is the return on the aggregate wealth portfolio which pays consumption as its dividends and $R_{t,t}$ is the return on an arbitrary asset available to the investor. For analytical tractability, we choose the discrete time Euler equation in (2) as a base of our analysis despite the existence of a continuous time analogue preference structure studied in Duffie and Epstein (1992a,b), Schroder and Skiadas (1999), among others.

Notice that the form of (1), or its continuous equivalent, makes the pricing kernel non-affine if the log-return on aggregate wealth $\ln R_{a,t}$ is non-linear. To maintain analytical tractability, therefore, we follow Campbell & Shiller (1988), Campbell (1993) and Bansal & Yaron (2004) among others to linearize the model\(^2\). Specifically, we Taylor expand the log return
\[ r_t = \ln \frac{P_{t+1} + D_{t+1}}{P_t} = \ln(e^{\ln \frac{P_{t+1}}{P_t}} + 1) - \ln \frac{P_t}{D_t} + \ln \frac{D_{t+1}}{D_t} \]
around the mean log price divided ratio to obtain
\[ r_{t+1} \approx k_0 + k_1 v_{t+1} - v_t + g_{d,t+1}, \]
for $g_{d,t+1} = \ln \frac{D_{t+1}}{D_t}$ and $v_t = \ln P_t - \ln D_t$. Further details are provided in appendix A.

The constants $k_0$ and $k_1$ depend on the mean log valuation ratio $Ev$:
\[ k_1 = \frac{e^{Ev}}{1 + e^{Ev}}, \]
\[ k_0 = -\log \left[(1 - k_1)^{1-k_1} k_1^k\right]. \]
\(^2\)Other approximations are possible - see for example Hansen, Heaton & Li (2004), Benzonii, Collin-Dufresne & Goldstein (2005).
In equilibrium, the model-implied mean price-dividend ratio $Ev$ should be consistent with the linearization coefficients $k_0$ and $k_1$. We show that this imposes a non-linear constraint on $k_1$, which can be solved recursively given the parameters of the model. This approach is similar to Bansal, Kiku & Yaron (2006).

Finally, to obtain the analytical solution to the model we conjecture that the valuation ratios are affine in the state variables:

$$v_t = A + B'X_t.$$  

(6)

Here, $X_t$ is a set of common state-variables which affect the dynamics of consumption growth as well as dividends of individual assets. In the next sections we model $X$ as a multi-variate affine jump diffusion and solve for the price-consumption loadings $A$ and $B$, which enables us to characterize the discount factor and the dynamics of the state variables under the risk-neutral measure.

2.1 State Variables

We follow here the presentation of Duffie, Pan & Singleton (2000) and assume that there is a set of $n$ state variables in the economy which follow the affine jump diffusion process. Specifically, we fix the probability space $\{\Omega, \mathcal{F}, \mathcal{P}\}$ and the information filtration $\mathcal{F}_t$, and suppose that $X_t$ is a Markov process in some state space $\mathcal{D} \subseteq \mathbb{R}^n$ with a stochastic differential equation representation

$$dX_t = \mu(X_t)dt + \Sigma(X_t)dW_t + \xi_t \cdot dN_t.$$  

(7)

$W_t$ is an $\mathcal{F}_t$ adapted Brownian motion in $\mathbb{R}^n$. The term $\xi_t \cdot dN_t$ (element-by-element multiplication) captures conditionally independent jumps arriving with intensity $l(X_t)$ and jump size distribution $\xi_t$ on $\mathcal{D}$. Intuitively, conditional on the path of $X$, the jump arrivals are the jump times of the Poisson distribution with possibly time-varying intensity $l(X_t)$. We further assume that jump sizes $\xi$ are i.i.d. in time and cross-sectionally; their distribution is specified through the "jump transform" (individual generating function) $\varphi : \mathbb{C} \rightarrow \mathbb{C}$,

$$Ee^{u\xi} = \varphi(u).$$

With a slight abuse of notations, we will sometimes evaluate $\varphi(\cdot)$ at a vector argument, which we take to mean a stack of element-by-element application of the jump transform. We assume that the moment-generating function of $\xi$ exists such that $\varphi$ is well defined for both complex and real arguments on some region of the complex plane. This is a somewhat restrictive assumption which rules out certain heavy tailed distributions including power-law ones.
We further impose an affine structure on the drift, diffusion and intensity functions:
\[
\mu(X_t) = \mathcal{M} + \mathcal{K}X_t,
\]
\[
\Sigma(X_t)\Sigma(X_t)' = h + \sum_i H_i X_{t,i},
\]
\[
l(X_t) = l_0 + l_1 X_t,
\]
for \((\mathcal{M}, \mathcal{K}) \in \mathbb{R}^n \times \mathbb{R}^{n \times n}, (h, H) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times n \times n}, (l_0, l_1) \in \mathbb{R}^n \times \mathbb{R}^{n \times n} \). For \(X\) to be well defined, there are additional joint restrictions on the parameters of the model, which are addressed in Duffie & Kan (1996) and Duffie & Singleton (1999).

We assume that the log consumption and dividend growth rates are linear in the states:
\[
d \ln C_t = \delta_c' dX_t
\]
\[
= \delta_c' (\mathcal{M} + \mathcal{K}X_t) dt + \delta_c' \Sigma(X_t) dW_t + \delta_c' (\xi_t \cdot dN_t),
\]
\[
d \ln D_t = \delta_d' dX_t
\]
\[
= \delta_d' (\mathcal{M} + \mathcal{K}X_t) dt + \delta_d' \Sigma(X_t) dW_t + \delta_d' (\xi_t \cdot dN_t).
\]
We typically structure the state variables so that the consumption growth is the first factor, while the dividend growth rate is the last one, so \(\delta_c \) and \(\delta_d\) become selection vectors \((1, 0, 0, \ldots)\) and \((\ldots, 0, 0, 1)\), respectively. This model setup follows Eraker (2006).

### 2.2 Equilibrium

To derive a continuous-time generalized pricing kernel, consider a mutual fund which continuously re-invests the proceeds from the dividend asset. The log-price of this mutual fund follows
\[
d \ln S_{t,t} = d \ln R_i(t).
\]
The discrete-time log-return from the mutual fund equals the cumulative continuous return, \(\ln S_{t+\delta}/S_t = \int_t^{t+\delta} dR_u\). We therefore can rewrite equation (2) as
\[
E_t \exp \left[ \theta \ln \delta - \frac{\theta}{\psi} g_{t+1} - (1 - \theta) \ln \frac{S_{c,t+1}}{S_{c,t}} + \ln \frac{S_{t,t+1}}{S_{t,t}} \right] = 1. \tag{8}
\]
Substituting the consumption asset, it now follows that the increments to the log pricing kernel are

\[ d \ln M_t = \theta \ln \delta dt - \frac{\theta}{\psi} d \ln C_t + (1 - \theta) d \ln R_{a,t}. \]  

To obtain a continuous version of the Euler equation (8), note that \( M_t S_t \) is a martingale:

\[ M_t S_t = E_t(M_{t+1} S_{t+1}), \]

which implies

\[ E_t[d(M_t S_t)] = 0. \]  

Now we can use the Euler equation above to solve for the loadings of the wealth to consumption ratio. Indeed, define \( \ln y_t = \ln M_t + \ln S_{c,t} \). Using the expressions for \( M_t, S_{c,t} \) and the log-linearized consumption return we obtain

\[
\begin{align*}
\ln y_t &= \theta \ln \delta dt - \frac{\theta}{\psi} d \ln C_t + \theta r_c(t) \\
&= \theta \ln \delta dt - \frac{\theta}{\psi} d \ln C_t + \theta [\delta_c + k_1 B]' dX_t + \theta(k_0 - (1 - k_1)(A + B'X_t)) dt \\
&= \left[ \theta \ln \delta + \theta \left( (1 - \frac{1}{\psi}) \delta_c + k_1 B \right)' (M + \Sigma X_t) + \theta k_0 - \theta (1 - k_1)(A + B'X_t) \right] dt \\
&\quad + \theta \left( (1 - \frac{1}{\psi}) \delta_c + k_1 B \right)' \Sigma(X_t) dW_t + \theta \left( (1 - \frac{1}{\psi}) \delta_c + k_1 B \right)' (\xi_t \cdot dN_t) \\
&= \mu_y(X_t) + \chi' \Sigma(X_t) dW_t + \chi' (\xi_t \cdot dN_t),
\end{align*}
\]

where \( \chi = \theta \left( (1 - \frac{1}{\psi}) \delta_c + k_1 B \right) \).

As \( y_t \) is a martingale, we use Ito lemma to set its drift to 0:

\[
\begin{align*}
\theta \ln \delta + \chi' (M + \Sigma X_t) + \theta k_0 - \theta (1 - k_1)(A + B'X_t) \\
&\quad + \frac{1}{2} \chi' \Sigma(X_t) \Sigma(X_t)' \chi + (\theta(\chi) - 1)' l(X_t) = 0.
\end{align*}
\]

\(^3\)Note that the pricing kernel is exactly the same as in discrete time, as

\[
\ln M_{t+1} - \ln M_t = \int_t^{t+1} d \ln M_s = \theta \ln \delta - \frac{\theta}{\psi} (\ln C_{t+1} - \ln C_t) + (\theta - 1) \int_t^{t+1} r_a(dt) \\
= \theta \ln \delta - \frac{\theta}{\psi} (\ln C_{t+1} - \ln C_t) + (\theta - 1) (\ln S_{c,t+1} - \ln S_{c,t}).
\]
Matching the coefficients on a constant and $X$, we obtain the following equations for $A$ and $B$:

\begin{align}
0 &= \mathcal{K}'\chi - \theta(1 - k_1)B + \frac{1}{2}\chi'H\chi + l'_1(\varphi(\chi) - 1), \quad (11) \\
0 &= \theta(\ln \delta + k_0 - (1 - k_1)A) + \mathcal{M}'\chi + \frac{1}{2}\chi'h\chi + l'_0(\varphi(\chi) - 1). \quad (12)
\end{align}

In general, these equations can yield multiple solutions to $A$ and $B$. In our numerical example, we generalize the criterion in Tauchen (2005) and select the root which ensures the non-explosiveness of the system as the contributions of stochastic volatility and jump components converge to zero. An alternative approach is to choose an "economically reasonable" solution which responds intuitively to model and preference parameters. We will provide more discussion in the empirical part of the paper.

We follow Bansal, Kiku and Yaron (2006) and solve for the linearization constants $k_0$ and $k_1$ as part of the equilibrium. Indeed, the model is very sensitive to the linearization parameters, so it is important to choose them in a consistent way. Specifically, let us rewrite the second equation above imposing the linearization restrictions between the mean price-to-consumption ratio $Ev$, $k_0$ and $k_1$ in equations (4) and (5).

From (6) and given the dynamics for $X$ in (21),

\[ Ev = A + B'\mu_X, \]

where $\mu_X$ is the vector with $i$th component

\[ \mu_{X,i} = \begin{cases} E(X_i) & \text{if } E(X_i) \text{ exists}, \\ 0 & \text{otherwise}. \end{cases} \]

Expanding $k_0$ in terms of $k_1$ we can show that

\[ k_0 + (k_1 - 1)A = k_0 - (1 - k_1)(Ev - B'\mu_X) \]

\[ = - \log k_1 + (1 - k_1)B'\mu_X. \quad (14) \]

Plugging this expression into (11), we obtain that the linearization coefficient $k_1$ satisfies the following non-linear equation:

\[ \theta \log k_1 = \theta (\ln \delta + (1 - k_1)B'\mu_X) + \mathcal{M}'\chi + \frac{1}{2}\chi'h\chi + l'_0(\varphi(\chi) - 1). \quad (15) \]

Given the parameters of the model, we numerically iterate on $k_1$ in the formula above starting from the initial value $\delta$, which is the exact solution for $k_1$ when $\psi = 1$. For the parameter values we consider, the algorithm converges very fast, in 2-5 iterations.
The evolution of the log pricing kernel can now be written as

\[ d \ln M_t = \theta \ln \delta dt - \frac{\theta}{\psi} d \ln C_t + (\theta - 1) r_c dt \]

\[ = (\theta \ln \delta - (\theta - 1) \log k_1 + (\theta - 1)(k_1 - 1) B'(X_t - \mu_X)) dt - \lambda dX_t \]

where

\[ \lambda = \gamma \delta_c + (1 - \theta) k_1 B. \]  \hfill (16)

Conjecture that the short rate \( r_t \) is affine in \( X_t \),

\[ r_t = \Phi_0 + \Phi'_t X_t. \]  \hfill (17)

As \( M_t e^{\int_0^t r(s)ds} \) is a martingale, we use Ito lemma to obtain

\[ \Phi_1 = (1 - \theta)(k_1 - 1)B + \mathcal{K}'\lambda - \frac{1}{2} \lambda' H \lambda - \lambda'_1 (\varrho(-\lambda) - 1), \] \hfill (18)

\[ \Phi_0 = -\theta \ln \delta + (\theta - 1)(\log k_1 + (k_1 - 1) B' \mu_X) + \mathcal{M}' \lambda 
- \frac{1}{2} \lambda' h \lambda - \lambda'_0 (\varrho(-\lambda) - 1). \] \hfill (19)

Thus, we can substitute the short rate to express the evolution of the discount factor in the following way:

\[ \frac{dM_t}{M_t} = -r_t dt - \Lambda'_t dW_t - \sum_i \left[ (1 - e^{-\lambda t} \xi_i^t) dN_t^i - (1 - \varrho(-\lambda^t)) \xi'_t(X_t) dt \right], \] \hfill (20)

where superscript \( i \) denotes the \( i \)th element in the vector, and \( \Lambda_t \) is defined by

\[ \Lambda_t = \Sigma(X_t)' \lambda. \]

The vector \( \Lambda_t \) is related to the price of jump risk of size \( \xi \) in \( i \)th state variable and literally is the price of Brownian motion risk. The following theorem is a slight generalization of Proposition 5, p. 1372 in Duffie, Pan, and Singleton (2000) and describes the evolution of the system under the risk-neutral measure.

**Theorem 1.** Under the risk-neutral measure \( Q \) induced by the discount factor \( M_t \), the state-variables follow

\[ dX_t = (\mathcal{M}^Q + \mathcal{K}^Q X_t) dt + \Sigma(X_t) dW_t^Q + \xi^Q_t \cdot dN_t^Q, \] \hfill (21)

where

\[ \mathcal{M}^Q = \mathcal{M} - h \lambda, \] \hfill (22)

\[ \mathcal{K}^Q = \mathcal{K} - H \lambda. \] \hfill (23)
The jump-arrival intensity is
\[ l^Q_t = l_t \cdot \varrho(-\lambda). \]  
(24)

The Q jump-size density is characterized by its Laplace transform \( \varrho^Q : \mathbb{C}^n \to \mathbb{C}^n \)
\[ \varrho^Q(u) = E^Q e^{u\xi} = \varrho(u - \lambda) / \varrho(-\lambda). \]  
(25)

Notice that if \( \lambda_i = 0 \), there is no difference in the jump measures and both market prices of diffusive and jump risks are zero. This pinpoints the importance of the parameters \( \lambda \) in generating risk premiums in our model.

The jump intensity is greater (smaller) under the equivalent measure \( Q \) whenever \( \lambda \) is negative (positive). The mean and standard deviation of jump size are greater under the risk neutral than objective measure when \( \varrho(-\lambda_i) \in (0, 1) \) and smaller if \( \varrho(-\lambda_i) > 1 \), as
\[ E^Q \xi = E(\xi) / \varrho(-\lambda_i), \]
\[ \text{Std}_Q(\xi_i) = \text{Std}_P(\xi_i) \varrho(-\lambda_i)^{-\frac{1}{2}}. \]

The following reward-to-risk ratio illustrates the equilibrium rewards for jump risks,
\[ \Lambda_i^J = \frac{E\xi_i - E^Q \xi_i}{\text{Std}(\xi_i)} = \frac{E\xi_i}{\text{Std}(\xi_i)} \left(1 - \frac{1}{\varrho(-\lambda_i)}\right). \]

It is somewhat misleading, although tempting, to coin this measure a market price of jump risk. Jump risks are characterized, and thus priced, not only according to their mean and standard deviations, but also higher order moments.

To build more intuition about the risk-neutral adjustment to the overall density of jump amplitudes, let us examine a particular case when the jump size \( \xi \) belong to an infinitely divisible class of distributions with finite variation Lévy measure, which includes Gaussian, gamma, \( \alpha \)-stable and tempered stable, compound Poisson distribution and others. In particular we can write down the moment generating function of jumps under the physical measure as \(^4\)
\[ \varrho(u) = e^{\mu u + \frac{1}{2} \sigma^2 u^2 + \int_\mathbb{R} (e^{ux} - 1 - ux) \nu(dx)}, \]
for certain \( \mu \) and \( \sigma \) and positive Radon measure \( \nu \). It is easy to see that if it exists, the risk-neutral distribution of jump sizes will remain infinitely-divisible, with the following parameters:
\[ \mu^Q = \mu - \lambda \sigma^2, \]
\[ \sigma^Q = \sigma, \]
\[ \nu^Q(dx) = e^{-\lambda x} \nu(dx). \]

\(^4\)The results apply under some technical existence and integrability conditions, see, for example, Cont & Tankov (2004).
For $\lambda < 0$, the risk-neutral adjustment shifts the distribution of jump sizes to the right and fattens its tails. Therefore, the investors adjust their perception of large negative jump-news in the economy by making them higher on average and more extreme under the risk-adjusted probabilities, while the opposite happens if $\lambda > 0$. For particular examples of the risk-neutral transformations of the jump size distribution, refer to the Table 1.

3 General asset prices

Consider the price of an asset which pays a continuous dividend stream $D_s, t < s < T$. The price of this asset (stock) is now obtained by taking the expectation under the risk neutral measure of its discounted payoffs $D(X_t)$:

$$P(\{D(X_s)\}_{s=t}^T) = \int_s^T E_t \left( \frac{M_t}{M_s} D_s \right) ds$$

$$\equiv \int_s^T E_t^Q \left( e^{-\int_t^s r(\tau)d\tau} e^{u'X_t} \right) ds.$$

To facilitate the computations, we follow Duffie, Pan and Singleton (2000) and compute a discounted characteristic function of $X_t$ under the risk-neutral measure:

$$\phi_X^Q(u, X_t, s) = E_t^Q \left( e^{-\int_t^s r(\tau)d\tau} e^{u'X_t} \right).$$

(26)

for $u \in \mathbb{C}^n$.

Under appropriate technical regularity conditions (see Duffie, Pan and Singleton (2000)), $\phi_X^Q$ is exponential affine in $X_t$,

$$\phi_X^Q(u, X_t, s) = e^{\alpha(s)+\beta(s)'X_t},$$

and its loadings satisfy the complex-valued ODEs

$$\dot{\beta} = -\Phi_1 + \mathcal{K}^Q\beta + \frac{1}{2} \beta' H \beta + \hat{t}_1^Q \left( \phi^Q(\beta) - 1 \right),$$

$$\dot{\alpha} = -\Phi_0 + \mathcal{M}^Q\beta + \frac{1}{2} \beta' h \beta + \hat{t}_0^Q \left( \phi^Q(\beta) - 1 \right),$$

(27)

subject to boundary conditions $\beta(0) = u, \alpha(0) = 0$.

In particular, setting $u = 0$ we immediately obtain that the yield on a discount bond with $s$ periods to maturity is given by,

$$y(X_t, s) = -\frac{1}{s} \left( \alpha(s) + \beta(s) X_t \right),$$

where $\alpha$ and $\beta$ solve the ODEs in (34) with boundary condition $\beta(0) = \alpha(0) = 0$. 
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<tr>
<th>Name</th>
<th>Physical measure</th>
<th>Risk-Neutral measure</th>
<th>Restrictions</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Infinitely Divisible Distributions</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Normal</td>
<td>$\varrho(u; \mu, \sigma) = \exp^{\mu + \frac{1}{2} u^2 \sigma^2}$</td>
<td>$\mu^Q = \mu - \lambda \sigma^2$</td>
<td>$r, \mu_v &gt; 0$</td>
</tr>
<tr>
<td></td>
<td>$\sigma^Q = \sigma$</td>
<td></td>
<td>$u &lt; \min \left( \frac{r}{\mu_v}, \frac{r}{\mu_v} + \lambda \right)$</td>
</tr>
<tr>
<td>Gamma</td>
<td>$\varrho(u; \tau, \mu, \nu) = (1 - \frac{\mu}{\tau} u)^{-r}$</td>
<td>$r^Q = \frac{r}{\tau + \lambda \mu_v}$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\mu^Q = \frac{\mu \tau}{\tau + \lambda \mu_v}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Tempered Stable</td>
<td>$\varrho(u; c, \alpha, r) = \exp^{\Gamma(-\alpha)((c-u)^\alpha - c^\alpha)}$</td>
<td>$r^Q = r$</td>
<td>$\alpha \in (0, 1)$</td>
</tr>
<tr>
<td></td>
<td>$\alpha^Q = \alpha$</td>
<td></td>
<td>$r, c &gt; 0$</td>
</tr>
<tr>
<td></td>
<td>$c^Q = c + \lambda$</td>
<td></td>
<td>$u &lt; \min(c, c + \lambda)$</td>
</tr>
<tr>
<td>Compound Poisson</td>
<td>$\varrho(u; c, f) = \exp^{\int (\exp^{\alpha x} - 1) f(x) dx}$</td>
<td>$c^Q = c \int \exp^{-\lambda x} f(x) dx$</td>
<td>$c &gt; 0$, $f(x)$ is pdf</td>
</tr>
<tr>
<td></td>
<td>$f^Q(x) = \exp^{-\lambda x} f(x) / \int \exp^{-\lambda x} f(x) dx$</td>
<td></td>
<td>$\int \exp^{ax} f(x) dx &lt; \infty$</td>
</tr>
<tr>
<td>Non-Infinitely Divisible Distributions</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Uniform</td>
<td>$\varrho(u; a, b) = \frac{\exp^{a(u-a)}}{u(b-a)}$</td>
<td>$g^Q(u; a, b) = \frac{\exp^{a(b-a)\lambda}}{\lambda - u}$</td>
<td>$b &gt; a$</td>
</tr>
</tbody>
</table>

Table 1: The risk-neutral adjustment to the jump size distribution.
3.1 Dividend Paying Assets

Consider an asset which dividend stream can be expressed as a linear function of the state-variables,

\[
\ln D_t = \delta_d X_t.
\]

From the discussion in the previous section, the price of an asset which pays a perpetual dividend \(D_t\), if it exists, is given by

\[
P_t(X_t) = \int_0^\infty \theta_X^Q(\delta_d, X_t, s) ds = \int_0^\infty e^{\alpha(s) + \beta(s) X_t} ds,
\]

(28)

where \(\alpha\) and \(\beta\) satisfy the ODEs in (34) subject to \(\beta(0) = \delta_d\) and \(\alpha(0) = 0\).

To build more intuition about the model, we can obtain an approximate solution to the price of this asset in the same way as for the aggregate market portfolio. It is straight-forward to show that if we assume that the equilibrium admits an exponential linear price dividend ratio,

\[
P_t = D_t \exp(A_d + B_d' X_t),
\]

we can recover the coefficients \(A_d\) and \(B_d\) by solving

\[
\mathcal{K}' \chi_d + (\theta - 1)(k_1 - 1) B + (k_{1,d} - 1) B_d + \frac{1}{2} \chi_d' H \chi_d + l_1' \rho(\chi_d) - 1 = 0,
\]

(30)

and

\[
\theta \log \delta - (\theta - 1) (\log k_1 + (k_1 - 1) B' \mu_X) - (\log k_{1,d} + (k_{1,d} - 1) B_d' \mu_X)
\]

\[
+ \mathcal{M}' \chi_d + \frac{1}{2} \chi_d' h \chi_d + l_0 \rho(\chi_d) - 1 = 0,
\]

(31)

where

\[
\chi_d = \delta_d + k_{1,d} B_d - \lambda,
\]

(32)

and \(k_{1,d}\) is the log-linearization coefficient for a dividend return. As before, we can solve the equations above for \(B_d\) and \(k_{1,d}\), and then obtain an intercept \(A_d\) from the chain of equalities

\[
A_d + B_d' \mu_X = E \log \frac{P_d}{D_t}
\]

\[
= \log \frac{k_{1,d}}{1 - k_{1,d}}.
\]

The first equality follows from the conjectured solution for the price-dividend ratio, while the second comes from the log-linearization procedure.

The main advantage of the formulae presented in this section is that we can obtain an exponential affine representation of the (approximate) equilibrium stock price dynamics. This facilitates semi-analytical computations of options prices, as illustrated next.
3.2 Option Pricing

Lewis (2000) and Carr and Madan (1999) discuss methods for computing options prices from the characteristic function of the underlying stock price. In the following we adapt the formula in Lewis (2000) to our setting.

The price of a call option is a function of the state variables, strike price and maturity of an option:

\[ C(X_t, K, s) = E_t^Q \left[ e^{-\int_t^t s^r(r)dr} \left( e^{\ln P_T} - K \right)^+ \right]. \]

In particular, we have shown that the short interest rate and log stock price are affine in state variables:

\[ r_t = \Phi_0 + \Phi_1' X_t, \]
\[ \ln P_t = \Lambda_d + (B_d + \delta_d)' X_t. \]

The assumed form of the price-dividend ratio follows if the stock pays a single terminal dividend \( D_T \) at some date \( T > t \) or if we apply the linearization described above for a stock with continuous dividend payments.

The generalized Fourier transform of the payoff function of the European option is equal to,

\[ \hat{w}(z) = \int_{-\infty}^{\infty} e^{izx} (e^x - K)^+ \, dx \]
\[ = \frac{K^{iz+1}}{z^2 - iz}, \]

for \( Im(z) > 1 \), and identical expression obtains for put options for \( Im(z) < 0 \).

Using the Parseval identity, we obtain

\[ C(X_t, K, s) = E_t^Q \left[ e^{-\int_t^t s^r(r)dr} \left( e^{\ln P_T} - K \right)^+ \right] = \frac{1}{2\pi} \! \! \! \int_{iz}^{iz} e^{-\int_t^t s^r(r)dr} e^{-iz\ln P_T} \hat{w}(z) \, dz \]
\[ = -K \frac{1}{2\pi} \! \! \! \int_{iz}^{iz} \varphi_p(-z, X_t, s) \frac{K^{iz}}{z^2 - iz} \, dz, \]

(33)

where \( z_i = Im(z) \), and \( \varphi_p(z, X_t, s) \) stands for the discounted risk-neutral characteristic function of the log stock price:

\[ \varphi_p(z, X_t, s) = E_t^Q e^{-\int_t^t s^r(r)dr} e^{iz\ln(P_t+s)}. \]
Now, it is easy to show that $\varrho$ is affine in $X_t$,

$$
\varrho_{t_0}(z, X_t, s) = e^{\alpha(s)+\beta(s)'X_t},
$$

where $\alpha$ and $\beta$ satisfy the same ODEs as before,

$$
\begin{align*}
\dot{\beta} &= -\Phi_1 + KQ^t \beta + \frac{1}{2} \beta' H \beta + t_1^Q \left( \varrho^Q(\beta) - 1 \right), \\
\dot{\alpha} &= -\Phi_0 + M^Q \beta + \frac{1}{2} \beta' h \beta + t_0^Q \left( \varrho^Q(\beta) - 1 \right),
\end{align*}
$$

subject to different boundary conditions $\beta(0) = i z(B_d + \delta_d)$, $\alpha(0) = i z A_d$.

The integration in (33) is performed on the intersection of the strips $z_i > 1$ for call option or $z_i < 0$ for puts, and the one parallel to the real $z$–axis. Notice that (33) requires a single numerical integration which is advantageous relative to the formulae in the extant literature (i.e, Heston (1993), Bates (1996), Duffie et al (2000)) which require two numerical integrations.

### 4 The Equilibrium Impact of Volatility Shocks

In an application of our model, we consider an economy in which consumption, dividends, and in the end asset prices, are influenced by a single state-variable, which is the conditional volatility of consumption growth. To this end we assume that log-consumption, $c = \ln C$ follows

$$
\begin{align*}
dc &= \mu dt + \sqrt{V} dW_z, \\
dV &= \kappa_V (\bar{V} - V) dt + \sigma_V \sqrt{V} dW_V + \xi_V dN, \\
\xi_V &\sim GA(r, \mu_V/r), \\
l_t &= l_0 + l_1 V_t.
\end{align*}
$$

The volatility process, $V$, is driven by the continuous Brownian motion, $dW_V$, as well as discontinuous process $\xi_V dN$ whose arrival intensity is $l_t$. Our assumption of Gamma distributed volatility jump sizes allows a fairly heavily tailed jump size distribution for small values of the scale parameter, $r$.

This model encompasses a number of models in the literature on stochastic volatility, including square root volatility model of Heston (1993), the exponential jump diffusion model of Duffie, Pan & Singleton (2001), Eraker (2004), among others. By removing the diffusive part, $\sigma_v = 0$, our volatility reduces to the gamma OU process ($l_1 = 0$ and $r = 1$). For a detailed treatment of Non-Gaussian OU processes refer to Barndorff-Nielsen and Shephard (2001). Our specification of the consumption process is a simplification of the Bansal & Yaron (2004) model in that the expected
log-consumption growth is constant. In the BY model, the expected consumption growth follows a mean-reverting AR(1) process. Fixing the expected growth rate in our model allows us to focus entirely on the equilibrium effects of stochastic volatility.

The first step in our analysis is to recover expressions for the coefficients $A$ and $B$ in eqns. (11) and (12). The "volatility factor loading" $B_v$ solves

\[
0 = -\theta \left[ \kappa_v k_1 + (1 - k_1) \right] B_v + \frac{1}{2} \theta^2 (1 - \frac{1}{\psi})^2 + \frac{1}{2} \theta^2 k_1^2 \sigma_v^2 B_v^2 + l_1 \left[ (1 - B_v \theta k_1 \mu_v / r)^{-r} - 1 \right].
\]

(39)

This equation admits an explicit solution only in special cases. In particular, if there are no state-dependent volatility jumps, $l_1 = 0$, then $B_v$ solves the quadratic $a + bB_v + cB_v^2 = 0$ for $a = \theta^2 (1 - \frac{1}{\psi})^2$, $b = -\theta (\kappa_v + (1 - k_1))$, $c = \theta^2 k_1^2 \sigma_v^2$. Tauchen (2005) points out that square root processes for volatility generally produce two roots for $B_v$. However, if $\theta < 0$, $b > 0$ and only the "right" root is non-explosive when the stochastic volatility parameter $\sigma_v$ converges to 0. By including state-dependent volatility jumps, we generally have more than two roots. In the case where the volatility is driven by pure jumps ($\sigma_v = 0$) and volatility jumps are exponentially distributed ($r = 1$), we can recover another quadratic equation for $B_v$, and we can use a similar argument to select the non-explosive solution when the jump contribution is converging to zero. When $\sigma_v$ is not zero, we typically get two real solutions for $B_v$, and we choose the "right" root near the one implied by a quadratic equation above.

Given the solution to $B_v$, we recover the dynamics of the state-variables under the risk neutral measure:

\[
dc = \left( \mu - \gamma V \right) dt + \sqrt{V} dW_c^Q
\]

\[
dV = \kappa_v (\bar{V} - V) dt - \lambda_v \sigma_v^2 V dt + \sigma_v \sqrt{V} dW_v^Q + \xi_v^Q dN^Q
\]

\[
\xi_v^Q \sim GA(r, \frac{\mu_v}{r + \lambda_v \mu_v})
\]

\[
l_t = \left( 1 + \frac{\lambda_v \mu_v}{r} \right)^{-r} (l_0 + l_1 V_t).
\]

(40)

(41)

(42)

(43)

This process is well defined whenever $r > \lambda_v \mu_v$ which places implicit restrictions on the permissible preference parameters. If this holds, and the market price of volatility risk is negative, $\lambda_v < 0$, then both jump sizes and arrival intensity are greater under the risk neutral measure,

\[
E^Q \xi_v > E^P \xi_v.
\]

\[
Var^Q \xi_v > Var^P \xi_v.
\]

\[
l_t^Q > l_t^P.
\]

In the case of exponentially distributed jump sizes, $r = 1$ and the jump size distribution as well as the jump intensity are scaled by the constant $(1 + \lambda_v \mu_v)^{-1}$.
which is greater than one whenever \( \psi > 1, \gamma > 1 \). Thus, we conclude, our equilibrium framework offers a very simple intuitive parameter adjustment from the objective to the risk neutral measure.

### 4.1 Dividends

We consider a stock whose perpetual dividend stream is

\[
d\ln D = \phi d\ln C + \sigma_d \sqrt{\mathcal{V}} dW_d,
\]

where the parameter \( \phi \) can be interpreted as a “consumption leverage” parameter or the OLS slope coefficient obtained by regressing \( d\ln D \) on \( d\ln C \). Whenever \( \phi > 1 \), we can think of the corporate dividends as being a levered position on total consumption output. The idea of dividends as a levered position on consumption is useful in reconciling the low consumption volatility with high volatilities of corporate earning and dividends. The term \( \sigma_d \sqrt{\mathcal{V}} dW_d \) represents asset specific noise which is not priced in equilibrium.

It is straightforward to recover an exact pricing formula for the price of a stock and the pricing formula takes the form \( D_t \int_0^\infty \exp(\alpha(s) + \beta(s)X_t)ds \). However, in order to recover exponential affine stock price dynamics we again employ the Campbell-Shiller approximation such that the assumed form of the price dividend ratio is \( \exp(A_d + B_dX_t) \). Equations (30) become

\[
- \kappa_v \left[ ((\theta - 1)k_1B_v + k_{1,d}B_{d,v}) - (\theta - 1)(1 - k_1)B_v - (1 - k_{1,d})B_{d,v}\right]
+ \frac{1}{2} \left[ (\gamma - \phi)^2 + \sigma_d^2 + \sigma_v^2 ((\theta - 1)k_1B_v + k_{1,d}B_{d,v})^2 \right]
+ l_1 \left[ \frac{1 - \mu_v(\theta - 1)k_1B_v}{r} - \frac{\mu_v k_{1,d}B_{d,v}}{r} \right]^{-r} - 1 = 0
\]  

(44)

which can be solved for \( B_{d,v} \) with similar caveats about the multiplicity of roots as in the previous section.

The dividend process follows

\[
d\ln D = \phi d\ln C + \sigma_d \sqrt{\mathcal{V}} dW_d^Q
= \phi(\mu - \gamma V)dt + \phi \sqrt{\mathcal{V}} dW_c^Q + \sigma_d \sqrt{\mathcal{V}} dW_d^Q
\]  

(45)

under the risk neutral measure. It is now straightforward to show that the stock price, which is given by \( \ln S = \ln D + (A + B_{d,v}V) \), evolves according to

\[
d\ln S = \left[ \phi(\mu - \gamma V) + B_{d,v}(\kappa_v(\bar{V} - V) - \lambda_v \sigma_v^2 V) \right] dt
+ \sigma_d \sqrt{\mathcal{V}} dW_d^Q + \phi \sqrt{\mathcal{V}} dW_c^Q + B_{d,v} \sigma_v \sqrt{\mathcal{V}} dW_v^Q + B_{d,v} \xi_d^Q dN^Q.
\]  

(46)
under the risk-neutral measure.

As the conditional variance of log price is proportional to $V_t$, the conditional correlation between log price and its variance is given by $\text{Corr}(d\ln S, dV)$. The latter is given by,

$$\text{Corr}(d\ln S, dV) = B_{d,v} \sqrt{\frac{\sigma_d^2 V + \mu^2 v^{-1}(l_0 + l_1 V)}{V(\sigma_d^2 + \phi^2 + B_{d,v}^2 \sigma_v^2) + B_{d,v}^2 \mu v^{-1}(l_0 + l_1 V)}},$$

(47)

where $l_0, l_1, \mu$, and $v$ can be under either the physical or risk-neutral measure. Thus, the stock price/volatility correlation is different under the objective $P$ and risk-neutral measure $Q$. The difference in $P$ and $Q$ correlation is driven by the magnitude of the jump-risk premia. In our model, the jump risk premia increases uniformly in the risk aversion parameter $\gamma$. Thus, larger values of $\gamma$ generate a larger dispersion in the correlation for the two measures. This effect is illustrated below.

Notice that the correlation is increasing (in absolute value) in the parameter $B_{d,v}$. This parameter again is a function of preferences, and typically takes on negative values for $\psi, \gamma > 1$. This implies that the correlation in 1 approaches negative one for large values of $\gamma$ and $\psi$. Figure 1 illustrates this effect. The figure shows the correlation under both the objective and the risk neutral measures decreases as a function of $\gamma$. The correlation becomes more negative when risk aversion increases. The correlation is less negative for all values of $\gamma$ when the value of the idiosyncratic dividend noise term, $\sigma_d$, is higher, illustrating that individual stocks (which contain a larger fraction of idiosyncratic noise) exhibit less pronounced volatility-stock price correlation than do equity indices.

The negative sign shows that increased macro-economic uncertainty leads to lower equilibrium stock valuations. Correlations are higher in absolute value under the risk neutral measure, and the difference between the two measures increases with the risk aversion coefficient. This is an important observation because researchers who attempt to fit reduced form no-arbitrage models to options price data often find that the magnitude of the stock-price volatility correlation well exceeds the correlations estimated from actual stock price/volatility estimates. For example, Bakshi, Cao and Chen (1997) calibrate jump diffusion models and find option implied correlations in the -0.6 to -0.8 range. This is generally outside the range found empirically relevant from returns data. Anderson, Benzoni and Lund (2002) estimate the correlation to be in the -0.5 to -0.6 range. Eraker, Johannes & Polson (2003) estimate it to be in the −0.4 to −0.5 range. Eraker (2004) fits various no-arbitrage jump diffusion models to both returns data and joint data of options and returns and finds that correlations are greater in magnitude when including options data. A difference of about ten percentage points in the $P$ and $Q$ correlations is consistent with a risk aversion, $\gamma$, of about nine in our model.
Figure 1: Correlation between innovations in (log) stock prices and volatility as a function of risk aversion, $\gamma$.

### 4.2 Price Patterns

In the following we discuss the properties of options prices computed with our model. The standard measure of empirical pricing patterns in options data is the options implied volatility. It is well known that the implied volatility of index options is convex over different strikes. To examine if our model generates implied volatility patterns that mimic those found in empirical data we compute implied volatilities by equating the theoretical model prices to prices computed through the Black & Scholes model using our model computed dividend yields and interest rates.

In figure 2 we plot the implied volatility as function of the strike price and maturity. The two surface plots are generated by varying the initial value of the volatility, $V_t$. The figure generates an implied volatility pattern which is fairly typical of those generated by models based upon no-arbitrage, with high negative correlation between volatility shocks and prices. The implied volatility has a more pronounced U shape at short maturities, and flattens out at the long end. Evidence of negative skewness is evident even for long maturity contracts.

Figure 3 depicts the implied volatility of our theoretical model prices computed over different values of risk aversion $\gamma$ and with a fixed value of $\psi = 4$. There are two
main effects of increasing the level of risk aversion. First, the stock prices become more volatile on average. This is reflected by an upward shift in implied volatility for higher values of $\gamma$. Secondly, and more interestingly, the convexity of the IV curve is much more pronounced for higher values of $\gamma$, reflecting a more heavily tailed equilibrium stock price distribution. Jumps to the volatility process always result in (negative) jumps in the stock price, as can be seen from equation (46). The negative price reactions to a jump in volatility can be significant, as evidenced by the increasing prices of far out-of-the-money put options in figure 3. Therefore, endogenizing the stock price in an economy where the volatility may increase suddenly can explain the high crash insurance premiums offered by out-of-the-money put options.

Does CRRA utility deliver implied volatility graphs that mimic those of the general Epstein-Zin model? Recall that the CRRA model obtains as a special case by imposing the constraint $\psi = 1/\gamma$. Figure 4 shows the implied volatility in this case. The figure illustrates that the implied volatility is low on average. This is consistent with the conventional insights on the failure of the CRRA model from Mehra & Prescott (1985), Hansen & Singleton (1982), and Hansen & Jaganathan (1991) among others who point out that the low volatility of consumption requires a large value of $\gamma$ to generate returns with mean and variance mimicking those of equity returns.
Another failure of this model is that it generates positive conditional skewness in the risk neutral stock price distribution, as can be seen from the fact that contracts with high strikes carry a higher premium. Note also that the fairly small difference in the implied volatilities across strikes indicate that the CRRA model produces much smaller departures from normality than the one based on Epstein Zin preferences.

It is possible to generate very steep equilibrium implied volatility functions in our example model. Recall that the parameter $r$ in the volatility jump size distribution determines the tail-behavior for the volatility jump sizes, and a small value of $r$ lead to heavier tails. Figure 5 illustrates the effect of changing $r$. The impact of out-of-the-money puts is significant, and prices increase uniformly with lower values of $r$. The case where $r = 0.15$ illustrates that the possibility of a very severe jump in volatility has a dramatic effect on the price of OTM puts.

Our discussion so far has mostly been relevant in the context options written on an index of stocks. Stock indices, unlike individual equities, are characterized by the fact that the risks are almost entirely systematic. In our model this is captured by a small value of $\sigma_d$ (we used 0.2 in our preceding discussion). Figure 6 illustrates the effect of increasing idiosyncratic risk, $\sigma_d$, on options prices. While the overall implied volatility increases, higher values of $\sigma_d$ significantly diminish conditional skewness and kurtosis as can be seen from the almost flat implied volatility curve. Increasing
Figure 4: Implied volatility for different levels of risk aversion, $\gamma$ in the case of CRRA utility.

$\sigma_d$ in our model thus has the effect of just adding Gaussian noise to the stock price process. Our model can easily be augmented to allow for company specific dividend jumps. This would generate additional kurtosis, and thus additional convexity in implied volatilities of individual equity options.

5 Concluding Remarks

The affine class of pricing models constitutes a significant and important class of models. They are typically derived under assumptions of no-arbitrage. This offers limited ability for economic interpretations of the market prices of risks that link the objective probability measure with the risk-neutral pricing measure. Our paper suggests a way to remedy this by constructing a pricing kernel which is based on Epstein-Zin preferences in discrete time. Importantly, our framework offers a convenient way to link the two measures in the presence of diffusive risks and jump risks. As all risk premia computations in our model are done through the specification of the three preference parameters $\delta, \psi, \gamma$, our model specification offers not only a tractability,
but also parameter parsimony. The latter is especially important in empirical studies of asset market behavior.

To see the benefits of our framework for empirical analysis, consider the following empirical strategy. First, it is possible to identify the parameters that govern the state-variables $X_t$ from empirical observations of a subset of assets and macroeconomic time series (e.g., consumption). By adding a limited number of data from equity and bond markets, it is additionally possible to obtain estimates of the preference parameters $\delta, \psi, \gamma$. Since these quantities are sufficient to compute the theoretical prices of any other asset available to the investor, we can now use additional asset market data, such as options, to test the model out-of-sample. Alternatively of course, we can use derivatives market data to estimate all the parameters in the model, and then subsequently use these estimates to evaluate the model’s out-of-sample performance using stock and bond data.

The model framework described in this paper is closely related to the long run risk model of Bansal & Yaron (2004). The main message of their paper is that a highly persistent component in expected consumption and dividend cash flows leads a considerable magnification of the risk premiums relative to i.i.d model economies. We focus on a model with only one state-variable, volatility, because this allows us to study the equilibrium effects of volatility shocks in isolation. It is straightforward to
use our framework to derive models with additional state-variables, including time-varying expected consumption growth, inflation, etc. More elaborate models for option prices are left for a future research.
Appendix

5.1 The Campbell-Shiller approximation

Let $dr_t$ denote the continuous time log return and let correspondingly $\int_t^{t+1} dr_s$ be the return over the interval $(t, t + 1]$. The Campbell-Shiller approximation to the discrete time return is

$$\int_t^{t+1} dr_s = k_0 + k_1 z_{t+1} - z_t + g_{t+1}$$  

where $z_t = A + B'X_t$ is the log-price dividend ratio, and $g_{t+1} = \delta_d X_t$ is the cash flow of the asset. Equation (48) is exactly the same as in discrete time models, as we showed in Section 2.

Now re-write (48) as

$$\int_t^{t+1} dr_s = k_0 + k_1 (z_{t+1} - z_t) - (1 - k_1) z_t + g_{t+1}$$  

$$= k_0 + k_1 \int_t^{t+1} B'dX_s - (1 - k_1) z_t + \delta_c' \int_t^{t+1} dX_s$$  

$$= \int_t^{t+1} k_0 ds + [\delta_c + k_1 B]' \int_t^{t+1} dX_s - \int_t^{t+1} (1 - k_1) z_t ds$$

We thus define the the continuous time log return

$$dr_t = k_0 dt + [\delta_c + k_1 B]' dX_t - (1 - k_1)(A + BX_t)dt.$$  

Note that our log-linearization mimics the standard Campbell-Shiller approximation in in discrete time modelling. This enables us to interpret $k_0$ and $k_1$ as linearization coefficients that are relevant over a unit of time in the model.
References


