Time Preference and the Penrose Effect in a Two-Class Model of Economic Growth

H. Uzawa

University of Chicago and University of Tokyo

1. Introduction

In the theory of economic growth, we are concerned with the analysis of those economic factors which crucially determine the process of growth for a national economy. Our primary interest is in the mechanisms by which aggregate variables such as national income, aggregate stock of capital, and others are interrelated and in how they change as time passes. Since Harrod (1948) first laid down the fundamental theorems for a dynamic economics, we have seen the emergence of an increasing number of aggregate growth models to clarify and extend these theorems, as aptly described in Hahn and Matthews' (1964) survey article. These growth models, however, have been mostly built upon premises directly involving aggregate variables, without specifying the postulates which govern the behavior of individual units comprising the national economy. In particular, the specifications of aggregate savings are seldom based upon analysis of individual behavior concerning savings and consumption; instead, they have been merely hypothesized in terms of historical and statistical observations. Similarly, the aggregative behavior of investment has been either entirely neglected, as has been typically the case with the so-called neoclassical models, or it has been postulated in terms of somewhat *ad hoc* relations involving market rate of interest, rate of profit, and other variables.

In the present paper, I should like to pay closer attention to the behavior of individual units concerning consumption, saving, and investment, and to build a formal model of economic growth for which the aggregate variables are described in terms of these microeconomic analyses.

This paper is part of a study in the theory of economic growth supported by the National Science Foundation. It has been presented at seminars at the University of Chicago and Cambridge University from whose participants I have received a number of valuable comments and suggestions. I should like particularly to acknowledge my indebtedness to Professors Milton Friedman, Arnold Harberger, Joan Robinson, and Lord Kahn.

A private-enterprise economy, with which the present paper is concerned, may be conveniently divided into two sectors; the household sector and the corporate sector. Decisions regarding the consumption of goods and services produced in the corporate sector are made by households; the households in turn are endowed with labor and possess, as assets, the securities issued by the corporate sector.

The analysis of the behavior of an individual household is carried out in terms of the Böhm-Bawerk-Fisherian theory of time preference. My presentation is based upon the mathematical formulation recently developed by Koopmans (1960), with certain modifications, and it may be regarded as an extension of a similar formulation introduced in a previous article of mine (1968*a*).[1] The marginal rate of substitution between current and future consumption is approximated by the schedule of time preference to be referred to as the Fisherian schedule. It relates the rate of time preference to the current level of consumption and to the utility level for all future consumption. Under the hypothesis that intertemporal preference orderings are homothetic, it will be possible to derive the optimum propensities to consume and save—both as functions of the expected market rate of interest alone—independently of the household's income.

To analyze the investment behavior of business firms in the corporate sector, it is necessary to re-examine the concept of real capital which will play a central role in the determination of the levels of output, employment, and investment in general. At each moment of time, a business firm consists of a complex of fixed factors of production, such as factories, machinery, and others, including managerial abilities and technological skills. Real capital is here introduced as an index to measure the productive capacity of such a complex of capital goods endowed within the firm at a particular time. Real capital, as the index of productive capacity of the firm, is then increased as the stock of fixed factors of production is accumulated. The relationships between the real value of investment and the resulting increase in the index of real capital will be described by the Penrose curve, discussed below in detail. It is assumed that the business firm plans the levels of employment and investment in order to maximize the present value of expected future net cash flows, which will be discounted by the market rate of interest. The desired level of investment per unit of real capital will be shown to depend upon the expected rate of profit and the market rate of interest.

Finally, these analyses are put together to formulate an aggregative model of capital accumulation and to briefly analyze the structure of short-run and long-run equilibrium processes.

[1] The paper referred to here contains a number of ambiguities both in its exposition and in its basic assumptions, particularly with respect to the shape of what I have called the Fisherian schedule of time preference.

2. Intertemporal Utility and the Rate of Time Preference

Let us consider a consumer unit which knows with certainty the stream of consumption over its lifetime. For the sake of simplicity, we assume that there is one kind of consumption good, remaining invariant over time, and the time horizon of the consumer unit is infinite.[2] It is assumed that the unit possesses an intertemporal preference ordering between any pair of consumption streams (subject to certain mathematical regularity conditions) and that indifference surfaces for such an intertemporal preference ordering are homothetic. In what follows, we restrict our attention to those intertemporal preference orderings which it is possible to represent by a utility functional of the following type:

$$U(0) = \int_0^\infty c(s)e^{-\Delta(s)}ds, \tag{1}$$

where $c(t)$ is the time path of the consumption stream over time $(0 \leqq t < \infty)$, and $\Delta(t)$ is the accumulated rate of time preference associated with the consumption path $c(t)$. In general, the utility functional is defined for any truncated part of the original consumption path:

$$U(t) = \int_t^\infty c(s)e^{-\Delta(s,t)}ds, \tag{2}$$

where $\Delta(s, t)$ is the accumulated rate of time preference between t and s. The rate of time preference $\Delta(s, t)$ depends upon the whole path of the consumption stream $c(\tau)$, $t \leqq \tau < \infty$, and the meaning of the definition (2) may be easily seen by differentiating it with respect to t:

$$\dot{U}(t) = \delta(t)U(t) - c(t), \tag{3}$$

where the (marginal) rate of time preference $\delta(t)$ is given by

$$\delta(t) = \left(\frac{\partial\Delta(t, t')}{\partial t'}\right)_{t'=t}. \tag{4}$$

At each moment of time t, $c(t)$ represents the level of current consumption at time t, while $U(t)$ may be regarded as the index measuring the utility of the consumption stream $c(t)$ in the future. Suppose the time preference ordering of the consumer unit in question is represented by indifference curves on the (c, U) plane, as indicated in Figure 1, where the horizontal axis measures the current consumption $c(t)dt$ and the utility of the future consumption $U(t + dt)$ is measured along the vertical axis. For any combination of current consumption $c(t)dt$ and future consumption $U(t + dt)$, to be represented by the point A, the marginal rate of substitution between present consumption and future consumption (that

[2] The consumer unit in consideration here is a perpetual institution of which each individual is responsible for the management during his lifetime.

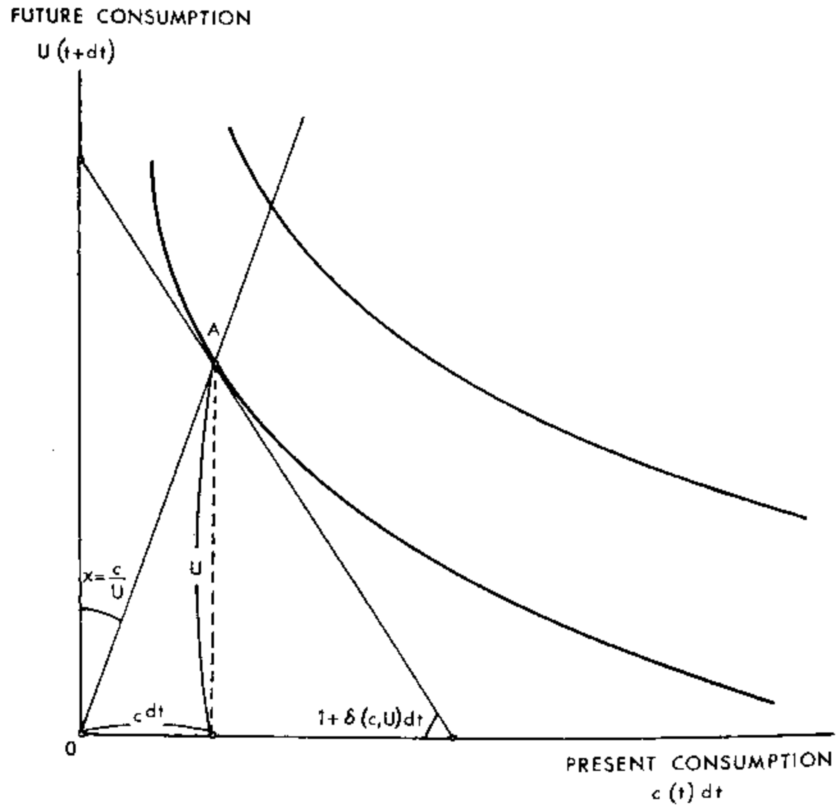

FIG. 1

is, the negative of the slope of the tangent line at A to the indifference curve) is equal to unity plus the rate of time preference, $1 + \delta(t)dt$. If we assume the intertemporal preference structure of the consumer unit remains identical over time, the schedule relating the rate of time preference $\delta(t)$ to the combination of current consumption $c(t)$ and the utility of future consumption $U(t)$ remains invariant, namely, $\delta(t) = \delta[c(t), U(t)]$ with a certain function $\delta(c, U)$.

Since we have assumed that the intertemporal preference ordering is homothetic, we may assume that indifference curves for c and U are also homothetic, so the schedule of the rate of time preference (c, U) is homogeneous of order zero with respect to c and U; namely,

$$\delta(c, U) = \delta(x), \qquad x = c/U. \tag{5}$$

The differential equation (3) may be written as

$$\frac{\dot{U}(t)}{U(t)} = \delta[x(t)] - x(t), \qquad x(t) = \frac{c(t)}{U(t)}. \tag{6}$$

The intertemporal preference ordering is subject to a diminishing marginal rate of substitution between current and future consumption if the time preference function $\delta(x)$ is convex. On the other hand, as is seen from Figure 1, an increase in the consumption-utility ratio, $x = c/U$, results in a decrease in $\delta(x)$. We may therefore assume that

$$\delta(x) > 0, \quad \delta'(x) < 0, \quad \delta''(x) > 0, \qquad \text{for all } x > 0 \tag{7}$$

as is typically illustrated in Figure 2, where x is measured along the horizontal axis and the rate of time preference δ along the vertical axis.

In general, it is difficult to find the level of the intertemporal utility functional for an arbitrary stream of consumption. However, if the rate of increase in the level of consumption $\dot{c}(t)/c(t)$ is constant over time, the intertemporal utility function $U(t)$ for the stream of consumption $c(s)$, $t \leqq s < \infty$, is easily derived. Let $U(0)$ be the utility of the consumption stream $c(s)$, $0 \leqq s < \infty$, for which the rate of increase in consumption $\dot{c}(t)/c(t)$ is constant over time, say

$$\dot{c}(t)/c(t) = \lambda. \tag{8}$$

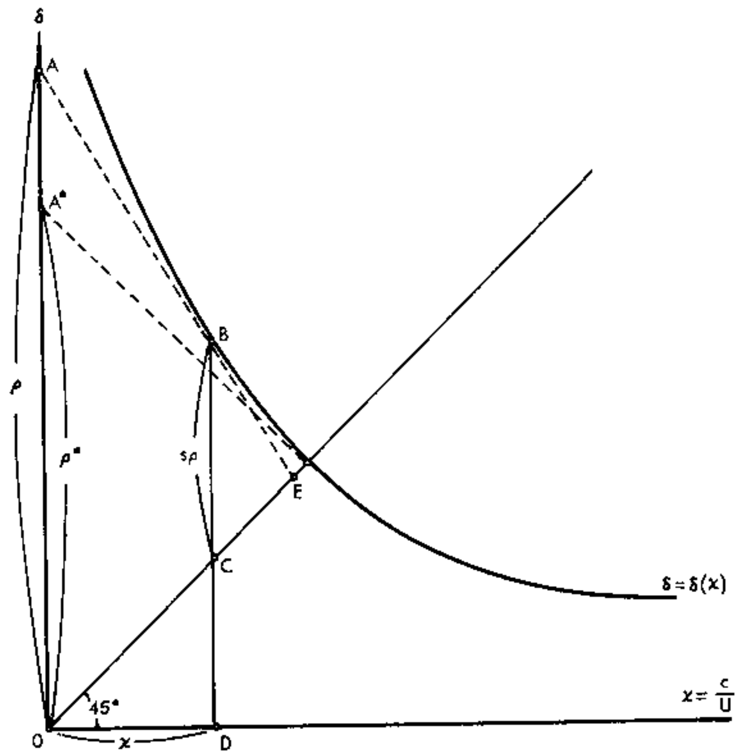

FIG. 2

The level of the intertemporal utility $U(0)$ now depends upon the initial level of consumption $c(0)$ and the rate of increase λ in consumption over time. Let us define the relationships by the functional notation:

$$U(0) = h[c(0), \lambda]. \tag{9}$$

Because of the linear homogeneity of the intertemporal utility functional (1), the function $h[c(0), \lambda]$ is linear homogeneous with respect to $c(0)$; namely,

$$h[\alpha c(0), \lambda] = \alpha h[c(0), \lambda]. \tag{10}$$

Since we have assumed that the intertemporal preference ordering remains invariant over time, we have

$$U(t) = h[c(t), \lambda], \tag{11}$$

where $U(t)$ is by definition the level of the intertemporal utility for the truncated consumption path $c(s)$, for $t \leqq s < \infty$. Hence, from (9), (10), and (11), we get

$$\frac{U(t)}{U(0)} = \frac{c(t)}{c(0)}. \tag{12}$$

Thus the consumption-utility ratio, $x(t) = c(t)/U(t)$, remains constant over time:

$$x(t) = x(0), \qquad \text{for all } t. \tag{13}$$

On the other hand, by differentiating $x(t)$ logarithmically and by taking (6) and (8) into account, we get

$$\frac{\dot{x}(t)}{x(t)} = \lambda + x(t) - \delta[x(t)], \tag{14}$$

which together with the constancy of $x(t)$ yields

$$x(t) = x^*, \tag{15}$$

where x^* is the solution to the following equation:

$$\lambda + x^* = \delta(x^*). \tag{16}$$

Since the schedule of the time preference $\delta(x)$ is negatively sloped, the consumption-utility ratio x^* satisfying (16) is uniquely determined, as illustrated in Figure 2. The level of the intertemporal utility $U(0)$ now is given by

$$U(0) = \frac{c(0)}{x^*} = \frac{c(0)}{\delta(x^*) - \lambda}. \tag{17}$$

As is seen from Figure 2, an increase in the rate of increase in consumption results with a lower rate of time preference $\delta(x^*)$ but with a higher consumption-utility ratio x^*. An increase in the rate of increase in consumption will lower the current level of consumption $c(0)$ relative to the future levels of future consumption, thus making a marginal increase in future consumption less desirable. It will then result in a higher rate of time preference, but an increase in the rate of increase in consumption, with given current consumption $c(0)$, will make the whole path of consumption more preferable, so the present value of consumption will be increased.

3. Time Preference and the Optimum Propensity To Save

The theory of homothetic intertemporal preference orderings may now be applied to examine the structure of each individual unit's consumption and saving functions. Let us consider a consumer unit which, at a certain moment of time 0, possesses an asset whose market value is $a(0)$ in real terms and which expects to receive wages and certain yields (dividends plus capital gains) for the asset it owns. We are interested in the way in which the consumer unit divides its income between current consumption and savings at each moment of time, and in the effects which changes in some of the basic variables exert upon the consumption and saving pattern. For the sake of simplicity, let us assume that the consumer unit's expectations concerning future real wages and interest rates are stationary, kept constant over time, say at w and ρ respectively. The current real income $y(0)$ then becomes

$$y(0) = w + \rho a(0). \tag{18}$$

Future real income is determined by the way real income is divided between consumption and savings; namely, real income $y(t)$ at time t is given by

$$y(t) = w + \rho a(t), \tag{19}$$

where $a(t)$ is the real value of assets held at time t, to be determined by

$$\dot{a}(t) = y(t) - c(t), \tag{20}$$

where $c(t)$ is real consumption planned at time t.

The differential equation (20) may be transformed to one involving real income $y(t)$:

$$\dot{y}(t) = \rho[y(t) - c(t)], \tag{21}$$

with the initial real income $y(0)$ given by (18).

The intertemporal preference ordering associated with the planned path of consumption $c(t)$, $0 \leqq t < \infty$, is represented by the utility functional $U(0)$ defined by (1), and the consumer unit is interested in choosing the time paths of consumption $c(t)$ and asset holdings $a(t)$, $0 \leqq t < \infty$, which satisfy the budget constraints (18–20) and for which the level of the intertemporal utility $U(0)$ is maximized among all feasible consumption paths starting with the initial quantity of assets $a(0)$.

We have assumed that the intertemporal utility functional $U(0)$ is (strictly quasi-) concave and homogeneous of order one, and the consumer unit's expectations concerning real wages and rates of interest are stationary. It is readily observed that, *for any initial real income y_0, the optimum paths of real consumption and asset holdings are uniquely determined and that the consumption-income ratio, $c(t)/y(t)$, remains constant along the optimum path.* This will be proved rigorously as follows:

First, let us suppose there exist two optimum paths of consumption, $c^0(t)$ and $c^1(t)$, starting with the given initial level of real income y_0. Then the weighted average of these two consumption paths:

$$c^\theta(t) = (1 - \theta)c^0(t) + \theta c^1(t), \qquad \text{for some } 0 < \theta < 1,$$

becomes feasible, and, in view of the (strict) convexity of intertemporal indifference surfaces, $c^\theta(t)$ will attain a higher value for the intertemporal utility functional $U(0)$, thus contradicting the optimality of the consumption path $c^0(t)$ or $c^1(t)$.

Second, to see the constancy of the consumption-income ratio $c(t)/y(t)$ along the optimum path, let us introduce the notations $c(t, y_0)$ and $y(t, y_0)$, which are, respectively, the optimum paths of real consumption and real income, starting with the initial level of real income y_0; $y(t, y_0)$ satisfies the differential equation (21) with the initial condition y_0 and consumption $c(t, y_0)$, and the value of the intertemporal utility functional (1) is maximized for the consumption path $c(t, y_0)$ among all feasible paths, starting with the same initial real income y_0. Let us consider another initial level of real income $\bar{y}_0$, and the corresponding optimum paths of consumption and real income, $c(t, \bar{y}_0)$ and $y(t, \bar{y}_0)$. Then the path of real income defined by $by(t, \bar{y}_0)$, where $b = y_0/\bar{y}_0$, becomes a feasible path of real income starting with initial real income y_0 and the consumption path $bc(t, \bar{y}_0)$. It is also seen that the value of the intertemporal utility functional $U(0)$ is maximized at $bc(t, \bar{y}_0)$ among all feasible consumption paths starting with initial income y_0; hence, by the uniqueness property of the optimum paths of consumption and real income, we get $c(t, y_0) = bc(t, \bar{y}_0)$, $y(t, y_0) = by(t, \bar{y}_0)$. That is

$$\frac{c(t, y_0)}{c(t, \bar{y}_0)} = \frac{y(t, y_0)}{y(t, \bar{y}_0)} = \frac{y_0}{\bar{y}_0}, \qquad \text{for all } t. \tag{22}$$

Let us now consider the truncated parts $c(s, y_0)$ and $y(s, y_0)$, $t \leqq s < \infty$ of the optimum path, $c(t, y_0)$ and $y(t, y_0)$. The time path $c(t + \tau, y_0)$ and $y(t + \tau, y_0)$, $0 \leqq \tau < \infty$, now becomes the optimum path for the initial real income $y(t, y_0)$; otherwise, it would be possible to find a feasible path of consumption starting with initial real income y_0. Therefore,

$$c(t + \tau, y_0) = c[\tau, y(t, y_0)], \qquad y(t + \tau, y_0) = y[\tau, y(t, y_0)]. \tag{23}$$

Hence, by applying (22) to (23), we get

$$c(t, y_0) = c[0, \; y(t, y_0)] = \frac{y(t, y_0)}{y(0, y_0)} c(0, y_0); \tag{24}$$

making the consumption-income ratio, $c(t)/y(t)$, constant along the optimum path.

The optimum path then may be found among those for which the consumption-income ratio, $c(t)/y(t)$, or equivalently the saving-income ratio, $1 - [c(t)/y(t)]$, remains constant over time. Let us consider a time path, $c(t)$ and $y(t)$, for which the saving-income ratio remains at the constant ratio s over time:

$$1 - s = c(t)/y(t), \tag{25}$$

then,

$$\dot{y}(t) = \rho s y(t), \tag{26}$$

and

$$\frac{\dot{c}(t)}{c(t)} = \frac{\dot{y}(t)}{y(t)} = \rho s. \tag{27}$$

The time path of consumption $c(t)$ then has a constant rate of increase, and the level of the intertemporal utility functional U_0 is determined as has been worked out in detail above. Let the consumption-utility ratio, x, be obtained by

$$\rho s + x = \delta(x). \tag{28}$$

Then,

$$U_0 = \frac{c_0}{x} = \frac{(1 - s)y_0}{x}, \tag{29}$$

which, together with (28), may be written as

$$\frac{\rho U_0}{y_0} = \frac{\rho - \delta(x)}{x} + 1. \tag{30}$$

The optimum saving-income ratio s may be obtained by first finding the consumption-utility ratio x which maximizes

$$\frac{\rho - \delta(x)}{x}, \tag{31}$$

Such an x may be obtained by solving the following equation:

$$\frac{\rho - \delta(x)}{x} = \delta'(x). \tag{32}$$

The determination of the optimum x may be easily seen from Figure 2, where the vertical axis measures the rate of time preference and the horizontal axis the consumption-utility ratio x. Let OA be equal to the rate of interest ρ and draw the tangent line from A to the schedule of the rate of time preference, intersecting at B. Then the slope of the line AB

with the horizontal axis, measured in the negative direction, gives us the maximized value of (31). If C and D are the points which the perpendicular line going through B intersects with the 45° ray and the horizontal axis, respectively, then $BD = \delta(x)$, $BC = s\rho$. Hence, the optimum saving-income ratio s is obtained by the ratio of BC over AO. Since the optimum saving-time ratio s is uniquely determined by the rate of interest alone, we may denote it as $s(\rho)$. Then the saving and consumption functions are given by:

$$\begin{cases} s(\rho, y) = s(\rho)y \\ c(\rho, y) = [1 - s(\rho)]y. \end{cases} \tag{33}$$

It is easily seen from Figure 2 that an increase in the rate of interest ρ results in an increase in $\rho s(\rho)$, but not necessarily in an increase in $s(\rho)$. The elasticity of the average propensity to save with respect to the rate of interest, then, is greater than -1, but not necessarily positive.

Let E = the intersection of the 45° ray with the schedule of the rate of time preference, A^* = the intersection of the tangent line at E with the vertical axis, and $\rho^* = OA^*$. Hence, if the rate of interest is given as ρ^*, then the optimum propensity to save is zero, and $s(\rho)$ is positive or negative, according to whether ρ is higher or lower than ρ^*. Namely, there exists what may be termed the natural rate of interest, ρ^* (uniquely determined by the schedule of the rate of time preference), for which the following conditions are satisfied for the schedule of the optimum propensity to save $s(\rho)$:

$$s(\rho) \gtreqless 0, \quad \text{according to whether} \quad \rho \gtreqless \rho^*. \tag{34}$$

4. Concept of Real Capital, the Production Function, and the Penrose Curve

The model of economic growth to be constructed in this paper focuses upon the analysis of the investment behavior of business firms, beginning with an examination of the concept of real capital upon which the production function of each individual firm is based.

Any business firm, as a productive agent, consists of a complex of machinery, equipment, tools, and other fixed factors of production, including managerial and administrative abilities, which, when combined with those factors of production readily available in the market, are employed to produce the output of the firm. At each moment of time, the available quantities of variable factors of production are assigned to various types of machinery, equipment, and other fixed means of production in such a way that the value of the resulting output is maximized. The maximum output thus obtained may generally depend upon the administrative, managerial, and engineering abilities of the firm, to be summarized by its short-run production function. Since the discussion throughout the present paper may be carried out in terms of a homogeneous output, as typically assumed in the standard neoclassical theory of economic growth, let us suppose that various kinds of output are always produced and consumed in fixed proportions. It will be further assumed that there is only one kind of variable factor of production—services of homogeneous labor—so that the short-run production function may be illustrated in Figure 3, where the quantity of output is measured along the vertical axis and that of available labor services along the horizontal axis. Let OQ_t represent the short-run production at a certain moment of time t when the firm possesses a certain quantity of each of the fixed factors of production; output AB thus stands for the maximum output which can be produced by the firm by properly allocating the quantity OA of labor services. Let us now consider the short-run production function of the same firm at the next instance, say $t + 1$, when the composition of fixed factors of production comprising the firm is different from that at time t. The maximum quantity of output $B'A$ which can be produced at time $t + 1$ from the available quantity of labor services OA in general differs

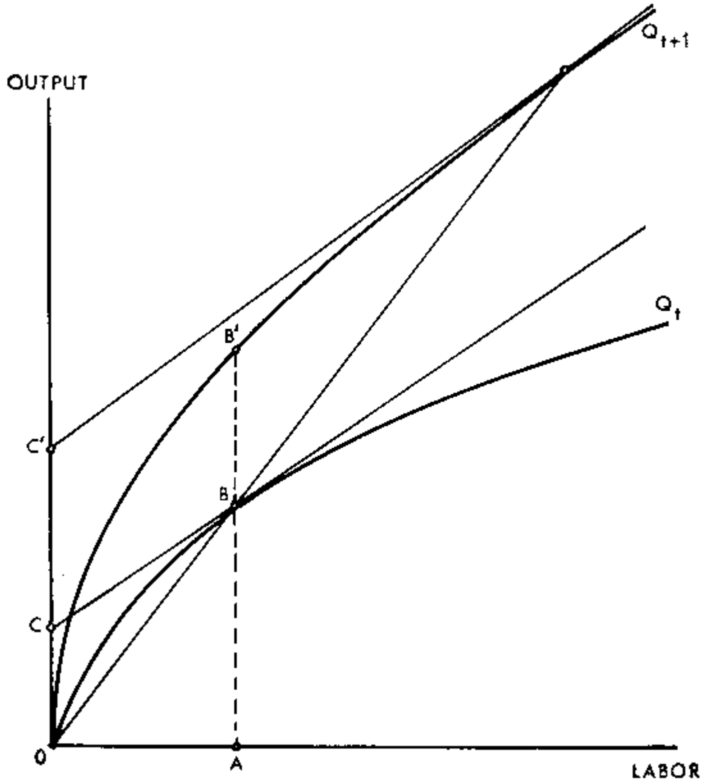

FIG. 3

from OB, thus resulting in a shift in the short-run production curve from OQ_t to OQ_{t+1}. The change in the composition of fixed factors of production from time t to time $t + 1$ involves changes not only in the number of readily measurable machinery, equipment, and so forth, but also in managerial and other abilities generally impossible to quantify. However, the question naturally arises whether it is possible to find an index to measure the extent to which the capacity of the firm as a productive agent has been increased by such changes in the composition of real capital. To examine the circumstances under which such an index may be unambiguously defined, let us consider the amount of the maximum profits the firm can obtain when it is faced with a competitive labor market. For a given real wage rate w, it employs labor services up to the level at which the marginal product of labor is equated to the real wage rate w, and the firm's earning capacity may be measured by the profits to be earned, as indicated by OC or OC' in Figure 3. Because of the shift in the short-run production curve, the profits obtained at time $t + 1$ would be larger than that at time t; the ratio OC' over OC may be used as an index of real capital at $t + 1$ with reference to the base year t; and we may define the index of real capital K_t at time t to satisfy:

$$K_{t+1}/K_t = OC'/OC. \tag{35}$$

The index of real capital thus defined, however, generally depends upon the real wage rate with respect to which total output is obtained. We may assume that *the ratio of profits at time t over that at time t + 1 is defined independently of the real wage rate with respect to which the labor employment is determined.*

The index of real capital at time t, K_t, is then uniquely determined with respect to the base year $t = 0$ when K_0 is assumed to be unity.

It is easily seen, from the way in which the index of real capital K_t is constructed, that the output Q_t at time t is determined by the index of real capital K_t at time t and the quantity of labor services L_t employed at time t:

$$Q_t = F(K_t, L_t), \tag{36}$$

where the production function F remains invariant over time. The above assumption then implies that the production function $F(K, L)$ is *homogeneous of degree one and that the isoquants are all convex toward the origin.*

The per capita output $q_t = Q_t/L_t$ is then a function of the capital-labor ratio $k_t = K_t/L_t$:

$$q_t = f(k_t),$$

for which it is assumed that

$$f(k) > 0, \quad f'(k) > 0, \quad f''(k) < 0, \qquad \text{for all } k > 0. \tag{37}$$

The shift in the short-run production function from OQ_t to OQ_{t+1} is primarily due to changes in the composition of real capital which have been induced by an investment of certain quantities of capital goods. Let us denote by Φ_t the value of such an investment measured in terms of the output produced by the firm (an increase in real investment Φ_t generally results in an increase in the index of real capital K_t, as defined above). The index of real capital K_t reflects the managerial and administrative abilities of the firm as well as the quantities of physical factors of production such as machinery and equipment. The actual increase in the index of real capital K_t due to a certain amount of investment is also constrained by the magnitude and quantities of managerial resources possessed by the firm at that moment. The schedule relating the rate of increase in K_t and the required level of investment will in general shift whenever there is a change in the quantity of real capital. If we suppose that the administrative, managerial, and other abilities which are required by the firm in the process of growth and expansion are present in proportion to the index of

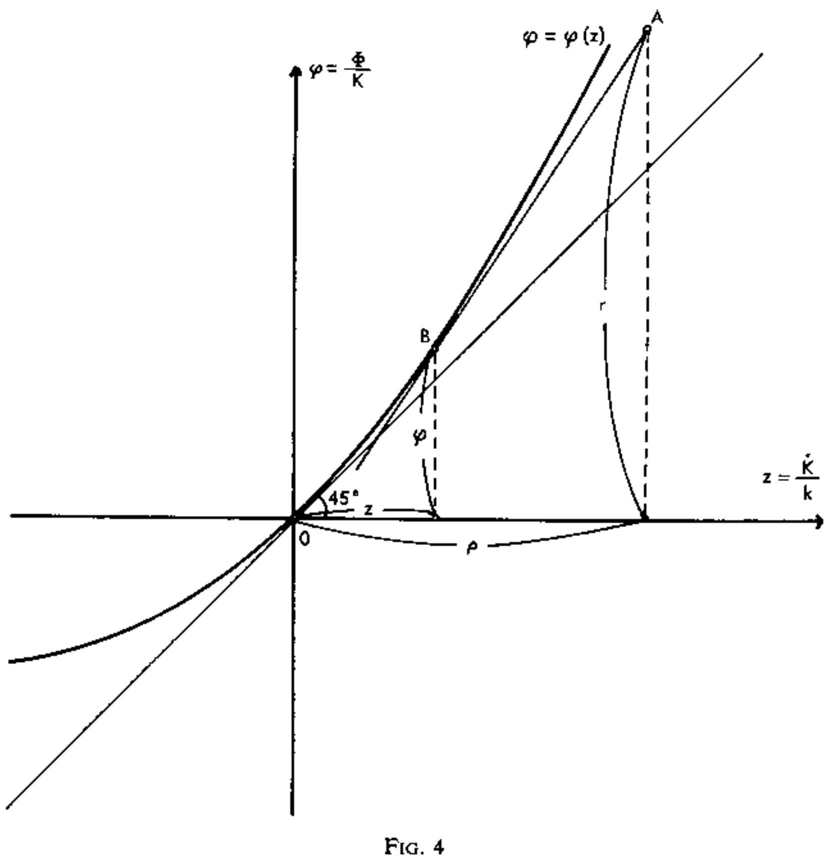

FIG. 4

real capital K_t, the schedule relating the rate of increase, $z = \dot{K}/K$, in real capital with the investment-capital ratio, $\varphi = \Phi/K$, may be assumed to remain invariant over time, independently of the level of real capital possessed by the firm at each moment of time. Such a schedule is in general described by a convex curve, as in Figure 4, where the rate of increase in real capital z and the investment-capital ratio φ are measured along the horizontal and vertical axes, respectively. For the sake of convenience, it will be referred to as the Penrose curve (see Penrose, 1959; and Uzawa, 1968*b*), to be denoted by $\varphi = \varphi(z)$, and it is assumed to satisfy the following conditions:

$$\varphi(z) \geqq 0, \quad \varphi'(z) > 0, \quad \varphi''(z) > 0, \qquad \text{for all } z, \tag{38}$$

particularly reflecting the scarcity of those factors which are indispensable to the firm in the process of growth. The conditions (38) indicate that the higher the rate of increase in real capital, the higher is the level of investment, and that the marginal cost of investment is increasing. We may without loss of generality assume that the Penrose curve has the slope of 45° when the rate of increase in real capital is zero:

$$\varphi(0) = 0, \qquad \varphi'(0) = 1. \tag{39}$$

5. Investment Function and Marginal Efficiency of Investment

The concepts of real capital and the Penrose effect as developed in section 4 will be used to discuss the investment behavior of a competitive firm. Let us consider a firm for which the index of real capital is measured at K_0 at a certain moment of time 0, and which possesses the production function $q = f(k)$ and the Penrose curve $\varphi = \varphi(z)$, as typically illustrated in Figures 3 and 4. To simplify the analysis, let us suppose that the firm has stationary expectations concerning future rates of real wages (w) and interest (ρ). Then the present value of future net cash flows becomes

$$\int_0^\infty [Q(t) - wL(t) - \Phi(t)]e^{-\rho t}dt, \tag{40}$$

where $Q(t) = F[K(t), L(t)]$ is the quantity of output, $K(t)$ and $L(t)$ are, respectively, the levels of real capital and labor employment, and $\Phi(t)$ is the level of investment, all planned for time t. The index of real capital $K(t)$ is increased by $Z(t)$:

$$\dot{K}(t) = Z(t), \tag{41}$$

where the investment-capital ratio $[\varphi(t) = \Phi(t)/K(t)]$ is related to the rate of increase in real capital $[z(t) = Z(t)/K(t)]$, by the Penrose curve:

$$\varphi(t) = \varphi[z(t)]. \tag{42}$$

The firm then is interested in finding the planned paths of real capital, labor employment, and investment for which the present value (40) of the expected future net cash flows is maximized subject to the constraint (41) with the initial level of real capital K_0.

In view of the assumptions (38), it can be shown that *if an optimum path of capital accumulation exists, then it is uniquely determined.* Indeed, suppose there were to exist two different paths of capital accumulation, $K^0(t)$ and $K^1(t)$, starting with K_0, for both of which the present value (40) is maximized. Let $Q^0(t), L^0(t), Z^0(t)$ and $Q^1(t), L^1(t), Z^1(t)$ be, respectively, the corresponding levels of output, labor employment, and increase in real capital. Define the path of capital accumulation $K^\theta(t)$ by $K^\theta(t) = (1-\theta)K^0(t) + \theta K^1(t)$, $0 < \theta < 1$, and the corresponding levels of employment and capital accumulation by $L^\theta(t) = (1-\theta)L^0(t) + \theta L^1(t)$, $Z^\theta(t) = (1-\theta)Z^0(t) + \theta Z^1(t)$. Then, because of the convexity assumption (38), the required level of investment $\Phi^\theta(t)$ satisfies the inequality, $\Phi^\theta(t) \leqq (1-\theta)\Phi^0(t) + \theta\Phi^1(t)$, for all t, with strict inequality for some time interval. Hence, the new path of capital accumulation $K^\theta(t)$ is also feasible and attains a higher present value, thus contradicting the optimality of $K^0(t)$ or $K^1(t)$.

To see the structure of the optimum path of capital accumulation, let us first observe that the optimum level of labor employment $L(t)$ at each moment of time t is so determined as to equate the marginal product of labor to the expected rate of real wages. Since the production function $F(K, L)$ is assumed to be linear homogeneous, the optimum capital-labor ratio $k = K(t)/L(t)$ at each moment of time t is uniquely determined, independently of time t, at the level satisfying the marginality condition:

$$f(k) - kf'(k) = w. \tag{43}$$

If we denote by $r = f'(k)$ the marginal product of real capital corresponding to the optimum capital-labor ratio k, the present value (40) may be reduced to

$$\int_0^\infty \{r - \varphi[z(t)]\}K(t)e^{-\rho t}dt, \tag{44}$$

with

$$\dot{K}(t)/K(t) = z(t), \qquad K_0 \text{ given}. \tag{45}$$

As we have seen above, the optimum path of capital accumulation is uniquely determined, and the optimum rate of increase in the index of real capital $z(0)$ at time 0 may be regarded as a function of $K(0)$, r, and ρ, to be denoted by

$$z(0) = G[K(0), w, \rho]. \tag{46}$$

Let $z(t)$ be the optimum path of the rate of increase in real capital starting with initial $K(0)$, then the truncated path $z(s)$, $t \leqq s < \infty$ also becomes optimum with respect to the initial level of capital $K(t)$, with respect to the same expected rates of real wages w and interest ρ. Hence, we have

$$z(t) = G[K(t), w, \rho]. \tag{47}$$

On the other hand, the present value of net cash flows (44) is a linear homogeneous functional and the function $G(K, w, \rho)$ is homogeneous of order zero, namely

$$z(t) = g(w, \rho), \qquad \text{for all } t, \tag{48}$$

with a certain function $g(w, \rho)$ of w and ρ alone.

Thus, we have shown that, *for any initial level of real capital* K_0 *and for any expected rates of real wages* w *and interest* ρ, *if an optimum path of capital accumulation exists, it is uniquely determined and the rate of capital accumulation,* $z = \dot{K}(t)/K(t)$, *is constant over time.*

The optimum path of capital accumulation now may be found among those paths for which the rate of increase in real capital is constant. If z is a constant rate of increase in K, then the present value (40) per $K(0)$ may be simply reduced to

$$v = \frac{r - \varphi(z)}{\rho - z}. \tag{49}$$

The maximization of the present value v given by (49) is easily done in terms of the Penrose curve in Figure 4. Let A be the point whose coordinates are (ρ, r). Then the present value v represents the slope of the line connecting A and the point on the Penrose curve B, $B = [z, \varphi(z)]$ corresponding to the planned rate of increase z in real capital. Therefore, the maximum value of v is attained when the point B is chosen in such a way that the line AB is tangent to the Penrose curve. Analytically, the optimum rate of increase in real capital z is obtained by solving the following marginality condition:[3]

$$\frac{r - \varphi(z)}{\rho - z} = \varphi'(z). \tag{50}$$

Since the optimum rate of increase in real capital z and the optimum investment-capital ratio φ are uniquely determined by the rate of interest ρ and the rate of profit r, we may use the functional notation:

$$z = z(\rho, r), \qquad \varphi = \varphi(\rho, r). \tag{51}$$

As is seen in Figure 4, an increase in the rate of profit r or a decrease in

[3] As is seen from (50), the deviation of the investment function is an application of the principle of marginal efficiency of investment of Keynes.

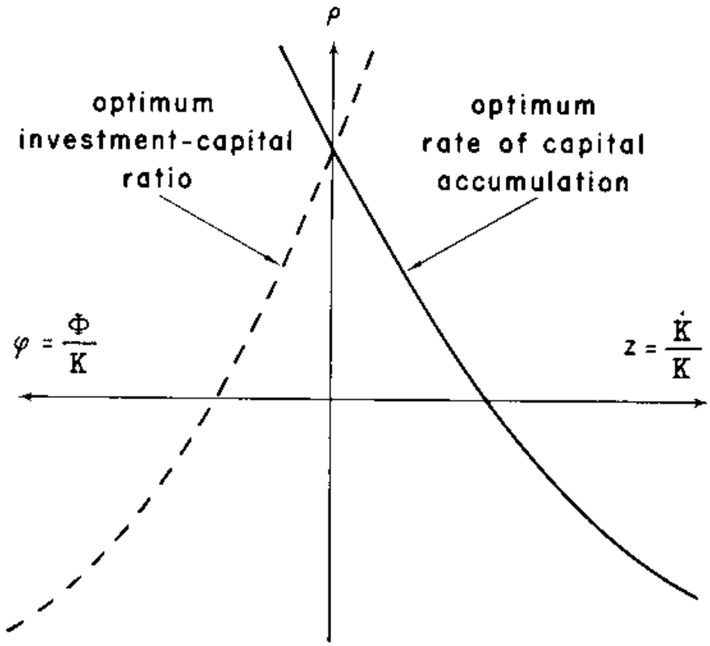

Fig. 5

the rate of interest ρ always increases both the optimum rate of capital accumulation and the optimum investment-capital ratio; namely,

$$\begin{cases} \dfrac{\partial z}{\partial \rho} < 0, & \dfrac{\partial z}{\partial r} > 0, \\ \dfrac{\partial \varphi}{\partial \rho} < 0, & \dfrac{\partial \varphi}{\partial r} > 0. \end{cases} \tag{52}$$

We can also see that the optimum rate of capital accumulation $z(\rho, r)$ is positive or negative according to whether the rate of interest ρ is smaller or larger than the rate of profit r:

$$z(\rho, r) \gtreqless 0, \qquad \text{according to } \rho \lesseqgtr r. \tag{53}$$

For the given rate of profit r, the schedule of the optimum investment-capital ratio and the optimum rate of increase in real capital are typically illustrated by the curves in Figure 5.

6. An Aggregate Model of Economic Growth

The analysis of the saving and investment behaviors of individual economic units as introduced in the previous sections may now be applied to examine the pattern of equilibrium growth for an aggregative economy. The basic structure of the model presented here is similar to the one discussed in detail in another paper (Uzawa, 1968*b*). At the risk of repetition, however, the premises on which the following model is built are briefly outlined.

The aggregative economy here is visualized as composed of two classes of economic units: households on the one hand, and business firms on the other. Households are the owners of labor and assets; their income consists of wages for the labor they provide and the interest and dividend payments for the assets they hold, and they divide their income between consumption and savings, the latter in the form of an increase in the asset holdings. On the other hand, business firms are engaged in the production of goods and services; they employ labor and other factors of production and determine the levels of output and employment in order to maximize the present value of future profits, to be discounted in terms of expected rates of interest. It is assumed that firms finance their investment through an issuance of shares.

At each moment of time, aggregate demand consists of the value of consumption goods demanded by households and the level of investment planned by business firms. The total output which the corporate sector plans to produce depends upon the level of the aggregate demand which it expects to get for its output, and the quantities of labor and other productive factors which the corporate sector desires to employ are accordingly determined. The rewards for these factors of production then constitute the national income, which in turn determines the actual level of the aggregate demand, together with actual consumption and investment. If we assume that the goods-and-services market, the labor market, and the share market are all perfectly competitive, then the economy attains a short-run equilibrium where the planned aggregate demand is equated to the actual level. The level of investment corresponding to such a short-run equilibrium then determines the rate by which real capital is accumulated.

In order to examine the process of capital accumulation in such an equilibrium system, let us postulate that the aggregate behavior of the household sector and that of the corporate sector are respectively explained in terms of the representative household and business firm, both of which possess the structure described in the previous sections.

It will be further assumed that output and labor are composed of homogeneous quantities, respectively, so it is possible to measure them in unambiguous terms. At each moment of time t, we assume as given the level of real capital K_t accumulated in the corporate sector (capital being defined in the way introduced in the above section), the quantity of labor available N_t, and the outstanding number of shares issued B_t, together with the conditions governing the processes of production and the choice of desirable consumption patterns. The aggregate income Y_t is composed of wages W_t, dividends D_t, and expected capital gains G_t^e:

$$Y_t = W_t + D_t + G_t^e. \tag{54}$$

Let p_t be the market price of a share. Then the value V_t of the share holdings at time t is given by

$$V_t = p_t B_t.$$

Hence, the rate of interest ρ_t prevailing in the securities market is denoted by

$$\rho_t = \frac{D_t + G_t^e}{V_t}, \tag{55}$$

and the national income may be written as

$$Y_t = W_t + \rho_t V_t. \tag{56}$$

The desired levels of consumption C_t and savings S_t are then described in terms of the consumption and saving functions derived in section 3, namely,

$$C_t = [1 - s(\rho_t)] Y_t, \qquad S_t = s(\rho_t) Y_t, \tag{57}$$

where $s(\rho_t)$ is the average propensity to save.

The number of new shares which the household sector as a whole desires to purchase, $\dot{B}_t^D$, is now given by

$$\dot{B}_t^D = \frac{S_t - G_t^e}{p_t}. \tag{58}$$

On the other hand, the desired level of investment Φ_t in the corporate sector is determined in the manner described in section 5. The investment per unit of real capital, $\varphi_t = \Phi_t/K_t$, depends upon the rate of profit, r_t, and the market rate of interest, ρ_t:

$$\Phi_t/K_t = \varphi(\rho_t, r_t), \tag{59}$$

where $\varphi(\rho, r)$ is the investment function derived from the Penrose theory of capital.

The aggregate supply of goods and services, Q_t, is determined in terms of the short-run production function: $Q_t = F(K_t, L_t)$, where labor is employed at the level L_t at which the marginal product of labor is equal to the real wage rate w_t.

The aggregate output Q_t is distributed as:

$$Q_t = W_t + D_t + RP_t, \tag{60}$$

where RP_t is the retained profit.

The number of new shares to be issued by the corporate sector, B_t^S, is then given:

$$\dot{B}_t^S = \frac{\Phi_t - RP_t}{p_t}. \tag{61}$$

The goods-and-services market is then at an equilibrium when the following condition is satisfied:

$$C_t + \Phi_t = Q_t. \tag{62}$$

The equilibrium conditions for the labor market and the share market are, respectively, given by:

$$L_t = N_t, \tag{63}$$

and

$$\dot{B}_t^D = \dot{B}_t^S. \tag{64}$$

The rate of capital accumulation z_t is determined relative to the equilibrium investment ratio φ_t through the Penrose curve, as described in section 5:

$$\dot{K}_t/K_t = z_t, \tag{65}$$

while the supply of labor is assumed to be inelastic and increasing at a certain rate, say n, to be exogenously given:

$$\dot{N}_t/N_t = n. \tag{66}$$

The dynamic structure of the equilibrium growth is now completely determined by the pair of differential equations (65) and (66). To analyze the structure of such a system, let us first reduce the equilibrium conditions (62–64) to those involving per capita quantities only.

7. Analysis of the Growth Equilibrium

It may be useful first to note that the equilibrium conditions (62) and (64) for the goods-and-services market and the securities market are interrelated in the sense that, if either one of these markets is in equilibrium, the other must automatically be in equilibrium. To verify this Walras Law, it suffices to rearrange the equations (54), (57), and (60) to obtain

$$p_t\dot{B}_t^S = (C_t + \Phi_t - Q_t) + (S_t - G_t^e),$$

which, together with the relation (58), shows that (62) and (64) are related to each other.

Let us introduce the following per capita variables:

$k_t = K_t/N_t$: the aggregate capital-labor ratio,
$q_t = Q_t/N_t$: per capita real net national product,
$y_t = Y_t/N_t$: per capita real national income,
$b_t = B_t/N_t$: the number of outstanding shares per capita,
$v_t = V_t/N_t$: the market value of shares per capita, $v_t = p_t b_t$.

Then, the real wage rate w_t and the rate of profit r_t under the full-employment condition (63) are determined respectively by

$$w_t = f(k_t) - k_t f'(k_t), \tag{67}$$

and

$$r_t = f'(k_t), \tag{68}$$

where $f(k)$ is the per capita production function.

The per capita net national product q_t and national income y_t are given by:

$$q_t = f(k_t), \tag{69}$$

and

$$y_t = w_t + \rho_t v_t. \tag{70}$$

The concept of real national income y_t which has been adopted here involves the expected capital gains, and it may not necessarily coincide with per capita real national product y_t. However, since we assume that whatever profits retained in the corporate sector are always reflected in capital gains, real income y_t in fact equals net national product q_t:

$$y_t = f(k_t). \tag{71}$$

It may be noted that the following conclusions remain valid for the case in which the expected capital gains are adapitively adjusted to the actual capital gains, provided the familiar qualifications are imposed upon the speed of adjustment.

Then the short-run equilibrium conditions discussed above are reduced to the following single equation:

$$\varphi(\rho_t, r_t) = s(\rho_t)\,\frac{y_t}{k_t}, \tag{72}$$

which corresponds to what Harrod (1948) termed the fundamental equation. The left-hand side of the equation (72) denotes the desired level of investment per unit of real capital, while the right-hand side indicates the amount of savings per unit of real capital which the community as a whole is willing to make when the market rate of interest is ρ_t. The short-run equilibrium is attained when the market rate of interest ρ_t is adjusted to equate both sides of (72). The dynamic system is then described by

$$\dot{k}_t/k_t = z(\rho_t, r_t) - n, \tag{73}$$

where $z(\rho_t, r_t)$ is the equilibrium rate of capital accumulation.

The analysis of the short-run equilibrium may be done in terms of the Hicksian technique as illustrated in Figure 6. Let the rate of interest ρ be measured along the vertical axis and let the investment-capital ratio or the saving-capital ratio be measured along the horizontal axis.

The desired level of investment per unit of capital, $\varphi(\rho, r_t)$, is increased whenever the market rate of interest ρ is increased as shown by curve II in Figure 6. In general, $s(\rho)y_t/k_t$ increases as the rate of interest ρ is increased; then it starts to decline, as typically illustrated by curve SS. Hence, the equilibrium rate of interest ρ_t is determined uniquely by the intersection of curves II and SS, leaving a certain possibility of multiple equilibria. In what follows, let us concentrate upon the case in which the equilibrium rate of interest is uniquely determined.

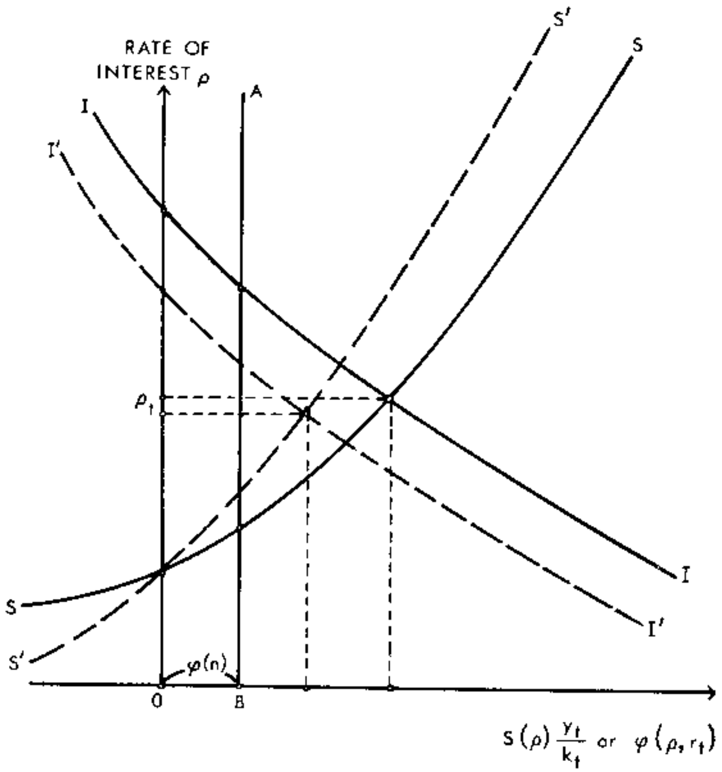

FIG. 6

As the aggregate capital-labor ratio k_t is increased, the average output y_t/k_t and the rate of profit r_t are both decreased, thus shifting curve II and curve SS to the left, as indicated by the dashed curves in Figure 6. Hence, the new equilibrium rate of investment is definitely decreased. The equilibrium rate of interest may be either increased or decreased depending upon the way these curves shift.

Since the dynamics of such an economy is described by the differential equation (73), it remains at a steady state if, and only if, the aggregate capital-labor ratio is kept at the level k^* for which the equilibrium rate of investment equals the level $\varphi(n)$ corresponding to the rate of labor growth n. Let us draw a vertical line AB in Figure 6 for the rate of investment $\varphi(n)$. Then we can easily see from Figure 6 that the aggregate capital-labor ratio k_t tends to increase ($\dot{k}_t > 0$) whenever k_t is lower than the long-run equilibrium ratio k^*, and it tends to decrease ($\dot{k}_t < 0$) if k_t is higher than k^*. If the initial aggregate capital-labor ratio k_0 is lower than the long-run ratio k^*, the equilibrium rate of investment is higher than the level which is required to maintain the capital-labor ratio intact, thus the aggregate capital-labor ratio tends to increase while the equilibrium rate of investment continues to decrease. The aggregate capital-labor ratio k_t then approaches the long-run ratio k^*, and the rate of investment continues to fall to approach the long-run equilibrium level $\varphi(n)$.

The long-run equilibrium capital-labor ratio k^* is determined by the following two equations:

$$\varphi(\rho^*, r^*) = \varphi(n), \tag{74}$$

$$s(\rho^*) \frac{y^*}{k^*} = \varphi(n), \tag{75}$$

when $y^* = f(k^*)$, $r^* = f'(k^*)$, and ρ^* is the long-run equilibrium rate of interest.

Equation (74) requires that the rate of investment the corporate sector desires to make must be equal to $\varphi(n)$. A higher rate of interest ρ^* has to be accompanied by a higher rate of profit r^*, thus by a lower capital-labor ratio k^*, in order to maintain the rate of investment at the level $\varphi(n)$. Therefore, the combinations (ρ^*, k^*) which satisfy (74) are described by a downward sloping curve AA' in Figure 7, where the two axes represent the

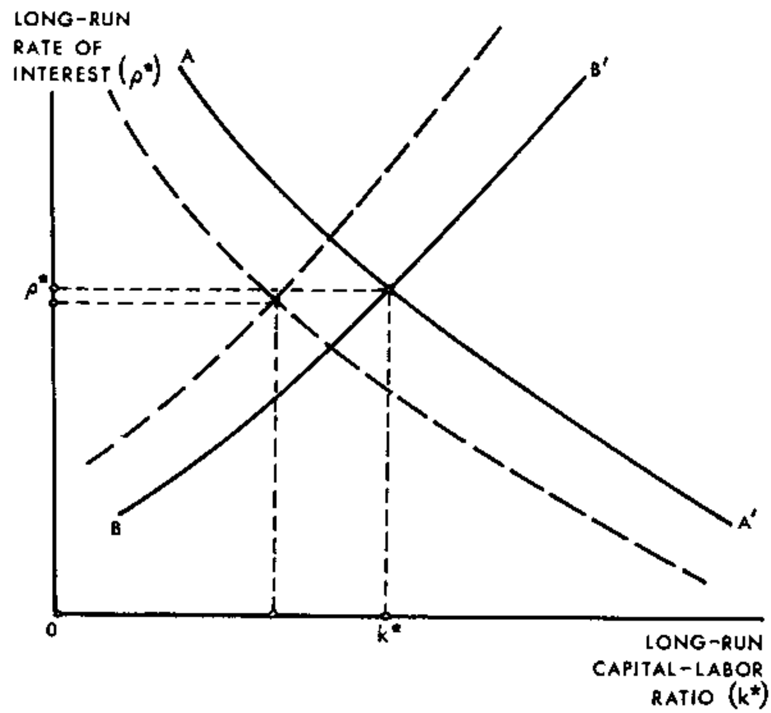

FIG. 7

long-run rate of interest and capital-labor ratio, respectively. On the other hand, equation (75) denotes that the desired level of savings per unit of capital has to be at $\varphi(n)$. The combinations of (ρ^*, k^*) which meet the requirement (75) are represented by an upward sloping curve such as BB', since a higher rate of interest ρ^* must be accompanied by lowering the average output y^*/k^*, hence in a higher capital-labor ratio k^*, in order to result in the long-run level $\varphi(n)$. The long-run rate of interest ρ^* and capital-labor ratio k^* are then uniquely determined.

In general, the long-run equilibrium is determined once we specify the Penrose curve, the production function, the Fisherian schedule of time preference, and the rate of labor growth n. The effect of a change or shift in one of these factors upon the long-run capital-labor ratio k^* may be analyzed by using curves AA' and BB' in Figure 7. For example, suppose the rate of labor growth n has been increased. Then, we can see from the structure of the investment and saving functions that curves AA' and BB' both shift to the left, thus resulting in a decrease in the long-run capital-labor ratio k^*.

8. Concluding Remarks

In this paper we have formulated an aggregative model of economic growth for which the postulates concerning investment and savings are derived from those involving the behavior of individual economic units. The saving behavior of the representative household has been examined in terms of the Fisherian theory of time preference along the lines of the Koopmans reformulation. The investment behavior of business firms in the corporate sector has been based upon a concept of real capital which measures the productive capacity of each business firm as an organic entity. The central role in the derivation of the investment function of a firm in a perfectly competitive market has been played by what has been termed the Penrose curve reflecting the amount of endowments within each firm of those productive factors which are limitational to the firm in the process of growth. Then a growth model has been constructed by postulating that the aggregative behavior of the household sector or of the corporate sector may be described by that of the representative household or business firm, each of which possesses the structure thus specified.

The model being constructed here is restricted to the aggregative national economy in which output, labor, and capital are all composed of homogeneous quantities. The nature of the securities market also has been limited to that of a share market in which no attention has been paid to the uncertainties with respect to expected capital gains. However, the most serious limitation of the present analysis is the hypothesis that the aggregative behavior of each of two major sectors of the national economy may be explained in terms of the representative unit which behaves itself in a way similar to each individual unit. It might be less objectionable for a static or stationary analysis, but an economic model which is purportedly analyzing the mechanism of a growing economy would be deemed questionable if enough attention were not paid to the process of aggregation.

References

Fisher, I. *The Theory of Interest.* New York: MacMillan, 1930.

Hahn, F. H. and Matthews, R. C. O. "The Theory of Economic Growth: A Survey," *Econ. J.*, LXXIV (1964), 779–902.

Harrod, R. F. *Towards A Dynamic Economics.* London: MacMillan, 1948.

Koopmans, T. C. "Stationary Ordinary Utility and Impatience," *Econometrica*, XXVIII (1960), 287–309.

Penrose, T. E. *The Theory of the Growth of the Firm.* Oxford: Blackwell, 1959.

Uzawa, H. "Time Preference, the Consumption Function, and Optimum Asset Holdings," in J. N. Wolfe (ed.). *Papers in Honour of Sir John Hicks: Value, Capital, and Growth.* Edinburgh: University Press, 1968. Pp. 485–504. (*a*)

Uzawa, H. "The Penrose Effect and Optimum Growth," *Econ. Studies Q.*, XIX (1968), 1–14. (*b*)