Disasters Implied by Equity Index Options

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Summary

Problem: disasters infrequent $\Rightarrow$ hard to estimate their distribution

Solution: infer from option prices (market prices of bets on disasters)

What we find

- disasters apparent in options data
- more modest than disasters in macro data

Outline

Preliminaries: entropy, AJ bound, cumulants

Disasters in macroeconomic data

Disasters in options data

Extensions

Entropy

Hans-Otto Georgii (quoted by Hansen and Sargent):

> *When Shannon had invented his quantity and consulted von Neumann on what to call it, von Neumann replied: "Call it entropy. It is already in use under that name and, besides, it will give you a great edge in debates because nobody knows what entropy is anyway."*

Alvarez-Jerman bound

Entropy: for $x > 0$

$$L(x) \;\; \equiv \;\; \log Ex - E \log x \;\; \geq \;\; 0$$

AJ bound

$$L(m) \;\; \geq \;\; E \left(\log r^{j} - \log r^{1} \right)$$

Cumulants

Cumulant generating function

$$k(s;x) \;=\; \log Ee^{sx} \;=\; \sum_{j=1}^{\infty} \kappa_j(x) s^j / j!$$

Cumulants are almost moments

$$\begin{aligned} \text{mean} &= \kappa_1 \\ \text{variance} &= \kappa_2 \\ \text{skewness} &= \kappa_3 / \kappa_2^{3/2} \\ \text{(excess) kurtosis} &= \kappa_4 / \kappa_2^2 \end{aligned}$$

Entropy and cumulants

Entropy of pricing kernel

$$L(m) \;=\; \log E e^{\log m} - E \log m \;=\; \sum_{j=2}^{\infty} \kappa_j(\log m)/j!$$

Zin's "never a dull moment" conjecture

$$L(m) \;=\; \underbrace{\kappa_2(\log m)/2!}_{\text{(log)normal term}} + \underbrace{\kappa_3(\log m)/3! + \kappa_4(\log m)/4! + \cdots}_{\text{high-order cumulants (incl disasters)}}$$

Plan of attack

Modelling assumptions

- iid
- Tight link between consumption growth and equity returns
- Representative agent with power utility [if needed]

Parameter choices

- Match mean and variance of log consumption growth
- Ditto log equity return
- Base “disasters” on Barro’s macroeconomic evidence
- Or on equity index options

Compare macro- and option-based examples

Macro disasters: environment

Consumption growth and "equity" return

$$\begin{aligned} g_{t+1} &= c_{t+1}/c_t \\ d_t &= c_t^{\lambda} \\ \log r^e_{t+1} &= \text{constant} + \lambda \log g_{t+1} \end{aligned}$$

Power utility

$$\log m_{t+1} = \log \beta - \alpha \log g_{t+1}$$

Yaron's "bazooka"

$$\kappa_j(\log m)/j! = \kappa_j(\log g)(-\alpha)^j/j!$$

Macro disasters: Poisson-normal mixture

Consumption growth

$$
\begin{aligned}
\log g_{t+1} &= w_{t+1} + z_{t+1} \\
w_{t+1} &\sim \mathcal{N}(\mu, \sigma^2) \\
z_{t+1}|j &\sim \mathcal{N}(j\theta, j\delta^2) \\
j &\geq 0 \text{ has probability } e^{-\omega}\omega^j/j!
\end{aligned}
$$

Parameter values

- Match mean and variance of log consumption growth
- Jump probability ($\omega = 0.01$), mean ($\theta = -0.3$), and variance ($\delta^2 = 0.15^2$) [similar to Barro, Nakamura, Steinsson, and Ursua]

Macro disasters: entropy

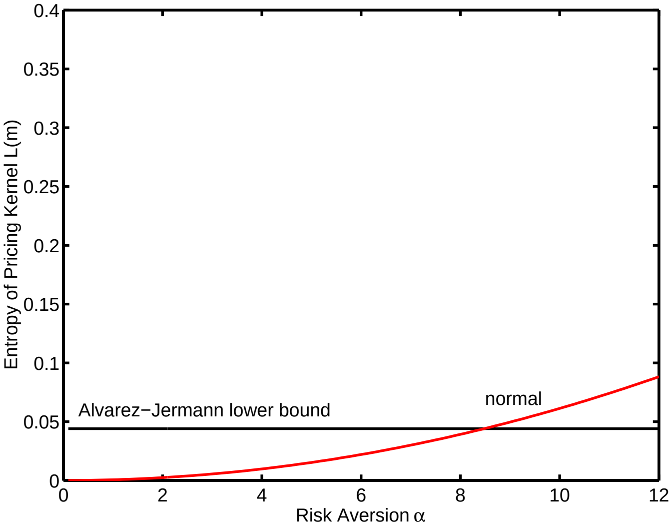
Cumulant generating functions

$$
\begin{aligned}
k(s;w) &\equiv \log E e^{sw} = s\mu + (s\sigma)^2/2 \\
k(s;z) &\equiv \log E e^{sz} = \omega\left(e^{s\theta+(s\delta)^2/2} - 1\right)
\end{aligned}
$$

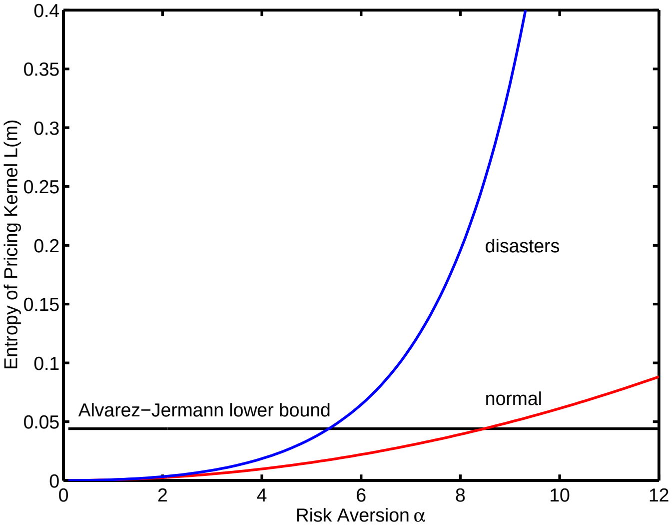
Entropy

$$
L(m) = (-\alpha\sigma)^2/2 + \omega\left(e^{-\alpha\theta+(\alpha\delta)^2/2} - 1\right) + \alpha\omega\theta,
$$

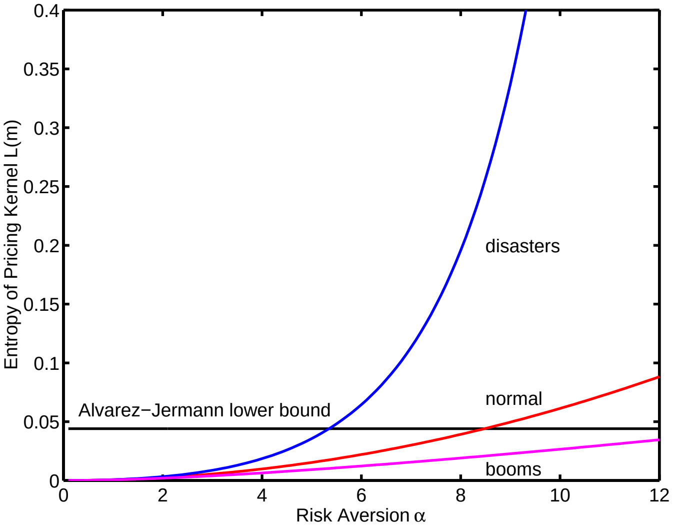
Macro disasters: entropy



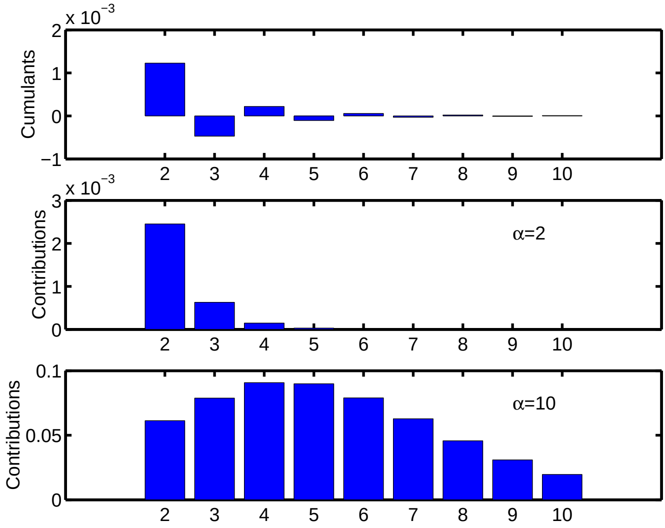
Macro disasters: entropy



Macro disasters: entropy



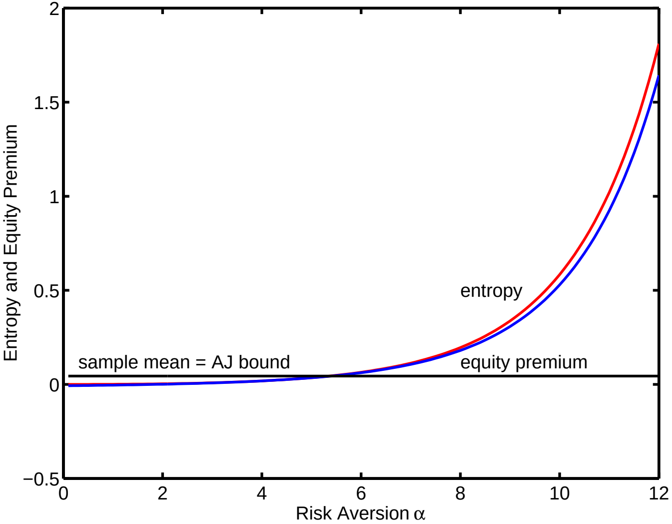
Macro disasters: cumulants



Macro disasters: cumulants

			High-Order Cumulants	
Model ($\alpha = 10$)	**Entropy**	Variance/2	Odd	Even
Normal	**0.0613**	0.0613	0	0
Poisson disaster	**0.5837**	0.0613	0.2786	0.2439
Poisson boom	**0.0266**	0.0613	**–0.2786**	0.2439

Macro disasters: equity premium



Option disasters: overview

Options an obvious source of information, but ...

- Options on equity, not consumption
- Determine risk-neutral, not true distribution
- True distribution has the usual lack of data problems

Plan of attack

- Estimate risk-neutral distribution from options
- Estimate true distribution two ways
- Compare options implied by macro-based disaster model

Option disasters: overview

Options an obvious source of information, but ...

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Risk-neutral probabilities

Notation: states x have (true) probabilities $p(x)$

Risk-neutral probabilities p^*

$$\begin{aligned} p^*(x) &= p(x)m(x)/q^1 \\ m(x) &= q^1 p^*(x)/p(x) \\ q^1 &= Em \quad \text{(1-period bond price)} \end{aligned}$$

Entropy (aka "relative entropy" or "Kullback-Leibler divergence")

$$L(m) = L(p^*/p) = E\log(p/p^*)$$

Risk-neutral probabilities: examples

Normal log consumption growth

- If $\log g \sim \mathcal{N}(\mu, \sigma^2)$ (true distribution)
- Then risk-neutral distribution also lognormal with $\mu^* = \mu - \alpha\sigma^2, \sigma^* = \sigma$

Poisson log consumption growth

- Jumps have probability ω and distribution $\mathcal{N}(\theta, \delta^2)$
- Risk-neutral distribution has same form with $\omega^* = \omega \exp[-\alpha\theta + (\alpha\delta)^2/2], \theta^* = \theta - \alpha\delta^2, \delta^* = \delta$

Option disasters: information in option prices

Put option (bet on low returns)

$$q_t^p \;=\; q_t^1 E_t^* (b - r_{t+1}^e)^+$$

Strategy

- Estimate p^* by varying strike price b (cross section)
- Estimate p and q^1 from time series data

Black-Scholes-Merton benchmark

- Quote prices as implied volatilities (high price $\Leftrightarrow$ high vol)
- Horizontal line if lognormal
- "Skew" suggests disasters

Option disasters: Merton model

Equity returns iid

$$
\begin{aligned}
\log r^e_{t+1} &= \log r^1 + w_{t+1} + z_{t+1} \\
w_{t+1} &\sim \mathcal{N}(\mu, \sigma^2) \\
z_{t+1}|j &\sim \mathcal{N}(j\theta, j\delta^2) \\
j &\geq 0 \text{ has probability } e^{-\omega}\omega^j/j!
\end{aligned}
$$

Risk-neutral distribution: ditto with *s

Option disasters: parameter values

Choose $(\mu, \sigma, \omega, \theta, \delta)$ to match distribution of equity returns

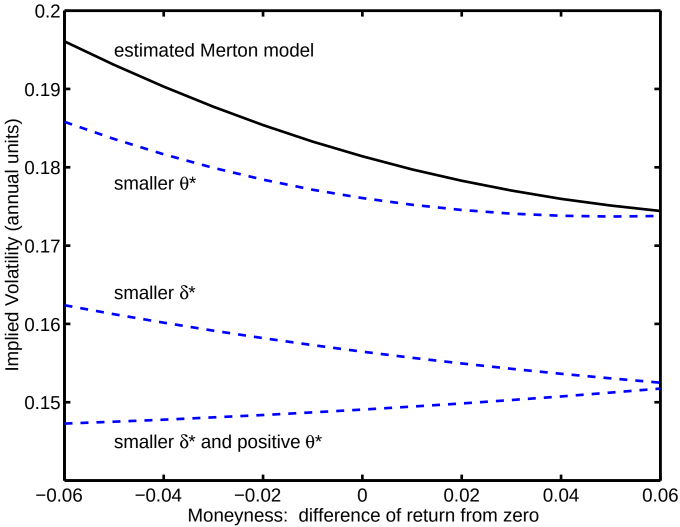
- Jumps: $\omega = 1.512$, $\theta = -0.0259$, $\delta = 0.0229$
- Equity premium: $\mu + \omega\theta$
- Variance of equity returns: $\sigma^2 + \omega(\theta^2 + \delta^2)$

Set $(\omega^*, \theta^*, \delta^*)$ to match option prices

- Jumps: $\omega^* = \omega$, $\theta^* = -0.0482$, $\delta^* = 0.0981$
- Set $\sigma^* = \sigma$
- Set μ^* to satisfy pricing relation $(q^1 E^* r^e = 1)$

All of this from Broadie, Chernov, and Johannes (JF, 2007)

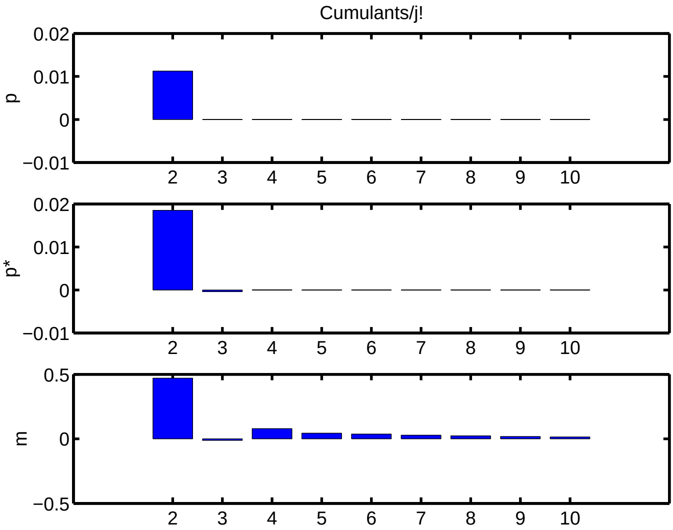
Option disasters: implied volatility



Option disasters: components of entropy

			High-Order Cumulants	
Model	Entropy	Variance/2	Odd	Even
Consumption-based models				
Normal ($\alpha = 10$)	0.0613	0.0613	0	0
Poisson ($\alpha = 10$)	0.5837	0.0613	0.2786	0.2439
Poisson ($\alpha = 5.38$)	0.0449	0.0177	0.0173	0.0099
Option-based model				
Option model	**0.7647**	0.4699	**0.1130**	**0.1819**

Option disasters: cumulants



Comparing macro- and option-based models

Entropy and cumulants of pricing kernel

Consumption growth implied by option prices

- Scale option-based p^* to consumption
- Find p using power utility
- Result: more modest skewness and kurtosis, tail probabilities

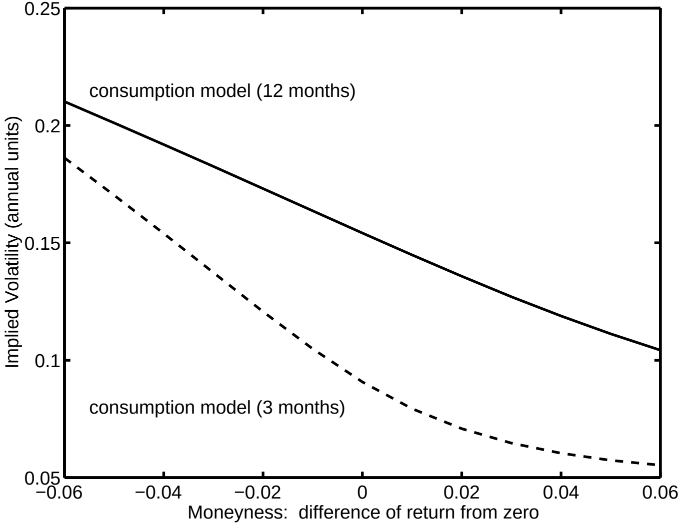
Option prices implied by consumption growth

- Find macro-based p^* using power utility
- Scale to equity returns
- Compute option prices
- Result: steeper volatility smile

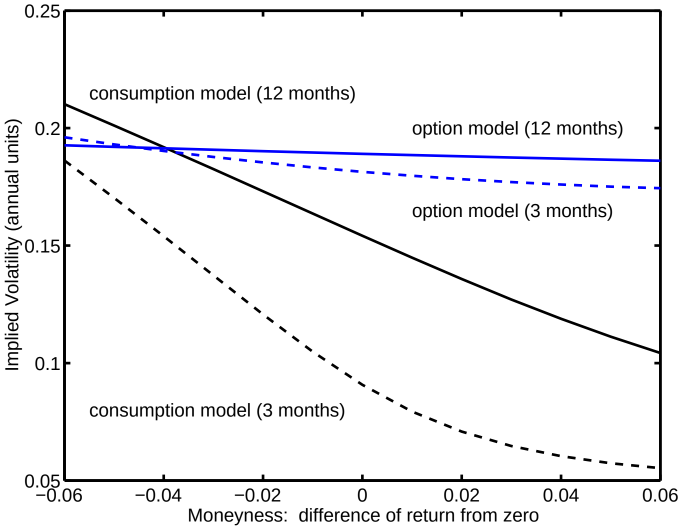
Comparing models: consumption implied by options

	Consumption Process Based on	
	Cons Growth	Option Prices
α	5.38	10.07
ω	0.0100	1.3864
θ	–0.3000	–0.0060
δ	0.1500	0.0229
Skewness	–11.02	–0.31
Excess Kurtosis	145.06	0.87
Tail prob (≤ -3 st dev)	**0.0090**	**0.0086**
Tail prob (≤ -5 st dev)	**0.0079**	**0.0002**

Comparing models: options implied by consumption



Comparing models: options implied by consumption



Risk aversion in the option model

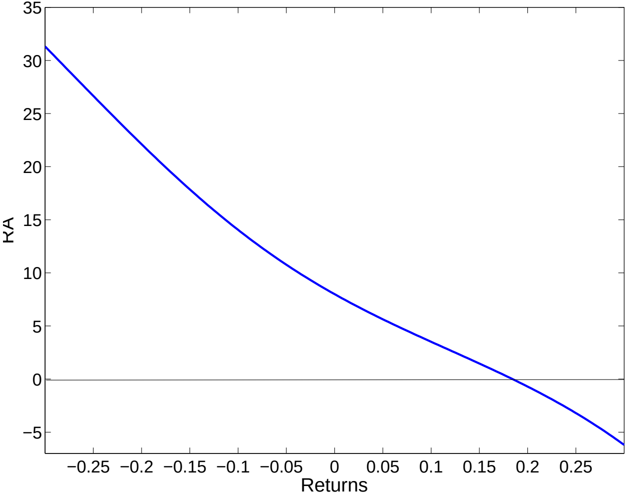
"Risk aversion" implied by arbitrary pricing kernel

$$\text{RA} \quad \equiv \quad -\frac{\partial \log m}{\partial \log g} \quad = \quad -\frac{\partial \log(p^*/p)}{\partial \log r^e} \cdot \frac{\partial \log r^e}{\partial \log g}$$

Obviously not power utility

- Risk aversion not constant ("state dependent")
- Parameters imply greater aversion to adverse risks

Risk aversion in the option model



Bottom line

Barro, Longstaff & Piazzesi, Rietz

- Disasters contribute to equity premium, entropy
- Evident in macro data

We look at options

- Smile/smirk suggests something like disasters
- But more modest than macro data
- High entropy suggests it's not enough to match equity premium

Open questions

Sources of apparent risk aversion

- Exotic preferences
- Heterogeneous agents
- Examples: Alvarez, Atkeson, and Kehoe; Bates; Chan and Kogan; Du; Guvenen; Lustig and Van Nieuwerburgh

Consumption and dividends

- Examples: Bansal and Yaron, Gabaix, Longstaff and Piazzesi

Time dependence

- Short rate, predictable returns, stochastic volatility
- Examples: Drechsler and Yaron, Wachter, Shaliastovich

Time dependence: irrelevant?

Cochrane and Hansen

$$\text{Var}(m) \quad = \quad E\text{Var}_t(m_{t+1}) + \underbrace{\text{Var}[E_t(m_{t+1})]}_{\text{small}}$$

Analog for entropy

$$L(m) \quad = \quad EL_t(m_{t+1}) + \underbrace{L[E_t(m_{t+1})]}_{\text{ditto}}$$

Picture something like this

$$\log m \quad = \quad \underbrace{\text{white noise}}_{\text{big}} + \underbrace{\text{predictable component}}_{\text{small}}$$

Time dependence: entropy

Alvarez-Jermann bound

$$L(m) \quad \geq \quad E\left(\log r^j - \log r^1\right) + L(q^1)$$

Conditional and unconditional entropy

$$L(m) \quad = \quad EL_t(m_{t+1}) + L(q^1)$$

Mean conditional entropy ("drift irrelevant")

$$EL_t(m_{t+1}) \quad \geq \quad E\left(\log r^j - \log r^1\right)$$

Time dependence: examples

Explore mechanisms for magnifying entropy

- Recursive preferences
- Habits
- Heterogeneous preferences [maybe later]

Exploits loglinearity of entropy and (many) asset pricing models

Thanks to Stan Zin for much of this

Recursive preferences: traditional version

Equations (Kreps-Porteus/Epstein-Zin/Weil)

$$
\begin{aligned}
U_t &= \left[(1-\beta)c_t^{\rho} + \beta\mu_t(U_{t+1})^{\rho}\right]^{1/\rho} \\
\mu_t(U_{t+1}) &= \left(E_t U_{t+1}^{\alpha}\right)^{1/\alpha} \\
IES &= 1/(1-\rho) \\
CRRA &= 1-\alpha \\
\alpha &= \rho \quad \Rightarrow \quad \text{additive preferences}
\end{aligned}
$$

Note: weakly separable, not additively separable

Recursive preferences: pricing kernel

Scale problem by c_t ($u_t = U_t/c_t$, $g_{t+1} = c_{t+1}/c_t$)

$$u_t = [(1-\beta) + \beta\mu_t(g_{t+1}u_{t+1})^\rho]^{1/\rho}$$

Pricing kernel (mrs)

$$\begin{aligned} m_{t+1} &= \beta \left(\frac{c_{t+1}}{c_t}\right)^{\rho-1} \left(\frac{U_{t+1}}{\mu_t(U_{t+1})}\right)^{\alpha-\rho} \\ &= \beta\, g_{t+1}^{\rho-1} \left(\frac{g_{t+1}u_{t+1}}{\mu_t(g_{t+1}u_{t+1})}\right)^{\alpha-\rho} \end{aligned}$$

Recursive preferences: loglinear approximation

Approximation

$$\begin{aligned}\log u_t &= \rho^{-1}\log\left[(1-\beta)+\beta\mu_t(g_{t+1}u_{t+1})^{\rho}\right]\\ &= \rho^{-1}\log\left[(1-\beta)+\beta e^{\rho\log\mu_t(g_{t+1}u_{t+1})}\right]\\ &\approx b_0+b_1\log\mu_t(g_{t+1}u_{t+1}).\end{aligned}$$

Exact if $\rho = 0$: $b_0 = 0$, $b_1 = \beta$

Solve by guess and verify

Example: Bansal-Yaron

Consumption growth

$$
\begin{aligned}
\log g_t &= g + \gamma(L) v_{t-1}^{1/2} w_{1t} \\
v_t &= v + \nu(L) w_{2t} \\
(w_{1t}, w_{2t}) &\sim \text{NID}(0, I)
\end{aligned}
$$

Guess value function

$$
\log u_t = u + \omega_g(L) v_{t-1}^{1/2} w_{1t} + \omega_v(L) v_t
$$

Solution includes

$$
\begin{aligned}
\omega_{g0} + \gamma_0 &= \gamma(b_1) \\
\omega_{v0} &= b_1(\alpha/2)\gamma(b_1)^2 \nu(b_1)
\end{aligned}
$$

Example: Bansal-Yaron (continued)

Pricing kernel

$$\begin{aligned}
\log m_{t+1} &= \log\beta + (\rho-1)g - (\alpha-\rho)(\alpha/2)\omega_{v0}^2 \\
&\quad + (\rho-1)[\gamma(L)/L]_+ v_{t-1}^{1/2} w_{1t} - (\alpha-\rho)(\alpha/2)\gamma(b_1)^2 v_t \\
&\quad + [(\rho-1)\gamma_0 + (\alpha-\rho)\gamma(b_1)] v_t^{1/2} w_{1t+1} \\
&\quad + (\alpha-\rho)\omega_{v0}^2 w_{2t+1}
\end{aligned}$$

Conditional entropy (monthly)

$$\begin{aligned}
L_t(m_{t+1}) &= [(\rho-1)\gamma_0 + (\alpha-\rho)\gamma(b_1)]^2 v_t/2 + (\alpha-\rho)^2\omega_{v0}^2/2 \\
0.0218 &= 0.0065 + 0.0153 \\
0.0026 &= 0.0026 + 0.0000 \text{ if } \rho = \alpha
\end{aligned}$$

Example: Wachter

Consumption growth

$$
\begin{aligned}
\log g_t &= g + \sigma w_{1t} + z_t \\
\lambda_t &= (1-\varphi)\lambda + \varphi\lambda_{t-1} + \sigma_\lambda w_{2t} \\
(w_{1t}, w_{2t}) &\sim \text{NID}(0, I) \\
z_t | j &\sim \mathcal{N}(j\theta, j\delta^2) \\
j &\geq 0 \text{ has jump intensity } \lambda_{t-1}
\end{aligned}
$$

Guess value function

$$
\log u_t = u + \omega_\lambda \lambda_t
$$

Solution includes

$$
\omega_\lambda = (1 - b_1\varphi)^{-1} b_1 \left[e^{\alpha\theta + (\alpha\delta)^2/2} - 1 \right] / \alpha
$$

Example: Wachter (continued)

Pricing kernel

$$\begin{aligned}\log m_{t+1} &= \log\beta + (\rho-1)x - (\alpha-\rho)(\alpha/2)[\sigma^2 + (\omega_\lambda\sigma_\lambda)^2] \\ &\quad - \lambda_t(e^{\alpha\theta+(\alpha\delta)^2/2} - 1)/\alpha \\ &\quad + (\alpha-1)(\sigma w_{1t+1} + z_{t+1}) + (\alpha-\rho)(\omega_\lambda\sigma_\lambda)w_{2t+1}\end{aligned}$$

Conditional entropy (monthly) [where's the bazooka?]

$$\begin{aligned}L_t(m_{t+1}) &= (\alpha-1)^2\sigma^2/2 + (\alpha-\rho)^2(\omega_\lambda\sigma_\lambda)^2/2 \\ &\quad + \lambda_t\left\{[e^{(\alpha-1)\theta+(\alpha-1)^2\delta^2/2} - 1] - (\alpha-1)\theta\right\} \\ 0.0100 &= 0.0001 + 0.0087 + 0.0012 \\ 0.0013 &= 0.0001 + 0.0000 + 0.0012 \text{ if } \rho = \alpha\end{aligned}$$

Example: Abel/Chan-Kogan external habit

Additive preferences with

$$v(c_t, x_t) = (c_t/x_t)^\alpha/\alpha$$
$$x_{t+1} = x + \chi(L)y_t \quad (\text{"predetermined"})$$

Pricing kernel

$$m_{t+1} = \beta(c_{t+1}/c_t)^{\alpha-1}(x_{t+1}/x_t)^\alpha$$

Entropy: habit irrelevant to conditional entropy [!?]

Example: Abel/Chan-Kogan external habit

Additive preferences with

$$\begin{aligned} v(c_t, x_t) &= (c_t/x_t)^\alpha/\alpha \\ x_{t+1} &= x + \chi(L) y_t \quad (\text{“predetermined”}) \end{aligned}$$

Pricing kernel

$$m_{t+1} = \beta (c_{t+1}/c_t)^{\alpha-1} (x_{t+1}/x_t)^\alpha$$

Entropy: habit irrelevant to conditional entropy [!?]

Example: recursive Abel/Chan-Kogan

Consumption growth

$$\log g_t \;=\; g + \gamma_0 w_t, \quad \{w_t\} \sim \mathrm{NID}(0,1)$$

Pricing kernel

$$\begin{aligned}\log m_{t+1} \;&=\; \text{constants} + \text{things dated } t \text{ and before} \\ &\quad + \; \{(\rho - 1) + (\alpha - \rho)[1 - b_1 \chi(b_1)]\}\, \gamma_0 w_{t+1}\end{aligned}$$

Note: habit introduces persistent component

Example: Campbell-Cochrane external habit

Additive preferences with

$$\begin{aligned} v(c_t, x_t) &= (c_t - x_t)^\alpha/\alpha \\ \log g_t &= g + w_{t+1}, \quad \{w_t\} \sim \text{NID}(0, \sigma^2) \end{aligned}$$

Approximation

$$\begin{aligned} s_t &= (c_t - x_t)/c_t \quad (\text{“surplus”}) \\ \log s_{t+1} &= (1-\varphi)\log s + \varphi \log s_t + \lambda(\log s_t) w_{t+1} \\ (1+\lambda)^2 &= \frac{[1 - 2(\log s_t - \log s)][1 - \varphi - b/(1-\alpha)]}{(1-\alpha)\sigma^2} \end{aligned}$$

Example: Campbell-Cochrane external habit (continued)

Pricing kernel

$$\begin{aligned} m_{t+1} &= \beta(c_{t+1}/c_t)^{\alpha-1}(s_{t+1}/s_t)^{\alpha-1} \\ \log m_{t+1} &= \text{constants} + \text{things dated } t \text{ and before} \\ &\quad - (1-\alpha)[1+\lambda(\log s_t)]w_{t+1} \end{aligned}$$

Conditional entropy (monthly)

$EL_t(m_{t+1})$	$= [(1-\alpha)(1-\varphi)-b]/2$
Campbell-Cochrane	0.0100
Wachter	0.0082
Verdelhan	0.0052

Time dependence: summary

Little time-dependence in pricing kernel

But: modest dynamics + recursive preferences can magnify entropy

Habits, too

Options: same devices can magnify the impact of disasters (Benzoni-Collin-Dufresne-Goldstein, Drechsler-Yaron, Eraker-Shaliastovich, & Campbell-Cochrane with jumps)