The market for crash risk

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Abstract

This paper examines the equilibrium when stock market crashes can occur and investors have heterogeneous attitudes towards crash risk. The less crash averse insure the more crash averse through options markets that dynamically complete the economy. The resulting equilibrium is compared with various option pricing anomalies: the tendency of stock index options to overpredict volatility and jump risk, the Jackwerth [Recovering risk aversion from option prices and realized returns. Review of Financial Studies 13, 433–451] implicit pricing kernel puzzle, and the stochastic evolution of option prices. Crash aversion is compatible with some static option pricing puzzles, while heterogeneity partially explains dynamic puzzles. Heterogeneity also magnifies substantially the stock market impact of adverse news about fundamentals.

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0. Introduction

The markets for stock index options play a vital role in providing a venue for redistributing and pricing various types of equity risk of concern to investors. Investors who like equity but are concerned about crash risk can purchase portfolio

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insurance, in the form of out-of-the-money put options. Direct bets on or hedges against future stock market volatility are feasible; most simply by buying or selling straddles, more exactly by the options-based bet on future realized variance proposed by Britten-Jones and Neuberger (2000) and analyzed further by Jiang and Tian (2005). By creating a market for these risks, the options markets should in principle permit the dispersion of these risks across all investors, until all investors are indifferent at the margin to taking on more or less of these risks given the equilibrium pricing of these risks. This idealized risk pooling underpins our theoretical construction of representative-agent models, and our pricing of risks from aggregate data sources – for instance, estimating the consumption CAPM based on aggregate consumption data.

How well do the stock index option markets operate? Empirical evidence on option returns suggests that the stock index options markets are operating inefficiently. Such evidence is based on observed substantial divergences between the ‘risk-neutral’ distributions compatible with observed post-1987 option prices, and the conditional distributions estimated from time-series analyses of the underlying stock index. Perhaps most important has been the fact that implicit (risk-neutral) standard deviations (ISDs) inferred from at-the-money options have been substantially higher on average than the volatility subsequently realized over the lifetime of the option. Furthermore, regressing realized volatility upon ISDs almost invariably indicates that ISDs are informative but biased predictors of future volatility, with bias increasing in the ISD level.

While the level of at-the-money ISDs is puzzling, the shape of the volatility surface across strike prices and maturities also appears at odds with estimates of conditional distributions. It is now widely recognized that the ‘volatility smirk’ that emerged after the 1987 crash implies substantial negative skewness in risk-neutral distributions, and various correspondingly skewed models have been proposed: implied binomial trees, stochastic volatility models with ‘leverage’ effects, and jump diffusions. And although these models can roughly match observed option prices, the associated implicit parameters do not appear especially consistent with the absence of substantial negative skewness in post-1987 stock index returns. To paraphrase Samuelson, the option markets have predicted nine out of the past five market corrections, generating surprisingly large returns from selling crash insurance via out-of-the-money put options. A further puzzle is that implicit jump risk assessments are strongly countercyclical. As shown below in Fig. 1, implicit jump risk over 1988–1998 was highest immediately after substantial market drops, and was low during the bull market of 1992–1996.

It is of course possible that the pronounced divergence between objective and risk-neutral measures represents risk premia on the underlying risks. The fundamental theorem of asset pricing states that provided there exist no outright arbitrage opportunities, it is possible to construct a ‘representative agent’ whose preferences are compatible with any observed divergences between the two distributions.

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1See Rubinstein (1994, pp. 774–775) or Bates (2000, Fig. 2).
However, Jackwerth (2000) and Rosenberg and Engle (2002) have pointed out that the preferences necessary to reconcile the two distributions appear rather oddly shaped, with sections that are locally risk loving rather than risk averse. Furthermore, the post-1987 Sharpe ratios from writing put options or straddles seem extraordinarily high – 2–6 times that of investing directly in the stock market. These speculative opportunities appear to have been present in the stock index options markets for almost 20 years.

It may be the stock index options markets are functioning more as insurance markets, rather than as genuine two-sided markets for trading financial risks. Viewing options markets as an insurance market for crash risk may be able to explain some of the option pricing anomalies – especially if the number of insurers is constrained. If crash risk is concentrated among option market makers, calibrations based upon the risk-taking capacity of all investors can be misleading. Speculative opportunities such as writing more straddles become unappealing when the market makers are already overly involved in the business. Furthermore, the dynamic response of option prices to market drops resembles the price cycles observed in insurance markets: an increase in the price of crash insurance caused by the contraction in market makers’ capital following losses.

This paper represents an initial attempt to model the dynamic interaction between option buyers and sellers. A two-agent dynamic general equilibrium model is constructed in which relatively crash-tolerant option market makers insure crash-averse investors. Heterogeneity in attitudes towards crash risk is modeled via heterogeneous state-dependent utility functions – an approach roughly equivalent to heterogeneous beliefs about the frequency of crashes. Crashes can occur in the

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3 Basak and Cuoco (1998) make a similar point regarding calibrations of the consumption CAPM when most investors do not hold stock.

4 Froot (2001, Fig. 3) illustrates the strong, temporary impacts of Hurricane Andrew in 1992 and the Northbridge earthquake in 1994 upon the price of catastrophe insurance.
model, given occasional adverse jumps in news about fundamentals. Derivatives are consequently not redundant in the model and serve the important function of dynamically completing the market. Given complete markets, equilibrium can be derived using an equivalent central planner’s problem, and the corresponding dynamic trading strategies and market equilibria are identified. Those equilibria are compared to styled facts from options markets.

There have been previous papers exploring heterogeneous-agent dynamic equilibria, some of which have explored implications for option pricing. These papers diverge on the types of investor heterogeneity, the sources of risk, and the choice between production and exchange economies. Back (1993) and Basak (2000) focus on heterogeneous beliefs. Grossman and Zhou (1996) explore the general-equilibrium implications of heterogeneous preferences (in particular, the existence of portfolio insurers) in a terminal exchange economy, given only one source of risk (diffusive equity risk). Options are redundant in this framework, but the paper does look at the implications for option prices. Weinbaum (2001) has a somewhat similar model, in which power utility investors differ in risk aversion. Bardhan and Chao (1996) examine the general issue of market equilibrium in exchange economies with intermediate consumption, with heterogeneous agents and jump diffusions with discrete jump outcomes. Dieckmann and Gallmeyer (2005) use a special case of the Bardhan and Chao structure to explore the general-equilibrium implications of heterogeneous risk aversion.

This paper assumes a terminal exchange economy, and sufficient sources of risk that options are not redundant. Perhaps the major divergence from the above papers is this paper’s focus on options markets. Whereas Bardhan and Chao (1996) and Dieckmann and Gallmeyer (2005) assume there are sufficient financial assets to dynamically complete the market, this paper focuses on the plausible hypothesis that options are the relevant market-completing financial assets. The paper develops some tricks for computing competitive equilibria using the short-dated options with overlapping maturities that we actually observe. Finally, the hypothesized source of heterogeneity – divergent attitudes towards crash risk – is plausible for motivating trading in option contracts that offer direct protection against stock market crashes.

The objective of the paper is not to develop a better option pricing model. That can be done better with ‘reduced-form’ option pricing models tailored to that objective; e.g., multi-factor option pricing models such as the Bates (2000) affine model or the Santa-Clara and Yan (2005) quadratic model. Furthermore, this paper ignores stochastic volatility, which is assuredly relevant when building option pricing models. Rather, the objective of this paper is to build a relatively simple model of the role of options markets in financial intermediation of crash risk, in order to examine the theoretical implications for prices and dynamic equilibria. Key issues include: what fundamentally determines the price of crash risk? Can we explain the sharp shifts we observe in the price of crash risk?

Section 1 of the paper recapitulates various stylized facts from empirical options research that influence model construction. Section 2 introduces the basic framework, and identifies a benchmark homogeneous-agent equilibrium. Section 3
explores the equilibrium when agents are heterogeneous, while Section 4 explores associated option pricing implications. Section 5 concludes.

1. Empirical option pricing anomalies and stylized facts

Three categories of discrepancies between objective and risk-neutral probability measures will be kept in mind in the theoretical section of the paper: volatility, higher moments, and the implicit pricing kernel that in principle reconciles the two measures. Furthermore, each category can be decomposed further into average discrepancies, and conditional discrepancies.

The unconditional volatility puzzle is that ISDs from stock index options are typically higher than realized stock market volatility. For instance, ISDs from 30-day at-the-money put and call options on S&P 500 futures over 1988–1998 have been on average 2% higher than the subsequent annualized daily volatility of stock market returns over the options’ lifetime.\(^5\) This discrepancy has generated substantial post-1987 profits on average from writing at-the-money puts or straddles, with Sharpe ratios roughly double that of investing in the stock market. See, e.g., Fleming (1998) or Jackwerth (2000).

The conditional volatility puzzle is that regressing realized volatility upon ISDs generally yields slopes that are significantly positive, but significantly less than one. For instance, the regressions using the 30-day ISDs and realized volatilities mentioned above yield volatility and variance results

\[
\sqrt{\frac{365}{T} \sum_{\tau=t+1}^{T} (\Delta \ln F_{\tau})^2} = 0.160 + 0.756 ISD_{t} + e_{t+T}, \quad R^2 = 0.45
\]

\[
\frac{365}{T} \sum_{\tau=t+1}^{T} (\Delta \ln F_{\tau})^2 = 0.027 + 0.681 ISD^2_{t} + e_{t+T}, \quad R^2 = 0.33
\]

with heteroskedasticity-consistent standard errors in parentheses.\(^6\) Since intercepts are small, the regressions imply that ISDs are especially poor forecasts of realized volatility when high. Straddle-trading strategies conditioned on the ISD level achieved Sharpe ratios almost triple that of investing directly in the stock market over 1988–1998.

The skewness puzzle is that the levels of skewness implicit in stock index options are generally much larger in magnitude than those estimated from stock index returns – whether from unconditional returns (Jackwerth, 2000) or conditional upon a time-series model that captures salient features of time-varying distributions (Rosenberg and Engle, 2002). Furthermore, implicit skewness remains pronounced

\(^5\)The puzzle is slightly exacerbated by the fact that at-the-money ISDs are downwardly biased predictors of the (risk-neutral) volatility over the lifetime of the options.

\(^6\)Jiang and Tian (2005) find similar results from regressions using the ‘model-free’ implicit variance measure of Britten-Jones and Neuberger (2000).
for longer maturities of stock index options of, e.g., 3–6 months. By contrast, the distribution of log-differenced stock indexes or stock index futures converges rapidly towards near-normality as one progresses from daily to weekly to monthly holding periods.

A further puzzle is the evolution of distributions implicit in option prices. Fig. 1 summarizes that evolution using updated estimates of the Bates (2000) 2-factor stochastic volatility/jump-diffusion model with time-varying jump risk. Sharp changes are occasionally observed both for total variance and for the instantaneous risk-neutral jump intensity $\lambda_t^j$. The graph indicates that the sharp market declines observed over 1988–1998 (in January 1988, October 1989, August 1990, November 1997, and August 1998) were accompanied by sharp increases in implicit jump risk. The puzzles here are the abruptness of the shifts (Bates (2000) rejects the hypothesis that implicit jump risk follows an affine diffusion), and the magnitudes of implicit jump risk achieved following the market declines. Since affine models assume the risk-neutral and objective jump intensities are proportional, these models imply objective crash risk is highest immediately following crashes. And while assessing the frequency of rare events is perforce difficult, Bates (2000, Table 9) finds no evidence that the occasionally high implicit jump intensities over 1988–1993 could in fact predict the intensity of subsequent stock return jumps.

Finally, there is the implicit pricing kernel puzzle discussed in Jackwerth (2000) and Rosenberg and Engle (2002). If the level of the stock index is viewed as a reasonably good proxy for the overall wealth of the representative agent, the implicit marginal utility function of the representative agent can be extracted directly from the divergence between the risk-neutral distribution inferred from prices of stock index options and the objective conditional distribution estimated from stock market returns. However, Jackwerth finds these implicit functions can appear oddly shaped, with marginal utility of wealth locally increasing in areas – risk loving, rather than risk averse.

There are currently three leading explanations for the above anomalies: a volatility risk premium, a jump risk premium, or demand pressures. Coval and Shumway (2001) and Bakshi and Kapadia (2003) attribute the substantial speculative opportunities from writing stock index options to a volatility risk premium. Pan (2002), by contrast, finds that the volatility risk premium necessary to reconcile

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7In options research, implicit skewness is roughly measured by the shape of the volatility ‘smirk,’ or pattern of ISDs across different strike prices (‘moneyness’). The skewness/maturity interaction can be seen by examined by the volatility smirk at different horizons, conditional upon rescaling moneyness proportionately to the standard deviation appropriate at different horizons. See, e.g., Bates (2000, Fig. 4). Tompkins (2001) provides a comprehensive survey of volatility surface patterns, including the maturity effects.

8Jackwerth’s results are disputed by Aït-Sahalia and Lo (2000), who find no anomalies when comparing average option prices from 1993 with the unconditional return distribution estimated from overlapping data from 1989 to 1993. The difference in results perhaps highlights the importance of using conditional rather than unconditional distributions, as in Rosenberg and Engle (2002). For instance, both conditional variance and implicit standard deviations are time varying; and a substantial divergence between the two because of mismatched data intervals can produce anomalous implicit marginal utility functions even in a lognormal environment.
objective and risk-neutral volatilities implies an excessively upward-sloped term structure of ISDs, while a substantial risk premium on time-varying jump risk fits the term structure better. Bates (2000) finds that this model can also match the maturity profile of implicit skewness better than models with constant implicit jump risk. The jump risk premium explanation sometimes appears in the guise of expectational error; e.g., Jackwerth’s (2000, p. 446) conjecture that OTM puts were overpriced because market participants overestimated the frequency of stock market crashes. Demand pressure explanations have appeared in Figlewski (1989), Jiang (2002), Bollen and Whaley (2004), Hodges et al. (2004), and Garleanu et al. (2005), sometimes accompanying the hypothesis that options markets are partly segmented from equity markets. These models attribute the overpricing of OTM puts to excess demand for those options, while Hodges et al. attribute the Jackwerth anomaly to excess demand for the long-shot positively skewed gambles provided by OTM calls.

The challenge for these explanations is in devising theoretical models of compensation for risk consistent with the magnitude of the speculative opportunities. The stochastic evolution of implicit jump risks from option prices also appears difficult to explain. This paper will focus on the jump risk premium explanation of option pricing anomalies, in an equilibrium model that also considers repercussions for equity markets. The apparent magnitude and evolution of the crash risk premium are the two central stylized facts that I will attempt to match.

2. A jump-diffusion economy with homogeneous agents

I consider a simple continuous-time endowment economy over \([0, T]\), with a single terminal dividend payment \(D_T\) at time \(T\). News about this dividend (or, equivalently, about the terminal value of the investment) arrives as a univariate Markov jump diffusion of the form

\[
d\ln D_t = \mu_d dt + \sigma_d dZ_t + \gamma_d dN_t, \tag{3}
\]

where \(Z_t\) is a standard Wiener process, \(N_t\) is a Poisson counter with constant intensity \(\lambda\), and \(\gamma_d < 0\) is a deterministic jump size or announcement effect, assumed negative. \(D_t = E_t D_T\) is the current signal about the terminal payoff and follows a martingale, implying \(\mu_d = -\frac{1}{2}\sigma_d^2 - \lambda(\bar{e}^{\gamma_d} - 1)\).

Financial assets are claims on terminal outcomes. Given the simple specification of news arrival, any three nonredundant assets suffice to dynamically span this economy; e.g., bonds, stocks, and a single long-maturity stock index option. However, it is analytically convenient to work with the following three fundamental assets:

1. a riskless numeraire bond in zero net supply that delivers one unit of terminal consumption in all terminal states of nature;
2. an equity claim in unitary supply that pays a terminal dividend \(D_T\) at time \(T\), and is priced at \(S_t\) at time \(t\) relative to the riskless asset; and
(3) a jump insurance contract in zero net supply that costs an instantaneous and
endogenously determined insurance premium \( \lambda^*_t \, dt \) and pays off 1 additional unit
of the numeraire asset conditional on each jump. The terminal payoff of one
insurance contract held to maturity is \( N_T - \int_0^T \lambda^*_t \, dt \).

Other assets such as options are redundant given these fundamental assets, and are
priced by no arbitrage given equilibrium prices for the latter two assets. Equivalently, the jump insurance (or crash insurance) contract can be synthesized
from the short-maturity options markets with overlapping maturities that we
actually observe. The equivalence between options and crash insurance contracts is
discussed below in Section 4.2.

Agents are assumed to have possibly state-dependent preferences over terminal
outcomes of the form

\[
U(W_t, N_t, t) = E_t[U(W_T, N_T)],
\]

where \( W_T = D_T \) is terminal wealth, \( N_T \) is the number of jumps over \([0, T]\), and
\( U(W_T, N_T) \) is assumed increasing and concave in \( W_T \). Particular specifications will
be discussed below.

Asset prices are determined by the terminal marginal utility of wealth
\( Z_T = U(W_T, N_T) \) and its current expectation \( E_t Z_T \) - the marginal utility of current
wealth. In particular, the price \( S_t \) of equity (in riskless bond units) is determined by
the Euler condition associated with exchanging \( S_t \) riskless bonds for an uncertain
terminal equity payoff \( D_T \):

\[
E_t [U_W(D_T, N_T)(D_T - S_t)] = 0
\]

implying

\[
S_t = \frac{E_t N_T D_T}{\eta_t}.
\]

The instantaneous equity premium can be derived from the martingale properties
of \( \eta_t \) and \( \eta_t S_t \), yielding

\[
E_t \left( \frac{dS_t}{S_t} \right) = -E_t \left( \frac{dS_t}{\eta_t} \frac{d\eta_t}{\eta_t} \right).
\]

Crash insurance can be priced comparably. Since crash insurance with
instantaneous cost \( \lambda^*_t \, dt \) pays off 1 unit of the numeraire conditional upon a jump
occurring in \((t, t + dt]\), its price is

\[
\eta_t \lambda^*_t \, dt = E_t [\eta_{t+dt}1_{N_t=1}] = \lambda \, d\eta_{t+dt}|_{N=1}.
\]

This can be rearranged to yield a crash risk premium of the form

\[
\frac{\lambda^*_t}{\lambda} = 1 + \frac{d\eta_t}{\eta_t}|_{N=1}.
\]

Thus, the precise evolution \( d\eta_t/\eta_t \) (or, equivalently, \( d\ln \eta_t \)) is of key importance
for determining equity and crash risk premia. The nature of that evolution, and
its dependency upon the functional form of \( U_W(\cdot) \), can be clarified by writing \( \ln \eta_t \) in
the form
\[
\ln \eta(D_t, N_t, t, T) = \ln E_t[U_W(D_t e^{\Delta d}, N_t + n)],
\]
(10)
where \(\Delta d \equiv \ln(D_T/D_t)\) and \(n \equiv N_T - N_t\) are future shocks with distributions independent of the current values of \((D_t, N_t)\). If terminal utility depends solely on terminal wealth \(D_T\), then both \(\eta_t\) and \(S_t\) are monotonic functions of current \(D_t\) but do not otherwise depend upon \(N_t\). The most popular utility specification has been power utility, which is the only wealth-dependent utility function consistent with stationary equity returns when \(D_t\) follows a geometric process such as (3) above.

This paper will explore a state-dependent expansion of power utility, of the form
\[
U(W_t, N_t, t) = E_t\left[ e^{YN_t} \frac{W_T^{1-R} - 1}{1 - R} \right], \quad R > 0.
\]
(11)
Associated with this ‘crash-averse’ utility specification is a current marginal utility
\[
\eta_t = E_t[e^{YN_t} D_t^{-R}] = e^{YN_t} D_t^{-R} E_t[e^{Yn-R\Delta d}].
\]
(12)
As this specification has not previously appeared explicitly in the finance literature, some motivation is necessary.

First, this specification makes explicit in utility terms what is implicit in the affine pricing kernels routinely used in the affine asset pricing literature. A typical affine approach for the pricing kernel \(Z_t\) specifies a linear structure in the underlying sources of risk:
\[
dl \ln Z_t = \mu \, dt + \sigma \, dZ_t + \gamma \, dN_t.
\]
(13)
See, e.g., Ho et al. (1996); or Wu (2006, Eq. (8)) for a recent application involving Lévy processes. Since \(\eta_t\) is nonnegative, such specifications are consistent with absence of arbitrage. For analytical tractability, affine models place functional-form constraints on how \(\sigma\) and the jump intensity can depend on any underlying state variables, but do not otherwise restrict the magnitudes of \(\sigma\) and \(\gamma\). However, by the jump-diffusion version of Itô’s lemma, any purely wealth-dependent utility specification severely constrains the sensitivities of \(\ln \eta(D_t, t)\) to diffusion and jump shocks. For instance, the power utility specification (12) with \(Y = 0\) implies
\[
dl \ln \eta_t = \frac{\partial \ln \eta_t}{\partial t} \, dt - R \, d \ln D_t = O(dt) - R \sigma_d \, dZ_t - R \gamma_d \, dN_t,
\]
(14)
implying relative pricing kernel sensitivities to large versus small shocks are constrained by the ratio \(\gamma_d/\sigma_d\). Any deviation of postulated relative sensitivities from this ratio is equivalent to introducing state dependency into the marginal utility function (12) of the form \(Y = \gamma_d + R \gamma_d\).

Perhaps the most intuitive justification is that the crash aversion parameter \(Y\) can be viewed a utility-based proxy for subjective beliefs about crash risk. Investors with crash-averse preferences \((Y > 0)\) are equivalent to investors with state-independent
preferences and a subjective belief that the jump intensity is \( \lambda \epsilon^Y \):

\[
E_0[\epsilon^{Y_NT}u(W_T)] = \sum_{N=0}^{\infty} \frac{e^{-\lambda T} (\lambda T \epsilon^Y)^N}{N!} E_0[u(W_T)|N \text{ jumps}]
= e^{\lambda T (\epsilon^Y - 1)} E_0[u(W_T)|\lambda^* = \lambda \epsilon^Y].
\] (15)

This reflects the general proposition that preferences and beliefs are indistinguishable in a terminal exchange economy. It should be recognized, however, that this interpretation involves very strong subjective beliefs, in that investors do not update their subjective jump intensities \( \lambda \epsilon^Y \) based on learning over time, or based on trading with other investors in the heterogeneous-agent equilibrium derived below.

A final and related justification is provided by Liu et al. (2005), who derive the marginal utility specification (12) from robust-control methods given uncertainty aversion to imprecise knowledge of the jump intensity. In the deterministic-jump special case of their model, investors consider alternate possibilities \( \lambda^\ast \) for the jump intensity parameter, and trade off the adverse utility consequences of higher \( \lambda^\ast \) against the divergence of \( \lambda^\ast \) from the benchmark \( \lambda \) – presumably the empirical point estimate. The outcome of that trade-off (Eqs. (28) and (3) in Liu et al.) is that cautious investors use an upwardly biased jump intensity assessment \( \lambda \epsilon^a \) – an approach observationally equivalent to crash-averse preferences for \( a^* = Y \).

It is also worth noting that crash-averse preferences (11) possess convenient properties: they retain the homogeneity of standard power utility, and the myopic investment strategy property of the log utility subcase (\( R = 1 \)). Furthermore, it will be shown below that crash-averse preferences generate stationary equity returns in a homogeneous-agent economy.

### 2.1. Equilibrium in a homogeneous-agent economy

The following lemma is useful for computing relevant conditional expectations.

**Lemma.** If \( d_t \equiv \ln D_t \) follows the jump-diffusion in (3) above, then

\[
E_t e^{\Phi d_{t+T} + \psi N_T} = \exp\{\Phi d_t + \psi N_t + (T-t)[\Phi \mu_d + \frac{1}{2} \Phi^2 \sigma_d^2 + \lambda(e^{\Phi \lambda_d + \psi} - 1)]\}. \tag{16}
\]

**Proof.** For \( \tau \equiv T - t \), there is a probability \( w_n = e^{-\lambda \tau} (\lambda \tau)^n / n! \) of observing \( n \equiv N_T - N_t \) jumps over \( (t, T) \). Conditional upon \( n \) jumps, \( \Delta d \equiv \ln D_T/D_t \) is normally distributed with mean \( \mu_d \tau + n \gamma_d \) and variance \( \sigma_d^2 \tau \). Consequently,

\[
E_t e^{\Phi d_{t+T} + \psi N_T} = e^{\Phi d_t + \psi N_t} E_t \exp[\Phi \Delta d + \psi n]
= e^{\Phi d_t + \psi N_t} E_t \exp[\Phi \Delta d|n=0 + n(\Phi \gamma_d + \psi)]
= e^{\Phi d_t + \psi N_t} \exp[(\Phi \mu_d + \frac{1}{2} \Phi^2 \sigma_d^2)\tau + \lambda \tau(e^{\Phi \lambda_d + \psi} - 1)]. \tag{17}
\]

The last line follows from the independence of the Wiener and jump components, and from the moment generating functions for Wiener and jump processes. \( \square \)
Using the lemma and Eqs. (12), (6), and (9) yield the following asset pricing equations:

\[ \frac{\lambda}{C_3} = \frac{\lambda e^{Y - R \gamma_d}}{C_0} \]  

(18)

\[ \eta_t = D_t^{-R g \gamma_d} e^{(T-t)(-R \mu_d + \frac{1}{2} R^2 \sigma_d^2 + (\lambda - \lambda^*)]} \]  

(19)

\[ S_t = D_t \exp\{ (T-t)(\mu_d + \frac{1}{2} \sigma_d^2 - R \sigma_d^2) + \lambda^*(e^{\gamma_d} - 1) \} \]  

(20)

The last equation implies that the price of equity relative to the riskless numeraire follows roughly the same i.i.d. jump-diffusion process as the underlying news about terminal value, with identical instantaneous volatility and jump magnitudes:

\[ dS_t/S_t = \mu \, dt + \sigma_d \, dZ_t + k(dN_t - \lambda \, dt) \]  

(21)

for \( k = e^{\gamma_d} - 1 \). The instantaneous equity premium

\[ \mu = R \sigma_d^2 + (\lambda - \lambda^*) k \approx R(\sigma_d^2 + \lambda \gamma_d^2) + (\lambda^2 - \lambda^* \gamma_d) Y \]  

(22)

reflects required compensation for two types of risk. First is the required compensation for stock market variance from diffusion and jump components, roughly scaled by the coefficient of relative risk aversion. Second, the crash aversion parameter \( Y \geq 0 \) increases the required excess return when stock market jumps are negative.

Crash aversion also directly affects the price of crash insurance relative to the actual arrival rate of crashes:

\[ \log(\lambda^* / \lambda) = -R \gamma_d + Y \]  

(23)

Finally, derivatives are priced as if equity followed the risk-neutral martingale

\[ dS_t/S_t = \sigma_d \, dZ_t^* + k(dN_t^* - \lambda^* \, dt) \]  

(24)

where \( N_t^* \) is a jump counter with constant intensity \( \lambda^* \). The resulting (forward) option prices are identical to the deterministic-jump special case of Bates (1991), given the geometric jump diffusion.

2.2. Consistency with empirical anomalies

The homogeneous crash aversion model can explain some of the stylized facts from Section 1. First, unconditional bias in implied volatilities is explained by the potentially substantial divergence between the risk-neutral instantaneous variance \( \sigma_d^2 + \lambda^* \gamma_d^2 \) implicit in option prices, and the actual instantaneous variance \( \sigma_d^2 + \lambda \gamma_d^2 \) of log-differenced asset prices. Second, the difference between the \( \lambda^* \) inferred from option prices and the estimates of \( \lambda \) from stock market returns is consistent with the observation in Bates (2000, pp. 220–221) and Jackwerth (2000, pp. 446–447) of too few observed jumps over 1988–1998 relative to the number predicted by stock index options. The extra parameter \( Y \) permits greater divergence in \( \lambda^* \) from \( \lambda \) than is feasible under standard power utility models.

To illustrate this, consider the following calibration: a stock market volatility \( \sigma_d = 15\% \) annually conditional upon no jumps, and adverse news of \( \gamma_d = -10\% \).
that arrives on average once every 4 years ($\lambda = .25$).\footnote{As $D_t$ is the signal regarding terminal stock market valuation, $\sigma_d$ and $\gamma_d$ are appropriately calibrated from stock market movements. By construction, this paper is using wealth- rather than consumption-based calibration, for two reasons. First, the empirical option pricing anomalies of Section 1 use wealth-based criteria. Second, stock market jumps are identified using high-frequency daily data, for which there do not exist comparable consumption data.} From Eqs. (22) and (23), the equity and crash risk premia are

$$\mu \approx .025R + .025Y, \quad \ln(\lambda^*/\lambda) = .10R + Y.$$ \hspace{1cm} (25)

For $R = 1$ and $Y = 1$, the equity premium is 5%/year, while the jump risk $\lambda^*$ implicit in option prices is three times that of the true jump risk. Thus, the crash aversion parameter $Y$ is roughly as important as relative risk aversion for the equity premium, but substantially more important for the crash premium. Achieving the observed substantial disparity between $\lambda^*$ and $\lambda$ using risk aversion alone ($Y = 0$) would require levels of $R$ that most would find unpalatable, and which would imply an implausibly high equity premium.

Since returns are i.i.d. under both the actual and risk-neutral distribution, the homogeneous-agent model is not capable of capturing the dynamic anomalies discussed in Section 1. The standard results from regressing realized on implicit variance cannot be replicated here, because neither is time varying in this model. Second, the model cannot match the observed tendency of $\lambda^*$ to jump contemporaneously with substantial market drops. Finally, the i.i.d. return structure implies that implicit distributions should rapidly converge towards lognormality at longer maturities, which does not accord with the maturity profile of the volatility smirk.

Furthermore, Jackwerth’s (2000) anomaly cannot be replicated under homogeneous crash aversion. As discussed in Rosenberg and Engle (2002), Jackwerth’s implicit pricing kernel involves the projection of the actual pricing kernel upon asset payoffs. E.g., stock index options with terminal payoff $V(S_t)$ have an initial price

$$v_0 = \frac{E_0[\eta_t V(S_t)]}{\eta_0} = E_0 \left[ V(S_t) \frac{E_0[\eta_t | S_t]}{E_0[\eta_t]} \right] = E_0[V(S_t)M(S_t)],$$ \hspace{1cm} (26)

where $M(S_t)$ has the usual properties of pricing kernels: it is nonnegative, and $E_0[M(S_t)] = 1$.

It is shown in Appendix A that for crash-averse preferences, this projection takes the form

$$M(S_t) = \kappa(t) S_t^{-R} \frac{p(S_t | \lambda e^Y)}{p(S_t | \lambda)},$$ \hspace{1cm} (27)

where $\kappa(t)$ is a function of time and $p(S_t | \lambda)$ is the probability density function of $S_t$ conditional upon a jump intensity of $\lambda$ over $(0, t)$. Implicit relative risk aversion is given by $-\partial \ln M(S)/\partial \ln S$. For $Y = 0$, one observes the strictly decreasing pricing kernel and constant relative risk aversion associated with power utility. For $Y > 0$, it is proven in Appendix A that $\ln M(S_t)$ is a strictly decreasing function of $\ln S_t$ that is illustrated below in Fig. 2. The result is relatively intuitive. The ratio $p(S_t | \lambda e^Y)/p(S_t | \lambda)$ in (27) is the change of measure from the crash-averse model to an equivalent economy.
discussed above in Eq. (15), in which homogeneous investors have strictly wealth-dependent preferences $u(W_T)$ and a subjective belief that the jump intensity is $\lambda e^Y$. Pricing kernels in this equivalent economy take the form $M_t = E_t^*[u'(D_t e^{Dd})]$ for $Dd = \ln(D_T/D_t)$. As this kernel is a strictly decreasing function of $D_t$ (or of $S_t$), it cannot replicate the negative implicit risk aversion (positive slope) estimated by Jackwerth (2000) and Rosenberg and Engle (2002) for some values of $S_t$.

Crash-averse preferences (or biased subjective beliefs) do replicate the higher implicit risk aversion (steeper negative slope) for low $\ln S_t$ values that was estimated by those authors and by Aït-Sahalia and Lo (2000). Correspondingly, crash aversion can generate apparently favorable empirical investment opportunities from put writing strategies. For instance, the instantaneous annualized Sharpe ratio on writing crash insurance is

$$\lambda^* - \frac{\lambda^* - \lambda}{\sqrt{\lambda(1 - \lambda \, dt)}} = \frac{\lambda^* - \lambda}{\sqrt{\lambda}}.$$

(28)

This can be substantially larger than the instantaneous Sharpe ratio $\mu/\sqrt{\sigma^2 + \lambda k^2}$ on equity given investors’ aversion to this type of risk. The put selling strategies examined in Jackwerth implicitly involve a portfolio that is instantaneously long equity and short crash insurance. Since adding a high Sharpe ratio investment to a market investment must raise instantaneous Sharpe ratios, this model is consistent with the substantial profitability of option-writing strategies reported in Jackwerth (2000), Coval and Shumway (2001), and Bakshi and Kapadia (2003). A corollary is that distorted subjective beliefs and the Liu et al. (2005) robust-control approach are equally incapable of explaining the Jackwerth anomaly.

Ex post Sharpe ratio estimates use instead of $\gamma$, where is the frequency of jumps observed over the data sample.

Coval and Shumway (2001, Table IV) also explicitly reject the hypothesis that option and stock index returns are jointly compatible with a power utility pricing kernel.
3. Equilibrium in a heterogeneous-agent economy

As this model is dynamically complete, equilibrium in the heterogeneous-agent case can be identified by examining an equivalent central planner’s problem in weighted utility functions. The solution to that problem is Pareto-optimal, and can be attained by a competitive equilibrium for traded assets in which all investors willingly hold market-clearing optimal portfolios given equilibrium asset price evolution. Section 3.1 below outlines the central planner’s problem, while Section 3.2 discusses the resulting asset market equilibrium. Section 3.3 identifies the supporting individual wealth evolutions and associated portfolio allocations, while Section 3.4 confirms the optimality of the equilibrium.

3.1. The central planner’s problem

For tractability, I assume all investors have common risk aversion \( R \), but differ in crash aversion \( Y \). Under homogeneous beliefs about state probabilities, the central planner’s problem of maximizing a weighted average of expected state-dependent utilities is equivalent to constructing a representative state-dependent utility function in terminal wealth (Constantinides, 1982, Lemma 2):

\[
U(W_T, N_T; \omega) \equiv \max_{W_{YT}} \sum Y \omega_Y f_Y(N_T) \frac{W_{YT}^{1-R} - 1}{1 - R}, \quad R > 0
\]

s. t. \( W_T = \sum Y W_{YT}, W_{YT} \geq 0 \quad \forall Y \) \hspace{1cm} (29)

for fixed weights \( \omega \equiv \{\omega_Y\} \) that depend upon the initial wealth allocation in a fashion determined below in Section 3.3. Since the individual marginal utility functions \( U_w(W_{YT}, N_T; Y) = +\infty \) at \( W_{YT} = 0 \) and the horizon is finite, the individual no-bankruptcy constraints \( W_{YT} \geq 0 \) are nonbinding and can be ignored. Optimizing the Lagrangian

\[
\max_{\{W_{YT}\}, \eta_T} \sum Y \omega_Y f_Y(N_T) \frac{W_{YT}^{1-R} - 1}{1 - R} + \eta_T \left[ W_T - \sum Y W_{YT} \right] \quad \text{for fixed } \omega = \{\omega_Y\} \hspace{1cm} (30)
\]

yields a terminal state-dependent wealth allocation

\[
w_Y(N_T, T; \omega) \equiv \frac{W_{YT}}{W_T} = \frac{[\omega_Y f_Y(N_T)]^{1/R}}{\sum Y [\omega_Y f_Y(N_T)]^{1/R}} \quad \text{for fixed } \omega = \{\omega_Y\} \hspace{1cm} (31)
\]

and a Lagrangian multiplier

\[
\eta_T = W_T^{-R} \left( \sum Y [\omega_Y f_Y(N_T)]^{1/R} \right)^{-R} \equiv W_T^{-R} f(N_T; \omega), \quad \text{for fixed } \omega = \{\omega_Y\} \hspace{1cm} (32)
\]

where \( f(\cdot) \) is a CES-weighted average of individual crash aversion functions \( f_Y(\cdot) \)'s. The Lagrangian multiplier \( \eta_T = U_w(W_T, N_T; \omega) \) is the shadow value of terminal wealth, and therefore determines the pricing kernel when evaluated at \( W_T = D_T \).
From the first-order conditions to (30), all individual terminal marginal utilities of wealth are directly proportional to the multiplier:

\[ U_W(W_{YT}, N_T, Y) = \frac{\eta_T}{\omega Y}. \]  

(33)

### 3.2. Asset market equilibrium

As in Eqs. (6)–(9) above, the pricing kernel \( \eta_T/\eta_I \) can be used to price all assets. That asset market equilibrium depends critically upon expectations of average crash aversion. Define

\[ g(N_t, t; \lambda') \equiv E_t[\tilde{f}(N_t + n)|\lambda'] = \sum_{n=0}^{\infty} \frac{e^{-\tilde{\lambda}'(T-t)}[\tilde{\lambda}'(T - t)]^n}{n!} \tilde{f}(N_t + n) \]  

(34)

as the conditional expectation of \( \tilde{f}(N_T) \) given jump intensity \( \lambda' \) over \((t,T]\) for future jumps \( n \equiv N_T - N_t \). It is shown in Appendix A that the resulting asset pricing equations are

\[ \eta_t = e^{\kappa_\eta(T-t)}D_t^{-R}g(N_t, t, \lambda e^{-R_i \lambda}) \]  

(35)

\[ \frac{\Delta S_t}{S_t} = e^{\kappa_S(T-t)} g(N_t, t, \lambda e^{(1-R_i \lambda)}}) g(N_t, t, \lambda e^{-R_i \lambda}) = e^{\kappa_S(T-t)} m(N_t, t), \]  

(36)

\[ \lambda^*(N_t, t) = \lambda e^{-R_i \lambda} \frac{g(N_t + 1, t, \lambda e^{-R_i \lambda})}{g(N_t, t, \lambda e^{-R_i \lambda})}, \]  

(37)

where

\[ \kappa_\eta = -R\mu_d + \frac{1}{2}R^2\sigma_d^2 + \lambda(\lambda e^{-R_i \lambda} - 1) \]

and

\[ \kappa_S = (\mu_d + \frac{1}{2}\sigma^2_d) - R\sigma_d^2 + \lambda e^{-R_i \lambda}(\lambda e^{-R_i \lambda} - 1). \]

The equilibrium equity price follows a jump-diffusion of the form

\[ \frac{\Delta S_t}{S_t} = \mu(N_t, t) dt + \sigma_d dZ_t + k(N_t, t)(dN_t - \lambda dt), \]  

(38)

where

\[ \mu(N_t, t) = -E_t \left[ \frac{\Delta S_t}{S_t} \frac{d\eta_t}{\eta_t} \right] = R\sigma_d^2 + [\lambda - \lambda^*(N_t, t)]k(N_t, t) \]  

(39)

and

\[ 1 + k(N_t, t) = e^{\kappa_d} \frac{m(N_t + 1, t)}{m(N_t, t)} \]  

(40)

for \( m(N_t, t) \) defined above in Eq. (36). The risk-neutral price process follows a martingale of the form:

\[ \frac{\Delta S_t}{S_t} = \sigma_d dZ_t + k(N^*_t, t)(dN^*_t - \lambda^*_t dt) \]  

(41)

for \( N^*_t \) a risk-neutral jump counter with instantaneous jump intensity \( \lambda^*(N^*_t, t) \), the functional form of which is given above in Eq. (37).
Several features of the equilibrium are worth emphasizing. First, conditional upon no jumps the asset price follows a diffusion similar to the news arrival process $D_t$, i.e., with identical and constant instantaneous volatility $\sigma_d$. This property reflects the assumption of common relative risk aversion $R$, and would not hold in general under alternate utility specifications or heterogeneous risk aversion. A further implication discussed below is that all investors hold identical equity positions.

Second, the equilibrium price process and crash risk premium depends critically upon the heterogeneity of agents. This is simplest to illustrate in the $R = 1$ case, for which equilibrium values can be expressed directly in terms of the weighted distribution of individual crash aversions. Define pseudo-probabilities

$$\pi_{Y_t} \equiv \frac{\omega_Y \exp[YN_t + \lambda \epsilon^{-\gamma_d}(T-t)(\epsilon^Y - 1)]}{\sum_Y \omega_Y \exp[YN_t + \lambda \epsilon^{-\gamma_d}(T-t)(\epsilon^Y - 1)]}$$

as the $N_t$-dependent weight assigned to investors of type $Y$ at time $t$, and define cross-sectional average $E_{CS}()$, variance $Var_{CS}()$, and covariance with respect to those weights. It is shown in Appendix A that the asset market equilibrium takes the form

$$\ln\left(\frac{\lambda^*}{\lambda}\right) = -\gamma_d + \ln E_{CS}[\epsilon^Y] \approx -\gamma_d + E_{CS}[Y]$$

(43)

$$\frac{\ln(S_t/D_t)}{T-t} = -\kappa + \ln E_{CS}[\epsilon^{\Phi(T-t)(\epsilon^Y - 1)}] \approx \mu + \frac{1}{2} \sigma_d^2 + \lambda \epsilon^{-\gamma_d} E_{CS}[\epsilon^Y](\epsilon^{\gamma_d} - 1),$$

(44)

$$\ln(1 + k_t) \approx \gamma_d [1 + \lambda \epsilon^{-\gamma_d}(T-t) Cov_{CS}(Y, \epsilon^Y)].$$

(45)

Heterogeneity has a divergent impact on the instantaneous level of prices versus the evolution of prices. To a first-order approximation, the crash risk premium in (43) and equity prices in (44) just replicate at any instant the homogeneous-agent equilibria of (18) and (20) at $R = 1$, using wealth-dependent weighted average values for $Y$ and $\epsilon^Y$, respectively. By contrast, the change in equity prices conditional upon a jump has two components: the direct impact of adverse news, and the indirect impact of the change in relative weights as wealth is transferred from crash-tolerant to crash-averse investors. The result is that relatively modest adverse news about terminal dividends can have a substantially magnified impact upon the stock market.

Fig. 3 below illustrates these impacts in the case of only two types of agents, conditional upon the initial wealth distribution and its impact on social weights $\omega$ (given below in Eq. (47)) and conditional upon an adverse news shock $\gamma_d = -0.03$. Crash-tolerant agents ($Y = 0$) can be viewed as knowing the true jump intensity $\lambda$. They trade with crash-averse agents ($Y = 1$), who can be viewed as having a subjective belief that crashes occur at $\epsilon^1 \approx 2.7$ times the true frequency. The presence of both types of agents in the economy has an extremely pronounced impact on the stock price response to jumps: a modest 3% drop in the terminal value signal can induce a 3–18% drop in the log price of equity! Crashes redistribute wealth, making

13Weinbaum (2001) and Dieckmann and Gallmeyer (2005) find that heterogeneous risk aversion increases stock market volatility relative to the underlying sources of risk.
the ‘average’ investor more crash-averse and exacerbating the impact of adverse news shocks. As indicated in Fig. 3, this magnification is also present for alternate values of the risk aversion parameter \( R \).

The crash risk premium \( \lambda^*_t \) is always between the \( e^{-R\gamma_d} \) value of the crash-tolerant investors \((Y = 0)\), and the \( e^{Y-R\gamma_d} \) value of the crash-averse investors. Its value depends monotonically upon the relative weights of the two types of investors. The equity premium \( \mu_t \) varies with \( R \) and with the magnitude of crash risk, but takes on generally reasonable values.

A final observation is that the asset market equilibrium depends upon the number of jumps \( N_t \), and is consequently nonstationary. This is an almost unavoidable feature of equilibrium models with a fixed number of heterogeneous agents. Heterogeneity implies agents have different portfolio allocations, implying their relative wealth weights and the resulting asset market equilibrium depend upon the nonstationary outcome of asset price evolution.\footnote{See Dumas (1989) and Wang (1996) for examples of the predominantly nonstationary impact of heterogeneity in a diffusion context. An interesting exception is Chan and Kogan (2002), who show that external habit formation preferences can induce stationarity in an exchange economy with heterogeneous agents.} In this model, the number of jumps \( N_t \) and time \( t \) are proxies for wealth distribution. Crashes redistribute wealth towards the more crash
averse, making the representative agent more crash averse. An absence of crashes has the opposite effect through the payment of crash insurance premia.

3.3. Supporting wealth evolution and portfolio choice

An investor’s wealth at any time \( t \) can be viewed as the value (or cost) of a contingent claim that pays off the investor’s share of terminal wealth \( W_T = D_T \) conditional upon the number of jumps:

\[
W_{Yt} = E_t \left[ \frac{\eta T}{\eta t} D_T w_Y(N_T, T; \omega) \right]
\]

\[
= S_t E_t \left[ \tilde{f}(N_T; \omega) \left( \omega \frac{1}{R} e^{Y N_T / R} / \sum Y \omega \frac{1}{R} e^{Y N_t / R} \right) \lambda e^{(1-R)Y_d} \right] \frac{E_t \left[ \tilde{f}(N_T; \omega) \lambda e^{(1-R)Y_d} \right]}{E_t \left[ \tilde{f}(N_T; \omega) \lambda e^{(1-R)Y_d} \right]}
\]

\[
= S_t w_Y(N_t, t; \omega),
\]

see Eq. (A.14) in Appendix A for details. The quantity \( W_{Yt} \) for \( t = n \), where \( n = \) the number of jumps up to \( t \), can be viewed as the value (or cost) of a jump-contingent evolution:

\[
w_Y(0, 0; \omega) = \kappa E_0 \omega \frac{1}{R} e^{Y N_T / R} \tilde{f}(N_T; \omega) \lambda e^{(1-R)Y_d}
\]

for \( \kappa = E_0 \tilde{f}(N_T; \omega) \lambda e^{(1-R)Y_d} \). In the \( R = 1 \) case the mapping between \( \omega \) and the initial wealth distribution is explicit, and takes the form

\[
w_Y(0, 0; \omega) = \kappa \omega Y e^{Y/\omega}. (48)
\]

The investment strategy that dynamically replicates the evolution of \( W_{Yt} \) can be identified using positions in equity and crash insurance that mimic the diffusion- and jump-contingent evolution:

\[
X_{Yt} = \frac{\partial W_{Yt}}{\partial S_t} = w_Y(N_t, t; \omega)
\]

\[
Q_{Yt} = [\Delta W_{Yt} - X_{Yt} \Delta S_t]_{d=1} = S(1 + k_t) [w_Y(N_t + 1, t; \omega) - w_Y(N_t, t; \omega)],
\]

where \( k_t = k(N_t, t) \) is the percentage jump size in the equity price given above in Eqs. (40) and (45). Thus, each investor holds \( X_{Yt} = W_{Yt} / S_t \) shares of equity (i.e., is 100% invested in equity), and holds a relative crash insurance position of

\[
q_{Yt} \equiv \frac{Q_{Yt}}{W_{Yt}} = (1 + k_t) \left[ \frac{w_Y(N_t + 1, t; \omega)}{w_Y(N_t, t; \omega)} - 1 \right].
\]

The wealth-weighted aggregate crash insurance positions \( \sum_y w_Y(N_t, t; \omega) q_{Yt} \) appropriately sum to 0.

Fig. 4 graphs the individual crash insurance demands \( (q_0, q_1) \) given crash aversions \( Y = 0 \) and 1, respectively, conditional upon the initial wealth share \( w_1 \) of the crash-averse investors and its impact upon equilibrium \( (\lambda^*, k_t) \) at time \( t = 0 \). The aggregate demand for crash insurance \( w_1 q_1 \) is also graphed, using the same calibration as in Fig. 3 above. At \( w_1 = 0 \), crash-tolerant investors \( (Y = 0) \) set a relatively low market-clearing price \( \lambda^* = \lambda e^{-\gamma_d} \) and sell little insurance. Crash-averse investors \( (Y = 1) \) insure heavily individually, but are a negligible fraction of the market. As \( w_1 \)
increases, $\lambda^*_t$ does as well (see Fig. 3 above) and the crash insurance positions of both investors decline. Aggregate crash insurance volumes are heaviest in the central regions where both types of investors are well represented. As $w_1$ approaches 1, the high price of crash insurance induces crash-tolerant investors to sell insurance that will cost them 61% of their wealth conditional upon a crash.

3.4. Optimality

The individual’s investment strategy yields a terminal wealth $W_{YT}$, and an associated terminal marginal utility of wealth $U_W(W_{YT}, N_T; Y)$ that from Eq. (33) is proportional to the Lagrangian multiplier $\eta_T$ that prices all assets. Therefore, no investor has an incentive to perturb his or her investment strategy given equilibrium asset prices and price processes. Furthermore, as noted above, the markets for equity and crash insurance clear, so the markets are in equilibrium. Since all individual state-dependent marginal utilities are proportional at expiration, the market is effectively complete. All investors agree on the price of all Arrow–Debreu securities, so their introduction would not affect the equilibrium.

4. Option markets in a heterogeneous-agent economy

4.1. Option prices

At time 0, European call options of maturity $t$ and strike price $X$ are priced at expected terminal value weighted by the pricing kernel:

$$c(S_0, t; X) = E_0 \left[ \frac{\eta_t}{\eta_0} \max(S_t - X, 0) \right] = E_0^D[\max(S_t - X, 0)].$$  \hspace{1cm} (51)$$

Conditional upon $N_t$ jumps over $(0, t]$, $\eta_t$ and $S_t$ have a joint lognormal distribution that reflects their common dependency on $D_t$ given above in Eqs. (35)
Consequently, it is shown in Appendix A that the risk-neutral distribution for $S_t$ is a weighted mixture of lognormals, implying European call option prices are a weighted average of Black–Scholes–Merton prices:

$$c(S_0, t; X) = \sum_{n=0}^{\infty} w_n^* e^{BSM(S_0, t; X, b_n, r = 0)}$$

$$= \sum_{n=0}^{\infty} w_n^* [S_0 e^{b_n t} N(d_{1n}) - XN(d_{1n} - \sigma_d \sqrt{t})]$$

where $\lambda' \equiv \lambda e^{-R_d t}$,

$$w_n^* \equiv \frac{e^{-\lambda' t} (\lambda')^n}{n!} \frac{g(n, t; \lambda')}{{g(0, 0; \lambda')}}$$

$$b_n = -\lambda'(e^{\sigma_d^2 t} - 1) + \{m_d + \ln[m(n, t)/m(0, 0)]\}/t$$

$$d_{1n} = [\ln(S_0/X) + b_n t + \frac{1}{2} \sigma_d^2 t]/\sigma_d \sqrt{t}$$

and $N(\cdot)$ is the standard normal distribution function. Put prices can be computed from call prices using put-call parity:

$$p(S_0, t, X) = c(S_0, t, X) + X - S_0.$$  (53)

Since jumps are always negative, the risk-neutral distribution of log-differenced equity prices implicit in option prices is always negatively skewed. Correspondingly, implicit standard deviations from options prices exhibit a substantial volatility smirk that is illustrated below in Fig. 5. However, this model’s volatility smirk flattens out at longer maturities. This is inconsistent with empirical evidence from Tompkins (2001) and Bates (2000, Fig. 4), who find that longer-maturity volatility smirks in stock index options are at least as pronounced as those from short-maturity options, when moneyness $\ln(X/S_0)$ is measured in maturity-specific standard deviation units.

### 4.2. Option replication and dynamic completion of the markets

Options can be dynamically replicated using positions in equity and crash insurance. Instantaneously, each call option has a price $c(S_t, N_t, t)$, and can be viewed as an instantaneous bundle of $c_S$ units of equity risk, and $[\Delta c - c_S \Delta S]_{d=1} > 0$ units of crash insurance.

This equivalence between options and crash insurance indicates how investors replicate the optimal positions of Section 3.3 dynamically, using the call and/or put options actually available. Crash-averse investors choose an equity/options bundle with unitary delta overall and positive gamma (e.g., hold $1^{1/2}$ stocks and buy one at-the-money put option with a delta of $-1^{1/2}$), while crash-tolerant investors take offsetting positions that also possess unitary delta (e.g., hold $1^{1/2}$ stock, and write 1 put option). Equity and option positions are adjusted in a mutually acceptable and offsetting fashion over time, conditional upon the arrival of news. As options expire,
new options become available and investors are always able to maintain their desired levels of crash insurance. All investors recognize that the price of crash insurance implicit in option prices will evolve over time, conditional on whether crashes do or do not occur, and take that into account when establishing their positions.

A further implication is that the crash-tolerant investors who write options actively delta-hedge their exposure, which is consistent with the observed practice of option market makers. As $\lambda_t^*/\lambda$ increases (e.g., because of wealth transfers to the crash averse from crashes), the market makers respond to the more favorable prices by writing more options as a proportion of their wealth. They simultaneously adjust their equity positions to maintain their overall target delta of 1. This strategy is equivalent to market makers putting their personal wealth in an index fund, and fully delta-hedging every index option they write.

4.3. Consistency with empirical option pricing anomalies

The heterogeneous-agent model explains unconditional deviations between risk-neutral and objective distributions analogously to the homogeneous-agent model. The divergence in the jump intensity $\lambda_t^*$ implicit in options and the true jump frequency

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15 As indicated above in Fig. 4, the aggregate positions (open interest) in crash insurance and therefore in options can either rise or fall as the wealth distribution varies.
can reconcile the average divergence between risk-neutral and objective variance, and between the predicted and observed frequency of jumps over 1988–1998. Both models generate volatility smirks that flatten out at longer maturities, contrary to the maturity profile of smirks observed in stock index options.

The advantage of the heterogeneous-agent model is that it partially explains some of the conditional divergences as well. First, the stochastic evolution of \( \lambda_t^* \) is qualitatively consistent with the evolution of jump intensity shown above in Fig. 1. \( \lambda_t^* \) depends directly upon the relative wealth distribution, which in turn follows a pure jump process given above in (46). Market jumps cause sharp increases in \( \lambda_t^* \); the crash insurance (or options) contracts transfer wealth to crash-averse investors, increasing demand (and reducing supply) for crash insurance. An absence of jumps steadily transfers wealth in the reverse direction, generating geometric decay in \( \lambda_t^* \) towards the lower level of crash-tolerant investors.

Fig. 6 illustrates the resulting evolution of instantaneous risk-neutral variance \( R\sigma_d^2 + \lambda_t^* \gamma_t^2 \) conditional on the five major shocks over 1988–1998, and conditional on starting with \( w_1 = .1 \) at end-1987. This behavior is qualitatively similar to the actual impact of jumps on overall variance and on jump risk shown above in Fig. 1. However, the absence of major shocks over 1992–1996 and the resulting wealth accumulation by crash-tolerant investors/option market makers implies that the shocks of 1997 and 1998 should not have had the major impact that was in fact observed. Furthermore, all simulated variance shocks are substantially smaller than the magnitudes seen in Fig. 1.

It is possible the heterogeneous model can explain the results from ISD regressions as well. The analysis is complicated by the fact that instantaneous objective and risk-neutral variance are nonstationary, with a nonlinear cointegrating relationship from their common dependency on the nonstationary variable \( N_t \):

\[
Var_t[d \ln S] = \left[ \sigma_d^2 + \lambda_t^* \gamma_t^2 \right] dt, \quad Var_t^*[d \ln S] = \left[ \sigma_d^2 + \lambda_t^*(N_t, t) \gamma_t^2 \right] dt
\]  

(54)

Fig. 6. Simulated instantaneous risk-neutral variance conditional upon jump timing matching 5 jumps observed over 1988–1998. Calibration: \( R = 1, w_1 = .1 \); i.e., crash-averse investors initially own 10% of total wealth at end-1987. Other parameters are the same as in Fig. 3.
for $\gamma_t \equiv \ln[1 + k(N_t, t)]$ and $\lambda^* > \lambda$. It is not immediately clear whether regressing realized on implied volatility is meaningful under nonlinear cointegration. However, the fact that implicit variance does contain information for objective variance but is biased upwards suggests that running this sort of regression on post-1987 data would yield the usual informative-but-biased results reported above in Eq. (2), with estimated slope coefficients less than 1 in sample.

It does not appear that the heterogeneous-agent model can explain the implicit pricing kernel puzzle. Using the same projection as in (26) above (in Appendix A, Section A.3), the projected pricing kernel is

$$
M(S_t) \equiv \frac{E_0[\eta_t|S_t]}{\eta_0} = \kappa(t)S_t^{-R} \sum_{N=0}^{\infty} w_N^{**} \frac{P(S_t|N)}{p(S_t)} \quad \text{where}
$$

$$
w_N = \frac{e^{-i\lambda t}N^N}{N!}, \quad w_N^{**} = \frac{w_N m(N, t)^R g(N, t, \lambda e^{(1-R)\eta_0})}{\sum_{N=0}^{\infty} w_N m(N, t)^R g(N, t, \lambda e^{(1-R)\eta_0})}
$$

and $\kappa(t)$ is a time-dependent scaling factor that does not affect implicit risk aversion. As illustrated in Fig. 7, this implicit pricing kernel appears to be a strictly decreasing function of $S_t$ — in contrast to the locally positive sections estimated in Jackwerth (2000) and Rosenberg and Engle (2002). However, the above implicit kernel can replicate those studies’ high implicit risk aversion for large negative returns, as indicated by the steep line in Fig. 7 for $\Delta s$ in the $-10\%$ to $-15\%$ range.

![Fig. 7. Log of the implicit pricing kernel conditional upon realized asset returns. Calibration: $w_1 = .3$, $t = 1/12$, $R = 1$ month. Other parameters are the same as in Fig. 3.](image)
5. Summary and conclusions

This paper has proposed a modified utility specification, labeled ‘crash aversion,’ to explain the observed tendency of post-1987 stock index options to overpredict realized volatility and jump risk. Furthermore, the paper has developed a complete-markets methodology that permits identification of asset market equilibria and associated investment strategies in the presence of jumps and investor heterogeneity. The assumption of heterogeneity appears to have stronger consequences than observed with diffusion models. In particular, stock market crashes become partly endogenous. Relatively small adverse announcement effects become substantially magnified by equilibrium wealth redistribution towards more crash-averse investors. The model in this paper consequently offers an explanation why we occasionally observe substantial crashes or ‘corrections’ in the stock market (e.g., the 1987 crash) despite no correspondingly large news about firms’ future prospects.

The model has been successful in explaining some of the stylized facts from stock index options markets. The specification of crash aversion is compatible with the tendency of option prices to overpredict volatility and jump risk, while heterogeneity of agents offers an explanation of the stochastic evolution of implicit jump risk and implicit volatilities. In this model, the two are higher immediately after market drops not because of higher objective risk of future jumps (as predicted by affine models), but because crash-related wealth redistribution has increased average crash aversion. Crash aversion is also consistent with the implicit pricing kernel approach’s assessment of high implicit risk aversion at low wealth levels, although the approach cannot replicate the local risk-loving behavior reported in Jackwerth (2000) and Rosenberg and Engle (2002).

While motivated by empirical option pricing regularities, the heterogeneous-agent model in the paper is unfortunately not suitable for direct estimation. First, jump risk is not the only risk spanned in the options markets. Stochastic variations in conditional volatility occur more frequently, and are also important to option market makers. Second, the nonstationary equilibrium derived here and characteristic of almost all heterogeneous-agent models hinders estimation. The purpose of the paper is to provide a theoretical framework for exploring the trading of jump risk through the options market, as an initial model of the option market making process.

The heterogeneous-agent model does, however, have some interesting implications for empirical equity and options research. In particular, the model indicates that implicit stock market crash magnitudes should follow stochastic jump processes, and that the magnitudes are related to the extent of investor heterogeneity at any given time. Models with time-varying jump distributions have not been extensively considered; perhaps they should be. And while there has been considerable work on developing measures of heterogeneity in beliefs (e.g., using analysts’ forecasts) and examining implications for equity markets, little of this has spilled over into options research – with the notable exception of Buraschi and Jiltsov (2006).
The framework in this paper can be expanded in various ways. For simplicity, this paper has focused on deterministic jumps, but extending the model to random jumps would be straightforward. A particularly interesting extension could be to explore the implications of portfolio constraints on positions in options and/or crash insurance. Selling crash insurance requires writing calls or puts – a strategy that individual investors cannot easily pursue. Further research will examine the impact of such constraints upon equilibria in equity and options markets.

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Appendix A

A.1. Asset market equilibrium in a heterogeneous-agent economy (Section 3.2)

Lemma. If the log signal $d_t \equiv \ln D_t$ follows the jump-diffusion given above in Eq. (3) and $\tilde{f}(N_T)$ is an arbitrary function,

$$E_t[D^m_t\tilde{f}(N_T)|\lambda] = D^m_t e^{\kappa(m)(T-t)} E_t[\tilde{f}(N_t + n)|\lambda e^{m\gamma}] ,$$

where $n \equiv N_T - N_t$, $\kappa(m) \equiv m\mu_d + \frac{1}{2}m^2\sigma_d^2 + \lambda(e^{m\gamma} - 1)$, and $E_t[\bullet|\lambda]$ denotes expectations conditional upon a jump intensity $\lambda$ over $(t, T]$. 

Proof. Define $\Delta d \equiv \ln(D_T/D_t)$ and $\tau \equiv T - t$. Then

$$E_t[D^m_t h(N_T)] = E_t[D^m_t e^{m\Delta d \tilde{f}(N_T)}] = D^m_t E_t[e^{m\Delta d_{\text{mono}} + m\gamma} \tilde{f}(N_t + n)]$$

$$= D^m_t e^{(m\mu_d + \frac{1}{2}m^2\sigma_d^2 + \lambda(e^{m\gamma} - 1))} E_t[\tilde{f}(N_t + n)|\lambda e^{m\gamma}] .$$

(A.1)

Define $g(N_t, t; \lambda') \equiv E[\tilde{f}(N_t + n)|\lambda']$. The asset pricing equations (35)–(37) follow directly from the lemma:

$$\eta_t = E_t[D_T^{-R} \tilde{f}(N_T)] \equiv D_T^{-R} e^{\kappa(-R)(T-t)} g(N_t, t; \lambda e^{-Rt}) ,$$

(A.3)
\[ S_t = \frac{E_t[D_T^{1-R}] f(N_T)}{\eta_t} = D_t e^{\eta [1-R(1-R)(1-R)^{(T-t)}]} g(N_t, t; \lambda e^{1-R_t \lambda}) 
\]
\[ = D_t e^{\kappa (T-t)} m(N_t, t), \quad (A.4) \]

\[ \lambda^*_t = \frac{\lambda_t \eta_{|\text{jumps}}}{\eta_t} = e^{-R_t \lambda} g(N_t + 1, t; \lambda e^{-R_t \lambda}) \frac{g(N_t, t; \lambda e^{-R_t \lambda})}{g(N_t, t; \lambda e^{-R_t \lambda})}. \quad (A.5) \]

### A.1.1. Asset pricing in the \( R = 1 \) subcase

In the special case \( R = 1 \) and for arbitrary \( \lambda', \bar{f}(N_T) \) is additively separable and \( g(\cdot) \) becomes

\[ g(N_t, t; \lambda') = E_t \left[ \sum Y^{(1)} \exp[YN_t + \lambda'(T - t)(e^Y - 1)] \right] \]
\[ = \sum Y^{(1)} \exp[YN_t + \lambda'(T - t)(e^Y - 1)]. \quad (A.6) \]

Define \( \lambda' \equiv \lambda e^{-R_t \lambda} \) and \( \lambda'' \equiv \lambda e^{1-R_t \lambda} \), and define pseudo-probabilities

\[ \pi_{Y_t} = \frac{\omega_Y \exp[YN_t + \lambda'(T - t)(e^Y - 1)]}{\sum Y^{(1)} \exp[YN_t + \lambda'(T - t)(e^Y - 1)]} \quad (A.7) \]

as the cross-sectional weight associated with investors of type \( Y \). Using (A.6) for \( g(\cdot) \), the equity pricing equation (A.4) becomes

\[ \frac{S_t}{D_t} = e^{\kappa (T-t) \sum Y^{(1)} \exp[YN_t + \lambda''(T - t)(e^Y - 1)]} \]
\[ \sum Y^{(1)} \exp[YN_t + \lambda'(T - t)(e^Y - 1)] \]
\[ = e^{\kappa (T-t) \sum Y^{(1)} \pi_{Y_t} \exp[(\lambda'' - \lambda')(T - t)(e^Y - 1)]} \]
\[ = e^{\kappa (T-t) E_{CS}[\exp(\Phi(T-t)(e^Y - 1)]} \quad (A.8) \]

for the cross-sectional expectation \( E_{CS}(\cdot) \) defined with regard to probabilities (A.7), and for \( \Phi \equiv \lambda'' - \lambda' = \lambda e^\omega (e^{-R_t \lambda} - 1) \). From (A.5), the jump risk premium has a similar representation:

\[ \frac{\lambda^*_t}{\lambda} = e^{-R_t \lambda} \frac{\sum Y^{(1)} \exp[YN_t + \lambda'(T - t)(e^Y - 1)]}{\sum Y^{(1)} \exp[YN_t + \lambda'(T - t)(e^Y - 1)]} \]
\[ = e^{-R_t \lambda} \sum Y^{(1)} \pi_{Y_t} e^Y = e^{-R_t \lambda} E_{CS}(e^Y). \quad (A.9) \]

The approximation for the log jump size follows from the following approximations:

\[ \ln m(N_t, t) \equiv \ln \left[ \frac{g(N_t, t; \lambda'')}{g(N_t, t; \lambda')} \right] \approx \frac{\partial}{\partial \lambda'} \ln \left[ \frac{g(N_t, t; \lambda')}{g(N_t, t; \lambda)} \right] (\lambda'' - \lambda'). \quad (A.10) \]
\[ \ln(1 + k_t) = R_t^{\gamma_d} + \ln \left[ \frac{m(N_t + 1, t)}{m(N_t, t)} \right] \approx R_t^{\gamma_d} + \frac{\partial \ln m(N_t; t)}{\partial N_t} \]
\[ \approx R_t^{\gamma_d} + \frac{\partial^2 \ln g(N_t; t; \lambda)}{\partial N_t \partial \lambda'}(\lambda'' - \lambda'). \]

From (A.6) and (A.7), the cross-derivative turns out to be
\[ \frac{\partial^2 \ln g(N_t; t; \lambda)}{\partial N_t \partial \lambda'} = (T - t) \left\{ \sum_y \pi_{Y_t} Y(e^Y - 1) - \sum_y \pi_{Y_t} Y \sum_y \pi_{Y_t}(e^Y - 1) \right\} \]
\[ = (T - t) \text{Cov}_{CS}(Y, e^Y). \] (A.11)

Consequently (from (A.11)),
\[ \ln(1 + k_t) \approx R_t^{\gamma_d} + (\lambda'' - \lambda')(T - t) \text{Cov}_{CS}(Y, e^Y) \]
\[ \approx R_t^{\gamma_d} + \lambda e^{\gamma d}(e^{-R_t^{\gamma_d}} - 1)(T - t) \text{Cov}_{CS}(Y, e^Y). \] (A.13)

Section 3.3, Eq. (46)

\[ V_t = E_t \left[ \eta_T^{\lambda} D_T^{(1/R)} \frac{e^{Y_{N_t}/R_T}}{f(N_T)} \right] = E_t \left[ \frac{D_t^{1-R} e^{Y_{N_t}/R_T} f(N_T)^{1-1/R}}{E_t[f(N_T)|\lambda e^{-R_t^{\gamma_d}}]} \right] \]
\[ = \frac{D_t^{1-R} e^{\kappa_{s(t-t)}} E_t[e^{Y_{N_t}/R_T} f(N_T)^{1-1/R}]}{E_t[f(N_T)|\lambda e^{-R_t^{\gamma_d}}]} \] (A.14)

Substituting in \[ S_t = D_t e^{\kappa_{s(t-t)}} E_t[f(N_T)|\lambda e^{-R_t^{\gamma_d}}]/E_t[f(N_T)|\lambda e^{-R_t^{\gamma_d}}] \] from (A.4) yields (46).

A.2. Objective and risk-neutral distributions

From (36), gross stock returns are
\[ \frac{S_t}{S_0} = e^{-\kappa_{s(t)}} D_t \frac{m(N_t, t)}{m(0, 0)} \] (A.15)
for \[ \kappa_s = (\mu_d + 1/2 \sigma_d^2) - R \sigma_d^2 + \lambda e^{-R_t^{\gamma_d}}(e^{\gamma_d} - 1). \] Since \( \Delta d = \ln(D_t/D_0) \) is an \( N_t \)-dependent mixture of normals, log-differenced stock prices \( \Delta s = \ln(S_t/S_0) \) are also a mixture of normals:
\[ p(\Delta s) = \sum_{N=0}^{\infty} w_N n(\Delta s|\mu_N, \sigma_d^2) \text{ for } w_N = \frac{e^{-\gamma d}(\gamma t)^N}{N!}, \] (A.16)
where \( n(z|\mu, \sigma^2) \) is the normal density function with mean \( \mu \) and variance \( \sigma^2 \), and
\[ \mu_N = (R - 1/2 \sigma_d^2)t + \lambda e^{-R_t^{\gamma_d}}(e^{\gamma_d} - 1) t + N \gamma_d + \ln[m(N, t)/m(0, 0)]. \]

Define \( 1(\Delta s = z) \) as the delta function that takes on infinite value when \( \Delta s = z \), zero value elsewhere, and integrates to 1. The objective density function
In the homogeneous-agent case, the risk-neutral density function is
\[
p^*(z) = E_0 \left[ \frac{\eta_0}{\eta} 1(\Delta s = z) \right] = \sum_{N=0}^{\infty} w_N E_0[\eta_0 1(\Delta s = z) | N \text{ jumps}] / \eta_0.
\] (A.17)

For any two normally distributed variables \(x\) and \(y\) and any arbitrary function \(h(y)\),
\[
E[e^y h(y)] = E[e^y]E[h(y + \sigma_{xy})],
\] (A.18)
where \(\sigma_{xy} = \text{Cov}(x, y)\). Conditional upon \(N\) jumps, \(\ln \eta_t\) and \(\Delta s\) are both normally distributed with covariance \(-R\sigma^2_d\). Consequently, (A.17) can be re-written as
\[
p^*(z) = \sum_{N=0}^{\infty} w_N E_0(\eta_0 | N \text{ jumps}) E_0[1(\Delta s - R\sigma^2_d = z) | N \text{ jumps}] / \eta_0
\]
\[
= \sum_{N=0}^{\infty} w_N E_0(\eta_0 | N \text{ jumps}) n(z | \mu_N - R\sigma^2_d, \sigma^2_d) / \eta_0
\]
\[
\equiv \sum_{N=0}^{\infty} w^*_N n(z | \mu_N - R\sigma^2_d, \sigma^2_d).
\] (A.19)

Since \(\eta_0 = E_0[\eta_0 | N \text{ jumps}] = \sum_N w_N E_0[\eta_0 | N \text{ jumps}]\), the weights \(w^*_N\) sum to 1. Furthermore, since
\[
\eta_t = e^{-\kappa_d(t-t)} D_0^{-R} \exp[-R(\Delta d)_{N_j=0} + N_0] g(N, t, \lambda \epsilon^{-R_i_d}),
\] (A.20)
it is straightforward to show that
\[
w^*_N = \frac{w_N e^{-R_i_d N} g(N, t, \lambda \epsilon^{-R_i_d})}{\sum_{N=0}^{\infty} w_N e^{-R_i_d N} g(N, t, \lambda \epsilon^{-R_i_d})} = \frac{e^{-\lambda t} (\lambda')^N g(N, t; \lambda')}{N! g(0, 0; \lambda')}
\] (A.21)
for \(\lambda' = \lambda \epsilon^{-R_i_d} \).

### A.3. Implicit pricing kernels (Eqs. (27) and (55))

Using Eqs. (19) and (20), the projection of the pricing kernel upon the asset price in the homogeneous-agent case is
\[
M(S_t) = \sum_{N=0}^{\infty} w^*_N n\left(\frac{S_t}{D_t}, R e^{YN_i} \kappa_0(t), |S_t|\right)
\]
\[
= S_t^{-R} E_0 \left[ \left( \frac{S_t}{D_t} \right)^{R e^{YN_i} \kappa_0(t)} |S_t| \right] = \kappa_1(t) S_t^{-R} E_0 e^{YN_i |S_t|},
\] (A.22)
where \(\kappa_0(t)\) and \(\kappa_1(t)\) capture time-dependent terms irrelevant to implicit risk aversion. The distribution of \(s_t \equiv \ln S_t\) is an \(N_r\)-dependent mixture of normals:
\[
p(s_t | N_i) = n(\mu_0 + N_0 \gamma_d^2, \sigma_d^2 t) \text{ with probability } w_{N_i} = \frac{e^{-\lambda t} (\lambda t)^{N_i}}{N_i!}
\] (A.23)
for an appropriate choice of $\mu_0$. Consequently, the conditional expectation in (A.22) can be evaluated using Bayes’ rule to evaluate the conditional probabilities

$$\text{Prob}[N_t = n | S_t] = \frac{w_n p(s_t | n)}{\sum_{n=0}^{\infty} w_n p(s_t | n)}$$  \hspace{1cm} (A.24)

yielding an implicit pricing kernel

$$M(S_t) = \kappa_1(t) S_t^{-R} \frac{\sum_{n=0}^{\infty} w_n p(s_t | n) e^{Y_n}}{\sum_{n=0}^{\infty} w_n p(s_t | n)} = \kappa(t) S_t^{-R} p(s_t | \lambda) e^{Y},$$  \hspace{1cm} (A.25)

where $p(s_t | \lambda)$ denotes the unconditional density of $s_t$ given a jump intensity of $\lambda$ over $(0, t]$. Taking partials with respect to $s_t$ and using the fact that $\partial p(s_t | n) / \partial s_t = -p(s_t | n)[s_t - (\mu_0 + n \gamma_d) / \sigma^2]$ yields (after some tedious calculations) an implicit risk-aversion value

$$- \frac{\partial \ln M(S_t)}{\partial S_t} = R + \frac{-\gamma_d}{\sigma^2 t} \frac{\text{Cov}^{**} (e^{Y_n}, \tilde{n})}{E^{**} [Y]}$$ \hspace{1cm} (A.26)

where $E^{**}$ and $\text{Cov}^{**}$ are defined with regard to the probabilities in (A.24). Since $e^{Y_n}$ and $n$ are both increasing functions of $n$, the covariance term is positive. Consequently, the implicit risk aversion is everywhere positive given $\gamma_d < 0$.

The heterogeneous-agent case is similar. From (35) and (36), the Lagrangian multiplier is

$$\eta_t = e^{\kappa_1(T-t) S_t^{-R} \left( \frac{S_t}{D_t} \right)} g(N_t, t; \lambda e^{-R \gamma_d}) = e^{(\kappa_1 + R \xi S)(T-t) S_t^{-R} m(N_t, t)^R g(N_t, t; \lambda e^{-R \gamma_d})}.$$  \hspace{1cm} (A.27)

This is of the same form as (A.22), with $m(N_t, t)^R g(N_t, \bullet)$ replacing $e^{Y N_t}$. Consequently, the implicit pricing kernel becomes

$$M(S_t) = \kappa_1(t) S_t^{-R} \frac{\sum_{n=0}^{\infty} w_N p(s_t | N) m(N, t)^R g(N, t; \lambda e^{-R \gamma_d})}{\sum_{n=0}^{\infty} w_N p(s_t | N)}$$  \hspace{1cm} (A.28)

for

$$w_N^{**} = \frac{w_N m(N, t)^R g(N, t; \lambda e^{-R \gamma_d})}{\sum_{n=0}^{\infty} w_N m(N, t)^R g(N, t; \lambda e^{-R \gamma_d})}.$$

References


