Campbell’s “Inspecting the mechanism” (JME, 1994) added some useful intuition to the stochastic growth model by giving quasi-analytic expression to many of its features. I thought it would be nice to do the same for some other models. These notes work through various linear and loglinear approximations to dynamic models and describe the relation between them. I thank Chris Edmond for getting me started on the right track.

**Canonical model: Lagrangian approach**

My canonical dynamic program goes like this: choose \( \{u_t\} \) to maximize

\[
\sum_{t=0}^{\infty} \beta^t f(x_t, u_t)
\]

subject to the law of motion

\[
x_{t+1} = g(x_t, u_t)
\]

and the initial value \( x_0 \). We call \( x \) the state and \( u \) the control. Think of both as scalars for now. Our goal: to find a linear approximation to the decision rule \( u = h(x) \), expressed in terms of properties (parameters) of the functions \( f \) and \( g \). The usual certainty equivalence property of LQ problems tells us that the same (approximate) solution applies to stochastic problems.

Each of our approaches has the same steps: (i) derive necessary conditions for an optimum, (ii) use these conditions to compute the steady state, and (iii) approximate the necessary conditions in the neighborhood of the steady state.

Here’s how the three steps look in this case. The Lagrangian is

\[
\mathcal{L} = \sum_{t=0}^{\infty} \beta^t \{ f(x_t, u_t) - \lambda_t [x_t - g(x_{t-1}, u_{t-1})] \}.
\]

In the initial term, think of \( g(x_{-1}, u_{-1}) \) as the given number \( x_0 \). The focs imply

\[
\begin{align*}
    u_t : & \quad 0 = f_u(x_t, u_t) + \beta \lambda_{t+1} g_u(x_t, u_t) \\
    x_t : & \quad 0 = f_x(x_t, u_t) - \lambda_t + \beta \lambda_{t+1} g_x(x_t, u_t) \\
    \lambda_t : & \quad 0 = x_t - g(x_{t-1}, u_{t-1}).
\end{align*}
\]

for \( t \geq 0 \).

*Working notes. No guarantee of accuracy or sense.*
Steady state. Kill the \( t \)s and solve the three equations for \( u \), \( x \), and \( \lambda \).

Linear approximation. The linearized focs are

\[
\begin{align*}
0 &= f_{ux}dx + f_{uu}du + \beta g_u d\lambda' \\
0 &= f_{xx}dx + f_{xu}du - d\lambda + \beta g_x d\lambda' \\
dx' &= g_x dx + g_u du.
\end{align*}
\]

The understanding is that all of these derivatives are evaluated at the steady state. [Comment: I’ve dropped terms involving second derivatives of \( g \). That’s tradition, and makes the algebra simpler, but we can put them in later if we want.]

Solution. Klein (JEDC 2000) suggests we put the three linear equations into first-order linear dynamic system in the the expanded state vector \( z = (u, \lambda, x) \). More commonly, we would use the first equation to substitute for \( u \), leaving us with the two-dimensional system in \((\lambda, x)\). The standard approach is to find the roots of the system, typically giving us one stable and one unstable root, which we solve backward and forward, resp. A good example is Hansen and Sargent (JEDC 1980, reprinted in the Lucas-Sargent volume), where this is done explicitly.

**Canonical model: recursive approach**

A recursive approach to the same problem starts with the Bellman equation

\[
J(x) = \max_u f(x, u) + \beta J[x' = g(x, u)].
\]

The foc and ec are, resp,

\[
\begin{align*}
0 &= f_u(x, u) + \beta J_x [g(x, u)] g_u(x, u) \\
J_x(x) &= f_x(x, u) + \beta J_x [g(x, u)] g_x(x, u).
\end{align*}
\]

These equations and the lom give us the solution.

Steady state. We find the steady state values of \( x \) and \( u \) by solving three equations (the foc, the ec, and the lom) for the three unknowns \((x, u, \text{ and } J_x)\) with \( x' = x \). [This is the same as before, with \( J_x(x_t) \) playing the role of \( \lambda_t \).] Assume we do this and have steady state values.

Linear approximation. Given the steady state, we can construct a linear approximation of the solution around the steady state by taking a linear approximation of the three equations:

\[
\begin{align*}
0 &= f_{ux}dx + f_{uu}du + \beta J_{xx} g_u [g_x dx + g_u du] \\
J_{xx} dx &= f_{xx} dx + f_{xu} du + \beta J_{xx} g_x [g_x dx + g_u du] \\
dx' &= g_x dx + g_u du.
\end{align*}
\]

Our goal is to find a scalar \( h_x \) such that \( du = h_x dx \). There are a couple different ways to attack this problem. Among them:
• Riccati equation for the unknown $J_{xx}$. Solve the first equation for the decision rule:

$$du = \left( \frac{f_{xx} + \beta J_{xx} g_u g_x}{f_{uu} + \beta J_{xx} (g_u)^2} \right) dx = h_x dx. \quad (1)$$

Then substitute into the second equation:

$$J_{xx} = f_{xx} + f_{xu} h_x + \beta J_{xx} (g_x)^2$$

If we do the algebra, this boils down to a quadratic in $J_{xx}$. We take the positive root. The equations should be familiar if you’ve seen linear-quadratic problems before.

• Guess and verify $h_x$. The trick here is to use (1) to substitute for $J_{xx}$. You get a quadratic in $h_x$ instead.

Example: LQ problem

A scalar LQ problem might be described using

$$f(x, u) = qu^2 + rx^2$$
$$g(x, u) = ax + bu$$

The parameters satisfy $r > 0$ and $(a, b) \neq 0$ ($q$ can be zero if we’re careful about dividing).

Lagrangian approach. Minimize

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t \left[ (qu_t^2 + rx_t^2) - 2\lambda_t (x_t - ax_{t-1} - bu_{t-1}) \right].$$

The focs imply

$$u_t : 0 = qu_t + \beta \lambda_{t+1} b$$
$$x_t : 0 = rx_t - \lambda_t + \beta \lambda_{t+1} a.$$ 

Using the foc for $u$, the lom becomes

$$x_{t+1} = ax_t + (b^2/q) \lambda_{t+1}.$$ 

This gives us the linear dynamic system

$$\begin{bmatrix} \beta a & 0 \\ \beta b^2/q & 1 \end{bmatrix} \begin{bmatrix} \lambda_{t+1} \\ x_{t+1} \end{bmatrix} = \begin{bmatrix} 1 & -r \\ 0 & a \end{bmatrix} \begin{bmatrix} \lambda_t \\ x_t \end{bmatrix}$$

As usual, the product of the two generalized eigenvalues is $1/\beta$. Both are positive, with one greater than $1/\beta$, and one less than $1/\beta$.

Recursive approach. The solution is characterized by the Bellman equation,

$$px^2 + c = \min_u \left\{ qu^2 + rx^2 + \beta p(ax + bu)^2 \right\},$$
for scalars \((p, c)\) to be determined. The first-order and envelope conditions are

\[
\begin{align*}
0 &= 2qu + 2\beta p(ax + bu)b \\
2px &= 2rx + 2\beta pa(ax + bu).
\end{align*}
\]

The foc gives us a decision rule \(u = -fx\) and controlled law of motion \(ax + bu = (a - bf)x = a^*x\) with

\[
\begin{align*}
f &= \frac{\beta pab}{q + \beta pb^2} \\
a^* &= \frac{aq}{q + \beta pb^2}.
\end{align*}
\]

The ec and foc together imply the Riccati equation

\[
p = r + \beta pa^2 - (\beta pab)^2/(q + \beta pb^2) = r + \beta pa^2q/(q + \beta pb^2).
\]

Once we know \(p\), we have the solution. Ultimately it’s a quadratic equation, which we can solve a number of ways. One is to solve the Riccati equation directly (iterating, for example). Another (which works in the scalar case) is to write it explicitly as a quadratic:

\[
\beta (b^2/q)p^2 + [1 - \beta (a^2 + rb^2/q)]p + r = 0.
\]

The final coefficient \((-r)\) is negative, so Descarte’s rule of signs tells us there’s a single positive root.

**Example: stochastic growth model**

Consider a loglinear approximation of the stochastic growth model, with endogenous state variable \(k\) and exogenous forcing process \(z\). We’re looking for a decision rule of the form

\[
\dot{c} = h_{ck}\dot{k} + h_{cz}\dot{z},
\]

where \(\dot{x} = \log x - \log \bar{x}\) for any variable \(x\) (namely, the differential of the log) and \(\bar{x}\) is the steady state value. We use a result noted by Hansen and Sargent (JEDC 1980) and Anderson, Hansen, McGrattan, and Sargent (handbook chapter 1995): in the LQ case, the coefficient \(h_{ck}\) does not depend on the process for \(z\). In that sense, the model has only one real state variable, and we need only solve one quadratic equation to find the solution \((h_{ck}, h_{cz})\).

Model. Consider the stationary stochastic growth model given by maximizing

\[
E_0 \sum_{t=0}^{\infty} \beta^t u(c_t),
\]

subject to the laws of motion

\[
\begin{align*}
k_{t+1} &= g(k_t, z_t) - c_t \\
\log z_{t+1} &= \varphi \log z_t + \varepsilon_{t+1}
\end{align*}
\]
with \( \{ \varepsilon_t \} \sim N(0, \kappa_2) \). We’ll generalize this later. Specific functions: \( u(c) = c^{1-\rho}/(1-\rho) \), \( y = zk^\theta \), \( g(k, z) = (1 - \delta)k + zk^\theta \). Then: \( u_c(c) = c^{-\rho} \) and \( g_k(k, z) = (1 - \delta) + \theta(y/k) \). The IES is \( \sigma = 1/\rho \). We drop \( E \) and \( \varepsilon \) in what follows on the usual certainty equivalence grounds.

**Recursive approach.** The Bellman equation is

\[
J(k, z) = \max_c u(c) + \beta J[g(k, z) - c, z'].
\]

The foc and ec are

\[
\begin{align*}
u_c(c) &= \beta J_k(k', z') \\
J_k(k, z) &= \beta J_k(k', z') g_k(k, z).
\end{align*}
\]

With our functional forms, the steady state satisfies

\[
\begin{align*}
c^{-\rho} &= \beta J_k \\
1 &= \beta g_k = \beta[(1 - \delta) + \theta z k^{\theta - 1}] = \beta[(1 - \delta) + \theta (y/k)] \\
\delta k &= zk^\theta - c,
\end{align*}
\]

which we solve for the unknowns \((c, k, J_k)\) for a given value of \( z \) (one, say). [Comment: Campbell expresses ratios like \((y/k)\) in terms of the underlying parameters. I leave them here, in part because the ratios are typically used to calibrate the parameters — ie, they’re the primitive. We can come back to that later if we want.] Consider ballpark numbers: \( \theta = 1/3, \delta = 0.025, \rho = 2, k/y = 10 \) (quarterly). Then

\[
10 = k/y = k^{1-\theta} \Rightarrow k = 10^{3/2} = 10.2362.
\]

Similarly, \( y = k^{1/3} = 3.162 \) and \( c = y - \delta k = (1 - \delta k/y)y = 2.372 \). The discount factor is

\[
\beta = [(1 - \delta) + \theta (y/k)]^{-1} = 0.9917,
\]

so \( J_k = c^{-\rho}/\beta = 0.1763 \).

Now loglinearize. The loms and return to capital \( g_k \) become (approximately)

\[
\begin{align*}
k' &= \beta^{-1}k + (y/k)z - (c/k)\hat{c} \\
z' &= \varphi \hat{z} \\
g_k &= -\beta(1 - \theta) \theta (y/k) \hat{k} + \beta \theta (y/k) \hat{z} = -g_{kk} \hat{k} + g_{kz} \hat{z}.
\end{align*}
\]

[This last is crucial: it introduces concavity of the production function into the solution.]

To solve the model, we conjecture

\[
-\hat{J}_k = p_{kk} \hat{k} + p_{kz} \hat{z}.
\]

[Negative to get concavity right with \( p > 0 \).] You might guess this is pretty close to conjecturing a loglinear decision rule or a logquadratic value function — and it is. The foc becomes (recall \( \sigma = 1/\rho \) is the IES)

\[
\hat{c} = -\sigma \hat{J}_k = \sigma (p_{kk} \hat{k}' + p_{kz} \hat{z}') = \sigma p_{kk} [\beta^{-1} \hat{k} + (y/k) \hat{z} - (c/k) \hat{c}] + \sigma p_{kz} \varphi \hat{z}.
\]
Solve this for \( \hat{c} \) to find the decision rule parameters,

\[
\begin{align*}
    h_{ck} &= \sigma \beta^{-1} p_{kk} / [1 + \sigma(c/k)p_{kk}] \\
    h_{cz} &= \sigma[(y/k)p_{kk} + \varphi p_{kz}] / [1 + \sigma(c/k)p_{kk}],
\end{align*}
\]

expressed as functions of (the unknown) \((p_{kk}, p_{kz})\).

Find \((p_{kk}, p_{kz})\). Loglinearize the ec and substitute in the foc:

\[
\sigma(p_{kk}\hat{k} + p_{kz}\hat{z}) = \hat{c} + \sigma g_{kk} \hat{k} - \sigma g_{kz} \hat{z}
\]

or

\[
\hat{c} = \sigma(p_{kk} - g_{kk})\hat{k} + \sigma(p_{kz} + g_{kz})\hat{z}.
\]

Lining up terms:

\[
\begin{align*}
p_{kk} - g_{kk} &= \beta^{-1} p_{kk} / [1 + \sigma(c/k)p_{kk}] \\
p_{kz} + g_{kz} &= [(y/k)p_{kk} + \varphi p_{kz}] / [1 + \sigma(c/k)p_{kk}].
\end{align*}
\]

The first equation is effectively the Riccati equation, and nails down \(p_{kk}\):

\[
\sigma(c/k)(p_{kk})^2 + [1 - \beta^{-1} - \sigma(c/k)g_{kk}]p_{kk} - g_{kk} = 0.
\]

The equation has one root of each sign (the last term is negative), we take the positive one.

[Comment: This is where the Hansen-Sargent result shows up — \(p_{kk}\) doesn’t depend on anything related to \(z\) (\(p_{kz}\) or \(\varphi\) for example).] The second equation nails down \(p_{kz}\) (given \(p_{kk}\)):

\[
p_{kz} = \frac{(y/k)p_{kk} - [1 + \sigma(c/k)p_{kk}]g_{kz}}{1 - \varphi + \sigma(c/k)p_{kk}}.
\]

Once we have the solutions \((p_{kk}, p_{kz})\), we find the decision rules from (4,5). To show: as \(\sigma\) goes to zero, \(h_{kk} = 1/\beta - (c/k)h_{ck}\) goes to one. In this sense, the IES is a key parameter for the dynamics of the model.

**Campbell approach.** Here we solve directly for the parameters of the decision rule. Combine the foc + ec to get an Euler equation:

\[
u_c(c) = \beta u_c(c')g_k(k', z').
\]

The log-linear version is

\[-\rho \hat{c} = -\rho \hat{c'} + \hat{g}_k' = -\rho \hat{c'} - g_{kk} \hat{k}' + g_{kz} \hat{z}'
\]

or

\[
\hat{c} = \hat{c'} + \sigma g_{kk} \hat{k}' - \sigma g_{kz} \hat{z}'.
\]

Substituting the conjectured decision rule and the loms, we get

\[
\hat{c} = (h_{ck} + \sigma g_{kk})\hat{k}' + (h_{cz} - \sigma g_{kz})\hat{z}'
\]

\[
= (h_{ck} + \sigma g_{kk})[\beta^{-1} \hat{k}' + (y/k)\hat{z'} - (c/k)\hat{c}] + (h_{cz} - \sigma g_{kz})\varphi \hat{z}'.
\]

6
or

\[ 1 + (h_{ek} + \sigma g_{kk})(c/k)(h_{ek} \dot{k} + h_{ez} \dot{z}) = (h_{ek} + \sigma g_{kk})\beta^{-1} \dot{k} + [(h_{ek} + \sigma g_{kk})(y/k) + (h_{ez} - \sigma g_{kz})\varphi] \dot{z} \]

Lining up the \( \dot{k} \) terms, we get

\[ (c/k)(h_{ek})^2 + [1 - \beta^{-1} + \sigma(c/k)g_{kk}]h_{ek} - \beta^{-1}\sigma g_{kk} = 0. \]

Another quadratic, similar to the earlier one. With some effort, you can show that this is equivalent to the earlier approach: not only do you get the same answer, but the other root for \( p_{kk} \) leads to the other root for \( h_{ek} \). Lining up the \( \dot{z} \) terms yields

\[ h_{ez} = \frac{(h_{ek} + \sigma g_{kk})(y/k) - \sigma \varphi g_{kz}}{1 - \varphi + (h_{ek} + \sigma g_{kk})(c/k)}. \]

Comment: I’ve verified these solutions numerically (the recursive and Campbell approaches give the same answers). See the Matlab file \texttt{rbcll.m}.

Value function. This will seem a little vague, but we have here a log-linear value function and can therefore likely approximate the solution with EZ preferences. Note that the derivative of the value function satisfies

\[ \log J_k = p_0 - p_{kk} \log k - p_{kz} \log z, \]

which implies

\[ J_k = e^{p_0} k^{-p_{kk}} z^{-p_{kz}}. \]

If we integrate wrt \( k \) we get (if \( p_{kk} \neq 1 \))

\[ J = \text{constant} + (p_{kk} + 1)^{-1} e^{p_0} k^{1-p_{kk}} z^{p_{kz}}. \]

I assume we can kill the constant. That leaves us with a “Cobb-Douglas” value function, which we can evaluate in the certainty equivalent function of EZ prefs.

Why different from Tallarini? He has (effectively) a quadratic approximation of the log of the value function in the logs of the variables. The derivatives are then loglinear, just as they are here.