Who Holds Risky Assets?

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Abstract: Preference heterogeneity is a natural explanation for portfolio heterogeneity. In a dynamic environment in which preference heterogeneity is as extreme as possible, we show that intuition about the relationship between risk aversion and holdings of risky assets derived from static choice problems can be very misleading. In equilibrium, an agent with recursive utility who is infinitely risk averse over static gambles will hold a portfolio composed almost entirely of risky assets. Conversely, an agent with recursive utility who is risk neutral over static gambles will hold a portfolio composed almost entirely of risk-free assets. Moreover, there is no added compensation for holding the risky asset since equilibrium asset prices will appear to be generated by a risk-neutral representative agent. This counter-intuitive result highlights the relative roles of static risk preferences and deterministic substitution preferences in recursive utility. Since portfolio choice is fundamentally a decision about intertemporal consumption lotteries, both characteristics are important. Specifically, we show that the preference for the timing of the resolution of uncertainty plays a major role in allocation of investments between risky and risk-free assets. A strong preference for the late resolution of uncertainty translates into a strong preference for smooth consumption paths and, hence, a portfolio choice heavily skewed toward risk-free assets. This strong preference can exist even when the agent is risk neutral with respect to static gambles. Conversely, a strong preference for the early resolution of uncertainty translates into a strong preference for smooth utility which can be achieved even when portfolio choice heavily skewed toward risky assets.

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1 Introduction

Blah, blah, blah...

2 Recursive Utility Pareto Problem

A social planner allocates income \( y_t \) across two agents. The dynamics of income are characterized by a Markov process with transition probabilities \( p(y_t | y_{t-1}) \). Each agent has a recursive utility function defined by

\[
U_t = \left[ (1 - \beta) c_t^\rho + \beta \mu_t(U_{t+1})^{\rho_1} \right]^{\frac{1}{\rho}}
\]

\[
\mu_t(U_{t+1}) = \left[ E_t U_{t+1}^a \right]^{\frac{1}{a}}
\]

where preference parameters \( \rho \leq 1, \alpha \leq 1 \) and \( 0 < \beta < 1 \) can differ across the two agents.

The marginal rate of intertemporal substitution is given by

\[
m_{t+1} = \beta \left( \frac{c_{t+1}}{c_t} \right)^{\rho - 1} \left( \frac{U_{t+1}}{\mu_t(U_{t+1})} \right)^{\alpha - \rho}
\] (1)

The two-agent Pareto problem is a sequence of consumption allocations for each agent \( \{c_{1,t}, c_{2,t}\} \) that maximizes the weighted average of date-0 utilities subject to the aggregate resource constraint which binds at each date and state:

\[
\max_{\{c_{1,t}, c_{2,t}\}} \lambda W_0 + (1 - \lambda) V_0
\]

s.t. \( c_{1,t} + c_{2,t} = y_t \) for all \( s^t \)

Note that even though each agent has recursive utility, the objective function of the social planner is not recursive. However, we can rewrite this as a recursive optimization problem following, lucas/stokey:84, and kan:95:

\[
J(V, s) = \max_{c_1, c_2, V'(s')} \left[ (1 - \beta)c_1^{\rho_1} + \beta \mu_{1,t}(J(V', s'))^{\rho_1} \right]^{\frac{1}{\rho_1}}
\]

s.t. \( [(1 - \beta)c_2^{\rho_2} + \beta \mu_{2,t}(V')^{\rho_2}]^{\frac{1}{\rho_2}} \geq V \)

\[
c_1 + c_2 = y(s)
\] (2)
where $s$ is the current information set, $\mu_{i,t}$ is the certainty equivalent based on information $s$, and $s'$ is next periods information set. The optimal policy involves choosing agent two’s consumption, $c_2$, and a future utility $V'$ (one for each future state, $s'$) to maximize agent one’s utility given an initial promise to agent 2 of utility, $V$. The promise serves as a law of motion for the state variable, $V$.

3 Linear-Leontief Preferences

Recursive preferences allow the flexibility specifying risk aversion, $\alpha$, and intertemporal substitution, $\rho$ separately. The difference in these two parameters, $\alpha - \rho$, determines the preference for the timing of the resolution of uncertainty. It turns out that an example with two agents who differ wildly along this dimension is explicitly solvable. To develop a better understanding of how the Pareto problem in equation eq:recursivepareto works, consider preferences

$$
W = \min\{(c_1, E[W'])\}
$$

$$
V = (1 - \beta)c_2 + \beta \min\{V'\}
$$

Agent 1 is risk neutral ($\alpha_1 = 1$) with perfectly inelastic deterministic substitution ($\rho_1 = -\infty$). In contrast, agent 2 has infinite risk aversion ($\alpha_2 = -\infty$) and perfectly elastic deterministic intertemporal substitution ($\rho_2 = 1$). Formally, we assume that $\beta_1 = \beta_2 = \beta$. However, note that the limit as the curvature parameter ($\rho_1$) goes to infinity eliminates the discount factor for agent 1. Similarly, as agent 2’s curvature parameter ($\alpha_2$) goes to infinity, the probabilities are eliminated. This leads to the “min” in their preferences. The agents, by construction, are very different. Agent 1 dislikes long-run risk, but tolerates short-run risk. Agent 2 is the opposite. In this extreme setting we can explicitly solve the Pareto problem to see how the consumption allocations vary over time. We can also, interestingly, solve for aggregate asset prices. Surprisingly (perhaps), this setting leads to a simple representative agent.
4 IID income endowment

To specify the recursive Pareto problem in this setting, let income be IID as $y_t = \bar{y} + \epsilon_t$ with $E\epsilon_t = 0$. The structure of the problem implies that no other features of the income process matter (the variance, for example).

$$J(y, V) = \max_{c, V'} \min \{y - c, EJ(y', V')\}$$

s.t. $V = (1 - \beta)c + \beta \min \{V'\}$

That is choose agent two’s consumption, $c$, and future utility, $V'$ (one for each future state) to maximize agent one’s utility from current consumption $y - c$ (applying the resource constraint) and future utility, $EW' = EJ(y', V')$.

This is given an initial utility promise to agent two. The utility promised to agent two, $V'$, defines the law of motion for the state variable.

We solve this problem with a guess and verification of the value function. Conjecture that the value function is linear (the obvious guess after you work out a few recursions).

$$J(y, V) = p_0 + p_y y - p_v V$$

for parameters $(p_0, p_y, p_v)$ to be determined. We solve for these parameters and the allocations in three steps. (i) Since agent 2 has infinite risk aversion, utility promises are constant across states, $V' = \bar{V}'$. (ii) The level of current promised utility imply a consumption allocation of

$$c = \frac{V}{1 - \beta} - \frac{\beta \bar{V}'}{1 - \beta}$$

(iii) The optimization is, therefore,

$$p_0 + p_y y - p_v V = \max_{\bar{V}'} \min \left\{y - \left(\frac{V}{1 - \beta} - \frac{\beta \bar{V}'}{1 + \beta}\right), p_0 + p_y \bar{y} - p_v \bar{V}'\right\}$$

Optimality occurs where the two quantities inside the min are equal (where current consumption equals future utility). This implies

$$\bar{V}' \left[p_v + \frac{\beta}{1 - \beta}\right] = p_0 + p_y \bar{y} - y + \frac{V}{1 - \beta}$$
Substitutes this relation into the Bellman equation and equate terms,

\[ p_0 = \beta \bar{y}, \quad p_y = 1 - \beta, \quad p_v = 1 \]

This solution defines the Pareto frontier as

\[ W = J(y, V) = (1 - \beta)\bar{y} + \beta y - V \]

The controlled decision rules and laws of motion include

\[ y - c = \bar{y} - V + (1 - \beta)(y - \bar{y}) \]
\[ c = V + \beta(y - \bar{y}) \]
\[ W' = \bar{y} - V + (1 - \beta)(y - \bar{y}) + (1 - \beta)e' \]
\[ \bar{V}' = V - (1 - \beta)(y - \bar{y}) \]

The exact solutions offer some insight into the optimal allocation rules. If there is an increase in income \( y \), agent 2 consumes a fraction \( \beta \) of it and agent 1 consumes fraction \( (1 - \beta) \). In agent 1’s case, optimality requires current consumption and expected future expected utility go up one-for-one; hence, the identical increase in \( W' \). In agent 2’s case, an increase in \( y \) leads to an increase in current consumption and a fall in future utility. This exploits the fact that agent 2 has infinite intertemporal substitution.

5 Persistent income endowment

The previous example shows how differences in attitude towards intertemporal risk affects the optimal income allocations. Therefore, it is interesting to see how the allocations behave if the endowment process follows a persistent process. Assume income \( y_t \) is AR(1) process

\[ y' = (1 - \varphi)\bar{y} + \varphi y + \epsilon' \]

where \( \bar{y} \) is the unconditional mean, \( \varphi \) is the persistence parameter, and \( E\epsilon' = 0 \). Again, the structure of the problem implies that no other features of the process matter (the conditional variance, for example).
We solve the Pareto problem with the same steps as above (a bit more algebra). This leads to the value function parameters

\[ p_0 = \frac{\beta(1-\varphi)}{1-\beta\varphi} \bar{y}, \quad p_y = \frac{(1-\beta)}{1-\beta\varphi}, \quad p_v = 1 \]

The controlled decision rules and laws of motion include

\[
\begin{align*}
    y - c &= \frac{1-\varphi}{1-\beta\varphi} y - V + \frac{1-\beta}{1-\beta\varphi} (y - \bar{y}) \\
    c &= V + \frac{\beta(1-\varphi)}{1-\beta\varphi} (y - \bar{y}) \\
    W' &= \bar{y} - V + \frac{(1-\beta)}{1-\beta\varphi} (y - \bar{y}) + \frac{(1-\beta)}{1-\beta\varphi} \epsilon' \\
    \bar{V}' &= V - \frac{(1-\beta)(1-\varphi)}{1-\beta\varphi} (y - \bar{y})
\end{align*}
\]

Comparing these results to the IID case above, shows the role of a persistent endowment process. Persistent income, \( \varphi > 0 \), increases the sensitivity of agent 1's consumption, \( y - c \), to current income. The persistence of the shock to income makes it easier to increase agent 1's future utility, \( EW' \). Recall, the inelastic intertemporal preferences of agent 1 require current and future utility to move one for one. The persistence reduces the sensitivity of agent 2's consumption, \( c \), to current income.

### 6 Aggregation

Having solved for the income allocations, we can now describe asset prices in these economies. Given the extreme nature of the Linear-Leontief preferences (\( \alpha_1 = 1, \rho_1 = -\infty, \alpha_2 = -\infty, \) and, \( \rho_2 = 1 \)), it is, perhaps, surprising that there exists a representative agent and that the representative agent has preferences in the class of recursive preferences. In fact, the representative agent in either economy (the income process does not matter) has linear preferences; that is \( \tilde{\alpha} = \tilde{\rho} = 1 \).

To start, consider the MRS of agent 2

\[ m'_2 = \beta \left( \frac{\epsilon'}{c} \right)^{\rho_2-1} \left( \frac{V'}{\mu_2(V')} \right)^{\alpha_2-\rho_2} \]
The first term in brackets is one since $\rho_2 = 1$. Optimality implies agent 2’s next period’s utility is a constant, $V' = \bar{V}'$. Hence, the second term in brackets is one. This implies, the pricing kernel for agent 2 is $\beta$. For Agent 1, $\alpha_1 = 1$ implies

$$m'_1 = \beta \left( \frac{y' - c'}{y - c} \right)^{\rho_1 - 1} \left( \frac{W'}{\mu_1(W')} \right)^{\alpha_1 - \rho_1}$$

Recall, that for agent 1, optimality requires that we equate current and future utility, hence $y - c = EW'$. Second, the Bellman equation implies $y' - c' = W'$ (a bit of algebra using the controlled laws of motion). This implies that the pricing kernel for agent 1 is, again, $\beta$. It is, of course, not surprising that optimal allocations imply $m'_1 = m'_2$.

Despite the extreme preferences in this example, asset prices are equivalent to an economy with a representative agent with linear preferences, $\hat{\alpha} = \hat{\rho} = 1$, implying a pricing kernel of $m' = \beta$. This holds for either of the income processes we considered. In this case, the representative agent preferences are independent of the income process. This is a feature that is specific to this example. In general, as we explore below, the representative agent preferences will reflect the preferences of the individual agents and features of the endowment process.

### 7 Income Distribution

An important characteristic in models with heterogeneous agents is the dynamic properties of the cross-section distribution of consumption (or wealth). If the two agents have consumption that grow at different rates, then the cross-section distribution degenerates to the point where one agent gets everything (in the limit). In the Linear-Leontief example does infinite risk aversion or inelastic substitution dominate the consumption cross-section? The answer is neither. Consider the case of an IID income endowment. The controlled laws of motion imply the consumption of both agents is a drift-less random walk. In addition, note that utility levels are also a random walk.
In the case of an income endowment with persistence, the consumption and income levels of both agents are stationary.

8 Decentralized Equilibrium

Equity price:

\[ P_t^s = E_t \sum_{j=1}^{\infty} \left( \prod_{i=1}^{j} m_{t+i} \right) y_{t+j} = \sum_{j=1}^{\infty} \beta^j E_t y_{t+j} \]

iid case: \( E_t y_{t+j} = \bar{y} \) so the equity price is \( P_t^s = [\beta/(1 - \beta)]\bar{y} \), constant for all \( t \).

AR(1) case: \( E_t y_{t+j} = \bar{y}(1 - \phi^j) + \phi^j y_t \), which implies an equity price of

\[ P_t^s = y_t \sum_{j=1}^{\infty} \beta^j \left[ \bar{y}(1 - \phi^j) + \phi^j \right] = \frac{\beta}{1 - \beta} \left[ \frac{\beta \varphi}{1 - \beta \varphi} \right] + \frac{\beta \varphi}{1 - \beta \varphi} y_t. \]

Bond price:

\[ P_t^b = E_t m_{t+1} = \beta \]

which is constant for all \( t \) for any income process.

Agent 2 portfolio choice in a stock-bond economy:

Date-\( t \) budget constraint:

\[ \theta_{t-1}^s (P_t^s + y_t) + \theta_{t-1}^b = c_t + \theta_t^s P_t^s + \theta_t^b P_t^b, \]

where \( \theta_t^s \) and \( \theta_t^b \) investments in equity and one-period bonds respectively.
Consider equilibrium prices in the iid case. The budget constraint becomes

\[ \theta_{s_{t-1}}(P^s + \bar{y} + \varepsilon_t) + \theta_{b_{t-1}}^b = c_t + \theta_{s_{t}}^s P^s + \theta_{b_{t}}^b \beta. \]

Conjecture a solution for equity holdings of \( \theta_{s_{t}}^s = \beta \) constant for all \( t \). The budget constraint becomes

\[ \beta \bar{y} + \beta \varepsilon_t + \theta_{b_{t-1}}^b = c_t + \theta_{b_{t}}^b \beta. \]

Recall that optimal consumption in this case satisfies

\[
\begin{align*}
\begin{align*}
c_t &= V_t + \beta \varepsilon_t \\
&= V_{t-1} + (\beta - 1) \varepsilon_{t-1} + \beta \varepsilon_t \\
&= V_0 + (\beta - 1) \sum_{j=1}^{t} \varepsilon_{t-j} + \beta \varepsilon_t,
\end{align*}
\end{align*}
\]

therefore any sequence of bond holdings such that

\[ \beta \bar{y} + \theta_{b_{t-1}}^b - \beta \theta_{b_{t}}^b = V_0 + (\beta - 1) \sum_{j=1}^{t} \varepsilon_{t-j}, \]

will support the optimal consumption allocation. In other words, the optimal bond holdings for Agent 2 solves the stochastic difference equation

\[ \theta_{b_{t}}^b = \kappa_0 + \kappa_1 \theta_{b_{t-1}}^b + \eta_t, \]

where \( \kappa_0 = \bar{y} - V_0 / \beta, \kappa_1 = 1 / \beta \) and \( \eta_t - \eta_{t-1} = [(1 - \beta) / \beta] \varepsilon_{t-1}. \)

Note that the asset holdings of Agent 1 can be found from the aggregate resource constraint, \( ie \), bonds are in zero net supply and there is one unit of the stock that pays a per-period dividend of \( y_t \).

The interesting feature of this competitive equilibrium is that the “infinitely risk averse” agent, Agent 2, holds almost all of the risky asset, \( \theta_{s_{t}}^s = \beta \), and the “risk neutral” agent, Agent 1, holds almost none, \( ie \), 1 - \( \beta \), of the risky asset. This example highlights the importance of the preference for the early resolution of uncertainty as a dimension of risk that can be captured with recursive utility.

Portfolio holdings in the AR(1) case...
Blah, blah, blah...

With recursive utility, “risk aversion” (ie, $\alpha$) is a statement about one-step-ahead utility lotteries. “Resolution of uncertainty” (ie, $\alpha - \rho$) is a statement about consumption lotteries. In the LL example, one guy (the 2nd guy, or the V guy) wants his utility next period to be perfectly predictable, but doesn’t care how much consumption has to fluctuate over time to deliver that. The other guy (the W guy) doesn’t care at all about how much utility fluctuates period to period, provided future consumption is perfectly predictable.

To see this, set $\beta$ equal to 1 (not technically, because things blow up, but intuitively, or think of a $\beta$ really close to 1). In the iid-income case, the infinitely risk averse (over utility lotteries!!!) guy absorbs all consumption risk: $c_t = V_0 + \varepsilon_t$, but by doing this, utility is always perfectly predictable: $V_t = V_0$. On the other hand, the other guy who has an infinite preference for early resolution of uncertainty (over consumption lotteries!!!) has perfectly predictable consumption every period: $y_t - c_t = \bar{y} - V_0$, which is constant because the iid case, the intercept is the only predictable component of future income (his utility is constant too, but that’s an artifact he doesn’t care about). You get similar intuition in the persistent-income case, and when $\beta$ is less than one.

I bet this intuition carries over to the less-extreme cases: One guy has a stronger preference for predictable utility and the other guy has a stronger preference for predictable consumption. Clearly they are related, but different. So we need to compare the $\varepsilon_{t+1}$ coefficient (ie, the conditional variance) in utility for one guy against the $\varepsilon_{t+1}$ coefficient in consumption for the other guy.