Lecture 19
Observability and state estimation

• state estimation
• discrete-time observability
• observability – controllability duality
• observers for noiseless case
• continuous-time observability
• least-squares observers
• example
State estimation set up

we consider the discrete-time system

\[ x(t + 1) = Ax(t) + Bu(t) + w(t), \quad y(t) = Cx(t) + Du(t) + v(t) \]

• \( w \) is state disturbance or noise
• \( v \) is sensor noise or error
• \( A, B, C, \) and \( D \) are known
• \( u \) and \( y \) are observed over time interval \([0, t - 1]\)
• \( w \) and \( v \) are not known, but can be described statistically, or assumed small (e.g., in RMS value)
State estimation problem

**state estimation problem**: estimate $x(s)$ from

$$u(0), \ldots, u(t-1), y(0), \ldots, y(t-1)$$

- $s = 0$: estimate initial state
- $s = t - 1$: estimate current state
- $s = t$: estimate (i.e., predict) next state

An algorithm or system that yields an estimate $\hat{x}(s)$ is called an *observer* or *state estimator*

$\hat{x}(s)$ is denoted $\hat{x}(s|t-1)$ to show what information estimate is based on (read, “$\hat{x}(s)$ given $t-1$”)
Noiseless case

Let’s look at finding \( x(0) \), with no state or measurement noise:

\[
x(t + 1) = Ax(t) + Bu(t), \quad y(t) = Cx(t) + Du(t)
\]

With \( x(t) \in \mathbb{R}^n \), \( u(t) \in \mathbb{R}^m \), \( y(t) \in \mathbb{R}^p \)

Then we have

\[
\begin{bmatrix}
y(0) \\
\vdots \\
y(t-1)
\end{bmatrix} = O_t x(0) + T_t
\begin{bmatrix}
u(0) \\
\vdots \\
u(t-1)
\end{bmatrix}
\]
where

$$\mathcal{O}_t = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{t-1} \end{bmatrix}, \quad \mathcal{T}_t = \begin{bmatrix} D & 0 & \cdots \\ CB & D & 0 & \cdots \\ \vdots & \ddots & \ddots & \ddots \\ CA^{t-2}B & CA^{t-3}B & \cdots & CB & D \end{bmatrix}$$

- $\mathcal{O}_t$ maps initials state into resulting output over $[0, t - 1]$
- $\mathcal{T}_t$ maps input to output over $[0, t - 1]$

hence we have

$$\mathcal{O}_t x(0) = \begin{bmatrix} y(0) \\ \vdots \\ y(t-1) \end{bmatrix} - \mathcal{T}_t \begin{bmatrix} u(0) \\ \vdots \\ u(t-1) \end{bmatrix}$$

RHS is known, $x(0)$ is to be determined
hence:

- can uniquely determine $x(0)$ if and only if $\mathcal{N}(O_t) = \{0\}$
- $\mathcal{N}(O_t)$ gives ambiguity in determining $x(0)$
- if $x(0) \in \mathcal{N}(O_t)$ and $u = 0$, output is zero over interval $[0, t - 1]$
- input $u$ does not affect ability to determine $x(0)$; its effect can be subtracted out
Observability matrix

by C-H theorem, each $A^k$ is linear combination of $A^0, \ldots, A^{n-1}$

hence for $t \geq n$, $\mathcal{N}(\mathcal{O}_t) = \mathcal{N}(\mathcal{O})$ where

$$
\mathcal{O} = \mathcal{O}_n = \begin{bmatrix}
C \\
CA \\
\vdots \\
CA^{n-1}
\end{bmatrix}
$$

is called the observability matrix

if $x(0)$ can be deduced from $u$ and $y$ over $[0, t - 1]$ for any $t$, then $x(0)$ can be deduced from $u$ and $y$ over $[0, n - 1]$

$\mathcal{N}(\mathcal{O})$ is called unobservable subspace; describes ambiguity in determining state from input and output

system is called observable if $\mathcal{N}(\mathcal{O}) = \{0\}$, i.e., $\text{Rank}(\mathcal{O}) = n$
Observability – controllability duality

let \((\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})\) be dual of system \((A, B, C, D)\), i.e.,

\[
\tilde{A} = A^T, \quad \tilde{B} = C^T, \quad \tilde{C} = B^T, \quad \tilde{D} = D^T
\]

controllability matrix of dual system is

\[
\tilde{C} = \left[ \tilde{B} \ A \tilde{B} \ T \cdot \tilde{A} \ n \ \tilde{B} \right]
\]
\[
= \left[ C^T \ A^T C^T \ ... \ (A^T)^n \ C^T \right]
\]
\[
= \mathcal{O}^T,
\]

transpose of observability matrix

similarly we have \(\tilde{O} = C^T\)
thus, system is observable (controllable) if and only if dual system is controllable (observable)

in fact,

\[ \mathcal{N}(\mathcal{O}) = \text{range}(\mathcal{O}^T)^\perp = \text{range}(\tilde{\mathcal{O}})^\perp \]

\text{i.e., unobservable subspace is orthogonal complement of controllable subspace of dual}
Observers for noiseless case

suppose $\text{Rank}(O_t) = n$ (i.e., system is observable) and let $F$ be any left inverse of $O_t$, i.e., $FO_t = I$

then we have the observer

$$x(0) = F \left( \begin{bmatrix} y(0) \\ \vdots \\ y(t-1) \end{bmatrix} - T_t \begin{bmatrix} u(0) \\ \vdots \\ u(t-1) \end{bmatrix} \right)$$

which deduces $x(0)$ (exactly) from $u, y$ over $[0, t-1]$

in fact we have

$$x(\tau - t + 1) = F \left( \begin{bmatrix} y(\tau - t + 1) \\ \vdots \\ y(\tau) \end{bmatrix} - T_t \begin{bmatrix} u(\tau - t + 1) \\ \vdots \\ u(\tau) \end{bmatrix} \right)$$

Observability and state estimation
i.e., our observer estimates what state was $t - 1$ epochs ago, given past $t - 1$ inputs & outputs

observer is (multi-input, multi-output) finite impulse response (FIR) filter, with inputs $u$ and $y$, and output $\hat{x}$
Invariance of unobservable set

**fact:** the unobservable subspace \( \mathcal{N}(\mathcal{O}) \) is invariant, *i.e.*, if \( z \in \mathcal{N}(\mathcal{O}) \), then \( Az \in \mathcal{N}(\mathcal{O}) \)

**proof:** suppose \( z \in \mathcal{N}(\mathcal{O}) \), *i.e.*, \( CA^k z = 0 \) for \( k = 0, \ldots, n-1 \)

evidently \( CA^k(Az) = 0 \) for \( k = 0, \ldots, n-2 \);

\[
CA^{n-1}(Az) = CA^n z = - \sum_{i=0}^{n-1} \alpha_i CA^i z = 0
\]

(by C-H) where

\[
\det(sI - A) = s^n + \alpha_{n-1}s^{n-1} + \cdots + \alpha_0
\]
Continuous-time observability

continuous-time system with no sensor or state noise:

\[ \dot{x} = Ax + Bu, \quad y = Cx + Du \]

can we deduce state \( x \) from \( u \) and \( y \)?

let’s look at derivatives of \( y \):

\[
\begin{align*}
    y &= Cx + Du \\
    \dot{y} &= C\dot{x} + D\dot{u} = CAx + CBu + D\dot{u} \\
    \ddot{y} &= CA^2x + CABu + CB\ddot{u} + D\ddot{u}
\end{align*}
\]

and so on
hence we have
\[
\begin{bmatrix}
y \\
y' \\
\vdots \\
y^{(n-1)}
\end{bmatrix}
= O x + T
\begin{bmatrix}
u \\
u' \\
\vdots \\
u^{(n-1)}
\end{bmatrix}
\]

where $O$ is the observability matrix and

\[
T = 
\begin{bmatrix}
D & 0 & \cdots \\
CB & D & 0 & \cdots \\
\vdots \\
CA^{n-2}B & CA^{n-3}B & \cdots & CB & D
\end{bmatrix}
\]

(same matrices we encountered in discrete-time case!)
rewrite as

$$Ox = \begin{bmatrix} y \\
y' \\
\vdots \\
y^{(n-1)} \end{bmatrix} - T \begin{bmatrix} u \\
u' \\
\vdots \\
u^{(n-1)} \end{bmatrix}$$

RHS is known; $x$ is to be determined

hence if $\mathcal{N}(O) = \{0\}$ we can deduce $x(t)$ from derivatives of $u(t), y(t)$ up to order $n - 1$

in this case we say system is observable

can construct an observer using any left inverse $F$ of $O$:

$$x = F \left( \begin{bmatrix} y \\
y' \\
\vdots \\
y^{(n-1)} \end{bmatrix} - T \begin{bmatrix} u \\
u' \\
\vdots \\
u^{(n-1)} \end{bmatrix} \right)$$
• reconstructs $x(t)$ (exactly and instantaneously) from

$$u(t), \ldots, u^{(n-1)}(t), y(t), \ldots, y^{(n-1)}(t)$$

• derivative-based state reconstruction is dual of state transfer using impulsive inputs
A converse

suppose \( z \in \mathcal{N}(\mathcal{O}) \) (the unobservable subspace), and \( u \) is any input, with \( x, y \) the corresponding state and output, \( i.e.\),

\[
\dot{x} = Ax + Bu, \quad y =Cx + Du
\]

then state trajectory \( \tilde{x} = x + e^{tA}z \) satisfies

\[
\dot{\tilde{x}} = A\tilde{x} + Bu, \quad y = C\tilde{x} + Du
\]

\( i.e.\), input/output signals \( u, y \) consistent with both state trajectories \( x, \tilde{x} \)

hence if system is unobservable, no signal processing of any kind applied to \( u \) and \( y \) can deduce \( x \)

unobservable subspace \( \mathcal{N}(\mathcal{O}) \) gives fundamental ambiguity in deducing \( x \) from \( u, y \)
least-squares observers

discrete-time system, with sensor noise:

\[ x(t + 1) = Ax(t) + Bu(t), \quad y(t) = Cx(t) + Du(t) + v(t) \]

we assume \( \text{Rank}(O_t) = n \) (hence, system is observable)

least-squares observer uses pseudo-inverse:

\[
\hat{x}(0) = O_t^\dagger \left( \begin{bmatrix} y(0) \\ \vdots \\ y(t - 1) \end{bmatrix} - T_t \begin{bmatrix} u(0) \\ \vdots \\ u(t - 1) \end{bmatrix} \right)
\]

where \( O_t^\dagger = (O_t^T O_t)^{-1} O_t^T \)
**interpretation:** $\hat{x}_{ls}(0)$ minimizes discrepancy between

- output $\hat{y}$ that *would be* observed, with input $u$ and initial state $x(0)$ (and no sensor noise), and

- output $y$ that *was* observed,

measured as

$$\sum_{\tau=0}^{t-1} \| \hat{y}(\tau) - y(\tau) \|^2$$

can express least-squares initial state estimate as

$$\hat{x}_{ls}(0) = \left( \sum_{\tau=0}^{t-1} \sum_{\tau=0}^{t-1} (A^T)^\tau C^T C A^\tau \right)^{-1} \sum_{\tau=0}^{t-1} (A^T)^\tau C^T \hat{y}(\tau)$$

where $\hat{y}$ is observed output with portion due to input subtracted:

$\tilde{y} = y - h \ast u$ where $h$ is impulse response
Least-squares observer uncertainty ellipsoid

since $\mathcal{O}_t \mathcal{O}_t^\dagger = I$, we have

$$\tilde{x}(0) = \hat{x}_{1s}(0) - x(0) = \mathcal{O}_t^\dagger \begin{bmatrix} v(0) \\ \vdots \\ v(t-1) \end{bmatrix}$$

where $\tilde{x}(0)$ is the estimation error of the initial state

in particular, $\hat{x}_{1s}(0) = x(0)$ if sensor noise is zero
(i.e., observer recovers exact state in noiseless case)

now assume sensor noise is unknown, but has RMS value $\leq \alpha$,

$$\frac{1}{t} \sum_{\tau=0}^{t-1} \|v(\tau)\|^2 \leq \alpha^2$$
set of possible estimation errors is ellipsoid

\[ \tilde{x}(0) \in \mathcal{E}_{\text{unc}} = \left\{ \mathcal{O}_t^\dagger \begin{bmatrix} v(0) \\ \vdots \\ v(t-1) \end{bmatrix} \mid \frac{1}{t} \sum_{\tau=0}^{t-1} \|v(\tau)\|^2 \leq \alpha^2 \right\} \]

\( \mathcal{E}_{\text{unc}} \) is ‘uncertainty ellipsoid’ for \( x(0) \) (least-square gives best \( \mathcal{E}_{\text{unc}} \))

shape of uncertainty ellipsoid determined by matrix

\[ (\mathcal{O}_t^T \mathcal{O}_t)^{-1} = \left( \sum_{\tau=0}^{t-1} (A^T)^\tau C^T C A^\tau \right)^{-1} \]

maximum norm of error is

\[ \|\hat{x}_{1s}(0) - x(0)\| \leq \alpha \sqrt{t} \|\mathcal{O}_t^\dagger\| \]
Infinite horizon uncertainty ellipsoid

the matrix

\[ P = \lim_{t \to \infty} \left( \sum_{\tau=0}^{t-1} (A^T)^{\tau} C^T C A^{\tau} \right)^{-1} \]

always exists, and gives the limiting uncertainty in estimating \( x(0) \) from \( u, y \) over longer and longer periods:

- if \( A \) is stable, \( P > 0 \)
  \( i.e., \) can’t estimate initial state perfectly even with infinite number of measurements \( u(t), y(t), t = 0, \ldots \) (since memory of \( x(0) \) fades . . . )

- if \( A \) is not stable, then \( P \) can have nonzero nullspace
  \( i.e., \) initial state estimation error gets arbitrarily small (at least in some directions) as more and more of signals \( u \) and \( y \) are observed
Example

- particle in $\mathbb{R}^2$ moves with uniform velocity
- (linear, noisy) range measurements from directions $-15^\circ, 0^\circ, 20^\circ, 30^\circ$, once per second
- range noises IID $\mathcal{N}(0, 1)$; can assume RMS value of $v$ is not much more than 2
- no assumptions about initial position & velocity

**problem:** estimate initial position & velocity from range measurements
express as linear system

\[
x(t + 1) = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} x(t), \quad y(t) = \begin{bmatrix} k_1^T \\ \vdots \\ k_4^T \end{bmatrix} x(t) + v(t)
\]

• \((x_1(t), x_2(t))\) is position of particle

• \((x_3(t), x_4(t))\) is velocity of particle

• can assume RMS value of \(v\) is around 2

• \(k_i\) is unit vector from sensor \(i\) to origin

true initial position & velocities: \(x(0) = (1 - 3 - 0.04 0.03)\)
range measurements (& noiseless versions):

measurements from sensors 1 – 4
• estimate based on \((y(0), \ldots, y(t))\) is \(\hat{x}(0|t)\)

• actual RMS position error is

\[
\sqrt{(\hat{x}_1(0|t) - x_1(0))^2 + (\hat{x}_2(0|t) - x_2(0))^2}
\]

(similarly for actual RMS velocity error)
RMS position error

RMS velocity error

Observability and state estimation
Continuous-time least-squares state estimation

assume \( \dot{x} = Ax + Bu, \ y = Cx + Du + v \) is observable

least-squares estimate of initial state \( x(0) \), given \( u(\tau), y(\tau), 0 \leq \tau \leq t \):
choose \( \hat{x}_{ls}(0) \) to minimize integral square residual

\[
J = \int_0^t \| \tilde{y}(\tau) - Ce^{\tau A}x(0) \|^2 \ d\tau
\]

where \( \tilde{y} = y - h \ast u \) is observed output minus part due to input

let’s expand as \( J = x(0)^T Q x(0) + 2r^T x(0) + s \),

\[
Q = \int_0^t e^{\tau A^T} C^T Ce^{\tau A} \ d\tau, \quad r = \int_0^t e^{\tau A^T} C^T \tilde{y}(\tau) \ d\tau,
\]

\[
s = \int_0^t \tilde{y}(\tau)^T \tilde{y}(\tau) \ d\tau
\]
setting $\nabla x(0)J$ to zero, we obtain the least-squares observer

$$\hat{x}_{ls}(0) = Q^{-1}r = \left( \int_0^t e^{\tau A^T} C^T C e^{\tau A} d\tau \right)^{-1} \int_0^t e^{A^T \tau} C^T \tilde{y}(\tau) d\tau$$

estimation error is

$$\tilde{x}(0) = \hat{x}_{ls}(0) - x(0) = \left( \int_0^t e^{\tau A^T} C^T C e^{\tau A} d\tau \right)^{-1} \int_0^t e^{\tau A^T} C^T v(\tau) d\tau$$

therefore if $v = 0$ then $\hat{x}_{ls}(0) = x(0)$