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VAR analysis, nonfundamental representations, Blaschke matrices

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Abstract

Macroeconomic models may produce ARMA structures where the determinant of the MA matrix polynomial has some roots inside the unit circle. This implies that the impulse-response functions are no longer identified and may vary in an infinite-dimensional space. This paper deals with this problem in the VAR, or structural VAR, framework. We provide a method to strongly limit the research for economically interesting nonfundamental impulse-response functions and show how to construct such representations from estimated VAR coefficients. We also give two empirical applications: GNP–unemployment, USA data, and interest rate–inflation, French data.

Key words: Nonfundamentalness; Alternative impulse-response functions
JEL classification: C22; E3

1. Introduction

Several papers have pointed out that economic models may produce moving average representations which are not fundamental, i.e., representations in which some of the roots of the MA determinant lie within the unit circle: see, e.g., Hansen and Sargent (1980, 1991), Futia (1981), Quah (1990), and Lippi and Reichlin (1991, 1993).

Such representations, although they imply the same autocovariance structure, cannot be obtained from inversion of estimated VARs. In fact, given an

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autocovariance structure, there exists an infinite- and infinite-dimensional variety of MA representations compatible with it; in such variety fundamental representations are only a thin subset. Consequently, when the economic model does not guarantee fundamentalness, the dynamic analysis based on standard impulse-response functions may be misleading.

This paper first proposes a criterion to limit the space of relevant MA representations; secondly, it shows how to construct such representations from estimated VAR coefficients. The method, which we illustrate on the basis of two examples, is easily implementable and can be employed to explore the range of the different dynamic impulse-responses to the shocks implied by a given model.

The paper is organized as follows. In Section 2 we give some background and terminology. In Section 3 we show that all nonfundamental representations in the ARMA class may be obtained from a fundamental one by means of Blaschke matrices. The latter are nontrivial matrices in the lag operator L which transform white noises into white noises. In Section 4 we introduce the distinction between basic and nonbasic nonfundamental representations with respect to a given ARMA representation: basic representations are finite in number up to multiplication by orthogonal matrices. We argue that the economic examples found in the literature, though providing a motivation for exploring nonfundamental representations, do not produce nonbasic nonfundamental representations. However, as we will show in Section 5, when our knowledge is limited to an estimated VAR, the distinction between basic and nonbasic representations is difficult. Nonetheless, if the estimated VAR is thought of as an approximation to an ARMA model, the MA roots of the latter should produce circles in the set of VAR roots. This feature is indeed quite evident in the two empirical cases analysed in Section 6, and is the basis of the criterion we propose to limit the exploration of nonfundamental representations. The results in Sections 3 and 4 are developments from Rozanov (1967) and Hannan (1970). Some of them can be found more or less explicitly in Hansen and Sargent (1991). The proofs are in the Mathematical Appendix.

2. Background and terminology

Let x_t be an n-dimensional stationary stochastic vector whose entries may be either I(0) or k-differences of I(k) processes. Assume for simplicity that x_t has rational spectral density. In this case x_t admits moving average representations:

$$x_t = B(L)u_t, \tag{1}$$

$$B(L) = \sum_{i=0}^{\infty} B_i L^i,$$

where (i) u_t is a white noise vector; (ii) $B(L)$ is a matrix of rational functions in L with no poles of modulus smaller or equal to unity. Representation (1) is called *fundamental* if, in addition, (iii) $\det B(L)$ has no roots of modulus smaller than unity. When this is the case u_t belongs to the linear space spanned by x_{t-k}, $k \geqslant 0$, call it I_t.

If some of the roots lie strictly inside the unit circle, then representation (1) is called *nonfundamental*. In that case, as we will show in the next section, u_t, while belonging to the space spanned by x_{t-k}, k being any integer, does not belong to I_t.

Macroeconomic models based on intertemporal maximization under rational expectations typically produce equilibrium solutions of the ARMA form:

$$M(L)x_t = P(L)w_t, \qquad (2)$$

where x_t is as in (1), w_t is a white noise, $M(L)$ and $N(L)$ are (finite) polynomial matrices whose coefficients are functions of the 'deep parameters', while $\det M(L)$ has no roots of modulus smaller or equal to unity.

It is important to point out that x_t is to be meant as the vector of *all* the variables observed by the econometrician. Accordingly, the space I_t is also called the *econometrician's information space*.

The white noise w_t belongs in general to the agents' information space. The latter may be larger than the econometrician's. As a consequence, representation (2), or its implied moving average

$$x_t = M(L)^{-1}N(L)w_t, \qquad (3)$$

is not necessarily fundamental. This means that the standard identification criterion for ARMA models, i.e., the assumption that all the roots of the MA determinant lie outside the unit circle does not have any economic justification, and therefore cannot be invoked to select one out of the multiple peaks of the likelihood function. Although some implication of the economic model can still be tested (see, e.g., Hansen and Sargent, 1991), this identification problem is a real difficulty when the aim is to study impulse-response functions and relative variances of components.

The problem is even more complicated when we take up the approach of VAR or structural VAR literature. Here, even though an intertemporal maximization under rational expectations is usually assumed as background, yet the dynamics are not fully specified. In particular, the theory does not produce information on the orders of the AR and MA matrix polynomials. In this case it is standard practice to estimate an unrestricted vector autoregression, while economic theory can still play a limited role in the subsequent identification step.

VAR procedure consists in, first, the estimation of the model:

$$S(L)x_t = u_t, \qquad (4)$$

where $S(L)$ is a finite polynomial matrix whose order is such that the hypothesis that u_t is a white noise is not rejected.[1] Secondly, the moving average representation is obtained by inversion:

$$x_t = S(L)^{-1} u_t. \tag{5}$$

Lastly, the theory is employed to identify the 'structural' white noise w_t, where this simply means to determine a matrix K such that $w_t = K u_t$. Finally, the structural impulse-response functions are determined as

$$x_t = [KS(L)]^{-1}(Ku_t) = [KS(L)]^{-1} w_t. \tag{6}$$

Representation (6), being obtained from the inversion of $S(L)$, has all the roots on the 'right side' of the unit circle, i.e., is fundamental. This has two consequences: first, its economic meaning is guaranteed only in the particular case in which the econometrician's and the agents' information spaces coincide. Secondly, it is only in this particular case that (6) can be thought of as an approximation to an underlying fully specified ARMA model.

3. Blaschke matrices and nonfundamental representations

In this section we show how to obtain all nonfundamental MA representations from a fundamental representation. For this we must introduce Blaschke matrices.

Let z denote a complex variable and $A(z)$ an $n \times n$ matrix whose elements are rational functions of z: $a_{ij}(z) = b_{ij}(z)/c_{ij}(z)$, where $b_{ij}(z)$ and $c_{ij}(z)$ are polynomials with no common roots. We shall assume throughout that det $A(z)$ has a finite number of zeroes (otherwise it would vanish over the whole complex field). Notice that we are not assuming real coefficients for $a_{ij}(z)$. Therefore the stochastic processes we will refer to below are complex. This will simplify the treatment by avoiding clumsy distinctions between real and pairs of complex conjugate roots. On the other hand, restricting to a real $A(z)$ and to real processes will not create any problem.

Definition 1. $A(z)$ is a Blaschke Matrix, BM henceforth, if:

(BM1) *$A(z)$ has no poles of modulus smaller or equal to unity.*

[1] When x_t is cointegrated, model (4) is misspecified and estimation must be performed on the basis of an Error Correction Model. For the purpose of the discussion here, there is no need to develop such case.

(BM2) *Denoting by $A^*(\cdot)$ the matrix obtained by transposing and taking conjugate coefficients, we have*

$$A(z)^{-1} = A^*(z^{-1}), \tag{7}$$

i.e.,

$$A(z)A^*(z^{-1}) = I. \tag{8}$$

(*Notice that BM are orthogonal matrices for $|z| = 1$.*)

Let u_t be a white-noise n-dimensional vector. If the covariance matrix of u_t is the identity we call u_t an *orthonormal* white noise. It follows immediately from (BM2) that if u_t is an orthonormal white noise, then $A(z)$ is a BM if and only if the vector

$$v_t = A(L)u_t \tag{9}$$

is also an orthonormal white noise.

Elementary examples of BM are:

(I) $A(z)$ is a constant orthogonal matrix;
(II) $A(z) = R(\alpha, z)$, where

$$R(\alpha, z) = \begin{pmatrix} \dfrac{z-\alpha}{1-\bar{\alpha}z} & 0 \\ 0 & I \end{pmatrix},$$

with $|\alpha| < 1$, while I here denotes the $(n-1)$-dimensional unit matrix.

Examples (I) and (II) combine to give

$$A(z) = R(\alpha_1, z)K_1 R(\alpha_2, z)K_2 \cdots R(\alpha_m, z)K_m, \tag{10}$$

with K_i orthogonal and m any positive integer. The theorem below states that *all* rational BM have the form (10).

Theorem 1. Let $A(z)$ be a BM. Then there exists an integer r and complex numbers α_i, $i = 1, r$, $|\alpha_i| < 1$, such that

$$A(z) = R(\alpha_1, z)K_1 R(\alpha_2, z)K_2 \cdots R(\alpha_r, z)K_r, \tag{11}$$

where K_i are orthogonal matrices.

Let us recall the definition of a fundamental representation for x_t, i.e., the moving average (1), with the properties (i), (ii), (iii). Since in this paper we are interested in identification restrictions which usually include at least orthogonality of the components of u_t (like in Sims' recursion schemes or in Blanchard and Quah, 1989) it will be convenient to add the condition:

(iv) *u_t is an orthonormal white noise.*

Now let

$$x_t = B(L)u_t \tag{12}$$

be fundamental. Nonfundamental representations can be immediately obtained from (12). Setting $v_t = A(L)^{-1}u_t$, $A(L)$ being a nonconstant BM, the representation

$$x_t = B(L)[A(L)v_t] = [B(L)A(L)]v_t$$

fulfills (i), (ii), and (iv), but not (iii). The converse is proved in the Appendix. Thus:

Theorem 2. Let x_t be stationary with rational spectral density. If $x_t = B(L)u_t$ is a fundamental representation of x_t and $x_t = C(L)v_t$ is any MA representation, i.e., one which fulfills (i), (ii), and (iv), but not necessarily (iii), then $C(L) = B(L)A(L)$, where $A(L)$ is a BM.

Theorem 2 implies that if $x_t = C(L)v_t$ is nonfundamental, the space generated by v_{t-k}, $k \geqslant 0$, is not contained in the econometrician's information space. In fact, applying Theorem 2, $x_t = B(L)A(L)v_t$, where $A(L)$ is BM. Assuming for simplicity that $B(L)$ is invertible: $A(L)v_t = B(L)^{-1}x_t$. Since the inversion of $A(L)$ can be obtained only by using the forward operator F, recovering v_t requires using not only the past but also the future of x_t.

Notice that $A(L)^{-1}$ does not possess an expansion in L valid in and on the unit circle. In fact, $A(L)^{-1}$ contains the ratios

$$\frac{1 - \bar{\alpha}_i L}{L - \alpha_i},$$

with $|\alpha_i| < 1$, which have poles inside the unit circle. However, setting $F = L^{-1}$, such terms can be rewritten as

$$\frac{F - \bar{\alpha}_i}{1 - \alpha_i F},$$

so that $A(L)^{-1}$ has a valid expansion in the forward operator F. Thus, if u_t is an orthonormal white noise, then $v_t = A(L)u_t$ is an orthonormal white noise lying in the past of u_t, while $u_t = A(L)^{-1}v_t$ lies in the future of v_t.

Let us lastly notice an important difference between fundamental and nonfundamental representations. We have:

U. *If* $x_t = B(L)u_t$ *and* $x_t = \tilde{B}(L)\tilde{u}_t$ *are two fundamental representations, then* $B(L) = \tilde{B}(L)K$, *where* K *is an orthogonal matrix.* (See Hannan, 1970, p. 66, Theorem 10′.)

This uniqueness result does not hold for nonfundamental representations, even when their determinants are equal. For instance, setting

$$C(L) = R(\alpha, L), \qquad \tilde{C}(L) = \frac{1}{2}\begin{pmatrix} \dfrac{L-\alpha}{1-\bar{\alpha}L} & \dfrac{L-\alpha}{1-\bar{\alpha}L} \\ -1 & 1 \end{pmatrix},$$

we have

$$\tilde{C}(L)C(L)^{-1} = \frac{1}{2}\begin{pmatrix} \bar{1} & \dfrac{L-\alpha}{1-\bar{\alpha}L} \\ -\dfrac{1-\bar{\alpha}L}{L-\alpha} & 1 \end{pmatrix},$$

although $\det C(L) = \det \tilde{C}(L) = (L-\alpha)/(1-\bar{\alpha}L)$.

4. Basic and nonbasic representations corresponding to a given vector ARMA

In this section we assume that an ARMA representation is given for the vector x_t, i.e., a couple of polynomial matrices $M(L)$ and $N(L)$ such that

$$M(L)x_t = N(L)u_t, \tag{13}$$

where u_t is a white noise, $\det M(L)$ has no zeroes of modulus smaller or equal to unity, $M(0) = I$.

Definition 2. The ARMA representation (13) of the stationary stochastic vector x_t *is fundamental if* $N(L)u_t$ *is fundamental.*

We now study the set of alternative representations which have the same AR polynomial as (13) but whose MA matrices may differ for a BM.[2] For an economic motivation of this problem we may go back to intertemporal maximization under rational expectations and assume that the restrictions produced by

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[2] As is well known, even assuming fundamentalness, whereas the moving average representation $M(L)^{-1}N(L)u_t$ is unique up to an orthogonal matrix, the couple of polynomial matrices in (13) is not unique. Here we do not deal with this problem.

the economic model are sufficient to completely identify the AR matrix while leaving the position of the MA roots (inside or outside the unit circle) undetermined. Most of the examples mentioned in the Introduction produce such a situation.

Now expand representation (13):

$$N(L) = N_0 + N_1 L + \cdots + N_q L^q,$$

and assume that it is fundamental, so that:

$$\det N(L) = \tau(1 - \alpha_1 L)(1 - \alpha_2 L) \cdots (1 - \alpha_h L), \tag{14}$$

with $|\alpha_i| \leqslant 1$ and $h \leqslant nq$, $\tau = \det N_0$.[3]

Let Ω be the subset of R^h whose elements ω fulfill the condition: ω_i is equal either to 1 or to -1.

Theorem 3. (a) For any given $\omega \in \Omega$, there exist representations

$$M(L)x_t = P(L)\omega_t,$$

where

$$P(L) = P_0 + P_1 L + \cdots + P_q L^q, \tag{15}$$

$$\det P(L) = \mu(1 - \beta_1 L)(1 - \beta_2 L) \cdots (1 - \beta_h L),$$

where w_t is an orthonormal white noise, while $\beta_i = \alpha_i$ if $\omega_i = 1$, $\beta_i = \bar{\alpha}_i^{-1}$ if $w_i = -1$, $\mu = \det P_0$. (b) If $P(L)$ and $Q(L)$ correspond to the same w, $P(L) = Q(L)K$, with K orthogonal.

Definition 3. The nonfundamental ARMA representations obtained in Theorem 3, which are finite in number up to orthogonal matrices, will be called basic *with respect to representation (13).*

The uniqueness result (b) has an important consequence. Consider all the basic moving averages compatible with a given ARMA and suppose that a given recursive scheme is chosen for identification. Then there is only one MA representation corresponding to that recursive scheme and having a given determinant. The same result holds if instead of a recursive scheme we choose a different identifying restriction that turns out in a linear transformation of the white noise through an orthogonal matrix (this is the case, for instance, with the restriction imposed in Blanchard and Quah, 1989).

[3] Given the structure of $N(L)$, the degree of $\det N(L)$ could be less than nq.

Remark 1. Basic nonfundamental representations are obtained from (13) by manipulating only the MA polynomial matrix, while the AR remains untouched. Moreover, if (13) is an ARMA (p,q), basic representations are ARMA (p,q).

Remark 2. The above definition is related to a given representation of x_t, not to x_t itself. Different representations, containing different MA polynomials, give rise to different sets of basic nonfundamental representations.

Now let us go back to representation (13) and suppose we substitute $A(L)w_t$ for u_t in the RHS, where

$$A(L) = KR(\gamma, L),$$

with $|\gamma| < 1$, $\gamma \neq \alpha_i$, for $i = 1, h$. In this case, irrespective of the orthogonal matrix K chosen, the representation

$$M(L)x_t = N(L)KR(\gamma, L)w_t \tag{16}$$

is not an ARMA, since the matrix $R(\gamma, L)$ is not a polynomial matrix, while $\det(N(L)K)$ does not contain the factor $1 - \bar{\gamma}L$. An ARMA representation corresponding to (16) can be obtained by multiplying both sides by the scalar $(1 - \bar{\gamma}L)$:

$$(1 - \bar{\gamma}L)M(L)x_t = N(L)K\begin{pmatrix} L - \gamma & 0 \\ 0 & (1 - \bar{\gamma}L)I \end{pmatrix}w_t.$$

This may be sufficient to highlight the features of nonbasic ARMA representations associated with (13), i.e., those obtained from (13) by using arbitrary BM to manipulate the MA polynomial matrix:

(A) The orders of the MA and AR polynomial matrices increase with respect to (13), according to the number of 'nonbasic' roots in the BM. Thus any order may be reached.
(B) If γ, $|\gamma| < 1$, is an MA root of a nonbasic representation associated with (13), $\gamma \neq \alpha_i$, $i = 1, h$, then $\bar{\gamma}^{-1}$ is a root of the AR polynomial.

In Lippi and Reichlin (1993) we have shown how far one can go when one allows for nonbasic representations. Precisely, discussing the decomposition of GNP into demand and supply components, we have proved that if arbitrary nonbasic representations are allowed, one can obtain any pre-assigned ratio between the variance of the two components.

On the other hand, property (B) of nonbasic nonfundamental ARMA representations is not likely to occur in models based on economic theory. Even in the case in which roots inside the unit circle cannot be excluded, both common sense and close examination of the examples in the literature, suggest that only by a fluke we could have γ, $|\gamma| < 1$, and $\bar{\gamma}^{-1}$ as roots of the determinants of the MA and the AR matrices, respectively. We conclude that the research of sensible nonfundamental representations associated with a given ARMA must be limited to basic representations.

5. The VAR case

The situation is much more complicated when we start with an estimated VAR, like in standard or structural VAR analysis. The difficulty is that we do not possess an immediate criterion to discriminate between basic and nonbasic representations. In fact, even if we stick to the idea that the underlying data generation process may be approximated by an ARMA structure, here the latter has undergone a further approximation by (4). Consequently, we have to deal with two problems. Firstly, whether there is some evidence of a nontrivial MA polynomial; secondly, in case of a positive answer, how to identify a reasonably small subset of the complex plane containing the roots of the MA polynomial.

To further clarify the issue, let us recall that a standard fundamental ARMA can always be obtained from (4) as

$$\det S(L)x_t = S_{\mathrm{ad}}(L)u_t. \tag{17}$$

Model (17) could be used to experiment with basic nonfundamental representations. This naive solution has been adopted in Lippi and Reichlin (1993). However, if for instance the structural model were an AR(2), and such a structure were correctly estimated as a VAR, the basic nonfundamental representations obtained from (17) would be completely illegitimate.

Here we try to move a step further and propose a criterion to detect a genuine MA component which is based on the fact that MA roots should generate circles in the set of the roots of the estimated VAR determinant.

This point can be illustrated by a two-dimensional model. Suppose the 'true' ARMA is

$$x_t = (N_0 - N_1 L)u_t.$$

Consider its fundamental standard version:

$$x_t = (I - NL)v_t. \tag{18}$$

Let α_1^{-1} and α_2^{-1} be the roots of $\det(I - NL)$ and assume for the moment that they are real. By fundamentalness $|\alpha_i| < 1$. If (18) were known, research of nonfundamental representations would be limited to the three basic ones obtained by taking the reciprocals of α_1^{-1}, of α_2^{-1}, or of both.

If instead we estimated a VAR, then, assuming that the moduli of the roots, α_i^{-1} are fairly greater than unity, and that the sample size is sufficiently large, the

VAR polynomial will be a close approximation to

$$G(L) = I + NL + N^2L^2 + \cdots + N^pL^p$$
$$= (I - N^{p+1}L^{p+1})(I - NL)^{-1}. \tag{19}$$

From elementary matrix algebra, the $2 \times p$ roots of $\det G(L)$ are $\alpha_1^{-1}\tau_k$, $\alpha_2^{-1}\tau_k$, where

$$\tau_k = \exp\left(k\frac{2\pi i}{p+1}\right),$$

for $k = 1, p$. Notice that when k varies between 1 and p, τ_k varies over all the (complex) $(k+1)$-roots of unity, with the exception of unity itself. Such roots form a regular $(p+1)$-polygon on the unit circle with a vertex coinciding with unity. The roots of $\det G(L)$ reproduce such polygon (excluding unity) on the circles of radii α_1^{-1} and α_2^{-1} respectively.

No difficulty would arise if the roots were complex conjugate. Whereas in the real case we have two circles, a couple of complex roots would generate only one circle.

Let us come to our proposed procedure. Keeping the assumption $n = 2$ will help to give a detailed presentation. Suppose, to fix our ideas, that the determinant of the estimated VAR,

$$S(L)x_t = u_t,$$

has one circle of complex roots. We proceed assuming that the VAR is an approximation to

$$M(L)x_t = N(L)\tilde{u}_t,$$

so that $T(L) = S(L)^{-1}$ approximates $M(L)^{-1}N(L)$, while the circle of complex roots is interpreted as a clue for the presence of a pair of complex conjugate roots in $N(L)$.

Step 1. Naturally, an empirical circle will be determined only approximately. Moreover, the position of the roots will not be sufficiently regular to permit identification of the MA roots responsible for the circle. Thus all the information we can retain is represented by the radius ρ of the circle.

Step 2. Consider now any β, with $|\beta| = \rho$. Then follow Theorem 3, firstly multiply $T(L)$ by

$$K = h^{-1}\begin{pmatrix} t_{12}(\beta) & \overline{t_{11}(\beta)} \\ -t_{11}(\beta) & \overline{t_{12}(\beta)} \end{pmatrix},$$

with $h = |t_{12}(\beta)|^2 + |t_{11}(\beta)|^2$. Secondly multiply by $R(\bar{\beta}^{-1}, L)$. Notice that we are proceeding as though β were a root of $T(L)$ and the basic nonfundamental representation corresponding to β were to be determined. In fact, in that case $T(L)K$ would contain the factor $1 - \beta^{-1}L$ in the first column, so that $R(\bar{\beta}^{-1}, L)$

would replace the root β with $\bar{\beta}^{-1}$. Lastly, as β is not real, the procedure has to be applied to $T(L)KR(\bar{\beta}^{-1}, L)$, with $\bar{\beta}$ instead of β. This will ensure that, after imposing the identifying restrictions, the resulting representation is real.

Step 3. Step 2 can be employed in a grid search over the interval $(0, \pi)$ for the argument of β. This means fixing a pace $\lambda = \pi/m$, m integer, and repeating Step 2 for $\beta = \rho(\cos\lambda k + i\sin\lambda k)$, $k = 1, m - 1$. For each argument the impulse-response functions can be displayed, their dynamics analysed and compared to the fundamental ones. In the next section we will apply the whole procedure to two empirical cases.

Two caveats on Step 2 are in order. Firstly, our choice of the matrix K is based on the fact that if β were a root of the MA polynomial, then using K, and then $R(\bar{\beta}^{-1}, L)$, would lead to approximate a basic nonfundamental representation. Although our rule follows naturally from the approximation argument, since we try all the β's on the circle, we are in fact dealing with nonbasic representations. Thus, as we have seen in Section 3, no uniqueness theorem can be invoked and the result is not independent of the rule adopted to choose K. Secondly, for $n > 2$ the situation is more complicated because even the assumption that β is a root of $T(L)$ does not lead to a unique K. In fact, the first column of K could be determined, but for the remaining ones we have only the conditions that they must be orthogonal to each other and to the first. Thus, for $n > 2$ no natural rule exists to fix K, and we are not presently able to suggest a way to avoid exploration of further dimensions, in addition to the observed circle of roots.

Lastly, it must be pointed out that circles of roots in the VAR determinant could arise in empirical cases from serious misspecification of the AR order. This may be shown by simulation. Thus, for a proper implementation of our method, it is important to perform the necessary diagnostic.

6. Empirical results

In this section, we will show how the procedure proposed in Section 5 works in practice. We perform two exercises on the basis of two bivariate VARs. After estimation, we first try to detect the presence of a nontrivial MA component by examining the position of the VAR roots in the complex plane. Having found evidence of such component in both cases, we compute and analyse alternative nonfundamental representations.

Example 1. The first example is the VAR estimated by Blanchard and Quah (1989) for the log of US real GNP in first differences, and the US unemployment rate. Data are quarterly for the period 1948.1 to 1987.4. We retain the authors' extimated coefficients for an eight-lag VAR and identification procedure, which consists in a long-run zero restriction on the structural MA matrix, unit variances, and orthogonality of the shocks. On the basis of the authors' identification restrictions, the shocks are interpreted as a demand and a supply shock.

Example 2. The second example is an Error Correction Model for the French nominal long-run interest rate and consumer inflation rate. Data are quarterly for the period 1946.1 to 1989.4. We estimate the model, compute the MA representation, and identify the model by imposing a triangular structure *à la* Sims with the interest rate preceding the inflation rate, so that the shocks can be interpreted as a shock to interest rate and a shock to inflation.

The first example produces sixteen roots, the second eighteen. Figs. 1 and 2 display their reciprocals in the complex plane.

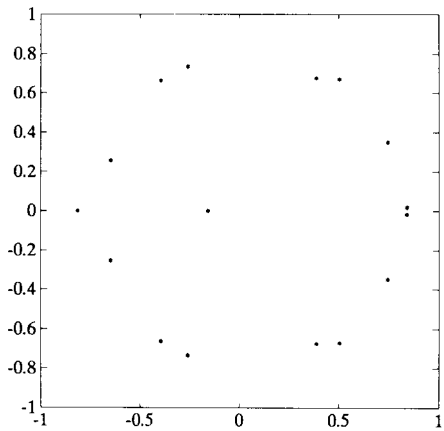

Fig. 1. *Example 1*: Reciprocals of the VAR roots in the complex plane.

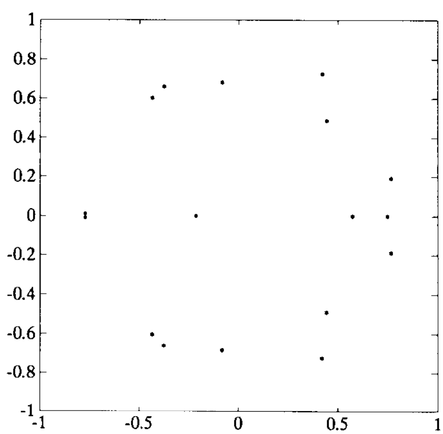

Fig. 2. *Example 2*: Reciprocals of the VAR roots in the complex plane.

In both examples, we can clearly detect one circle of roots, the radii being approximately 1.25 and 1.33, respectively.

Let us first analyse Fig. 1. We can see that there are fourteen complex roots around the circle and two real roots, one of which is isolated. The circle suggests an MA component of order one with two complex roots. The isolated root should belong to an AR component, and since the latter must have at least two roots, we may infer that the real roots belong to an AR(1) component. These considerations suggest that the VAR approximates an ARMA (1, 1).

In Example 2 (Fig. 2) we have eighteen roots, fourteen of which are complex and disposed around a circle, while the remaining four are real. The unit root is due to cointegration. The same reasoning as in Example 1 leads us to the conclusion that the approximated model is an ARMA (2, 1).

As each experiment produces four impulse-response functions, we only show in Fig. 3 some of the responses of output to a demand shock for Example 1, and in Fig. 4 some of the responses of interest rate to a shock to interest rate for Example 2. In both cases the solid line corresponds to the fundamental representation, while the dashed and the dotted lines correspond, respectively, to $\beta = 1.25\exp(i\pi/8)$ and $\beta = 1.25\exp(1.7i\pi/8)$ for Example 1; to $\beta = 1.33\exp(i\pi/4)$ and $\beta = 1.25\exp(i\pi/2)$ for Example 2.

Two features common to both examples are worth reporting. First, there is a band for $\arg(\beta)$, between 0 and $\pi/2$, producing interesting impulse-response functions. For $\arg(\beta)$ greater than $\pi/2$ the functions are almost identical to the fundamental ones, while the difference become more and more substantial as β approaches zero.

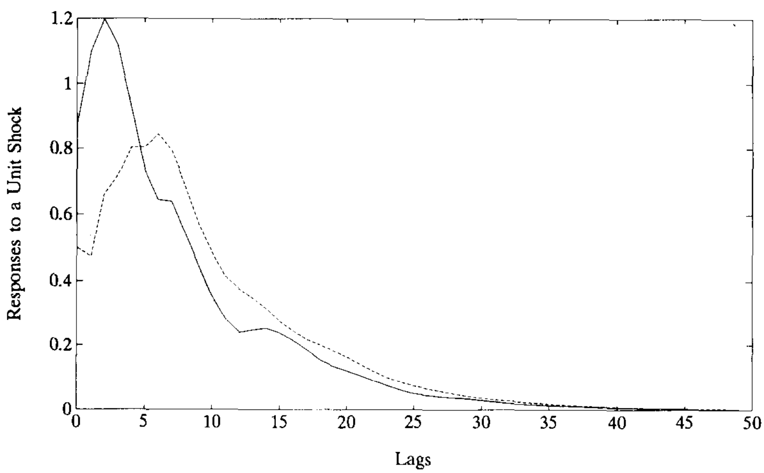

Fig. 3. *Example 1*: Response of output to a demand shock; solid line: fundamental representation; dashed line: $\beta = 1.25\ \exp(i\pi/8)$; dotted line: $\beta = 1.25\ \exp(1.7i\pi/8)$.

Secondly, for arg(β) inside the band, the first impact of the shock is considerably smaller than the first impact of the fundamental function. Moreover, the size of the first impact is strictly linked to the relative variance of the corresponding component.

In Tables 1 and 2 we report: (1) In the first row, the value of the first impact of, respectively, a unit shock to demand on output, a unit shock to interest rate on

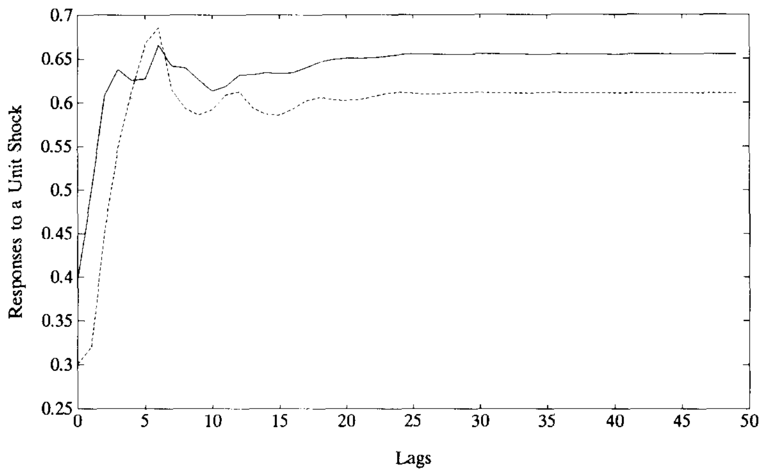

Fig. 4. *Example 2*: Response of interest rate to a shock to interest rate; solid line: fundamental representation; dashed line: $\beta = 1.33\ \exp(i\pi/4)$; dotted line: $\beta = 1.33\ \exp(i\pi/2)$.

Table 1
Example 1

	Fundamental	$\arg(\beta) = \pi/8$	$\arg(\beta) = 1.7\pi/8$
FI	0.86	0.50	0.60
VR	9.4	0.48	1.21

FI: First inpact of a unit demand shock on output.
VR: Variance ratio of demand component to supply component in output.

Table 2
Example 2

	Fundamental	$\arg(\beta) = \pi/4$	$\arg(\beta) = \pi/2$
FI	0.39	0.30	0.30
VR	18	2.16	4

FI: First inpact of a unit interest rate shock on interest rate.
VR: Variance ratio of interest rate component to inflation component in interest rate.

interest rate. (2) In the second row, the relative variance of demand and supply components in output (Table 1), of the interest rate and inflation component in interest rate (Table 2).

As we can see, the relative importance of the demand component is by far greater in the fundamental function than in the two nonfundamental ones. Furthermore, the order of the relative importance is the same as for the first impact values. The first statement holds for Example 2 as well, whereas the second applies to the comparison between the fundamental function and each of the nonfundamental ones.

Summing up, the application of our method to Examples 1 and 2 has produced nonfundamental functions whose shapes are not very different from the corresponding fundamental. However, both first impact values and relative variances vary considerably. In particular, the difference in first impact values suggests that exogenous information on the first lag responses might provide an additional choice criterion among alternative functions.

7. Concluding remarks

Relevant economic examples show that the standard assumption on the roots of MA representations is not warranted. When the economic theory is sufficiently informative to produce an ARMA structure, but not to determine the position of the roots of the MA polynomial (inside or outside the unit circle), we have argued that the exploration of nonfundamental representations can be limited to a finite number of alternatives. On the other hand, when only a VAR representation is available, a choice among the alternatives is much more difficult. Naturally, the choice should be limited to economically sensible impulse-response functions. However, a more formal strategy can be attempted. We have observed that if the VAR were an approximation to an ARMA with a nontrivial MA polynomial, the roots of the latter would generate circles of complex roots in the VAR determinant. This leads us to propose a criterion to limit the space of possible MA representations, which is based only on the information contained in the determinant of the VAR matrix. Application of this criterion to two empirical cases gives encouraging results.

Mathematical appendix

The results in Lemmas 1 and 2, although preparatory to Theorem 1, are of some autonomous interest.

Lemma 1. Let $A(z)$ *be BM and* $\det A(z)$ *be a nonzero constant. Then* $A(z)$ *is a constant orthogonal matrix.*

Proof. Consider the stochastic vector

$$v_t = A(L)u_t = (A_0 + A_1 L + A_2 L^2 + \cdots)u_t, \tag{20}$$

where u_t is an orthonormal white noise. By (BM1) the expansion above is valid in an open set containing the unit disk. Since det $A(L)$ never vanishes $A(L)$ is invertible, so that

$$u_t = (C_0 + C_1 L + \cdots)v_t.$$

By (20), for $k \geqslant 0$, $\mathrm{cov}(v_t, u_{t-k}) = A_k$. Furthermore, by (BM2), v_t is a white noise, i.e., $\mathrm{cov}(v_t, v_{t-k}) = 0$, for any k. Thus,

$$A_k = \mathrm{cov}(v_t, u_{t-k}) = \mathrm{cov}\left(v_t, \sum_{s=0}^{\infty} C_s v_{t-k-s}\right) = 0,$$

for $k > 0$ (for this argument see Lütkepohl, 1984). Thus $A(z) = A_0$. Q.E.D.

Lemma 2. Let $A(z)$ be a BM. There exists an integer r and complex α_i, $i = 1, r, \tau$, $|\alpha_i| < 1$, $|\tau| = 1$, such that

$$\det A(z) = \tau \frac{(z - \alpha_1)(z - \alpha_2) \cdots (z - \alpha_r)}{(1 - \bar{\alpha}_1)(1 - \bar{\alpha}_2) \cdots (1 - \bar{\alpha}_r)}.$$

Proof. Let

$$\det A(z) = p(z)/q(z)$$

be a nonredundant representation of det $A(z)$. Suppose $\alpha \neq 0$, $p(\alpha) = 0$. From (7),

$$p(z)\bar{p}(z^{-1}) = \bar{q}(z^{-1})q(z).$$

Since $q(\alpha)$ cannot vanish, $\bar{q}(\alpha^{-1}) = 0$, i.e., $q(\bar{\alpha}^{-1}) = 0$. In the same way, if $\beta \neq 0$, $q(\beta) = 0$, then $p(\bar{\beta})^{-1} = 0$. Since $A(z)$ cannot have poles of modulus smaller or equal to unity, $|\alpha_i| < 1$. Setting $|z| = 1$, we get $|\tau| = 1$. Q.E.D.

Notice that Lemma 2 is a representation theorem for a scalar BM. We are now ready to prove the general representation theorem:

Proof of Theorem 1. If det $A(z)$ has no roots, then $A(z)$ is a constant by Lemma 1. Assume that β is a root of multiplicity m for det $A(z)$. By Lemma 2, $|\beta| < 1$. Consider the system of equations

$$A(\beta)y = 0$$

in the unknown y. As det $A(\beta)$ vanishes, there exists a nontrivial solution g. Let K be an orthogonal matrix whose first column is proportional to g. Now define

$\tilde{A}(z) = A(z)K$, and notice that $\tilde{a}_{i1}(z)$, $i = 1, n$, contain the factor $z - \beta$. Then define

$$\hat{A}(z) = \tilde{A}(z)\begin{pmatrix} \dfrac{1-\bar{\beta}z}{z-\beta} & 0 \\ 0 & I \end{pmatrix} = \tilde{A}(z)R(\beta, z)^{-1}.$$

Firstly, $\hat{A}(z)$ has no poles of modules smaller or equal to unity. Secondly,

$$\hat{A}(z)\hat{A}^*(z^{-1}) = \tilde{A}(z)\begin{pmatrix} \dfrac{1-\bar{\beta}z}{z-\beta} & 0 \\ 0 & I \end{pmatrix}\begin{pmatrix} \dfrac{1-\beta z^{-1}}{z^{-1}-\bar{\beta}} & 0 \\ 0 & I \end{pmatrix}\tilde{A}^*(z^{-1}) = I.$$

Therefore $\hat{A}(z)$ is a BM. Moreover, β is a root of $\det \hat{A}(z)$ of multiplicity $m - 1$. Finally,

$$A(z) = \hat{A}(z)R(\beta, z)K^{-1}.$$

Thus the Theorem may be proved by induction on the number of roots of $\det A(z)$. Q.E.D.

Proof of Theorem 2. We proceed along the line of the proof of Theorem 1. Let α be a root of $\det C(L)$, $|\alpha| < 1$. Find an orthogonal matrix K such that $C(L)K$ has the factor $L - \alpha$ in all the elements of the first column. Then multiply by $R(\alpha, L)^{-1}$. This eliminates one factor, $L - \alpha$. Repeating this procedure one eventually obtains $\tilde{B}(L) = C(L)\tilde{A}(L)^{-1}$, with $\tilde{B}(L)$ fulfilling condition (3), while $\tilde{A}(L)$ is a BM. From $x_t = C(L)v_t$ we have

$$x_t = (C(L)\tilde{A}(L)^{-1})\tilde{A}(L)v_t = \tilde{B}(L)w_t,$$

the last being a fundamental representation because $\tilde{B}(L)$ fulfills condition (3). Thus, by (5), $\tilde{B}(L) = B(L)K$, so that $C(L) = B(L)K\tilde{A}(L)$. Q.E.D.

Proof of Theorem 3. (a) In order to construct a matrix $P(L)$ corresponding to a given ω, suppose for instance that $w_1 = -1$, so that we want to substitute the root $\bar{\alpha}_1$ for the root α_1^{-1} [the latter is the first root of $\det N(L)$]. The procedure is the same as in Theorem 1. Firstly, find an orthogonal matrix K such that $N(L)K$ contains the factor $1 - \alpha_1 L$ in the first column. Then multiply by $R(\bar{\alpha}_1, L)$, thus obtaining a new representation $\tilde{N}(L)\tilde{u}_t$ where $\tilde{u}_t$ is an orthonormal white noise, while

$$\det \tilde{N}(L) = \tilde{k}(1 - \bar{\alpha}_1^{-1}L)(1 - \alpha_2 L) \cdots (1 - \alpha_h L).$$

The subsequent steps are now clear. Notice that the procedure is ready for straightforward computer implementation.

(b) Let $P(L)$ and $Q(L)$ be the moving average matrices of two representations corresponding to the same w and let $\bar{\alpha}_1, \bar{\alpha}_2, \ldots, \bar{\alpha}_j$ be the roots inside the unit circle of $\det P(L)$ and $\det Q(L)$. We can assume that all the roots are simple;

continuity of the roots allows extension to the case when some of them are multiple. We proceed by induction on j. If the number of the roots inside the unit circle is zero, we have fundamental representations. In this case the uniqueness theorem (U), Section 3, applies. Assume that $j > 0$ and that statement (b) holds for $j - 1$. Firstly, determine the orthogonal matrices K_1 and K_2 such that both $\tilde{P}(L) = P(L)K_1$ and $\tilde{Q}(L) = Q(L)K_2$ contain the factor $1 - \bar{\alpha}_j^{-1}L$ in all the elements of the first column. Then consider

$$\hat{P}(L) = \tilde{P}(L)R(\bar{\alpha}_j, L)^{-1}, \qquad \hat{Q}(L) = \tilde{Q}(L)R(\bar{\alpha}_j, L)^{-1}.$$

The roots inside the unit circle are $\bar{\alpha}_1, \bar{\alpha}_2, \dots, \bar{\alpha}_{j-1}$ for both $\hat{P}(L)$ and $\hat{Q}(L)$. Moreover, they both contain the factor $1 - \alpha_j L$ in all the elements of the first column. By the inductive hypothesis $\hat{P}(L)$ and $\hat{Q}(L)$ differ for a constant orthogonal matrix: $\hat{P}(L) = \hat{Q}(L)K$. Considering the first column and putting $L = \alpha_j^{-1}$,

$$0 = \hat{p}_{i1}(\alpha_j^{-1}) = \hat{q}_{i2}(\alpha_j^{-1})k_{21} + \cdots + \hat{q}_{in}(\alpha_j^{-1})k_{n1}, \tag{21}$$

for $i = 1, n$. Since α_j^{-1} is a root of the first column of $\hat{Q}(L)$, the simplicity assumption implies that at least one of the square submatrices of the $n \times (n-1)$ matrix on the RHS of (21) is nonsingular. Thus $k_{i1} = 0$, for $i = 2, n$. Orthogonality of K implies

$$K = \begin{pmatrix} k_{11} & 0 \\ 0 & H \end{pmatrix}, \tag{22}$$

with $|k_{11}| = 1$ and H orthogonal. Matrices having the shape of K in (22) commute with matrices $R(\alpha, L)$. Thus: $P(L) = Q(L)K_2 K K_1^{-1}$. Q.E.D.

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