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Bond Arithmetic

0. Overview

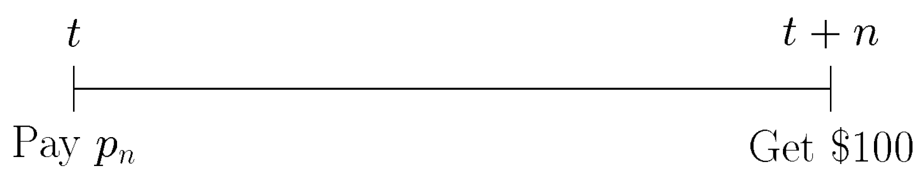
- Zeros and coupon bonds
- Spot rates and yields
- Day count conventions
- Replication and arbitrage
- Forward rates
- Yields and returns

1. Why Are We Doing This?

- Explain nitty-gritty of bond price/yield calculations
- Remark: "The devil is in the details"
- Introduce principles of replication and arbitrage

2. Zeros or STRIPS

- A *zero* is a claim to \$100 in n periods (price = p_n)



- A *spot rate* is a yield on a zero:

$$p_n = \frac{100}{(1 + y_n/2)^n}$$

- US treasury conventions:
  - price quoted for principal of 100
  - time measured in half-years
  - semi-annual compounding

2. Zeros (continued)

- A *discount factor* is a price of a claim to one dollar:

$$d_n = \frac{p_n}{100} = \frac{1}{(1 + y_n/2)^n}$$

- Examples (US treasury STRIPS, May 1995)

Maturity (Yrs)	Price ($)	Discount Factor	Spot Rate (%)
0.5	97.09	0.9709	5.99
1.0	94.22	0.9422	6.05
1.5	91.39	0.9139	
2.0	88.60	0.8860	6.15

3. Compounding Conventions

- A yield convention is an arbitrary set of rules for computing yields (like spot rates) from discount factors

- US Treasuries use semiannual compounding:

$$d_n \;=\; \frac{1}{(1+y_n/2)^n}$$

  with n measured in half-years

- Other conventions with n measured in years:

$$d_n \;=\; \begin{cases} (1+y_n)^{-n} & \text{annual compounding} \\ (1+y_n/k)^{-kn} & \text{“k” compounding} \\ e^{-ny_n} & \text{continuous compounding } (k \to \infty) \\ (1+ny_n)^{-1} & \text{“simple interest”} \\ (1-ny_n) & \text{“discount basis”} \end{cases}$$

- All of these formulas define rules for computing the yield y_n from the discount factor d_n, but of course they're all different and the choice among them is arbitrary. That's one reason discount factors are easier to think about.

4. Coupon Bonds

- Coupon bonds are claims to fixed future payments (c_n, say)

- They're collections of zeros and can be valued that way:

$$\begin{aligned} \text{Price} &= d_1 c_1 + d_2 c_2 + \cdots + d_n c_n \\ &= \frac{c_1}{(1+y_1/2)} + \frac{c_2}{(1+y_2/2)^2} + \cdots + \frac{c_n}{(1+y_n/2)^n} \end{aligned}$$

- Example: Two-year "8-1/2s"
  Four coupons remaining of 4.25 each

$$\begin{aligned} \text{Price} &= 0.9709 \times 4.25 \ + \ 0.9422 \times 4.25 \\ &\quad + 0.9139 \times 4.25 \ + \ 0.8860 \times 104.25 \\ &= 104.38. \end{aligned}$$

- Two fundamental principles of asset pricing:
  - Replication: two ways to generate same cash flows
  - Arbitrage: equivalent cash flows should have same price

5. Spot Rates from Coupon Bonds

- We computed the price of a coupon bond from prices of zeros

- Now reverse the process with these coupon bonds:

Bond	Maturity (Yrs)	Coupon	Price
A	0.5	8.00	100.97
B	1.0	6.00	99.96

- Compute discount factors "recursively":

$$100.97 = d_1 \times 104 \quad \Rightarrow \quad d_1 = 0.9709$$

$$\begin{aligned} 99.96 &= d_1 \times 3 + d_2 \times 103 \\ &= 0.9709 \times 3 + d_2 \times 103 \quad \Rightarrow \quad d_2 = 0.9422 \end{aligned}$$

- Spot rates follow from discount factors:

$$d_n = \frac{1}{(1 + y_n/2)^n}$$

5. Spot Rates from Coupon Bonds (continued)

- Do zeros and coupon bonds imply the same discount factors and spot rates?

- Example: Suppose Bond B sells for 99.50, implying $d_2 = 0.9377$ (B seems cheap)
  - Replication of B's cash flows with zeros:

$$\begin{aligned} 3 &= x_1 \times 100 \quad \Rightarrow \quad x_1 = 0.03 \\ 103 &= x_2 \times 100 \quad \Rightarrow \quad x_2 = 1.03 \end{aligned}$$

  - Cost of replication is

$$\text{Cost} = 0.03 \times 97.09 + 1.03 \times 94.22 = 99.96$$

  - Arbitrage strategy: buy B and sell its replication
    Riskfree profit is 99.96 – 99.50 = 0.46

- **Proposition.** If (and only if) there are no arbitrage opportunities, then zeros and coupon bonds imply the same discount factors and spot rates.

- Presumption: markets are approximately "arbitrage-free"

- Practical considerations: bid/ask spreads, hard to short

5. Spot Rates from Coupon Bonds (continued)

- Replication continued
  - Replication of coupon bonds with zeros seems obvious
  - Less obvious (but no less useful) is replication of zeros with coupon bonds
  - Consider replication of 2-period zero with x_a units of A and x_b units of B:

$$\begin{aligned} 0 &= x_a \times 104 + x_b \times 3 \\ 100 &= x_a \times 0 + x_b \times 103 \end{aligned}$$

    Remark: we've equated the cash flows of the 2-period zero to those of the portfolio (x_a, x_b) of A and B

    * Solution:
      - $x_b = 0.9709 = 100/103$: hold slightly less than one unit of B, since the final payment (103) is larger than the zero's (100)
      - $x_a = -0.0280$: short enough of A to offset the first coupon of B
    * We can verify the zero's price:

$$\text{Cost} = -0.0280 \times 100.97 + 0.9709 \times 99.96 = 94.22.$$

  - Remark: even if zeros didn't exist, we could compute their prices and spot rates.

6. Yields on Coupon Bonds

- Spot rates apply to specific maturities

- The *yield-to-maturity* on a coupon bond satisfies

$$\text{Price} = \frac{c_1}{(1+y/2)} + \frac{c_2}{(1+y/2)^2} + \cdots + \frac{c_n}{(1+y/2)^n}$$

- Example: Two-year 8-1/2s

$$104.38 = \frac{4.25}{(1+y/2)} + \frac{4.25}{(1+y/2)^2} + \frac{4.25}{(1+y/2)^3} + \frac{104.25}{(1+y/2)^4}$$

The yield is $y = 6.15\%$.

- Comments:
  - Yield depends on the coupon
  - Computation: guess y until price is right

7. Par Yields

- We've found prices and yields for given coupons

- Find the coupon that delivers a price of 100 (par)

$$\text{Price} \;=\; 100 \;=\; (d_1 + \cdots + d_n)\text{Coupon} + d_n 100$$

The annualized coupon rate is

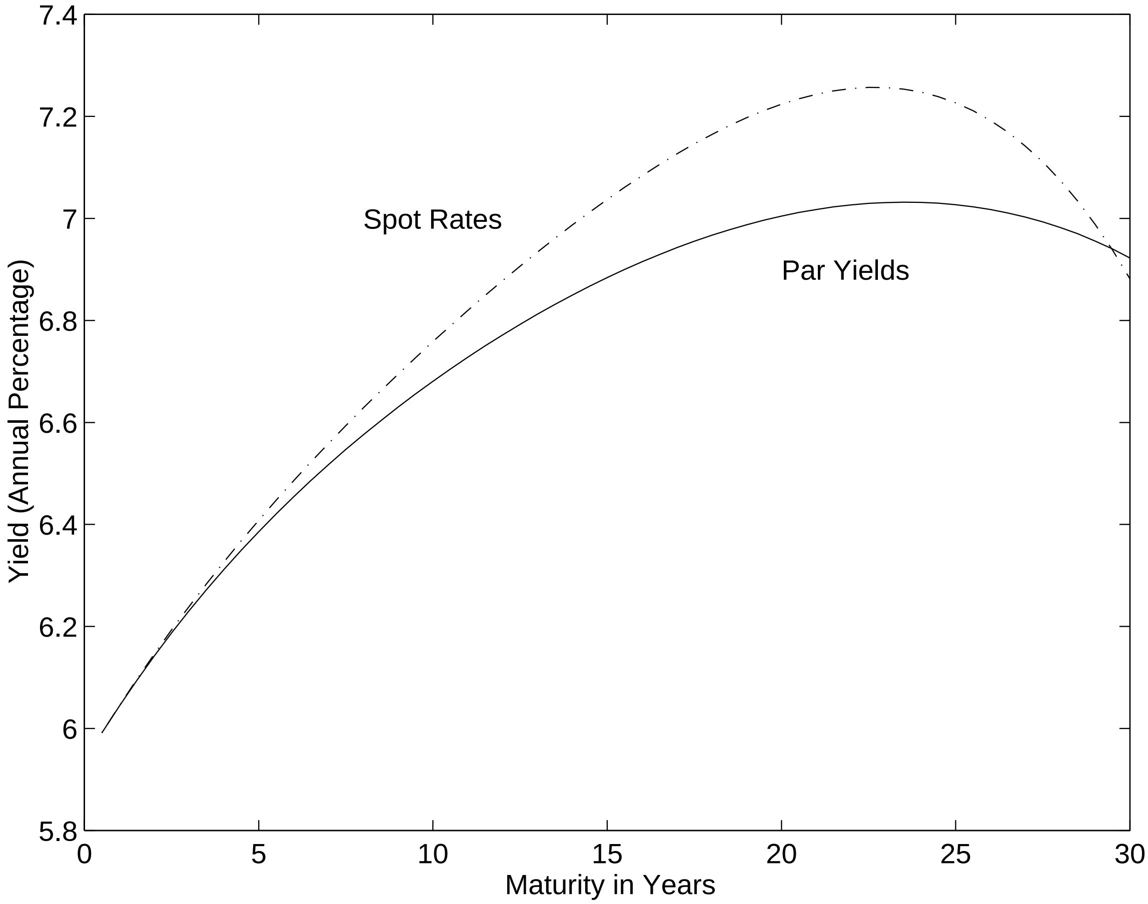
$$\begin{aligned}\text{Par Yield} &= 2 \times \text{Coupon} \\ &= 2 \times \frac{1 - d_n}{d_1 + \cdots + d_n} \times 100\end{aligned}$$

- This obscure calculation underlies the initial pricing of bonds and swaps (we'll see it again)

8. Yield Curves

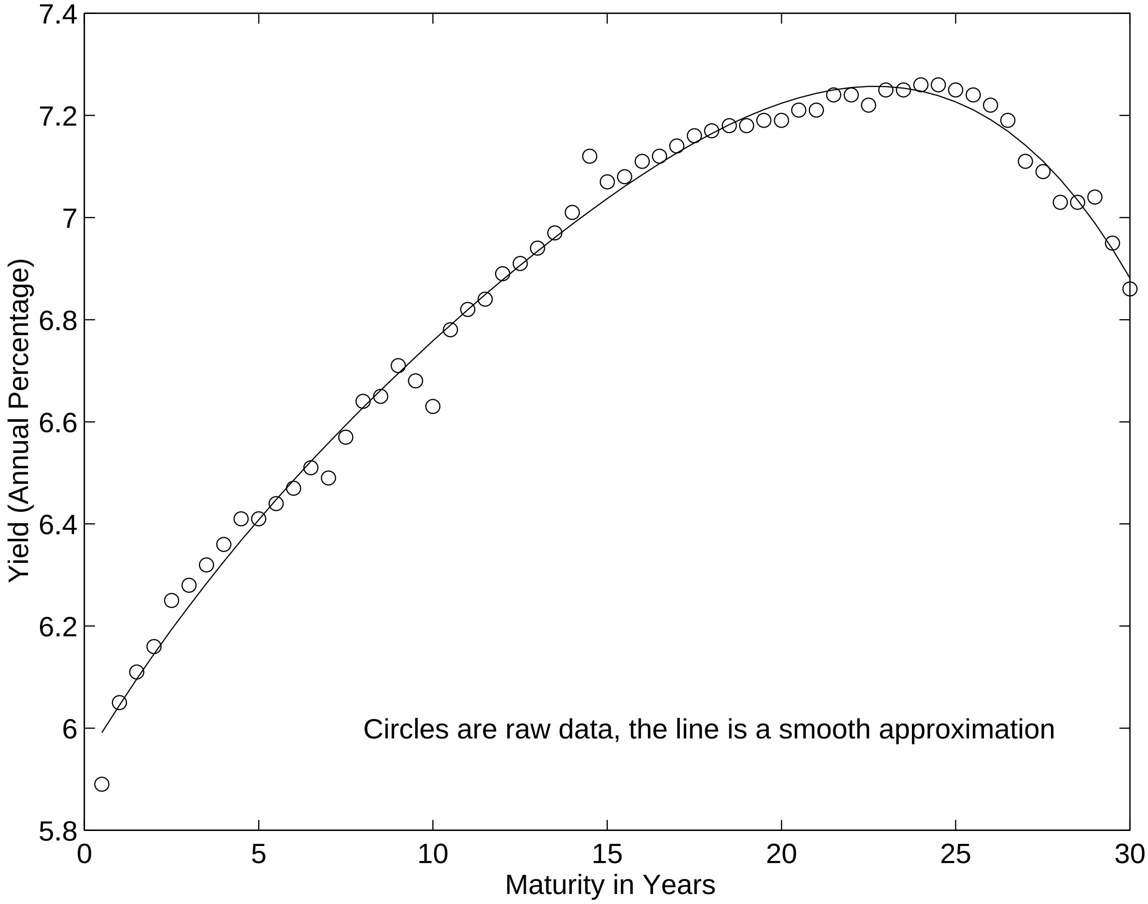
- A *yield curve* is a graph of yield y_n against maturity n

- May 1995, from US Treasury STRIPS:



9. Estimating Bond Yields

- Standard practice is to estimate spot rates by fitting a smooth function of n to spot rates or discount factors



- "Noise": bid/ask spread, stale quotes, liquidity (on/off-the-runs), coupons, special features
- Reminder that the frictionless world of the proposition is an approximation

10. Day Counts for US Treasuries

- Overview
  - Bonds typically have fractional first periods
  - You pay the quoted price plus a pro-rated share of the first coupon (accrued interest)
  - Day count conventions govern how prices are quoted and yields are computed

- Details
  - Invoice price calculation (what you pay):

$$\begin{aligned} \text{Invoice Price} &= \text{Quoted Price} + \text{Accrued Interest} \\ \text{Accrued Interest} &= \frac{u}{u+v}\,\text{Coupon} \\ u &= \text{Days Since Last Coupon} \\ v &= \text{Days Until Next Coupon} \end{aligned}$$

  - Yield calculation ("street convention"):

$$\begin{aligned} \text{Invoice Price} &= \frac{\text{Coupon}}{(1+y/2)^{w}} + \frac{\text{Coupon}}{(1+y/2)^{w+1}} \\ &\quad + \cdots + \frac{\text{Coupon} + 100}{(1+y/2)^{w+n-1}} \\ w &= v/(u+v) \\ n &= \text{Number of Coupons Remaining} \end{aligned}$$

10. Day Counts for US Treasuries (continued)

- US Treasuries use "actual/actual" day counts for u and v (ie, we actually count up the days)

- Example: 8-1/2s of April 97 (as of May 95)

Issued	April 16, 1990
Settlement	May 18, 1995
Matures	April 15, 1997
Coupon Freq	Semiannual
Coupon Dates	15th of Apr and Oct
Coupon Rate	8.50
Coupon	4.25
Quoted Price	104.19

Time line:

$$u =$$

$$v =$$

$$n =$$

10. Day Counts for US Treasuries (continued)

Price calculations:

$$\begin{aligned} \text{Accrued Interest} &= \\ \text{Invoice Price} &= \end{aligned}$$

Yield calculations:

$$\begin{aligned} w &= \\ d &= 1/(1+y/2) \qquad \text{(to save typing)} \\ 104.95 &= d^w(1+d+d^2+d^3)\,4.25 + d^{w+3}100 \\ \Rightarrow &\quad d = 0.97021 \\ &\quad y = 6.14\% \end{aligned}$$

(The last step is easier with a computer)

11. Other Day Count Conventions

- US Corporate bonds (30/360 day count convention) (roughly: count days as if every month had 30 days)

  Example: Citicorp's 7 1/8s

Settlement	June 16, 1995
Matures	March 15, 2004
Coupon Freq	Semiannual
Quoted Price	101.255

  Calculations:

$$n =$$
$$u =$$
$$v =$$
$$w =$$
$$\text{Accrued Interest} =$$
$$\text{Invoice Price} =$$

$$\text{Invoice Price} = d^w \left(\frac{1-d^n}{1-d} \right) \text{ Coupon } + d^{w+n-1} 100$$

$$\Rightarrow \quad y = 6.929\%$$

  Remark: the formula works, don't sweat the details!

11. Other Day Count Conventions (continued)

- Eurobonds (30E/360 day count convention)
  (ie, count days as if every month has 30 days)

  Example: IBRD 9s, dollar-denominated

Settlement	June 20, 1995
Matures	August 12, 1997
Coupon Freq	Annual
Quoted Price	106.188

  Calculations:

$$n =$$
$$u =$$
$$v =$$
$$w =$$
$$\text{Accrued Interest} =$$
$$\text{Invoice Price} =$$

$$\text{Invoice Price} = d^w \left(\frac{1-d^n}{1-d} \right) \text{ Coupon } + d^{w+n-1}100$$

$$\Rightarrow \quad y = 5.831\%$$

  Remark: the formula works for all coupon frequencies with the appropriate modification of d [here $d = 1/(1+y)$]

11. Other Day Count Conventions (continued)

- Flow chart for computing bond yields
  - Determine: Coupon Rate and Coupon Frequency (k per year, say)
  - Compute:

    $$\text{Coupon} = \text{Coupon Rate}/k$$

  - Compute Accrued Interest, Invoice Price, and w using appropriate day-count convention
  - Computing the yield
    - Define

      $$d = \frac{1}{1 + y/k}$$

    - Find the value of d that satisfies

      $$\text{Invoice Price} = d^w \left(\frac{1 - d^n}{1 - d}\right) \text{ Coupon } + d^{w+n-1} 100$$

    - Compute y from d:

      $$y = k(1/d - 1)$$

11. Other Day Count Conventions (continued)

- Eurocurrency deposits
  (generally actual/360 day count convention)

  Example: 6-month dollar deposit in interbank market

Settlement	June 22, 1995
Matures	December 22, 1995
Rate (LIBOR)	5.9375

  Cash flows: pay (say) 100, get 100 plus interest

  Interest computed by

$$\begin{aligned} \text{Interest} &= \text{LIBOR} \times \frac{\text{Actual Days to Payment}}{360} \\ &= 5.9375 \times \left(\frac{183}{360}\right) = 3.018. \end{aligned}$$

  Remarks

  - Can be denominated in any currency
  - "Interest" is analogous to $y/2$ in treasury formulas

12. Forward Rates

- A one-period ***forward rate*** f_n at date t is the rate paid on a one-period investment arranged at t (“trade date”) and made at $t + n$ (“‘settlement date”)

trade	settlement	maturity
t	$t+n$	$t+n+1$

- Representative cash flows (scale is arbitrary)

t	$t+n$	$t+n+1$
0	$-F$	100

with $F = 100/(1 + f_n/2)$ (verify that rate is f_n)

- Replication with zeros

t	$t+n$	$t+n+1$
$-p_{n+1}$		100
xp_n	$-100x$	

Choose x to replicate cash flows of forward contract

$$0 = -p_{n+1} + xp_n$$

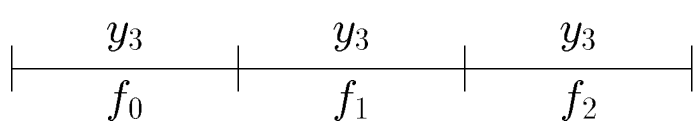
$$\Rightarrow \quad 1 + f_n/2 = \frac{p_n}{p_{n+1}} = \frac{d_n}{d_{n+1}}$$

12. Forward Rates (continued)

- Sample forward rate calculations:

Maturity	Price ($)	Spot Rate (%)	Forward Rate (%)
0.5	97.09	5.99	5.99
1.0	94.22	6.05	6.10
1.5	91.39	6.10	6.20
2.0	88.60	6.15	6.29

- Forward rates are the marginal cost of one more period:



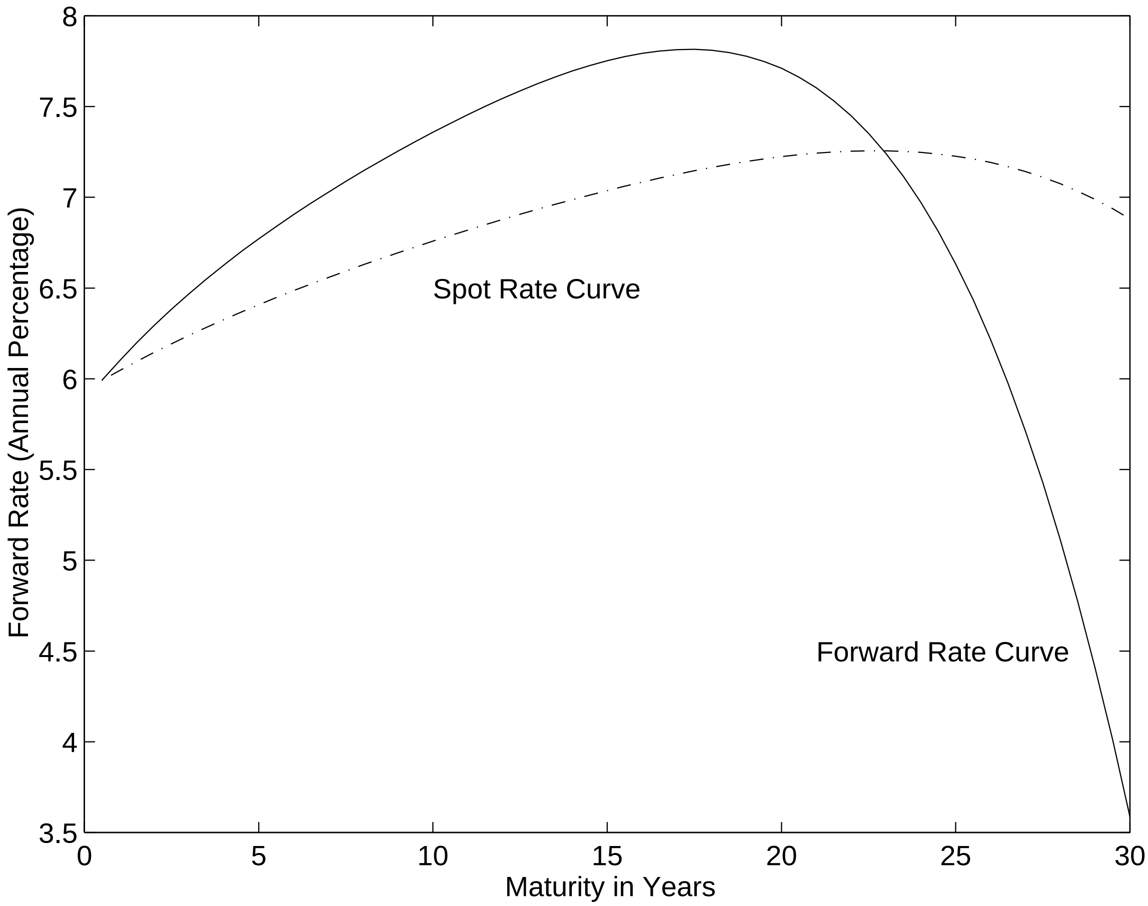
- Spot rates are (approximately) averages:

$$
\begin{aligned}
y_n &\cong n^{-1}(f_0 + f_1 + \cdots + f_{n-1}) \\
&\cong n^{-1} \sum_{j=1}^{n} f_{j-1}
\end{aligned}
$$

(This is exact with continuous compounding.)

12. Forward Rates (continued)

- Forward and spot rates in May 1995:



13. Yields and Returns on Zeros

- Example: Six-month investments in two zeros

Zero	Maturity	Price	Spot Rate (%)	Forward Rate (%)
A	0.5	97.56	5.00	5.00
B	1.0	94.26	6.00	7.00

- Scenarios for spot rates in six months

Spot Rates (%)	Scenario 1	Scenario 2	Scenario 3
y_1	8.00	5.00	2.00
y_2	9.00	6.00	3.00
	"up 3"	"no change"	"down 3"

- One-period returns on zeros

$$1 + h/2 = \frac{\text{Sale Price}}{\text{Purchase Price}}$$

(h for holding period, which is six months here)

- Scenario 2 returns

$$\text{(A)} \quad 1 + h/2 = \frac{100}{97.56} = 1.025 \quad \Rightarrow \quad h = 0.0500$$

$$\text{(B)} \quad 1 + h/2 = \frac{97.56}{94.26} = 1.035 \quad \Rightarrow \quad h = 0.0700$$

13. Yields and Returns on Zeros (continued)

- Six-month returns (h):

	Return on A (%)	Return on B (%)
Scenario 1	5.00	4.02
Scenario 2	5.00	7.00
Scenario 3	5.00	10.08

- Remarks
  - Return on A is the same in all scenarios
    Standard result when holding period equals maturity:

$$(1 + h/2)^n \ = \ \frac{100}{p_n} \ = \ (1 + y/2)^n$$

  - Return on B depends on interest rate movements

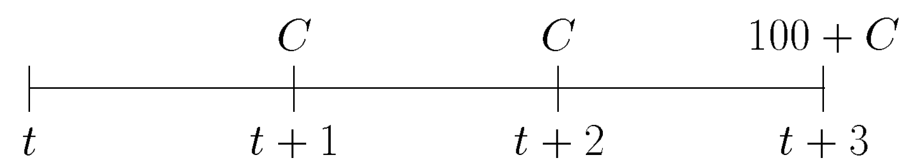
14. Yields and Returns on Coupon Bonds

- One-period returns

$$1 + h/2 = \frac{\text{Sale Price + Coupon}}{\text{Purchase Price}}$$

(buy and sell just after coupon payment)

- Return when held to maturity
  Needed: return r on reinvested coupons
  - Three-period example



$$(1+h/2)^3 = \frac{(1+r)^2C + (1+r)C + C + 100}{\text{Purchase Price}}$$

  - Return depends on reinvestment rate r (arbitrary)
  - No simple connection between return and yield

- Bottom line: yields are not returns

Summary

- Bond prices and discount factors represent the time-value of money
- Spot rates do, too
- Conventions govern the calculation of spot rates from discount factors
- Yields on coupon bonds are a common way of representing prices, but are not otherwise very useful
- Cash flows of coupon bonds can be "replicated" with zeros, and vice versa
- Replication and arbitrage relations apply to frictionless markets, but hold only approximately in practice
- Yields and returns aren't the same thing