

## State-Contingent Claims

### 0. Overview

- Functions of Financial Markets
- Fruit
- States
- State-Contingent Claims
- Valuing Fixed Income Derivatives
- Models: Uses and Misuses

## 1. Functions of Financial Markets

- Borrowing and lending:
  - People with funds (lenders) lend to those with investment opportunities (borrowers)
  - Competition for funds tends to locate the best projects
  
- Risk management
  
- Fixed income derivatives
  - callable bonds
  - prepayable mortgages
  - interest rate caps
  - floating rate notes with caps and floors
  
- Remarks:
  - Payments are uncertain — “contingent”
  - Can’t be valued with discount factors
  - Options are everywhere

## 2. Fruit

- Overview: more abstract, but same approach (decompose assets into component parts)
- Why fruit? To show you how simple this is.
- Example 1
  - Basket 1: 25 apples, 100 bananas, \$150
  - Basket 2: 50 apples, 50 bananas, \$150
  - Guess: apples worth \$2, bananas \$1
  - Math: solve

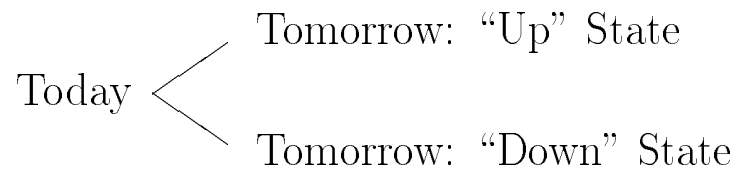
$$150 = q_a \times 25 + q_b \times 100$$

$$150 = q_a \times 50 + q_b \times 50$$

- Example 2
  - Basket 1: 50 apples, 50 bananas, \$100
  - Basket 2: 25 apples, 25 bananas, \$80
  - Analysis: buy basket 1, short 2 basket 2's, pocket profit of  $2 \times 80 - 100 = 60$  (arbitrage!)
  - What are apples and bananas worth? Not clear.
- **Proposition.** If (and only if) there are no arbitrage opportunities, we can derive prices of individual fruits consistent with the prices of baskets.

### 3. States

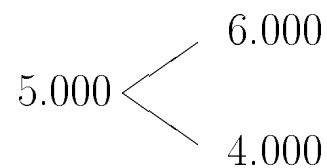
- *States* are situations or scenarios
- Uncertainty: several situations or states are possible
- Generic representation over two periods:



- Examples of states

Up State	Down State
Yankees Win	Yankees Lose
Yields Rise	Yields Fall
Russia defaults	Russia Does Not Default

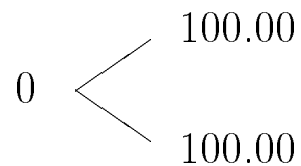
- Short rate tree (the example we care about):



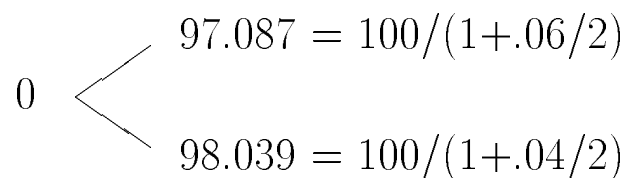
(Short rate  $r = y_1$ : 6-month LIBOR or Treasury yield)

#### 4. Two-Period State-Contingent Claims

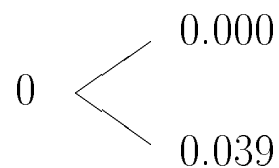
- Assets are like fruit baskets: collections of components  
(Analogy: coupon bonds are collections of zeros)
- *State-contingent claims*: cash flows depend on the state
- *Assets* are collections of state-contingent cash flows
- Example 1: One-period zero



- Example 2: Two-period zero  
(we use the short rate tree from the previous page)



- Example 3: Option with strike = 98



#### 4. Two-Period State-Contingent Claims (cont'd)

- For valuation, we need:
  - List of states:  $s$ , say
  - Cash flow associated with state  $s$ :  $c_s$
  - Price of one dollar in state  $s$ :  $q_s$

- Valuation:

$$\begin{aligned}\text{Asset Price} &= \sum_s q_s c_s \\ &= q_u c_u + q_d c_d\end{aligned}$$

(Just like fruit: sum value of components)

- **Proposition.** If (and only if) there are no arbitrage opportunities, we can derive state prices consistent with the prices of assets.

#### 4. Two-Period State-Contingent Claims (cont'd)

- Approach 1 (analogous to fruit and coupon bonds):  
If prices of zero and option are 97.561 and 0.019, we find state prices from

$$97.561 = q_u \times 100 + q_d \times 100$$

$$0.019 = q_u \times 0 + q_d \times 0.039$$

Solution:  $q_u = q_d = 0.4878$  (to greater accuracy)

- Approach 2 (standard shortcut in fixed income):
  - State prices sum to one-period discount factor:

$$d_1 = q_u \times 1 + q_d \times 1$$

$$= q_u + q_d$$

- Split discount factor in half (fifty-fifty rule):

$$q_u = q_d = 0.5 \times d_1 = \frac{0.5}{1 + r/2}$$

- In our examples,  $r = 0.05$  and

$$q_u = q_d = 0.5/(1 + .05/2) = 0.4878$$

#### 4. Two-Period State-Contingent Claims (cont'd)

- Applications of Approach 2

- Example 1 (one-period zero):

$$\begin{aligned}\text{Price} &= q_u c_u + q_d c_d \\ &= 0.4878 \times 100 + 0.4878 \times 100 = 97.561\end{aligned}$$

- Example 2 (two-period zero):

$$\begin{aligned}\text{Price} &= q_u c_u + q_d c_d \\ &= 0.4878 \times 97.087 + 0.4878 \times 98.039 = 95.184\end{aligned}$$

- Example 3 (option):

$$\begin{aligned}\text{Price} &= q_u c_u + q_d c_d \\ &= 0.4878 \times 0 + 0.4878 \times 0.039 = 0.019\end{aligned}$$

- Summary of theory:

- Asset: collection of state-contingent cash flows
- State prices: fifty-fifty rule
- Asset prices: sum product of cash flows and state prices

## 5. Other Versions of State Prices (optional)

- State prices come in other forms (but do the same thing)
- *Pricing kernel*
  - Defined by  $m$  in

$$q_s = \pi_s m_s$$

where  $\pi_s$  is the probability that state  $s$  occurs

- Probabilities down-weight value of unusual events
- Asset pricing follows

$$\begin{aligned} \text{Asset Price} &= \sum_s q_s c_s \\ &= \sum_s \pi_s m_s c_s = E(mc) \end{aligned}$$

- Discount factor:  $d_1 = E(m)$  (set  $c_s = 1$  for all  $s$ )
- Kernel  $m$  captures effects of risk:
  - \* Constant kernel  $m = d_1$  (why?) means

$$\text{Asset Price} = d_1 E(c),$$

which applies if investors are neutral to risk.

- \* In general covariance with cash flows governs price:

$$\text{Asset Price} = d_1 E(c) + \text{Cov}(m, c)$$

( $m$  is like the market return in the CAPM)

## 5. Other Versions of State Prices (continued)

- *Risk-neutral probabilities*

- Defined by

$$\pi_s^* = \frac{q_s}{\sum_{s'} q_{s'}} = \frac{q_s}{d_1} = \frac{\pi_s m_s}{d_1}$$

- Note that  $\{\pi_s^*\}$  are positive if state prices are and sum to one: this is the sense in which they're probabilities

- Asset pricing follows

$$\text{Asset Price} = d_1 \sum_s \pi_s^* c_s = d_1 E^*(c)$$

where  $E^*$  means the expectation using probabilities  $\pi_s^*$

- Risk-neutral is a misnomer both ways:

- \* Effects of risk are built in

- \* Probabilities only in the technical sense

- All three versions are equivalent:

- State prices

- Pricing kernel

- Risk-neutral probabilities

## 6. Multi-Period Contingent Claims

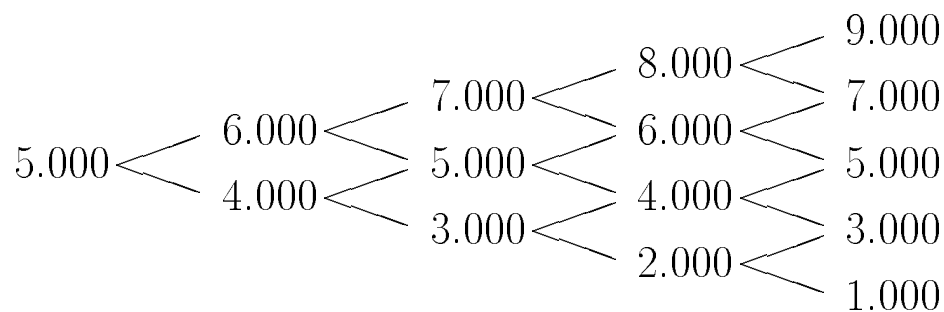
- Changes in short rate  $r = y_1$  (Ho and Lee version):

$$r_{t+1} = r_t + \mu_{t+1} + \varepsilon_{t+1}$$

with

$$\varepsilon_{t+1} = \begin{cases} +\sigma & \text{with probability one-half} \\ -\sigma & \text{with probability one-half} \end{cases}$$

- Parameters
  - $\mu_{t+1}$  is expected change in  $r$  (“drift”)
  - $\sigma$  is standard deviation (“volatility”)
- Short rate tree ( $r_0 = 5\%$ ,  $\mu_{t+j} = 0$ ,  $\sigma = 1\%$ ):



- Paths for the short rate:
  - *Path A* (*up, down, up, down*). The short rates are 5, 6, 5, 6, and 5.
  - *Path B* (*down, down, down, up*). The short rates are 5, 4, 3, 2, and 3.

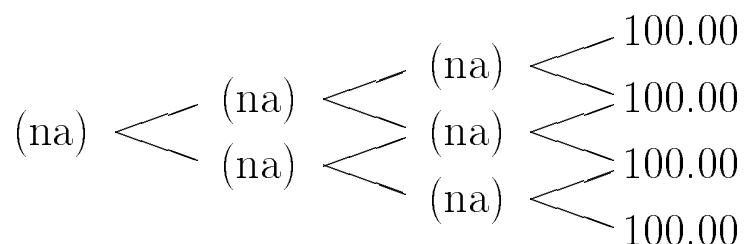
## 6. Multi-Period Contingent Claims (continued)

- Recursive valuation: treat each “node” as a two-period tree
- Apply these equations at each node, starting at the end:

$$q_u = q_d = 0.5 \times d_1 = \frac{0.5}{1 + r/2} \quad (1)$$

$$\text{Price } p = \text{Current Cash Flow} + q_u p_u + q_d p_d \quad (2)$$

- Remember these equations: they control everything
  - $p_s$  is price of asset in state  $s$
  - “Current cash flow” is a new subtlety that we’ll deal with when it arises
- Example 1: 3-period zero
    - At maturity, bond has cash flow of 100 in all states:

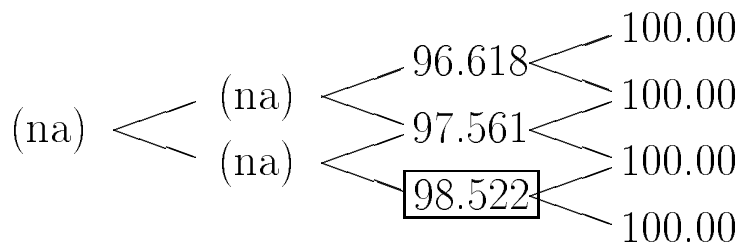


(This is the easy part: we know the value at maturity)

## 6. Multi-Period Contingent Claims (continued)

- Example 1: 3-period zero (continued)

- In the previous period, prices are



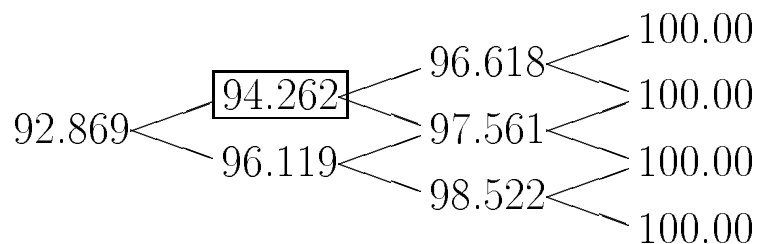
Details for “boxed” node (short rate 3%):

- \* State prices are  $q_u = q_d = 0.5/(1 + .03/2) = 0.4926$

- \* Zero’s price is

$$\text{Price} = 0.4926 \times 100 + 0.4926 \times 100 = 98.522$$

- The rest of the tree is



Details for “boxed” node (short rate 6%):

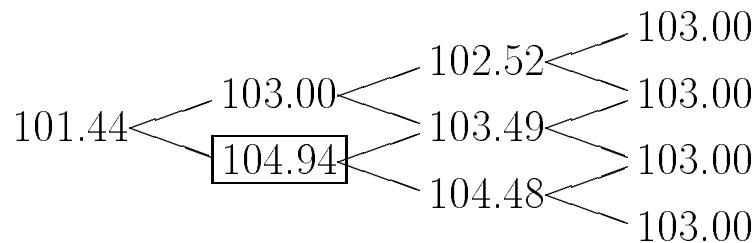
- \* State prices are  $q_u = q_d = 0.5/(1 + .06/2) = 0.4854$

- \* Zero’s price is

$$\text{Price} = 0.4854 \times 96.618 + 0.4854 \times 97.561 = 94.262$$

## 6. Multi-Period Contingent Claims (continued)

- Example 2: 3-period 6% bond
  - Differs in having cash flows at each date
  - The complete price tree is

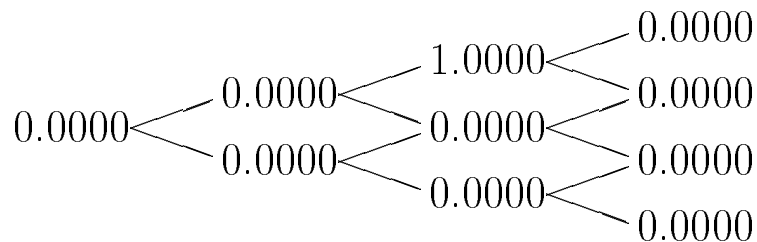


- Details for “boxed” node (short rate 4%):
  - \* State prices are  $q_u = q_d = 0.5/(1 + .04/2) = 0.4902$
  - \* Price includes “current cash flow” of 3:
 
$$\begin{aligned} \text{Price} &= 3 + 0.4902 \times 103.49 + 0.4902 \times 104.48 \\ &= 104.94 \end{aligned}$$
  - \* Since the price includes the current cash flow, we should think of it as including the accrued interest.
- Other nodes follow the same method: starting at the end of the tree, apply fifty-fifty rule (1) and pricing relation (2) to compute preceding nodes.

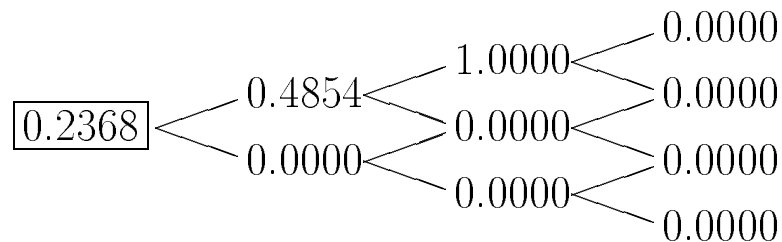
## 6. Multi-Period Contingent Claims (continued)

- Example 3: pure contingent claim

- Cash flow tree



- Price path is



- Details for “boxed” node (short rate 5%):

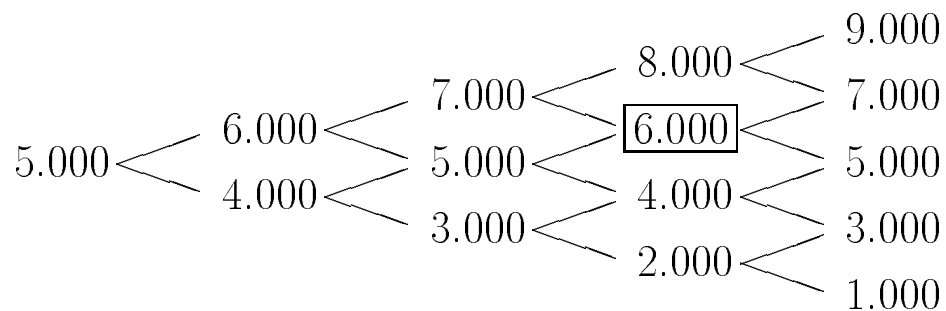
$$\begin{aligned} \text{Price} &= 0.4878 \times 0.4854 + 0.4878 \times 0 \\ &= 0.2368 \end{aligned}$$

- Meaning: claim to one dollar if “up-up” occurs is worth \$0.2368 now. (Remember this number for later.)

- Prices of such claims are analogous to discount factors: they summarize value across time and states.

## 7. Labelling Nodes/States

- It's helpful to have a label for a node/state
- Consider (again) this short rate tree:



- Labelling system:

$i$  = number of up moves

$n$  = number of periods since the start

$(i, n)$  defines the state (the node)

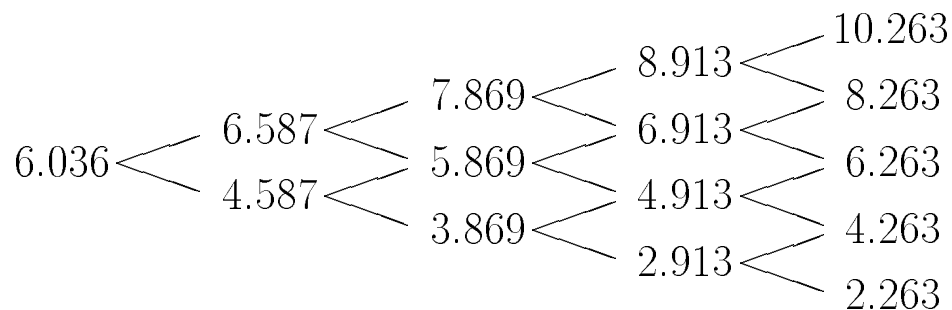
- Examples:

$(i, n) = (0, 0)$  is the initial node, the starting point

$(i, n) = (2, 3)$  is the node with the box

## 8. Valuing Fixed Income Derivatives

- Value a series of fixed income derivatives
- Input: short rate tree designed to reproduce spot rates of (6.036, 5.809, 5.824, 5.839, 5.914) (June 1995 LIBOR-based rates) (more on this later)



- Recursive valuation summary: Apply these equations at each node, starting at the end:

$$q_u = q_d = 0.5 \times d_1 = \frac{0.5}{1 + r/2} \quad (1)$$

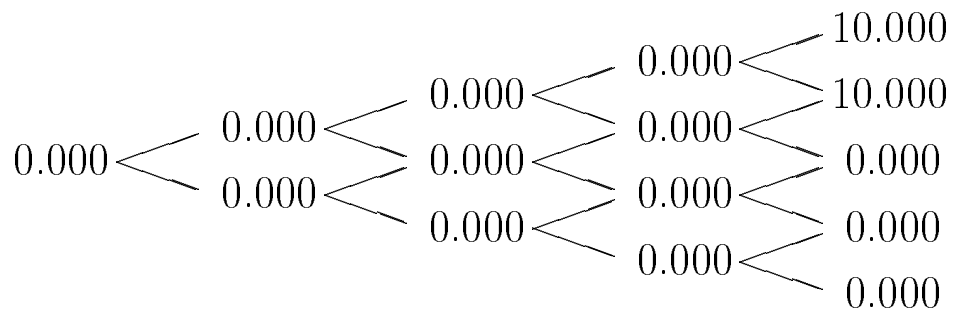
$$\text{Price } p = \text{Current Cash Flow} + q_u p_u + q_d p_d \quad (2)$$

## 8. Valuing Fixed Income Derivatives (continued)

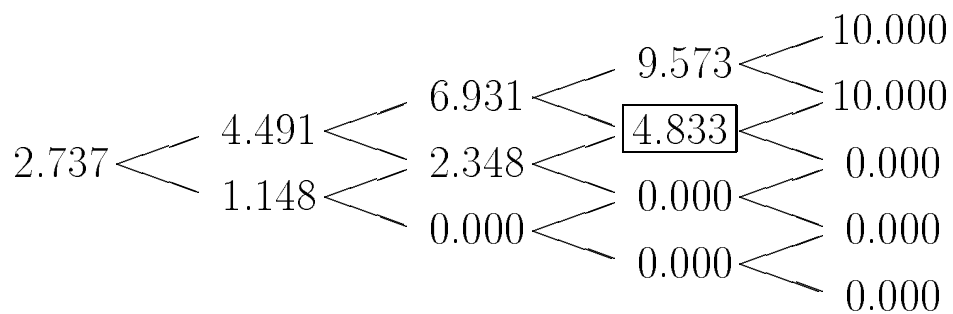
- Example 1: Bet on short rate (“digital option”)

Get \$10 if 6-m LIBOR  $>$  7% in 2 years, zero otherwise

- Cash flow tree:



- Find price with eqs (1,2), starting at end:



- Details for state (2,3) (box, short rate 6.913%):

\* State prices:  $q_u = q_d = 0.5 / (1 + .06913/2) = 0.4833$

\* Price is

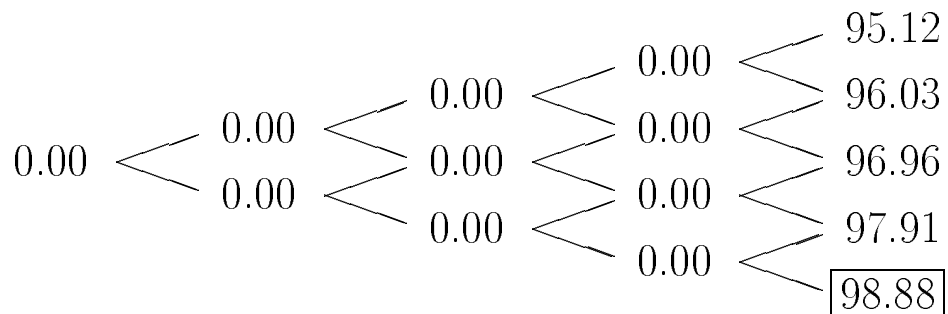
$$4.833 = 0.4833 \times 10 + 0.4833 \times 0$$

- Remark: note interest-sensitivity through tree

## 8. Valuing Fixed Income Derivatives (continued)

- Example 2: 5-period (30-month) zero

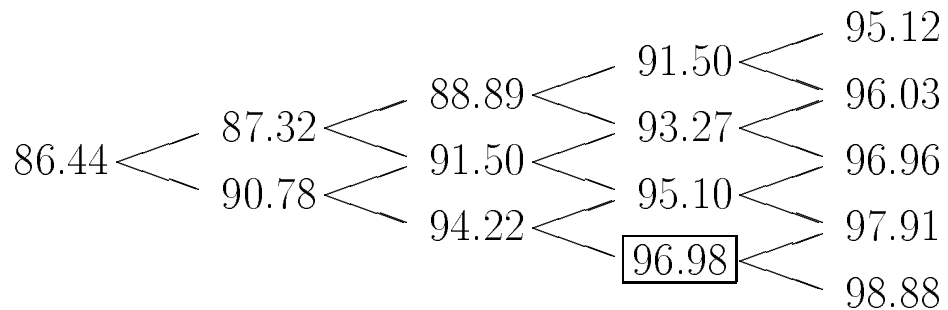
– “Cash flows” over four periods:



Details for state (0,4) (box, short rate 2.263%):

$$98.88 = 100 / (1 + .02263/2)$$

– Price path:



Details for state (0,3) (box, short rate 2.913%):

$$q_u = q_d = 0.5 / (1 + .02913/2) = 0.4928$$

$$\text{Price} = 0.4928 \times 97.91 + 0.4928 \times 98.88 = 96.98$$

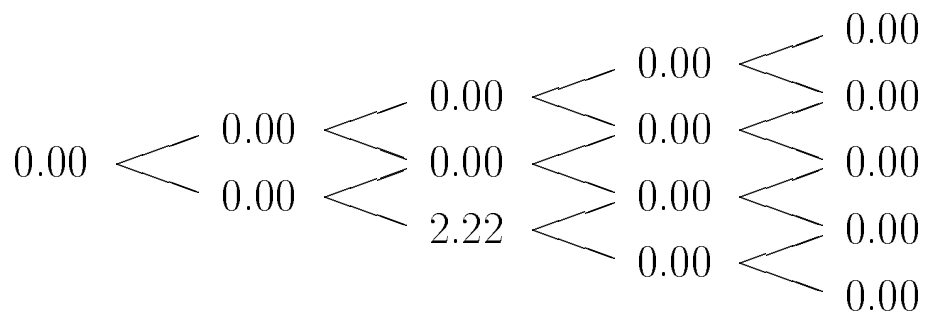
- Remark: tree is more work than spot rate, but gives scenarios for future, too

## 8. Valuing Fixed Income Derivatives (continued)

- Example 3: Option on the zero

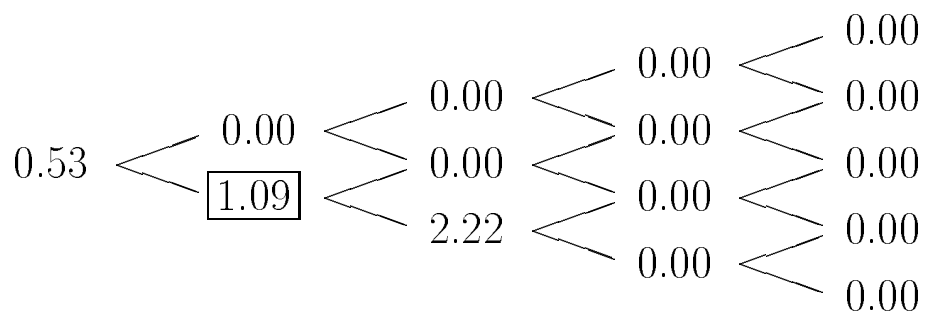
Option to buy zero for 92 in two periods (European call)

– Cash flow tree:



Remark:  $2.22 = 94.22 - 92$ .

– Price path for option:



Details for state (0,1) (box, short rate 4.587%):

$$q_u = q_d = 0.5 / (1 + .04587/2) = 0.4888$$

$$\text{Price} = 0.4888 \times 0.00 + 0.4888 \times 2.22 = 1.09$$

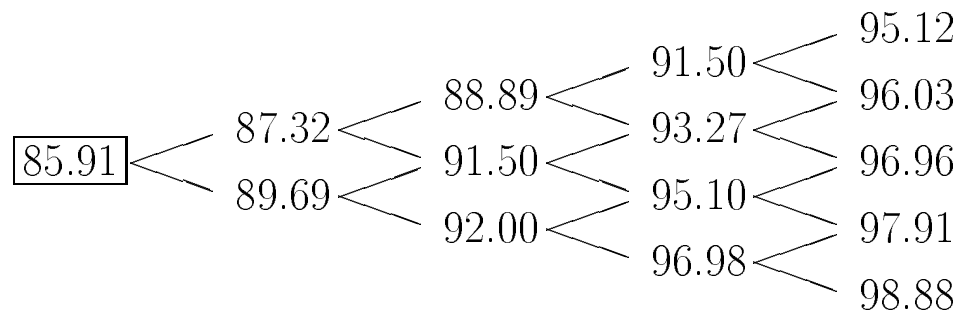
– Remark: this should be getting familiar!

## 8. Valuing Fixed Income Derivatives (continued)

- Example 4: Replication of a callable zero

- Callable zero: long the zero, short the option

- Price tree (difference between two previous trees):



Details for boxed node:  $85.91 = 86.44 \text{ (zero)} - 0.53 \text{ (option)}$ .

- Replication basics

- \* We can replicate the cash flows of an asset with those of a combination of two other assets (two because each node has two branches stemming from it).

- \* Similar to fruit and coupon bonds

- \* Replication is dynamic: it's different at each node in the tree.

## 8. Valuing Fixed Income Derivatives (continued)

- Example 4: Callable zero (continued)
  - Replicate cash flows of the callable zero with underlying zero ( $l$  for long) and one-period zero ( $s$  for short).
  - Compute quantities  $x_l$  of the long and  $x_s$  of the short that generate the same cash flows as the callable zero in the up and down states from a particular node:

$$\text{Callable}_u = x_l \times \text{Long}_u + x_s \times \text{Short}_u$$

$$\text{Callable}_d = x_l \times \text{Long}_d + x_s \times \text{Short}_d$$

- For the initial node the numbers are

$$87.32 = x_l \times 87.32 + x_s \times 100$$

$$89.69 = x_l \times 90.78 + x_s \times 100$$

Solution:  $x_l = 0.69$ ,  $x_s = 0.27$ .

- For node  $(0, 1)$  the numbers are

$$91.50 = x_l \times 91.50 + x_s \times 100$$

$$92.00 = x_l \times 94.22 + x_s \times 100$$

Solution:  $x_l = 0.18$ ,  $x_s = 0.75$  (call dominates here).

- For node  $(1, 1)$  the numbers are

$$88.89 = x_l \times 88.89 + x_s \times 100$$

$$91.50 = x_l \times 91.50 + x_s \times 100$$

Solution:  $x_l = 1.00$ ,  $x_s = 0.00$  (call irrelevant here).

## 8. Valuing Fixed Income Derivatives (continued)

- Example 4: Callable zero (continued)

- Interest sensitivity

- \* Callable has shorter duration than underlying: replication includes short zero.
- \* Duration varies with interest rates: compare the replicating strategies for nodes (0,1) and (1,1).

- Pricing by replication

- \* Callable is worth (at initial node)

$$\begin{aligned}\text{Price} &= 0.69 \times \text{Price of Long} + 0.27 \times \text{Price of Short} \\ &= 0.69 \times 86.44 + 0.27 \times 100 / (1 + 0.06036/2) \\ &= 85.91\end{aligned}$$

(answer has greater accuracy than components)

- \* We get the same answer as before.
- \* This has to be true: state pricing precludes arbitrage opportunities (see the proposition).
- \* Replication based on the model; the world may be different.

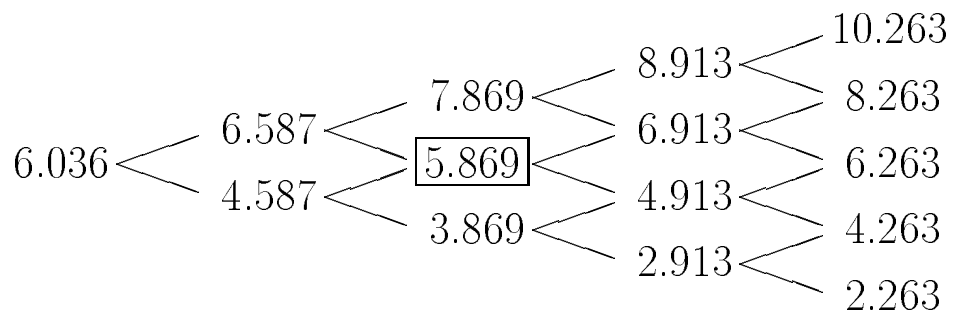
## 8. Valuing Fixed Income Derivatives (continued)

- Example 5: 2-Year FRN

(semiannual payments of 6-m LIBOR)

– Issue: how do we get the cash flows into the tree?

Rate tree is (still):



Since interest is paid one period after quote, what rate is paid in state (1,2) (box)?

- Trick: discount payments and shift back one period
  - \* Cash flow for boxed node:  $5.869/2$  (semiannual) discounted at same rate is worth 2.851
  - \* Remark: same value, but we adjust timing to fit into tree

## 8. Valuing Fixed Income Derivatives (cont'd)

- Example 5: 2-Year FRN (continued)

- Cash flows at the end (in four periods)

- \* Principal plus interest, discounted back one period

- \* Value is 100 for usual reason (FRNs trade at par)

- \* Example: state (3,3) with  $r = 8.913\%$

$$\text{Interest} = (8.913/2)/(1 + 0.08913/2) = 4.266$$

$$\text{Principal} = 100/(1 + 0.08913/2) = 95.734$$

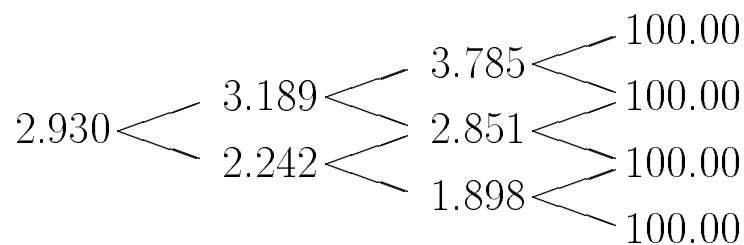
$$\text{Total} = 100.00$$

- Earlier cash flows

- \* Interest only

- \* Example: state (1,2) with  $r = 5.869\%$  has “cash flow”  $(5.869/2)/(1 + 0.05869/2) = 2.851$

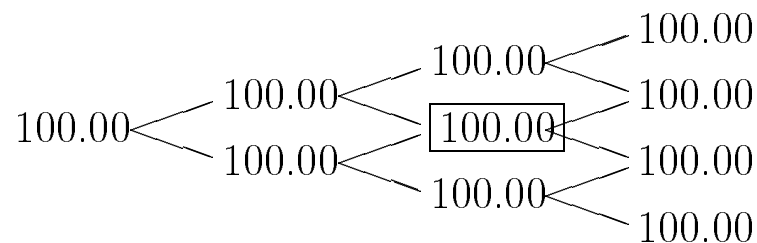
- Complete set of cash flows:



## 8. Valuing Fixed Income Derivatives (cont'd)

- Example 5: 2-Year FRN (continued)

- Price path is



- Details for state (1,2) (box, short rate 5.869%):

$$q_u = q_d = 0.5 / (1 + .05869/2) = 0.4857$$

$$\begin{aligned} \text{Price} &= 2.851 + 0.4857 \times 100.00 + 0.4857 \times 100.00 \\ &= 100.00 \end{aligned}$$

- Remarks:

- \* Note that we have used the “current cash flow”
- \* Useful trick for other short-rate related derivatives
- \* Answer no surprise

## 8. Valuing Fixed Income Derivatives (cont'd)

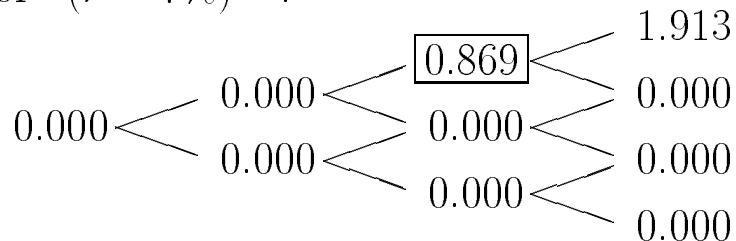
- Example 6: 2-Year 7% Interest Rate Cap

- Terms:

- \* Payments based on  $(r - 7\%)^+$  ( $x^+$  means  $\max(0, x)$ )

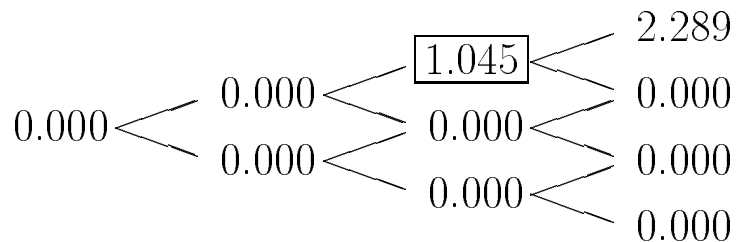
- \* Notional principal 250mm

- Tree for “ $(r - 7\%)^+$ ”:



Details for state (2,2):  $0.869 = 7.869 - 7.000$

- Cash flow tree:



Details:

- \* General idea: apply interest at semiannual rate to notional, and shift it back one period

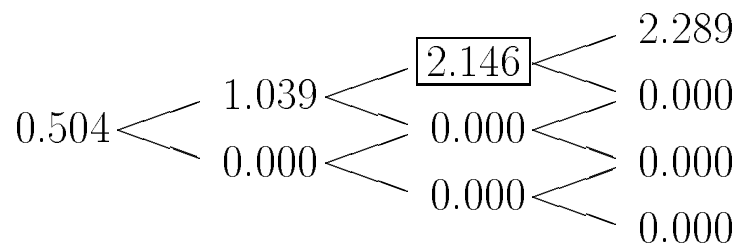
- \* State (2,2):

$$250 \times \frac{0.869/2}{1 + .07869/2} = 1.045$$

## 8. Valuing Fixed Income Derivatives (cont'd)

- Example 6: 2-Year 7% Interest Rate Cap (continued)

– Price path:



– Details for state (2,2) (box, short rate 7.869%):

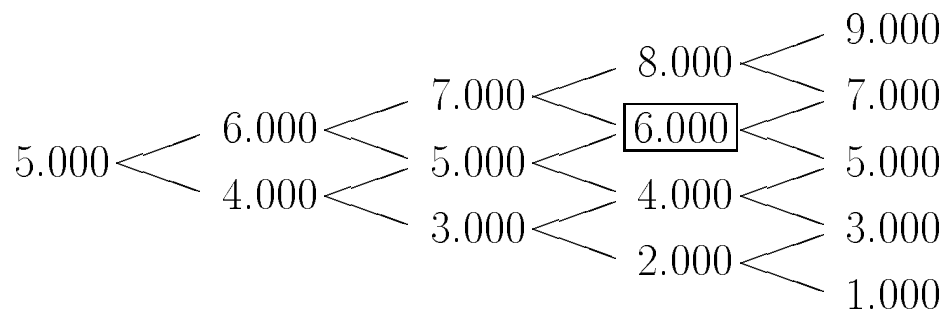
$$\begin{aligned}
 q_u &= q_d = 0.5 / (1 + .07869/2) = 0.4811 \\
 \text{Price} &= 1.045 + 0.4811 \times 2.289 + 0.4811 \times 0.000 \\
 &= 2.146
 \end{aligned}$$

– Remarks:

- \* This is a fair amount of effort
- \* But the hard part is adjusting the cash flows
- \* And the principles of valuation are the same

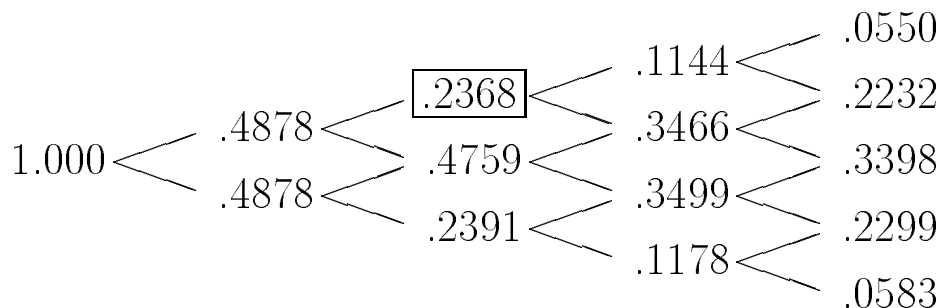
## 9. Duffie's Formula

- How do we construct a short rate tree consistent with current spot rates?
  - Hard way: compute prices of all relevant zeros
  - Easy way: use Duffie's formula to compute state prices
  - Warning: this is a complex digression, but it saves us time and effort later on.
- Remark: Don't focus on this section until you need it — it's far less important than what we've already done.
- Sample interest rate tree (could be anything):



## 9. Duffie's Formula (continued)

- Multiperiod state prices
  - $q(i, n)$  is the price now of one dollar in state  $(i, n)$
  - Note the difference between  $q(i, n)$  and  $(q_u, q_d)$ :
    - \* The latter (2-period state prices) pertains to a node and the two branches out of it.
    - \* The former (multiperiod state prices) to any node in the tree relative to the start.
  - Example:  $q(2, 2) = 0.2368$  (computed earlier)
  - We can list the complete set of  $q(i, n)$  in a tree:



- Remark: these are prices now for one dollar payable in the relevant state/node (think about this: it's not a price path)

## 9. Duffie's Formula (continued)

- Computing spot rates from short rate tree:
  - State prices — compute them somehow!
  - Discount factors. A claim to one dollar in all states in  $n$  periods is worth the sum of the state prices for that date:

$$d_n = \sum_{i=0}^n q(i, n) \times 1$$

(We are summing down column  $n$  here)

- Big picture: we have subdivided the discount factor into state-specific components
- Spot rates are

$$d_n = \frac{1}{(1 + y_n/2)^n} \Rightarrow y_n = 2(1/d_n^{1/n} - 1)$$

- Discount factors and spot rates for our example:

Maturity $n$	1	2	3	4
Discount Factor $d_n$	0.9756	0.9518	0.9287	0.9062
Spot Rate $y_n$ (%)	5.00	5.00	4.99	4.97

- Remark: the hard part is finding the state prices (next)

## 9. Duffie's Formula (continued)

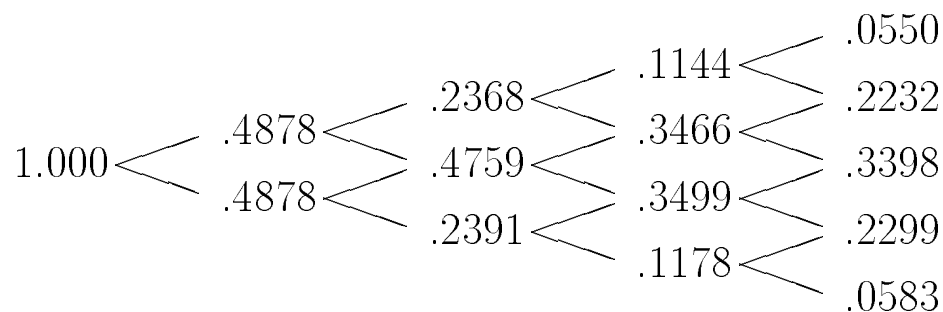
- Duffie's formula is a short cut for computing state prices:

$$q(i, n+1) = \begin{cases} \frac{0.5q(i, n)}{1 + r(i, n)/2} & \text{if } i = 0 \\ \frac{0.5q(i, n)}{1 + r(i, n)/2} + \frac{0.5q(i-1, n)}{1 + r(i-1, n)/2} & \text{if } 0 < i < n+1 \\ \frac{0.5q(i-1, n)}{1 + r(i-1, n)/2} & \text{if } i = n+1 \end{cases}$$

- Remarks:
  - Start with  $q(0, 0) = 1$  (a dollar today is worth a dollar) and increase  $n$  one period at a time.
  - The formula differs between the edges (where only one node leads in) and the interior (two nodes lead in).
  - This is a lot less work than doing each state-contingent claim on its own.
  - Don't worry about the details — this is something that works, but it's not important otherwise.

### 9. Duffie's Formula (continued)

- Applying Duffie's formula
- State price tree is (again)



- Details

– State (0,0):  $q(0,0) = 1$

– State (1,1):

$$q(1,1) = \frac{0.5 \times q(0,0)}{1 + r(0,0)/2} = \frac{0.5 \times 1.000}{1 + .0500/2} = .4878$$

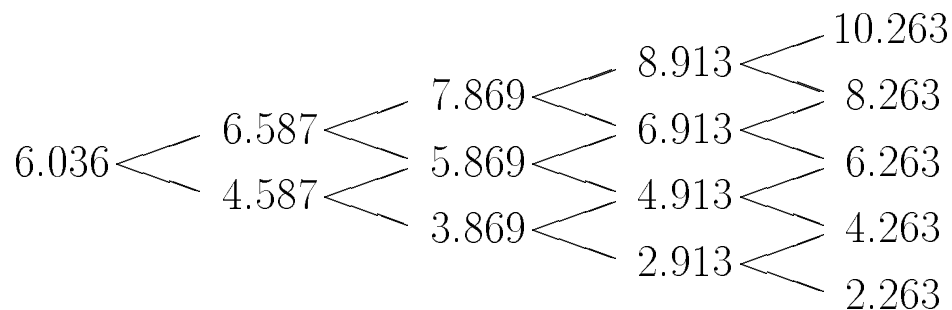
– State (1,2):

$$\begin{aligned} q(1,2) &= \frac{0.5 \times q(1,1)}{1 + r(1,1)/2} + \frac{0.5 \times q(0,1)}{1 + r(0,1)/2} \\ &= \frac{0.5 \times 0.4878}{1 + .0600/2} + \frac{0.5 \times 0.4878}{1 + .0400/2} = .4759 \end{aligned}$$

- Whew!

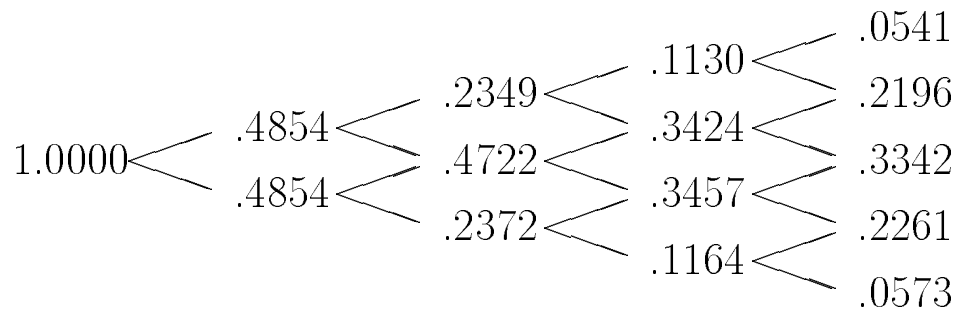
## 10. Choosing Parameters

- Volatility  $\sigma$ 
  - Estimate from recent data (1% is a ballpark number for a six-month period)
  - Infer from options (interest rate caps, options on eu-rodollar futures, swaptions)
- Mean or “drift” parameters  $\mu_t$ :
  - Choose to reproduce current spot rates – exactly!
  - Remark: Absolutely essential. If the model prices zeros incorrectly, why would we believe it for derivatives?
- Example: spot rates of (6.036, 5.809, 5.824, 5.839, 5.914) (June 1995 LIBOR-based rates) and  $\sigma = 1$  (all percents)
  - Drift parameters are  $(-0.45, 0.28, 0.04, 0.35)$  (percent)
  - Rate tree is the one we used earlier:



## 10. Choosing Parameters (continued)

- State price tree (via Duffie's formula):



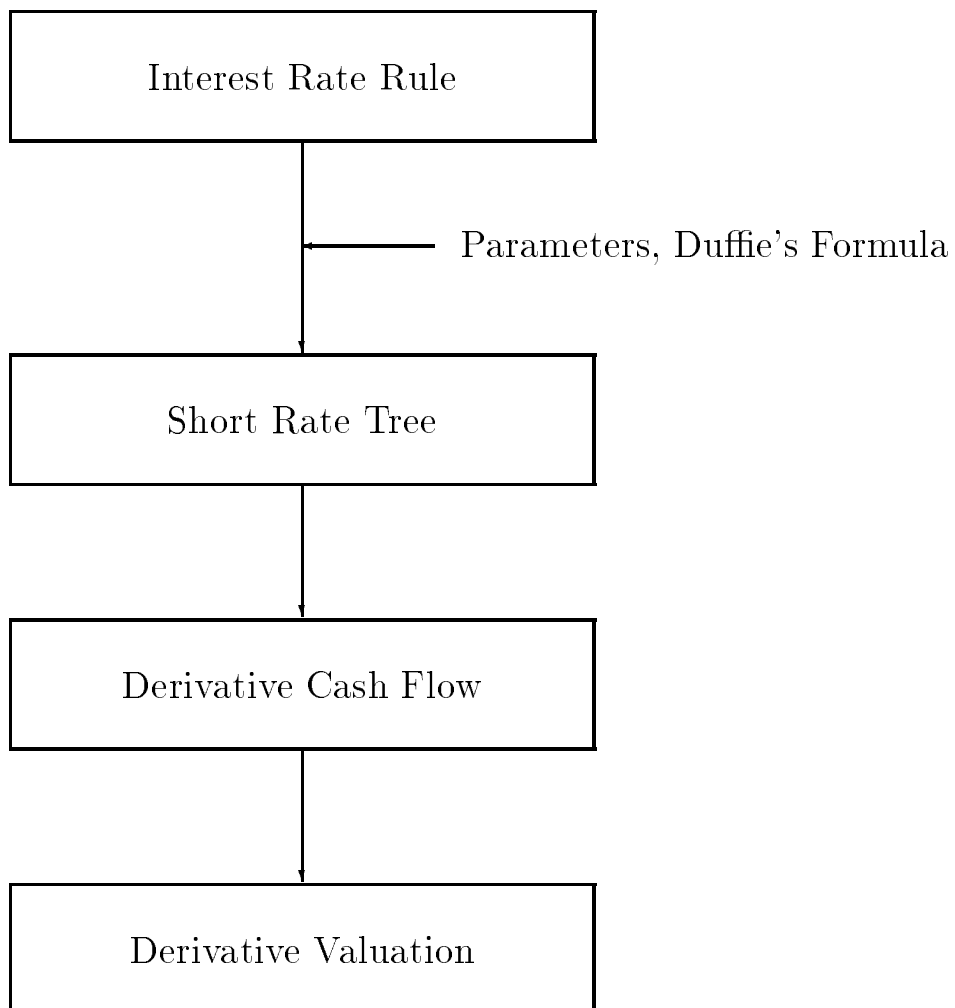
- Discount factors and spot rates:

Maturity $n$	1	2	3	4	5
Disc Factor $d_n$	0.9707	0.9443	0.9175	0.8913	0.8644
Spot Rate $y_n$ (%)	6.036	5.809	5.824	5.839	5.914

- Summary:
  - Base  $\sigma$  on history or options
  - Choose  $\{\mu_{t+j}\}$  to match spot rates
  - Duffie's formula makes this easier

## 10. Choosing Parameters (continued)

- Flow chart for derivatives valuation



## 11. Modeling Issues

- Our model is Ho and Lee, the first of an approach that is now the industry standard. Ho was here when he developed it and remains an active participant in Stern activities. But it's not the last word.

- Discrete set of states (either up or down).

Response. We can get as much refinement as we like by using a short time interval, but discretization is nevertheless a crude approximation. An alternative is “continuous” models, which are not based on trees.

- Interest rates can become negative in our model.

Response. Many people model the log of the short rate, as in

$$\log r_{t+1} = \log r_t + \mu_{t+1} + \varepsilon_{t+1}$$

Once a short rate tree is generated this way, the rest of the analysis is the same (state prices, Duffie's formula).

- Volatility varies over time.

Response. The popular Black-Derman-Toy model allows  $\sigma$  to vary with  $t$ , just as  $\mu$  does in the Ho and Lee model. Other models allow  $\sigma$  to vary randomly as an additional factor.

## 11. Modeling Issues (continued)

- Mean reversion. Our model (and the one above) exhibits no tendency for interest rates to return to some typical value. Long trees therefore exhibit a wide range of possible interest rates, which many of us consider unreasonable.  
Response. Models with mean-reversion are easily and commonly built, less easily put into a tree structure.
- Multiple factors. We argued earlier that the behavior of spot rates requires more than one factor: the spot curve shifts up and down, twists, etc.  
Response. Models with multiple factors. Easier said than done.
- Complexity. Bells and whistles are motivated by realism (the world is more complex than our model), but too much makes a model hard to understand and use. Emanuel Derman, physicist turned Goldman exec, writes:

Go as far as you can with one factor. ... The disadvantage of having two or more factors is that it requires you to ask traders things like: “If you tell me the asset’s volatility, the volatility of volatility, the correlation of volatility with return, and the mean reversion coefficient, I can tell you what it’s worth.”

## Summary

- States are examples of possible future events.
- Pricing states is like pricing fruit or zeros.
- Assets are claims to uncertain cash flows: the cash flows, we say, are contingent on the state.
- Quantitative asset pricing consists of concocting a useful list of states and deducing (somehow) the prices of payments in each of them.
- We developed a theoretical framework that identified states with the short rate and priced state-contingent claims “recursively” using the “fifty-fifty rule.”
- Given a model, we can value an enormous range of fixed income derivatives — even those with mind-boggling complexity — with the same tools.
- Models require both sensible structure and sensible parameters.
- Models are not reality. Connecting the two is as much art as science.