

State-Contingent Claims

0. Overview

- Fruit
- States
- State-Contingent Claims
- Examples
- Models: Their Uses and Misuses

1. Fruit

- Why? To show you how simple this is.

- Example 1

- Basket 1: 25 apples, 100 bananas, \$150
- Basket 2: 50 apples, 50 bananas, \$150
- Guess: apples worth \$2, bananas \$1
- Math: solve

$$150 = q_a \times 25 + q_b \times 100$$

$$150 = q_a \times 50 + q_b \times 50$$

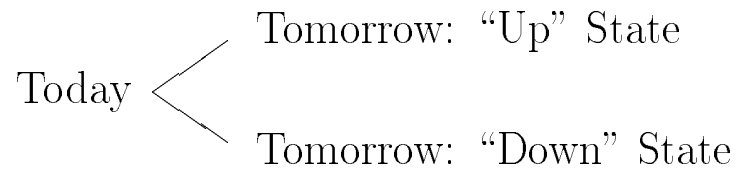
- Example 2

- Basket 1: 50 apples, 50 bananas, \$100
- Basket 2: 25 apples, 25 bananas, \$80
- Analysis: buy basket 1, short 2 basket 2's, pocket profit of $2 \times 80 - 100 = 60$ (arbitrage!)
- What are apples and bananas worth? Not clear.

- **Proposition.** If (and only if) there are no arbitrage opportunities, we can derive prices of individual fruits consistent with the prices of baskets.

2. States

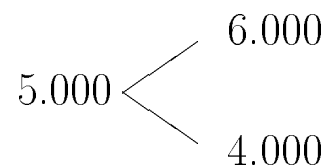
- *States* are situations or scenarios
- Uncertainty: several situations or states are possible
- Generic representation over two periods:



- Examples of states

Up State	Down State
Knicks Win	Knicks Lose
Yields Rise	Yields Fall
GM Defaults	GM Does Not Default

- Short rate tree:

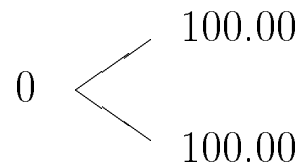


(Short rate $r = y_1$: 6-month LIBOR or Treasury yield)

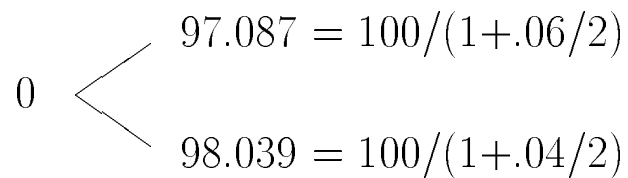
3. Two-Period State-Contingent Claims

- Assets are like fruit baskets
- *State-contingent claims*: cash flows depend on the state
- *Assets* are collections of state-contingent cash flows

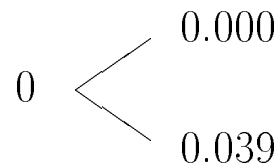
- Example 1: One-period zero



- Example 2: Two-period zero



- Example 3: Option with strike = 98



3. Two-Period State-Contingent Claims (cont'd)

- We need:
 - List of states: s , say
 - Cash flows associated with states: c_s
 - State prices: q_s

- Asset prices:

$$\begin{aligned}\text{Asset Price} &= \sum_s q_s c_s \\ &= q_u c_u + q_d c_d\end{aligned}$$

(Just like fruit: sum value of components)

- **Proposition.** If (and only if) there are no arbitrage opportunities, we can derive state prices consistent with the prices of assets.

3. Two-Period State-Contingent Claims (cont'd)

- If prices of zero and option are 97.561 and 0.019, then

$$97.561 = q_u \times 100 + q_d \times 100$$

$$0.019 = q_d \times 0.039$$

Solution: $q_u = q_d = 0.4878$ (to greater accuracy)

- Standard approach is a little different:

– State prices sum to one-period discount factor:

$$d_1 = q_u \times 1 + q_d \times 1$$

$$= q_u + q_d$$

– Split discount factor in half (fifty-fifty rule):

$$q_u = q_d = 0.5 \times d_1 = \frac{0.5}{1 + r/2}$$

– In our examples, $r = 0.05$ and

$$q_u = q_d = 0.5/(1 + .05/2) = 0.4878$$

- Example 1 (one-period zero):

$$\text{Price} = q_u c_u + q_d c_d$$

$$= 0.4878 \times 100 + 0.4878 \times 100 = 97.561$$

3. Two-Period State-Contingent Claims (cont'd)

- Example 2 (two-period zero):

$$\begin{aligned}\text{Price} &= q_u c_u + q_d c_d \\ &= 0.4878 \times 97.087 + 0.4878 \times 98.039 = 95.184\end{aligned}$$

- Example 3 (option):

$$\begin{aligned}\text{Price} &= q_u c_u + q_d c_d \\ &= 0.4878 \times 0 + 0.4878 \times 0.039 = 0.019\end{aligned}$$

- Summary of theory:
 - State prices: fifty-fifty rule
 - Asset prices: sum value of contingent cash flows

4. Other Versions of State Prices (optional)

- State prices come in other forms (but do the same thing)
- *Pricing kernel*
 - Defined by m in

$$q_s = \pi_s m_s$$

where π_s is the probability that state s occurs

- Probabilities down-weight value of unusual events
- Asset pricing follows

$$\begin{aligned}\text{Asset Price} &= \sum_s q_s c_s \\ &= E(mc)\end{aligned}$$

- Discount factor: $d_1 = E(m)$ (set $c_s = 1$ for all s)
- Kernel m captures effects of risk:
 - * Constant kernel $m = d_1$ (why?) means

$$\text{Asset Price} = d_1 E(c),$$

which applies if investors are neutral to risk.

- * In general covariance with cash flows governs price:

$$\text{Asset Price} = d_1 E(c) + \text{Cov}(m, c)$$

(m is like the market return in the CAPM)

4. Other Versions of State Prices (continued)

- *Risk-neutral probabilities*

- Defined by

$$\pi_s^* = \frac{q_s}{\sum_{s'} q_{s'}} = \frac{q_s}{d_1} = \frac{\pi_s m_s}{d_1}$$

- Note that $\{\pi_s^*\}$ are positive if state prices are, and sum to one: this is the sense in which they're probabilities
- Asset pricing follows

$$\text{Asset Price} = d_1 \sum_s \pi_s^* c_s = d_1 E^*(c)$$

where E^* means the expectation using probabilities π_s^*

- Risk-neutral is a misnomer both ways:
 - * Effects of risk are built in
 - * Probabilities only in the technical sense

- All three versions are equivalent:

- State prices
- Pricing kernel
- Risk-neutral probabilities

5. Multi-Period Contingent Claims

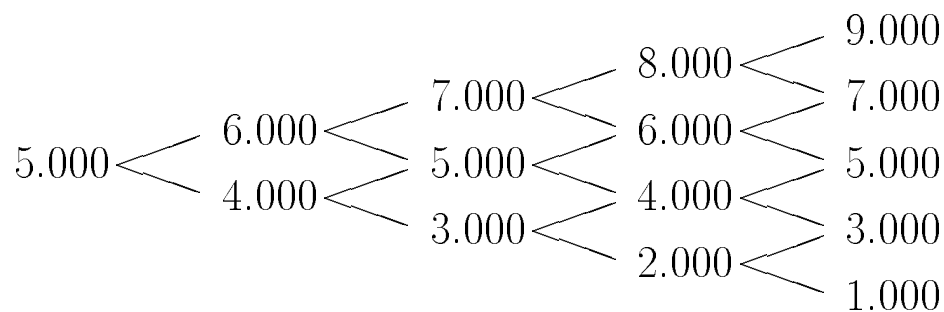
- Changes in short rate $r = y_1$ (Ho and Lee version):

$$r_{t+1} = r_t + \mu_{t+1} + \varepsilon_{t+1}$$

with

$$\varepsilon_{t+1} = \begin{cases} +\sigma & \text{with probability one-half} \\ -\sigma & \text{with probability one-half} \end{cases}$$

- Parameters
 - μ_{t+1} is expected change in r
 - σ is standard deviation
- Short rate tree ($r_0 = 5\%$, $\mu_{t+j} = 0$, $\sigma = 1\%$):



- Paths for the short rate:
 - *Path A* (*up, down, up, down*). The short rates are 5, 6, 5, 6, and 5.
 - *Path B* (*down, down, down, up*). The short rates are 5, 4, 3, 2, and 3.

5. Multi-Period Contingent Claims (continued)

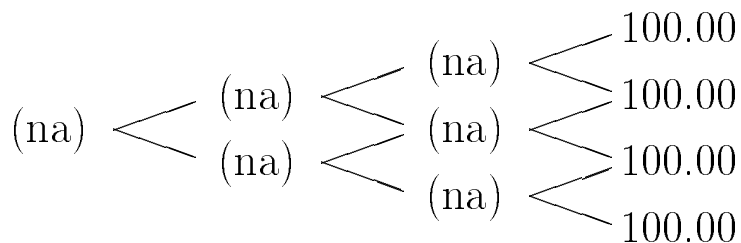
- Approach: treat each “node” as a two-period tree
- Apply these equations at each node, starting at the end:

$$\text{Asset Price} = \text{Current Cash Flow} + q_u c_u + q_d c_d \quad (1)$$

$$q_u = q_d = 0.5 \times d_1 = \frac{0.5}{1 + r/2} \quad (2)$$

(“Current cash flow” is a new subtlety that we’ll deal with when it arises, but ignore otherwise)

- Example 1: 3-period zero
 - At maturity, bond has cash flow of 100 in all states:

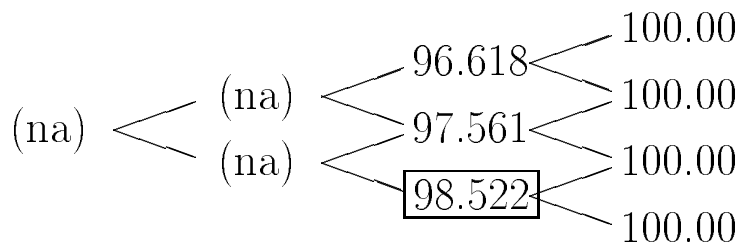


(This is the easy part: we know the value at maturity)

5. Multi-Period Contingent Claims (continued)

- Example 1: 3-period zero (continued)

- In the previous period, prices are



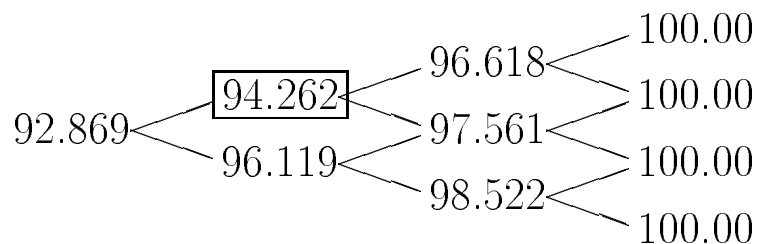
Details for “boxed” node (short rate 3%):

- * State prices are $q_u = q_d = 0.5/(1 + .03/2) = 0.4926$

- * Zero’s price is

$$\text{Price} = 0.4926 \times 100 + 0.4926 \times 100 = 98.522$$

- The rest of the tree is



Details for “boxed” node (short rate 6%):

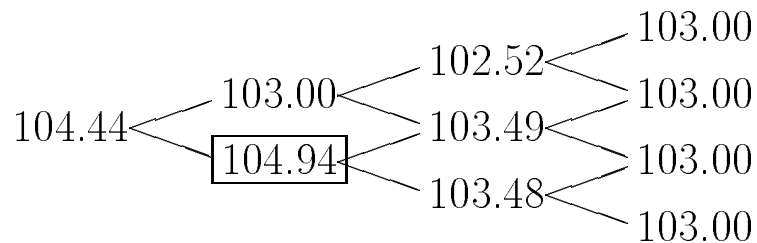
- * State prices are $q_u = q_d = 0.5/(1 + .06/2) = 0.4854$

- * Zero’s price is

$$\text{Price} = 0.4854 \times 96.618 + 0.4854 \times 97.561 = 94.262$$

5. Multi-Period Contingent Claims (continued)

- Example 2: 3-period 6% bond
 - Differs in having cash flows at each date
 - The complete price tree is



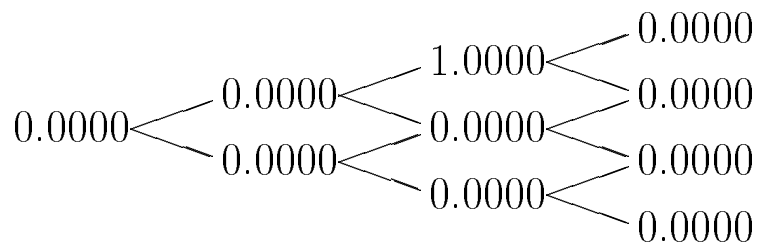
- Details for “boxed” node (short rate 4%):
 - * State prices are $q_u = q_d = 0.5/(1 + .04/2) = 0.4902$
 - * Price includes “current cash flow” of 3:

$$\begin{aligned} \text{Price} &= 3 + 0.4902 \times 103.49 + 0.4902 \times 104.48 \\ &= 104.94 \end{aligned}$$
 - * Since the price includes the current cash flow, we should think of it as including the accrued interest.
- Other nodes follow the same method: starting at the end of the tree, apply pricing relation (1) and fifty-fifty rule (2) to compute preceding nodes.

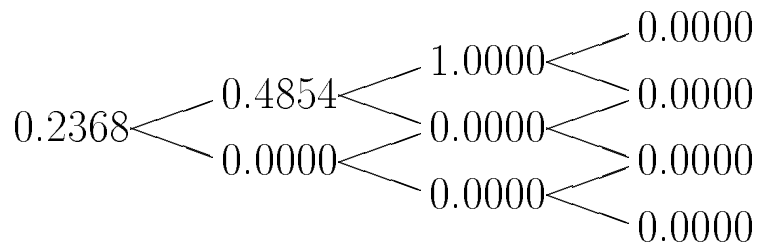
5. Multi-Period Contingent Claims (continued)

- Example 3: pure contingent claim

- Cash flow tree



- Complete price tree is

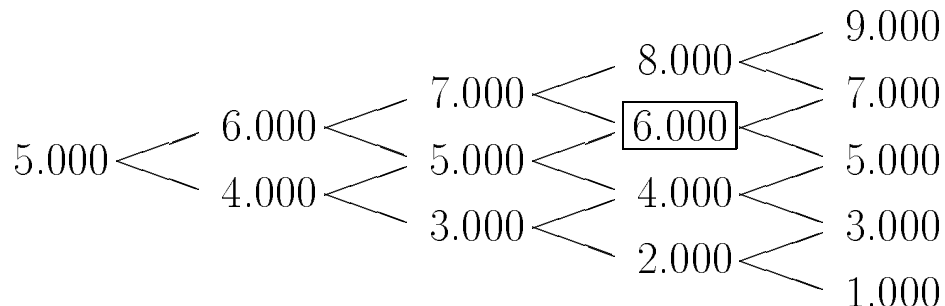


- Meaning: claim to one dollar if “up-up” occurs is worth \$0.2368 now. (Remember this number for later.)

- Prices of such claims analogous to discount factors: they summarize value across time and states.

6. Duffie's Formula

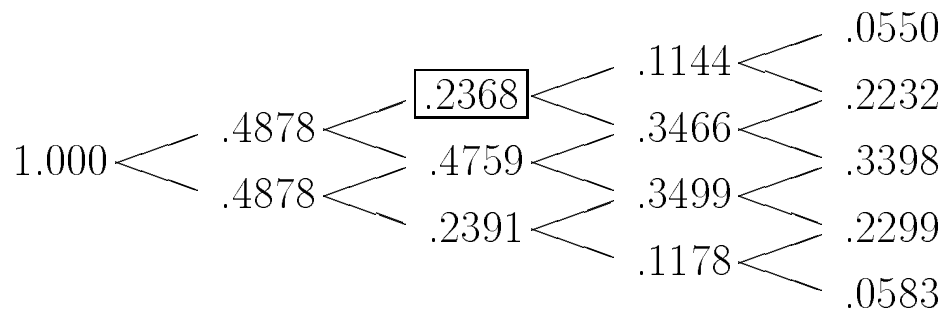
- Objective: compute spot rates associated with rate tree
- Solutions:
 - Hard way: compute prices of all relevant zeros
 - Easy way: use Duffie's formula to compute state prices
 - Warning: not hard to use, but takes work to get there
- Preliminaries:
 - Sample interest rate tree:



- Labelling system for states: i = number of up moves, n = number of periods since the start, (i, n) defines the state (the node)
- Examples:
 - $(i, n) = (0, 0)$ is the initial node, the starting point
 - $(i, n) = (2, 3)$ is the node with the box

6. Duffie's Formula (continued)

- (Multiperiod) state prices
 - $q(i, n)$ is the price now of one dollar in state (i, n)
 - Note the difference between $q(i, n)$ and (q_u, q_d) : The latter (2-period state prices) pertains to a node and the two branches out of it. The former (multiperiod state prices) to any node in the tree relative to the start.
 - We can list the complete set of $q(i, n)$ in a tree:



- The entry in the box was computed earlier
- Discount factors. A claim to one dollar in all states in n periods is worth the sum of the state prices for that date:

$$d_n = \sum_{i=0}^n q(i, n) \times 1$$

- Spot rates are

$$d_n = \frac{1}{(1 + y_n/2)^n} \Rightarrow y_n = 2(1/d_n^{1/n} - 1)$$

6. Duffie's Formula (continued)

- Duffie's formula is a short cut for computing state prices:

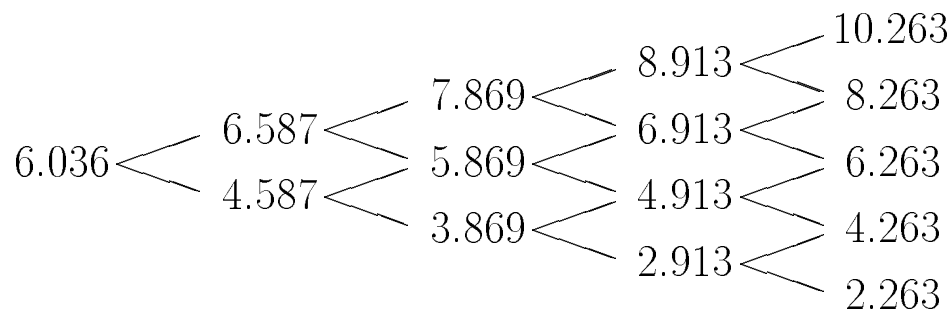
$$q(i, n+1) = \begin{cases} \frac{0.5q(i, n)}{1 + r(i, n)/2} & \text{if } i = 0, n+1 \\ \frac{0.5q(i, n)}{1 + r(i, n)/2} + \frac{0.5q(i-1, n)}{1 + r(i-1, n)/2} & \text{if } 0 < i < n+1 \end{cases}$$

- Remarks:
 - This is a lot less work than doing each state-contingent claim on its own.
 - The formula differs between the edges (where only one node leads in) and the interior (two nodes lead in).
 - Don't worry about the details.
- Discount factors and spot rates for our example:

Maturity n	1	2	3	4
Discount Factor d_n	0.9756	0.9518	0.9287	0.9062
Spot Rate y_n (%)	5.00	5.00	4.99	4.97

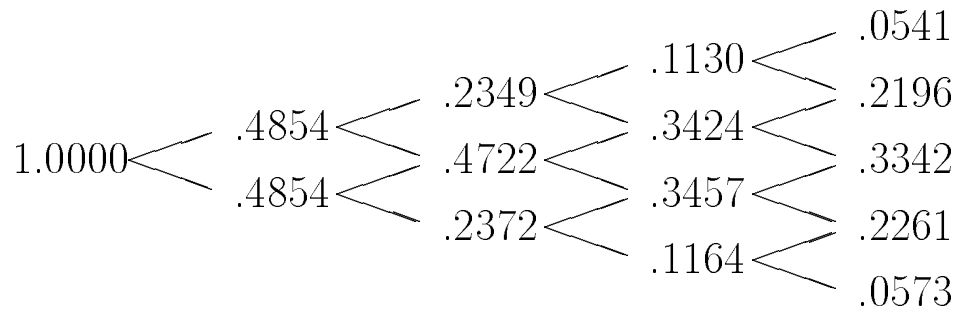
7. Choosing Parameters

- Volatility σ
 - Estimate from recent data (1% is a ballpark number for a six-month period)
 - Infer from options (interest rate caps, options on eu-rodollar futures, swaptions)
- Mean or “drift” parameters μ_t :
 - Choose to reproduce current spot rates – exactly!
 - Remark: Absolutely essential. If the model prices zeros incorrectly, why would we believe it for derivatives?
- Example: spot rates of (6.036, 5.809, 5.824, 5.839, 5.914) (June 1995 LIBOR-based rates) and $\sigma = 1$ (all percents)
 - Drift parameters are $(-0.45, 0.28, 0.04, 0.35)$ (percent)
 - Rate tree is



7. Choosing Parameters (continued)

- State price tree:



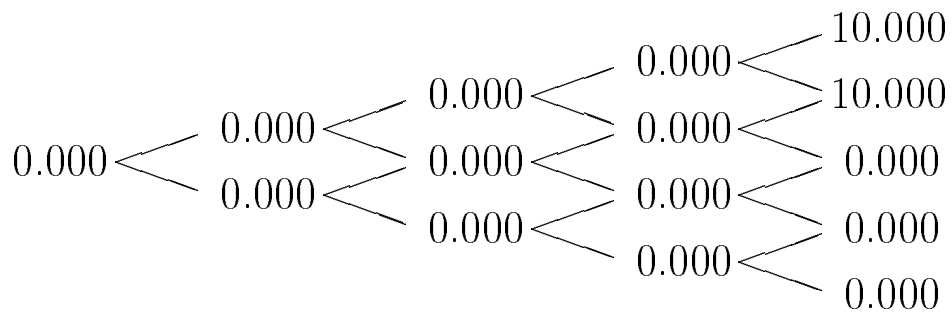
- Summary:
 - Base σ on history or options
 - Choose $\{\mu_{t+j}\}$ to match spot rates
 - Duffie's formula makes this easier

8. Examples

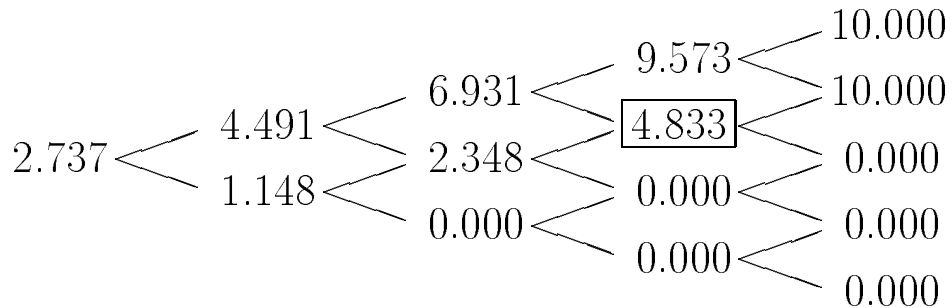
- Example 1: Bet on short rate

Get \$10 if 6-m LIBOR > 7% in 2 years, zero otherwise

– Cash flow tree:



– Find price with eqs (1,2), starting at end:



Details for state (2,3) (box, short rate 6.913%):

* State prices are $q_u = q_d = 0.4833$

* Price is

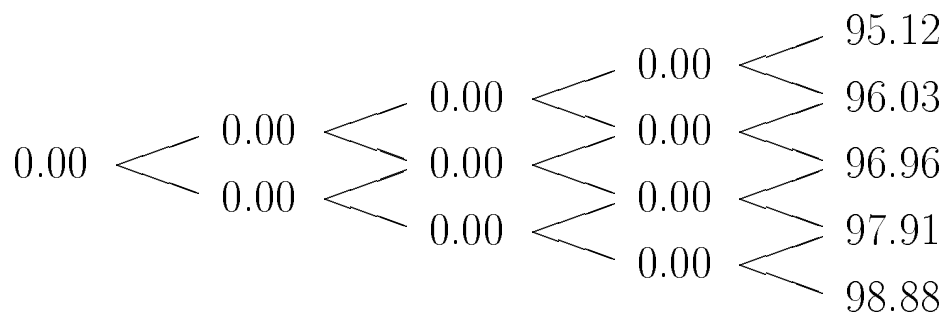
$$4.833 = 0.4833 \times 10 + 0.4833 \times 0$$

– Short cut: compute price from (multiperiod) state prices

$$2.737 = 0.0541 \times 10 + 0.2196 \times 10$$

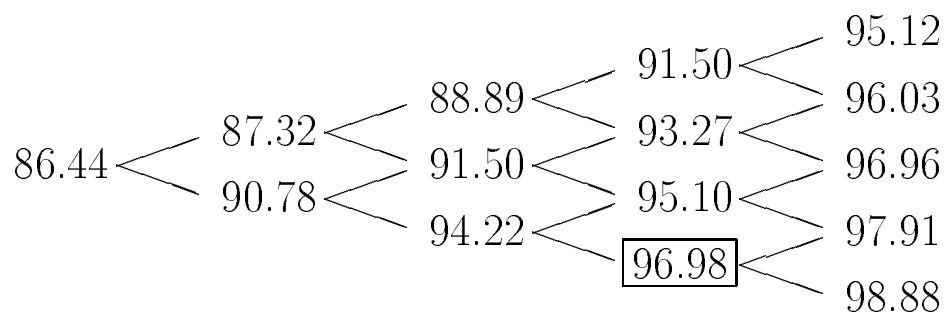
8. Examples (continued)

- Example 2: 5-period (30-month) zero
Pays \$100 in all states fives periods from now.
- “Cash flows” in four periods:



Compute by discounting 100 back one period:
eg, $98.88 = 100 / (1 + .02263/2)$

- Price tree:



Details for state (0,3) (box, short rate 2.913%):

$$q_u = q_d = 0.5 / (1 + .02913/2) = 0.4928$$

$$\text{Price} = 0.4928 \times 97.91 + 0.4928 \times 98.88 = 96.98$$

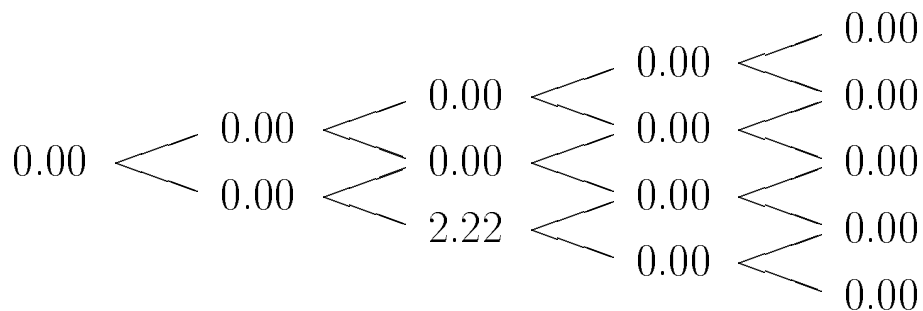
- Remark: Tree gives us scenarios for future prices.

8. Examples (continued)

- Example 3: Option on the zero

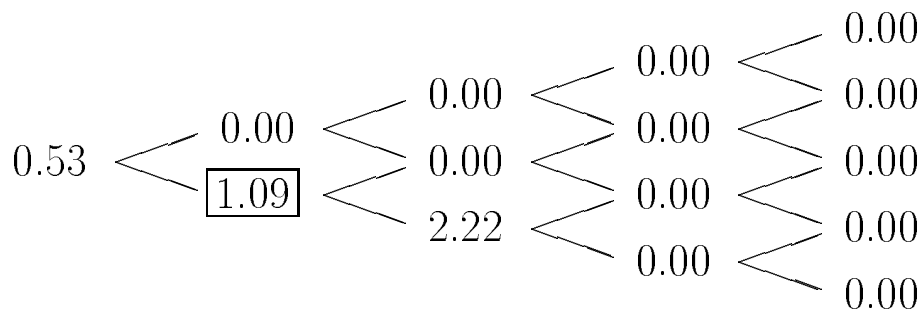
Option to buy zero for 92 in two periods (european call)

– Cash flow tree:



Remark: $2.22 = 94.22 - 92$.

– Price tree:



Details for state (0,1) (box, short rate 4.587%):

$$q_u = q_d = 0.5 / (1 + .04587/2) = 0.4888$$

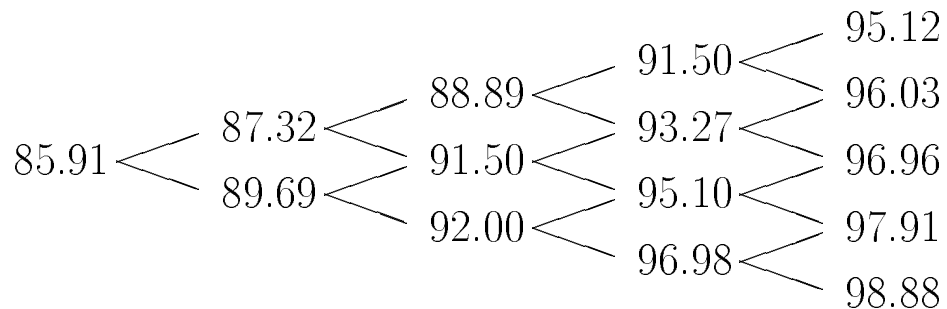
$$\text{Price} = 0.4888 \times 0.00 + 0.4888 \times 2.22 = 1.09$$

8. Examples (continued)

- Example 4: Replication of a callable zero

Callable zero: long the zero, short the option

- Price tree (difference between two previous trees):



Remark: $85.91 = 86.44 - 0.53$.

- Replication basics

- * We can replicate the cash flows of an asset with those of a combination of two other assets (two because each node has two branches stemming from it).
- * Replication is dynamic: it's different at each node in the tree.

8. Examples (continued)

- Example 4: Callable zero (continued)
 - Replicate cash flows of the callable zero with underlying zero (l for long) and one-period zero (s for short).
 - Compute quantities x_l of the long and x_s of the short that generate the same cash flows as the callable zero in the up and down states from a particular node:

$$\text{Callable}_u = x_l \times \text{Long}_u + x_s \times \text{Short}_u$$

$$\text{Callable}_d = x_l \times \text{Long}_d + x_s \times \text{Short}_d$$

- For the initial node the numbers are

$$87.32 = x_l \times 87.32 + x_s \times 100$$

$$89.69 = x_l \times 90.78 + x_s \times 100$$

Solution: $x_l = 0.69$, $x_s = 0.27$.

- For node $(0, 1)$ the numbers are

$$91.50 = x_l \times 91.50 + x_s \times 100$$

$$92.00 = x_l \times 94.22 + x_s \times 100$$

Solution: $x_l = 0.18$, $x_s = 0.75$ (call dominates here).

- For node $(1, 1)$ the numbers are

$$88.89 = x_l \times 88.89 + x_s \times 100$$

$$91.50 = x_l \times 91.50 + x_s \times 100$$

Solution: $x_l = 1.00$, $x_s = 0.00$ (call irrelevant here).

8. Examples (continued)

- Example 4: Callable zero (continued)
 - Interest sensitivity
 - * Callable has shorter duration than underlying: replication includes short zero.
 - * Duration varies with interest rates: compare the replicating strategies for nodes (0,1) and (1,1).

- Pricing by replication

- * Callable is worth (at initial node)

$$\begin{aligned}\text{Price} &= 0.69 \times \text{Price of Long} + 0.27 \times \text{Price of Short} \\ &= 0.69 \times 86.44 + 0.27 \times 100 / (1 + 0.06036/2) \\ &= 85.91\end{aligned}$$

(answer has greater accuracy than components)

- * We get the same answer as before.
 - * This has to be true: state pricing precludes arbitrage opportunities (see the proposition).
 - * Replication based on the model; the world may be different.

9. Modeling Issues

- Our model is Ho and Lee, the first of its kind. Ho was here when he developed it, and remains an active participant in Stern activities. But it's not the last word.
- Discrete set of states (either up or down).
Response. We can get as much refinement as we like by using a short time interval.

- Interest rates can become negative in our model.
Response. Many people model the log of the short rate, as in

$$\log r_{t+1} = \log r_t + \mu_{t+1} + \varepsilon_{t+1}$$

Once a short rate tree is generated this way, the rest of the analysis is the same (state prices, Duffie's formula).

- Volatility varies over time.
Response. The popular Black-Derman-Toy model allows σ to vary with t , just as μ does in the Ho and Lee model. Other models allow σ to vary randomly as an additional factor.

9. Modeling Issues (continued)

- Mean reversion. Our model (and the one above) exhibits no tendency for interest rates to return to some typical value. Long trees therefore exhibit a wide range of possible interest rates, which many of us consider unreasonable.
Response. Models with mean-reversion are easily and commonly built, less easily put into a tree structure.
- Multiple factors. We argued earlier that the behavior of spot rates requires more than one factor: the spot curve shifts up and down, twists, etc.
Response. Models with multiple factors. Easier said than done.
- Complexity. Bells and whistles are motivated by realism (the world is more complex than our model), but too much makes a model hard to understand and use. Emanuel Derman, physicist turned Goldman exec, writes:

Go as far as you can with one factor. ... The disadvantage of having two or more factors is that it requires you to ask traders things like: “If you tell me the asset’s volatility, the volatility of volatility, the correlation of volatility with return, and the mean reversion coefficient, I can tell you what it’s worth.”

Summary

- States are examples of possible future events.
- Pricing states is like pricing fruit.
- Assets are claims to uncertain cash flows: the cash flows, we say, are contingent on the state.
- Quantitative asset pricing consists of concocting a useful list of states and deducing (somehow) the prices of payments in each of them.
- We developed a theoretical framework that identified states with the short rate, and priced state-contingent claims using the “fifty-fifty rule.” The results are summarized in Duffie’s formula.
- Models require both sensible structure and sensible parameters.
- Models are not reality. Connecting the two is as much art as science.