PORTFOLIO THEORY WHEN INVESTMENT RELATIVES ARE LOGNORMALLY DISTRIBUTED

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PORTFOLIO THEORY as first presented by Markowitz [10] and Tobin [16] has been shown to be correct under either of two assumptions:[1]

(1) returns are normally distributed,

(2) the investor's utility function is quadratic.

The assumption of a quadratic utility function has been questioned by a number of authors because of its implications of decreasing absolute risk aversion.[2] On the other hand, the assumption of normally distributed returns has been questioned because return distributions must be bounded from below and because empirical evidence suggests that returns are not normally distributed.[3] The current controversy over the probability distribution of returns revolves around whether one plus return is lognormally distributed or follows a stable paretian distribution.[4] Under either of these assumptions, combined with the unrealistic implications of the quadratic utility function standard portfolio theory is inapplicable.[5]

However, if investment relatives are distributed either lognormally or distributed according to a stable paretian distribution, then portfolio theory can be reformulated. Fama [4] [5] has gone a long way towards a reformulation under the assumption of stable paretian distributions. In this paper, we propose to reformulate portfolio theory under the assumption that investment relatives are lognormally distributed. This reformulation is extremely important given both the multitude of areas within economics where portfolio models have found applications and the increasing acceptance of the lognormality of price relatives in the economic literature.

This paper is divided into three sections. The first derives an efficient set theorem (analogous to the original Markowitz theorem) when portfolios are located in log space. The second section provides an algorithm for calculating such an efficient frontier. Finally, the last section proves that if utility func-

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1. See Feldstein [7] and Tobin [16].

2. See Pratt [15], and Mossin [13].

3. See [6].

4. One plus return is usually defined as the investment relative assuming re-investment of dividends. We shall refer to this as an investment relative throughout the rest of this paper. For a discussion of this controversy, See [6], [8], [9], [11] and [14].

5. For example, if price relatives follow a non-normal stable paretian distribution standard, deviations are not a meaningful measure of risk (variances are infinite). On the other hand, as Feldstein [7] has shown, if price relatives are lognormally distributed, then utility functions are not necessarily convex from below in arithmetic mean variance space, and a unique maximum need not exist.

tions exhibit decreasing marginal utility of wealth, then utility curves are convex from below in log space.

I. The Efficient Set Theorem

The infinite set of possible portfolios is represented diagrammatically in Figure 1. Tobin [16] has shown that if returns are normally distributed and

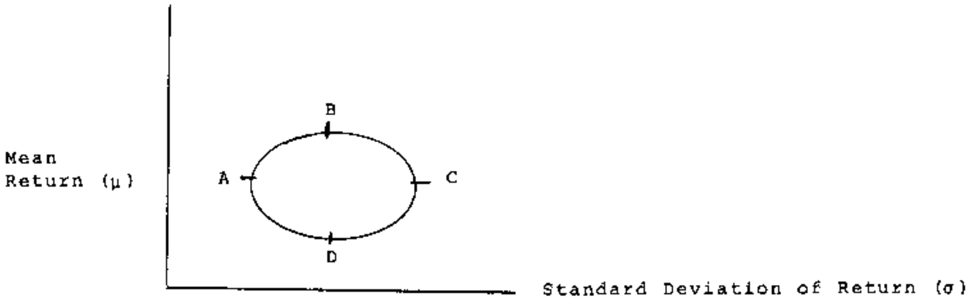

Figure 1

investors exhibit positive marginal utility of wealth, the only portfolios which could possibly maximize the one period utility of wealth lie on the upper left hand border AB.[6] This subset of portfolios is characterized by portfolios that:

1. Maximize expected return for any level of standard deviation of return.
2. Minimize standard deviation of return for any level of expected return.

These characteristics are called the efficient set theorem.

As discussed above, theory and empirical evidence suggest that investment relatives are more likely to be lognormally distributed than normally distributed. Consequently, an efficient set theorem, when investment relatives are lognormally distributed, is highly desirable.[7] In this section we will prove that when investment relatives are lognormally distributed, there is an analogous efficient set theorem which can be derived in mean log investment relative (m), standard deviation of log investment relative (s) space. In order to derive this efficient set theorem, it is only necessary to assume that utility functions exhibit positive marginal utility of wealth. With this assumption, it can be shown that the optimum portfolio lies on the boundary of all portfolios examined in mean log investment relative, standard deviation of log investment relative space. Or that (see Figure 2) there is always a portfolio somewhere on the curve A B C D which is preferred to an interior point. In this section, we shall first prove that some boundary point always maximizes expected

6. Other assumptions can be used to derive the efficient set theorem. They involve restricting the utility function so that it is concave and increasing to the left, (e.g., Fama and Miller [4] pp. 224).

7. Empirical evidence that the investment relatives are lognormally distributed for portfolios is provided by Kendall [8], Moore [12] and Osborne [14] in their examination of the distribution of price changes of indices. These same authors have also provided evidence that investment relatives for securities are lognormally distributed. Alternative sets of theoretical conditions under which lognormally distributed portfolio returns are consistent with lognormally distributed security returns are set forth by Merton [11] and Aitchison and Brown [1] page 110.

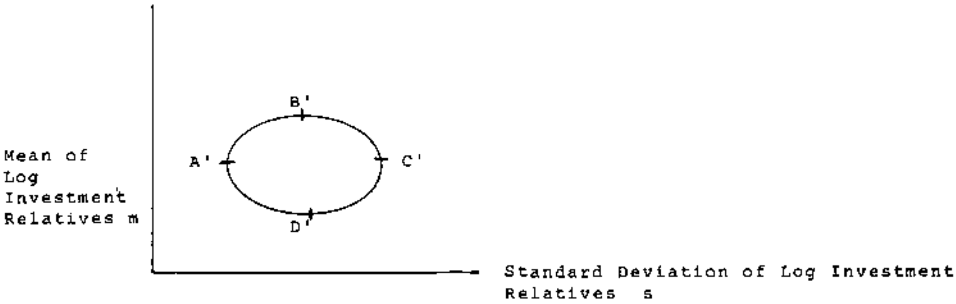

FIGURE 2

utility and then we shall determine that portion of the boundary curve A B C D which dominates all other portfolios.

If the investment relative is lognormally distributed, m is the mean of log r and s is the standard deviation of log r, then $z = \frac{\log r - m}{s}$ is normally distributed with mean of zero and standard deviation of one. The frequency function of z is well known and stationary in z, f(z) space.[8]

In other words, if we standardize the log of a lognormal variable by subtracting m and dividing by s, the standardized variable z has a known distribution that is only dependent on the value of the standardized variable itself. Having derived a standardized variate, it is now possible to prove an efficient set theorem. Let:

1. $U(w_1)$ represent a one period utility function for terminal wealth.
2. E[ ] be the expected value operator.
3. w_t be wealth at t.

Since r is one plus the return on the investment portfolio, w_1 can be written as $w_0 r$.

If we let $z = \frac{\log r - m}{s}$, then $\log r = m + sz$ and w_1 is equal to[9]

$$w_1 = w_0 r = w_0 e^{m+sz}. \tag{1}$$

Expected utility of wealth can be expressed as:

$$E[U(w_1)] = \int_{-\infty}^{+\infty} U(w_0 e^{m+sz})\, f(z) dz. \tag{2}$$

Note once again that the frequency function for the standardized variate only depends on z. Therefore, differences between the expected utilities of various portfolios are only a function of m and s, or formally $E[U(w_1)] = f(m, s)$.

8. This statement might at first seem contrary to Feldstein [7]. Feldstein's point is that if z were set equal to the investment relative minus its mean divided by its standard deviation $\frac{r - \mu}{\sigma}$, the distribution of z would not be the same for all lognormal distributions. But by working in log space and defining z in terms of log returns, we have defined a variable which is the same for all lognormal distributions.

9. w_1 is lognormally distributed since w_0 is a known constant and r is lognormally distributed. The product of a constant and a lognormal variate is a new variable which is lognormally distributed.

Using the chain rule to take the derivative of (2) with respect to m for a fixed value of s yields

$$\frac{\partial E[U(w_1)]}{\partial m} = w_0 \int_{-\infty}^{+\infty} \frac{\partial U(w_1)}{\partial w_1} e^{m+sz} f(z)\, dz.$$

This integral is positive for all values of m: since e to any power is positive, $\frac{\partial U(w_1)}{\partial w_1}$ is positive since investors are assumed to prefer more wealth to less, and f(z) can only take on positive values.

A positive integral means that expected utility is maximized if m is maximixed. We have completed our derivation of the analogy to the first part of the efficient set theorem, namely: maximize the expected value of the log of relative price (m) for any level of standard deviation of log price (s).

Taking the derivative of (2) with respect to s for a fixed value of m yields:

$$\frac{\partial E[U(w_1)]}{\partial s} \int_{-\infty}^{+\infty} \frac{\partial U(w_1)w_1}{\partial w_1} z f(z)\, dz. \tag{3}$$

Since f(z) is symmetric about zero, (3) can be positive, negative or zero depending on whether $\frac{\partial U(w_1)w_1}{\partial w_1}$ increases, decreases or is constant as wealth changes.

Consider Figure 2. The derivative of expected utility with respect to m is positive; therefore, only the points on the border A' B' C' can be efficient. The derivative of expected utility with respect to s has the same sign as the derivative of $\frac{\partial U(w_1)w_1}{\partial w_1}$. If the derivative of $\frac{\partial U(w_1)w_1}{\partial w_1}$ is negative, then A' B' is the efficient set; if it is zero, then B' is the efficient *point*; and if it is positive, then B' C' is the efficient set.[10] This completes the derivation of an efficient set theorem.

Before leaving this section it is worthwhile to examine briefly the types of utility functions for which $\frac{\partial U(w_1)w_1}{\partial w_1}$ is increasing, constant or decreasing. Taking the derivative of this expression yields:

$$\frac{\partial U(w_1)}{\partial w_1} + w_1 \frac{\partial^2 U(w_1)}{\partial^2 w_1} \gtreqless 0.$$

Rearranging yields:

$$\frac{-w_1 U''(w_1)}{U(w_1)} \lesseqgtr 1.$$

The above is the function for relative risk aversion as formulated by Pratt [15] and Arrow [2]. For utility functions which exhibit constant relative risk aversion, the relationship of the above to one is constant. If an investor

10. An algorithm for determining the composition of portfolio B is contained in Elton and Gruber [3].

has a utility function which does not exhibit constant relative risk aversion, the relative risk aversion function might pass through one. This implies that for these functions an optimum can occur on different sections of the efficient frontier depending on the investor's wealth level.

II. Computing an Efficient Frontier

In the previous section we showed that utility is maximized if m is made as large as possible for any value of s. It is well known that a one-to-one mapping exists between points in m, s space, and points in μ, σ space.[11] The relationships are

$$\mu = e^{m + 1/2\, s^2}, \text{ and} \tag{4}$$

$$\sigma^2 = (e^{m + 1/2\, s^2})^2 \, (e^{s^2} - 1). \tag{5}$$

The loci of points of constant s can be plotted in μ, σ space by examining the equation

$$s = \left[\ln \left(\frac{\sigma^2}{\mu^2} + 1 \right) \right]^{1/2} = c \ \text{(a constant)}$$

for different values of c. Rearranging yields

$$\mu = \frac{1}{[e^{c^2} - 1]^{1/2}} \sigma$$

which is the equation of a family of straight lines emanating from the origin in μ, σ space (see Figure 3). The preferred portfolio on a ray of constant s

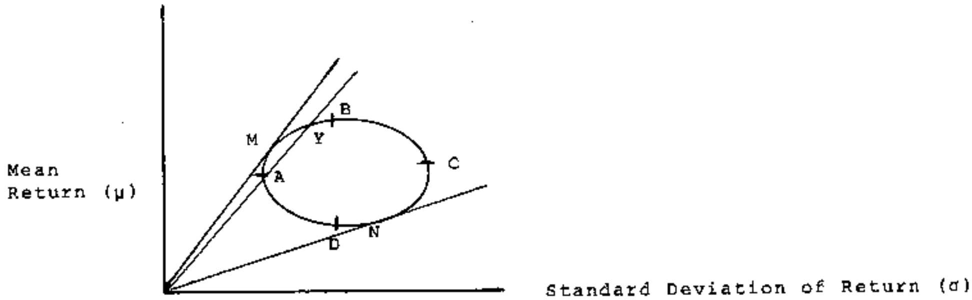

FIGURE 3

is the one that maximized m.

$$m = \log \mu - 1/2\, s^2.$$

If s is constant, then m is made as large as possible by maximizing μ. Thus, μ is maximized by moving as far from the origin as possible along a ray of constant s. Examine the ray O A Y. Portfolio Y dominates portfolio A and all other portfolios along the line O A Y, for although it has the same value of s, it has a higher value of μ and thus a higher value of m. Similarly, the efficient set in μ, σ space can be defined by examining all rays emanating from

11. See (1) for the derivation of this relationship.

the origin, and for each ray selecting that feasible portfolio which lies furthest from the origin. Thus in Figure 3 the efficient set lies along the boundary curve M B C N. The points M and N are the two tangency points of the rays of constant s and the boundary curve of all feasible portfolios. The point M can be alternatively defined as that portfolio with the maximum attainable ratio of μ to σ. The point N can be alternatively defined as that portfolio with the lowest attainable ratio of μ to σ. The efficient set in μ, σ space has now been defined as M B C N. Note that the efficient set in μ, σ space, when returns are lognormally distributed, is considerably different from the efficient set when returns are normally distributed. The efficient set with lognormal returns may not contain some portfolios which are efficient when returns are normally distributed (the region A M) but it contains additional portfolios in the region B C N. Since all of the portfolios which are efficient in m, s space lie on a boundary in μ, σ space, the efficient frontier in m, s space can be constructed by first constructing the boundary in μ, σ space.

The boundary in μ, σ space can be calculated by solving three quadratic programming problems.[12]

1. Maximize $\mu - \lambda\sigma$.
2. Maximize $\mu + \lambda\sigma$.
3. Maximize $-\mu + \lambda\sigma$.

The efficient set can then be found by employing the tangency condition discussed above to delete part of the boundary.

III. The Convexity of Utility Curves

In this section we will prove that if investors exhibit decreasing marginal utility of wealth, utility functions must be convex from below in expected log investment relative (m), standard deviation of log investment relative (s) space.

Figure 4 represents a hypothetical indifference map in m, s space. While we do not as yet know that utility functions are convex from below, we do

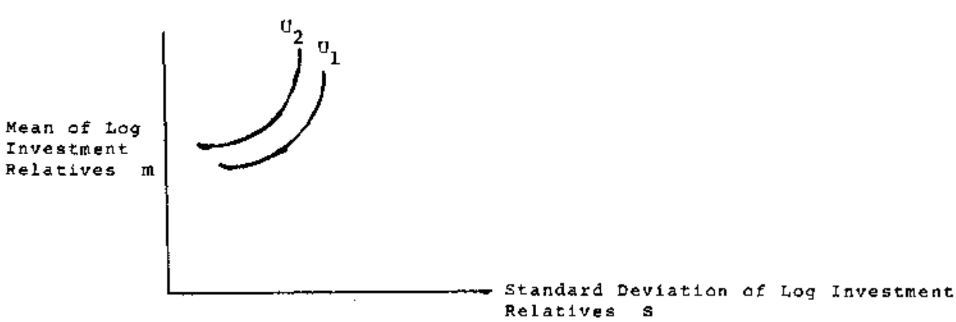

Figure 4

12. We have resorted to this algorithm for constructing the efficient set since there is no algorithm (short of complete enumeration) for directly constructing the efficient set in log space. For example, substituting the log r_i for r_i and variance of log r_i for variance r_i for each security and solving the quadratic programming problem will not lead to optimum portfolios since the $\log r_i + \log r_j \neq \log (r_i + r_j)$.

know from a previous section (see Section I), that utility increases as we move up in m, s space. For example, points along U_2 represent a higher level of utility than points along U_1.[13] Let us select two points on U_1 and call them (m, s), (m′, s′). If we let m″ s″ represent a portfolio on the line connecting m′ s′ and m, s, then the portfolio will have coordinates:

$$[m'', s''] = [xm + (1 - x)m', xs + (1 - x)s'] \; 0 < x < 1. \tag{6}$$

If the utility of this combined portfolio is greater than U_1, then utility functions are convex from below.

Let w_{1i} be a particular outcome of w_1, the terminal wealth from investing in the portfolio with an expected log investment relative and standard deviation of log investment relative given by (m, s). Then from equation (1):

$$w_{1i} = w_0 e^{m + sz_i} \tag{7}$$

Similarly:

$$w'_{1i} = w_0 e^{m' + s' z_i} \tag{8}$$

Define w''_{1i} as shown in equation (9)

$$\log w''_{1i} = x \log w_{1i} + (1 - x) \log w'_{1i} \tag{9}$$

As will be apparent shortly, this definition of w''_{1i} yields a point on the line connecting m s and m′ s′. Substituting equation (7) and (8) into (9) and employing equation (6) yields:

$$w''_{1i} = w_0 e^{m'' + s'' z_i} \tag{10}$$

This is a point on the line connecting m s and m′ s′. Subtracting $\log w_0$ from both sides of equation (9) and noting that $(x) + (1 - x) = 1$, we have:

$$\log w''_{1i} - \log w_0 = x \log w_{1i} - x \log w_0 + (1 - x) \log w'_{1i} - (1 - x) \log w_0.$$

This simplifies to

$$\log \frac{w''_{1i}}{w_0} = x \log \frac{w_{1i}}{w_0} + (1 - x) \log \frac{w'_{1i}}{w_0}. \tag{11}$$

From the assumption that investors exhibit decreasing marginal utility of wealth employing equation (11) we know that[14]

$$U\left[\log \frac{w''_{1i}}{w_0}\right] > xU\left[\log \frac{w_{1i}}{w_0}\right] + (1 - x)\, U\left[\log \frac{w'_{1i}}{w_0}\right]. \tag{12}$$

Substituting equations (7), (8) and (10) into equation (12) yields

$$U(m'' + s'' z_i) > x\, U(m + sz_i) + (1 - x)\, U(m' + s' z_i). \tag{13}$$

13. As shown in Section I, the utility could equally well increase as we move up to the right rather than up to the left. The proof of convexity is the same in either case.

14. This statement is true since $\log \frac{w''_1}{w_0}$ is a weighted average of $\log \frac{w_1}{w_0}$ and $\log \frac{w'_1}{w_0}$. We know that for a utility function which involves decreasing marginal utility, the utility of a weighted average will always be greater than the weighted average of the utility of the two numbers being averaged. Note that a proof like Tobins [16] proof holds only when all outcomes are positive in the appropriate space.

Taking the expected value of (13) cannot change the sign of the inequality since the density function of z is non-negative for all values of z. Therefore:

$$E[U(m'' + s''z)] > x\,E[U(m + sz)] + (1 - x)\,E[U(m' + s'z)]. \qquad (14)$$

Since both terms on the right side of the inequality have the same utility (U_1), an average of the two terms must have the same utility U_1.
Therefore,

$$E[U(m'' + s''z)] > U_1$$

or the utility curves are convex from below in m, s space.[15]

IV. Conclusion

In this paper under the assumptions of lognormally distributed investment relatives and positive marginal utility of wealth, we have:

1. derived an efficient set theorem in log investment relative, standard deviation of log investment relative space.
2. presented an algorithm for determining the portfolios in this efficient set.
3. derived the surface in μ, σ space along which the optimum portfolios must lie.
4. demonstrated that utility curves in log investment relative, standard deviation of log investment relative space must be convex from below if investors exhibit decreasing utility of wealth.

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15. We should point out a seeming inconsistency between our proof and Feldstein's [6] proof that utility functions are not necessarily convex from below in mean standard deviation space when returns are lognormally distributed. The basis of Feldstein's proof is that the unit normal variate only exists in mean, standard deviation space when returns are normally distributed. But when returns are lognormally distributed, the unit normal variate does exist in mean log and standard deviation log space. Just as utility functions are convex in mean return standard deviation space when returns are normally distributed, they are convex in mean log return, standard deviation log return space when returns are lognormally distributed.

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