MIMO RADAR: AN IDEA WHOSE TIME HAS COME

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ABSTRACT

It has been recently shown that multiple-input multiple-output (MIMO) antenna systems have the potential to dramatically improve the performance of communication systems over single antenna systems. Unlike beamforming, which presumes a high correlation between signals either transmitted or received by an array, the MIMO concept exploits the independence between signals at the array elements. In conventional radar, target scintillations are regarded as a nuisance parameter that degrades radar performance. The novelty of MIMO radar is that it takes the opposite view, namely, it capitalizes on target scintillations to improve the radar's performance. In this paper, we introduce the MIMO concept for radar. The MIMO radar system under consideration consists of a transmit array with widely-spaced elements such that each views a different aspect of the target. The array at the receiver is a conventional array used for direction finding (DF). The system performance analysis is carried out in terms of the Cramer-Rao bound of the mean-square error in estimating the target direction. It is shown that MIMO radar leads to significant performance improvement in DF accuracy.

I. Introduction

The idea of active direction finding for radar or active sonar is not new (see, for example [1, 2]). In radar or active sonar, a known waveform is transmitted by an omnidirectional antenna, and a target reflects some of the transmitted energy toward an array of sensors that is used to estimate some unknown parameters, e.g., bearing, range, or speed. There are two common approaches for estimating the unknown parameters. In the first approach, high resolution techniques, e.g., MUSIC or maximum likelihood (ML) [3], are used to estimate parameters of the target of interest. In the second approach, the array of sensors is used to steer a beam toward a certain direction in space and look for some energy,

essentially the same way as a conventional radar with a directional antenna. It is well known that an array of receivers can steer a beam toward any direction in space by using a process known as *beamforming* [4]. Unlike high resolution techniques, beamforming is based on a fixed transformation.

The advantages of using an array of closely spaced sensors at the receiver are well known (see, for example [5, 4, 3, 6, 7]). Among these advantages are: the lack of any mechanical elements in the system, the ability to use advanced signal processing techniques for improving performance, and the ability to steer multiple beams at once. In this paper we are concerned with radars employing multiple antennas both at the transmitter and at the receiver.

Transmit arrays have been proposed in the form of electronic steered arrays (ESA). With an ESA, phase shifts at the transmit antennas form and steer the transmit beam similar to a directional antenna, except that the steering is electronic rather than mechanical. Before introducing a new concept for radar with multiple transmit antennas, a fair question to ask is whether an ESA has any processing gain (in addition to its mechanical advantages). The ESA essentially mimics the scanning operation of a directional antenna. However, as we show next, ESA's have no advantage over systems that use a single omnidirectional antenna at the transmitter. To that end, we note that the error in that angle of arrival estimation is a function of the total received energy. Assume that the total transmitted power is independent of the number of transmit antennas. For the single transmit antennas case, say that the average transmitted power is P and the duration of the transmitted waveform is T seconds. The energy received from the target is σPT, where σ represents the target's radio cross section (RCS). Now, assume an ESA that creates a beam with beamwidth ϕ. With the beamwidth ϕ, the transmitter can realize a gain of $\frac{2\pi}{\phi}$ (in linear scale). However, since the transmitter needs to scan the whole space in T seconds, it can illuminate the target for only $\frac{T\phi}{2\pi}$ seconds. The total received energy is $\sigma P \frac{2\pi}{\phi} \frac{T\phi}{2\pi} = \sigma PT$. This demonstrates that the amount of energy received by the radar is independent of the exact number of elements of the ESA. Therefore, the ESA has

no advantage over a radar system with a single transmit antenna. It follows that in later sections, we are justified to compare the new radar concept with a single transmit antenna system.

Every target is characterized by its RCS function [8]. A target's RCS function represents the amount of energy reflected from the target toward the receiver as a function of the target aspect with respect to the transmitter/receiver pair. It is well known that this function is rapidly changing as a function of the target aspect [8]. Both experimental measurements and theoretical results demonstrate that scintillations of 10 dB or more in the reflected energy can occur by changing the target aspect by as little as one milliradian. These RCS scintillations are responsible for signal fading, which can cause large degradations in the system's detection and estimation performances.

Motivated by recent developments in communication theory [9], we introduce the new concept of multiple-input multiple-output (MIMO) radar. MIMO communication systems overcome the problems caused by fading by transmitting different streams of information from several decorrelated transmitters [10]. Since the transmitters are decorrelated, different signals undergo independent fading. In MIMO communications systems, the receiver enjoys the fact that the average (over all information streams) signal to noise ratio (SNR) is more or less constant, whereas in conventional systems, which transmits all their energy over a single path, the received SNR varies considerably.

Our proposed MIMO radar enjoys the same benefits enjoyed by MIMO communication systems. Specifically, our proposed system overcomes target RCS scintillations by transmitting different signals from several decorrelated transmitters. The received signal is a superposition of independently faded signals, and the average SNR of the received signal is more or less constant. This is in marked contrast to conventional radar, which under classical Swerling models suffers from large variations in the received power.

The reader should note that the whole notion of MIMO radar is new, and the main purpose of this paper is to introduce this concept. In addition, we present one specific scenario in which our proposed system improves the performance considerably. For clarity of presentation, in this paper, the treatment of MIMO radar focuses on direction finding (DF), ignoring range and Doppler effects. In subsequent work, we will elaborate on these and many other aspects associated with this concept.

The rest of the paper is organized as follows: Section II introduces the MIMO radar signal and channel models, including a classification of various MIMO radar systems. Section III presents an analysis of a MIMO DF system. The essence of the analysis is to determine the Cramer-Rao bound (CRB) associated with the MIMO radar and compare it with that of a single-antenna system. This section contains a specific example with a uniform linear antenna array at the receiver. Numerical results are provided. Finally, Section IV draws conclusions.

II. MIMO Radar Signal Model

In this section, we describe a general signal model for the MIMO radar. The model focuses on the effect of the target spatial properties ignoring range and Doppler effects. The signal model separates the contributions of the transmit array, target, and receive array, to the received signal. By doing so it provides insight into the principles of MIMO radar.

Not surprisingly, the radar MIMO signal and channel model is related to MIMO channel models for communications, for example [11]. The signal model can be used to describe both conventional radar systems and our proposed MIMO radar system. Assume a radar system that utilizes an array with M antennas at the transmitter, and N sensors at the receiver. The transmitter and the receiver are not necessarily collocated (bistatic radar). Assume also a far field complex target that consists of many (say, Q) independent scatterers with approximately the same RCS. This assumption corresponds to a target composed of many small reflectors. The target is illuminated by narrowband signals whose amplitude does not change appreciably across the target (roughly, that means a bandwidth smaller than c/D, where c is the speed of light and D is the target length). Each scatterer is assumed to have isotropic reflectivity modeled by zero-mean, unit-variance per dimension, independent and identically distributed (i.i.d.) complex random variables ζ_q. The target is then modeled by the diagonal matrix $\boldsymbol{\Sigma} = \left(1/\sqrt{2Q}\right) \mathrm{diag}\left(\zeta_0, \ldots, \zeta_{Q-1}\right)$, where the normalization factor makes the target RCS $E\left[\mathrm{Tr}\left(\boldsymbol{\Sigma}\boldsymbol{\Sigma}^*\right)\right] = 1$ (the superscript denotes complex conjugate) independent of the number of scatterers in the model. With RCS fluctuations that are fixed during an antenna scan, but vary independently scan to scan, our target model is a classical Swerling case I [8].

For simplicity, we assume that the target scatterers are laid out as a linear array, and that this array and the arrays at the transmitter and receiver are parallel. This scenario is depicted in Fig. 1. The signal radiated by the m-th transmit antenna impinges as a planewave on the Q scatterers at angles $\theta_{m,q}, q = 0, \ldots, Q - 1$ (measured with respect to the normal to the array). This assumption holds when the distance to target R is much larger than the transmitter array aperture. The signal vector induced by the m-th transmit antenna is given by

$$\mathbf{g}_m = \left[1, e^{-j2\pi \sin\theta_{m,2}\Delta_2/\lambda}, \ldots, e^{-j2\pi \sin\theta_{m,Q}\Delta_Q/\lambda}\right]^T,$$

where Δ_q is the spacing between the first and $(q+1)$-th scatterer, λ is the carrier wavelength, and the superscript

denotes vector/matrix transposition. Assume that the target scatterers are uniformly spaced, i.e., $\Delta_q = q\Delta$. The common phase shift between the transmitter and the target has no bearing on performance and is left out. With MIMO radar, we seek to exploit the spatial diversity of the target. To achieve spatial diversity, it is required that different transmit antennas see uncorrelated aspects of the target. Mathematically, this is expressed as orthogonality between signal vectors. For an arbitrary antenna element m, the condition for orthogonality with the signal vector induced by element $m+1$, is given by

$$\mathbf{g}_m^\dagger \mathbf{g}_{m+1} = \sum_{q=0}^{Q-1} e^{j2\pi[(\sin\theta_{m+1,q} - \sin\theta_{m,q})\Delta_q/\lambda]} = 0, \quad (1)$$

where the superscript denotes complex conjugate and transpose. The signal vectors are organized in the $M \times Q$ transmit matrix $\mathbf{G} = [\mathbf{g}_1\mathbf{g}_2\ldots\mathbf{g}_M]^T$. Assuming that the range to target is much larger than the inter-element separation at the transmitter, the range is assumed independent of the transmitter antenna. Geometric considerations lead to the relation

$$\sin\theta_{m+1,q} - \sin\theta_{m,q} \approx d_t/R, \quad (2)$$

where d_t is the inter-element spacing at the transmitter. Using this in (1), and noting that the right hand side of (2) is independent of the scatterer index q, we obtain

$$\sum_{q=0}^{Q-1} e^{j2\pi(d_t/R)q\Delta/\lambda} = 0. \quad (3)$$

A necessary condition for (3) to be met is for the angles to complete at least a turn of the unit circle, i.e.,

$$\frac{d_t\Delta}{\lambda R} \geq \frac{1}{Q-1}. \quad (4)$$

This condition obtained solely from geometric considerations also has an appealing intuitive physical interpretation. The beamwidth of the energy backscattered from the target towards the transmitter is approximately given by λ/D, where $D = (Q-1)\Delta$ is the target size. The target presents different aspects to adjacent transmit antennas if the inter-element spacing at the transmitter is greater than the target beamwidth coverage at distance R, namely

$$d_t \geq \frac{\lambda R}{D}, \quad (5)$$

which turns out to be the same as (4).

To complete the transmitter model, we assume phase shifts imposed on the transmitted signals represented by the length$-M$ vector $\mathbf{b}(\theta')$, where the m-th element is given by $e^{j2\pi(m-1)d_t\sin\theta'/\lambda}$. Lowpass equivalents of the transmitted waveforms are listed along the diagonal of the matrix $\mathbf{S} = \text{diag}(s_1,\ldots,s_M)$. The transmitted waveforms are normalized such that $|s_i|^2 = 1/M$. The normalizing factor ensures that the transmitted power is independent of the number of transmit antennas. In case all antennas transmit the same waveform, $\mathbf{S} = s\mathbf{I}_M$, where the subscript denotes the order of the unity matrix.
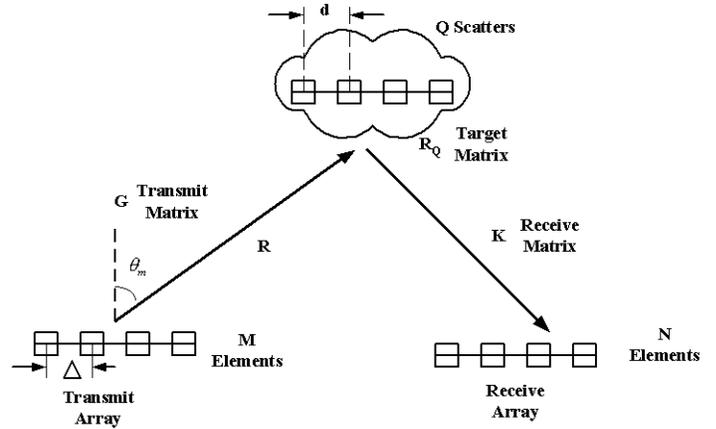

Fig. 1. Bistatic radar scenario. The target consists of multiple scatterers organized in the form of a linear array.

The model for the array at the receiver is developed similar to that at the transmitter, resulting in an $Q \times N$ channel matrix $\mathbf{K}$, where each row $\mathbf{a}_n^T$, $n = 0,\ldots,N-1$, constitutes a signal vector from a scatterer of the target to the receiver array. The bearing between scatterer q and antenna n is denoted $\theta_{n,q}$. Orthogonality conditions for target signatures at the receiver can be developed similar to (4), with the inter-element spacing at the receiver d_r replacing d_t. A case of special interest is a receiver array with $d_r = \lambda/2$. This makes possible unambiguous DF. Since the range to target is assumed much larger than the array aperture, $\theta_{n,q} \simeq \theta_n$ for all q. Consequently, the receive matrix $\mathbf{K} = \mathbf{1}_Q \otimes \mathbf{a}^T(\theta_0)$, where the operation is the Kronecker product (each element of the first operand is multiplied by the second operand), and the vector $\mathbf{1}_Q$ is a $Q \times 1$ vector of ones. Finally, the $N \times 1$ steering vector at the receiver is denoted $\mathbf{a}(\theta)$ and its definition is analogous to $\mathbf{b}(\theta')$ defined earlier.

Putting it all together, the MIMO radar channel model is given by the $M \times N$ matrix

$$\mathbf{H} = \mathbf{G\Sigma K}. \quad (6)$$

The element h_{ij} of the channel matrix provides the complex-valued channel gain from transmitter antenna i to receiver antenna j. The signal vector received by the MIMO radar (after demodulation and matched filtering) is given by

$$\mathbf{r} = \mathbf{H}^T\mathbf{Sb}^*(\theta') + \mathbf{v}, \quad (7)$$

where the superscript denotes complex conjugate, and the additive (spatially) white Gaussian noise vector $\mathbf{v}$ consists of i.i.d., zero-mean complex normal random variables with

variance $1/\text{SNR}$, where SNR is the average signal-to-noise ratio per antenna element at the receiver. The channel model in (6) and the signal model in (7) can be used to represent a variety of scenarios for MIMO radar.

Model Classification

MIMO radar signal models can be classified into three general groups:

- Conventional radar array modeled with an array at the receiver and a single antenna or an array at the transmitter. The array elements are spaced at half-wavelengths to enable beamforming and DF.

- MIMO radar for DF. Transmit antenna elements are widely spaced to support spatial diversity aspects of the target. Receiver array performs DF.

- MIMO radar for detection of multiple targets. In this scenario, there are multiple targets in the range cell under test. Antennas at the transmitter are less separated than in the previous scenario, such that scatterers belonging to the same target are not individually resolved. Yet, the separation is sufficient to resolve multiple targets in the same range cell.

The first two signal models are detailed in the sequel. The third will be discussed in a subsequent publication.

Conventional Radar Array

Conventional systems are systems in which the elements of the transmitting and receiving arrays are closely spaced. At the transmitter, that means that the inter-element spacing does not meet (5) or, equivalently, that multiple elements are contained within one target beamwidth. At the receiver, the spacing is $d_r \leq \lambda/2$ to enable unambiguous estimation of the angle of arrival.

Let the target bearing with respect to the transmit and receive arrays be respectively, θ_0' and θ_0. The transmit matrix is given by $\mathbf{G} = \mathbf{b}(\theta_0') \otimes \mathbf{1}_Q^T$. The receive matrix is given by $\mathbf{K} = \mathbf{1}_Q \otimes \mathbf{a}^T(\theta_0)$. It follows that the channel matrix is given by

$$\mathbf{H} = \frac{1}{\sqrt{2}} \alpha \mathbf{b}(\theta_0') \mathbf{a}^T(\theta_0), \qquad (8)$$

where $\alpha = (1/\sqrt{Q}) \mathbf{1}^T \zeta$, and ζ is the vector formed by the target gains ζ_q, $\zeta = [\zeta_0, \ldots, \zeta_{Q-1}]^T$. By assumption, ζ_q are zero-mean, unit-variance per dimension, i.i.d.; hence by the central limit theorem α approaches a zero-mean, complex normal distribution. Subsequently, the target's RCS $|\alpha|^2$, follows a χ_2^2 chi-square distribution with 2 degrees of freedom. Note that with this model, there is no diversity 'gain' in the target RCS.

With a conventional radar array, all antennas transmit the same waveform s. Beamforming at the transmitter is

represented by the vector $\mathbf{b}^*(\theta')$. The signal model in this case is given by

$$\mathbf{r} = \left(1/\sqrt{2}\right) \mathbf{a}(\theta_0) \mathbf{b}^T(\theta_0') \mathbf{b}^*(\theta') \alpha s + \mathbf{v}, \qquad (9)$$

Now, if the receiver uses a beamformer to steer towards direction θ, then the output of the beamformer is

$$\begin{aligned} y &= \left(1/\sqrt{2}\right) \mathbf{a}^\dagger(\theta_0) \mathbf{r} + v' \\ &= \left(1/\sqrt{2}\right) \mathbf{a}^\dagger(\theta_0) \mathbf{a}(\theta_0) \mathbf{b}^T(\theta_0') \mathbf{b}^*(\theta') \alpha s + v' \quad (10) \end{aligned}$$

This model represents a bistatic radar where $\mathbf{b}^T(\theta_0') \mathbf{b}^*(\theta')$ plays the role of the transmit antenna pattern, whereas $\mathbf{a}^\dagger(\theta_0) \mathbf{a}(\theta_0)$ is the receive antenna pattern. Since, $E[\alpha] = 0$, $E\left[|\alpha|^2\right] = 2$, $|s|^2 = 1/M$, the signal power in (10), is given by $\left|\mathbf{a}^\dagger(\theta_0) \mathbf{a}(\theta_0) \mathbf{b}^T(\theta_0') \mathbf{b}^*(\theta') s\right|^2 \leq MN^2$. Note that the instantaneous signal power $\left|\mathbf{a}^\dagger(\theta_0) \mathbf{a}(\theta_0) \mathbf{b}^T(\theta_0') \mathbf{b}^*(\theta') \alpha s\right|$ has a χ_2^2 distribution (chi-square with 2 degrees of freedom).

MIMO Radar: Direction Finding

In MIMO radar for direction finding (DF), the transmit antennas are sufficiently separated to meet the orthogonality condition (5) for targets of interest. The columns of the transmit matrix $\mathbf{G}$ meet the orthogonality condition in (1). In contrast, elements of the receive array are closely separated to enable DF measurements. Assume that the target is at angle θ_0 with respect to the receive array normal. The receive matrix is given by $\mathbf{K} = \mathbf{1}_Q \otimes \mathbf{a}^T(\theta_0)$. Since the goal is to illuminate the target to achieve spatial diversity, phase shifts at the transmitter are set to zero, $\mathbf{b}(\theta') = \mathbf{1}_M$. From (6), it follows that the channel matrix is given by

$$\mathbf{H} = \frac{1}{\sqrt{2}} \boldsymbol{\alpha} \otimes \mathbf{a}^T(\theta_0), \qquad (11)$$

where the component α_m, of the $M \times 1$ vector $\boldsymbol{\alpha}$, is defined $\alpha_m = \left(1/\sqrt{Q}\right) \mathbf{g}_m^T \zeta$, and ζ was defined previously. Due to the orthogonality among the transmit vectors $\mathbf{g}_m$, the variates α_m are uncorrelated. Moreover, for $Q \to \infty$, the random variables $\left(1/\sqrt{Q}\right) \alpha_m$ are zero-mean, unit-variance (per dimension), i.i.d. complex normal.

The signal model is given by

$$\begin{aligned} \mathbf{r} &= \frac{1}{\sqrt{2}} \left(\boldsymbol{\alpha}^T \otimes \mathbf{a}(\theta_0)\right) \mathbf{S} + \mathbf{v} \\ &= \frac{1}{\sqrt{2}} \mathbf{a}(\theta_0) \sum_{i=0}^{M-1} \alpha_i s_i + \mathbf{v}. \qquad (12) \end{aligned}$$

The signal model, with all normalization factors specified so far, ensures that the average transmitted power $E\left[\left|(1/\sqrt{2}) \sum_{i=0}^{M-1} \alpha_i s_i\right|^2\right] = 1$. Conditioned on the target

vector α, the received vector $\mathbf{r}$ is complex, multivariate normal with correlation matrix $(2M)^{-1} \|\alpha\|^2 \mathbf{a}(\theta_0) \mathbf{a}^\dagger(\theta_0) + \text{SNR}^{-1} \mathbf{I}_N$.

To gain better insight, consider a specific case with $M = 2$, $N = 1$. The signal model is given by

$$r = \frac{1}{\sqrt{2}} (\alpha_1 s_1 + \alpha_2 s_2) + v. \qquad (13)$$

If both antennas transmit the same waveform, $s_1 = s_2 = s$, the received signal is given by

$$r = \frac{1}{\sqrt{2}} (\alpha_1 + \alpha_2) s + v. \qquad (14)$$

Since the channel parameters $\alpha_{1,2}$ are unknown at the receiver, it is impossible to take advantage of the target spatial diversity. This system fails to achieve the target diversity sought.

Conversely, for orthogonal transmitted waveforms such that $s_1^* s_2 = 0$, $|s_1|^2 = |s_2|^2 = 1/2$, the received signal can be processed to yield the test statistic

$$\begin{aligned} \xi &= |s_1^* r|^2 + |s_2^* r|^2 \\ &= \frac{1}{4} \left(|\alpha_1|^2 + |\alpha_2|^2 \right) + v''. \end{aligned} \qquad (15)$$

Since for $Q \to \infty$, the random variables $|\alpha_i|^2$, $i = 1, 2$, have a χ^2 distribution, and they are i.i.d. (due to the orthogonality between $\mathbf{g}_1$ and $\mathbf{g}_2$), the target component in (15) has a χ_4^2 (chi-square with 4 degrees of freedom) distribution. This is a consequence of the different and uncorrelated RCS presented by the target to the different elements of the transmitting antenna. Thus MIMO radar results in a *diversity gain* that manifests itself through a more advantageous distribution of the target component in the received signal.

III. MIMO DF Analysis

In a radar DF system, an omnidirectional antenna illuminates the space, and based on the energy reflected from the target, the receiver estimates the target's bearing. In this section we examine the achievable performance of a MIMO radar when used as a DF system. For simplicity and mathematical tractability we make the following assumptions:

1. The transmitted signal vector $\mathbf{s}$ is random with a complex normal distribution, and a spatially white, stationary power spectral density with correlation matrix $(1/M) \, \mathbf{I}_M$.

2. The elements of both the transmitting and the receiving arrays are omnidirectional.

3. Multiple, independent snapshots of the received signal are available for processing.

A common figure of merit for comparing the performance of different systems is the estimator mean square error (MSE). The system's MSE depends on the exact estimation method, e.g., ML, MUSIC, beamforming, used. In order to have a fair comparison between different systems, for each system, we evaluate a lower bound on the performance of all unbiased estimators.

Cramer-Rao Bound

In what follows, we analyze the performance of a MIMO radar used as an active direction finder with $M \geq 1$ transmitting elements. The received signal is given by the model (12). Conditioned on α, $\sum_{m=1}^M \alpha_m s_i$ is a zero mean, white, complex normal random variable with variance $(1/M) \|\alpha\|^2$. In our model there are three unknown parameters, the direction parameter θ, the target parameters α, and the SNR. Let the vector denoting the unknown parameters, $\psi = [\theta, \text{SNR}, \|\alpha\|^2]$.

The Cramer-Rao bound is probably the best known lower bound on the MSE of unbiased estimators [12]. Denote by $p(\mathbf{r}|\psi)$ the family of distributions of the received signal parameterized by the vector of unknown parameters ψ. The Cramer-Rao lower bound for estimating ψ is given by,

$$\text{CRB}(\psi) = \mathbf{J}^{-1} = \left(-E\left[\frac{\partial^2 \log p(\mathbf{r}|\psi)}{\partial \psi \partial \psi^T} \right] \right)^{-1}. \qquad (16)$$

We are interested only in the direction θ, whereas the others are nuisance parameters. We denote the corresponding bound $\text{CRB}(\theta|\alpha)$, where the notation indicates the conditioning of the bound on the unknown parameters α. As already noted, conditioned on α, $\mathbf{r}$ is a complex normal random vector with correlation matrix $(2M)^{-1} \|\alpha\|^2 \mathbf{a}(\theta) \mathbf{a}^\dagger(\theta) + \text{SNR}^{-1} \mathbf{I}_N$. The CRB conditioned on α can be computed, and it is given by [13, 12],

$$\begin{aligned} \text{CRB}(\theta|\alpha) = \frac{N}{2L} &\left[\frac{2M}{N\text{SNR}\|\alpha\|^2} + \frac{4M^2}{(N\text{SNR})^2 \|\alpha\|^4} \right] \cdot \\ &\cdot \left(\left\| \dot{\mathbf{a}}(\theta) \right\|^2 - \left\| \dot{\mathbf{a}}^\dagger(\theta) \mathbf{a}(\theta) \right\|^4 / N \right)^{-1}, \end{aligned}$$

where L is the number of snapshots used by the array for estimating θ.

We can lower bound the MSE of any unbiased estimator by averaging the CRB with respect to α. We denote this bound by $\text{ACRB}(\theta) = E_\alpha [\text{CRB}(\theta|\alpha)]$. By using (16) the $\text{ACRB}(\theta)$ is given by

$$\begin{aligned} \text{ACRB}(\theta) &= E_\alpha [\text{CRB}(\theta|\alpha)] \\ &= \frac{N}{2L} E_\alpha \left[\frac{2M}{N\text{SNR}\|\alpha\|^2} + \frac{4M^2}{(N\text{SNR})^2 \|\alpha\|^4} \right] \cdot \\ &\quad \cdot \left(\left\| \dot{\mathbf{a}}(\theta) \right\|^2 - \left\| \dot{\mathbf{a}}^\dagger(\theta) \mathbf{a}(\theta) \right\|^4 / N \right)^{-1}. \qquad (17) \end{aligned}$$

The following lemma is essential for deriving a closed form expression for ACRB(θ).

Lemma 1 $E_{\boldsymbol{\alpha}}\left[\frac{1}{||\boldsymbol{\alpha}||^2}\right] = \frac{1}{2(M-1)}$ and
$E_{\boldsymbol{\alpha}}\left[\frac{1}{||\boldsymbol{\alpha}||^4}\right] = \frac{1}{2^2(M-1)(M-2)}$

Proof 1 *We first note that $||\boldsymbol{\alpha}||^2$ is distributed as a chi-square random variable with $2M$ degrees of freedoms. This allows us to write the expectations of interest as $E_{\boldsymbol{\alpha}}\left[\frac{1}{||\boldsymbol{\alpha}||^2}\right] = E_{\chi_{2M}^2}\left[\frac{1}{X}\right]$ and $E_{\boldsymbol{\alpha}}\left\{\frac{1}{||\boldsymbol{\alpha}||^4}\right\} = E_{\chi_{2M}^2}\left[\frac{1}{X^2}\right]$.*

Given the density function of the χ_{2M}^2 random variable

$$p_X(x) = \frac{x^{2M/2-1}e^{-x/2}}{\Gamma(2M/2)\,2^{2M/2}}, \qquad (18)$$

where Γ denotes the Gamma function, evaluate

$$
\begin{aligned}
E_{\chi_{(2M)}^2}\left[\frac{1}{X}\right] &= \int_0^\infty \frac{1}{x}\frac{x^{2M/2-1}e^{-x/2}}{\Gamma(2M/2)\,2^{2M/2}}dx \\
&= \int_0^\infty \frac{x^{(2M-2)/2-1}e^{-x/2}}{\Gamma(2M/2)\,2^{2M/2}}dx \\
&= \frac{\Gamma((2M-2)/2)\,2^{(2M-2)/2}}{\Gamma(2M/2)\,2^{2M/2}} = \frac{1}{2(M-1)} \quad (19)
\end{aligned}
$$

and

$$
\begin{aligned}
E_{\chi_{2M}^2}\left[\frac{1}{X^2}\right] &= \int_0^\infty \frac{1}{x^2}\frac{x^{2M/2-1}e^{-x/2}}{\Gamma(2M/2)\,2^{2M/2}}dx \\
&= \int_0^\infty \frac{x^{(2M-4)/2-1}e^{-x/2}}{\Gamma(2M/2)\,2^{2M/2}}dx \\
&= \frac{\Gamma((2M-4)/2)\,2^{(2M-4)/2}}{\Gamma(2M/2)\,2^{2M/2}} \\
&= \frac{1}{2^2(M-1)(M-2)} \quad (20)
\end{aligned}
$$

Using the results of the lemma, the ACRB is given by

$$\text{ACRB}(\theta) = \left(\left\|\dot{\mathbf{a}}(\theta)\right\|^2 - \left\|\dot{\mathbf{a}}^{\dagger}(\theta)\mathbf{a}(\theta)\right\|^2/N\right)^{-1}\frac{N}{2L}$$
$$\frac{M}{(M-1)N\text{SNR}}\left[1 + \frac{M}{(M-2)(N\text{SNR})}\right].\,(21)$$

It is easy to verify that if the target's RCS is constant, that is if $||\boldsymbol{\alpha}||^2 = 2M$ deterministically, the CRB is independent of M. Having this in mind, it is only natural to define the system's fading loss as the additional SNR necessary to achieve the same MSE as a system that is not subject to fading.

By using the results of Lemma 1, it is easy to verify that the fading loss (in dB) as a function of the number of elements in the transmitting array is lower and upper bounded

as

$$10\log_{10}\frac{M}{M-1} \le \text{FL}(M) \le 10\log_{10}\frac{M}{\sqrt{(M-1)(M-2)}}.$$
$$(22)$$

Consider the case $M = 1$. In this case the fading loss is infinite. Furthermore, the ACRB in (21) is infinite. However, when, the unknown angle parameter θ is estimated using say, the maximum likelihood estimation, the resulting MSE approaches zero as the SNR approaches infinity. This discrepancy deserves additional consideration. The CRB bound is a small error bound, that is, it predicts the MSE based on the behavior of the log-likelihood function in the vicinity of the true parameter vector. Since it is a small error bound, this bound ignores the full structure of the parameter space [12], which may result in nonsensical values. For example, in the problem at hand, if $||\alpha||^2$ is low, the CRB, (16), might be much higher than π^2. However, since $\theta \in [-\pi, \pi]$ the MSE of any estimator is lower than π^2. Hence in this case, the CRB is useless.

The CRB approaches infinity at the rate of $\frac{1}{||\alpha||^2}$, that is, $\text{CRB} = \mathcal{O}(||\boldsymbol{\alpha}||^{-2})$. Therefore in order for the ACRB to yield a finite result, the density function $p_X(x)$, where $X = ||\boldsymbol{\alpha}||^2$ is a chi-square random variable with $2M$ degrees of freedom, should approach zero faster than $(||\boldsymbol{\alpha}||^2)$ as $||\boldsymbol{\alpha}|| \to 0$. By examining the probability density function of X in (18), it is easy to see that this happens only for $M \ge 3$. But in reality, since the MSE of any estimator is smaller than π^2, averaging the performance of any estimator with respect to $\boldsymbol{\alpha}$ yields a finite result for *any* M.

Now, consider the case $M \to \infty$. Here, the fading loss approaches zero, that is, the variations in the target's SNR do not affect the system's MSE. This phenomena can be explained using the following intuitive argument. Without fading, the received signal is $\mathbf{r} = \mathbf{a}(\theta)s + \mathbf{v}$, where s is zero-mean, $(1/M)$ variance, complex normal. With fading, the received signal is $\mathbf{r} = \left(1/\sqrt{2}\right)\mathbf{a}(\theta)\sum_{m=1}^M \alpha_m s + \mathbf{v}$. However, according to the central limit theorem, as M approaches infinity, $\sum_{m=1}^M \alpha_m s$ approaches a zero-mean, $(2/M)$ variance, complex normal random variable. Hence, as M approaches infinity the received signal is equal to the received signal of a non-fading system.

Uniform Linear Array

In this section we specifically consider the case of a uniform linear array (ULA) with omnidirectional antennas at the receiver. Consider a ULA with N elements with half a wavelength spacing. The n-th element of the array steering vector equals

$$[\mathbf{a}(\theta)]_n = e^{jn\pi\sin\theta}. \qquad (23)$$

The n-th element of $\dot{\mathbf{a}}(\theta)$ is

$$\left[\dot{\mathbf{a}(\theta)}\right]_n = -jn\pi\cos\theta[\mathbf{a}(\theta)]_n. \qquad (24)$$

¿From these relations it follows that the squared norm of the steering vector is given by $\mathbf{a}^\dagger(\theta)\mathbf{a}(\theta) = N$, and the squared norm of the derivative of the steering vector is given by

$$
\begin{aligned}
\dot{\mathbf{a}}^\dagger(\theta)\dot{\mathbf{a}}(\theta) &= \pi^2 \cos^2\theta \sum_{n=0}^{N-1} n^2 \\
&= \frac{(N-1)\,N\,(2N-1)}{6} \pi^2 \cos^2\theta. \quad (25)
\end{aligned}
$$

Finally, we have

$$
\begin{aligned}
\left\| \dot{\mathbf{a}}^\dagger(\theta)\mathbf{a}(\theta) \right\|^2 &= \left(\sum_{n=0}^{N-1} jn\pi \cos\theta\, \mathbf{a}^\dagger(\theta)\mathbf{a}(\theta) \right)^2 \\
&= \frac{(N-1)^2\, N^2}{4} \pi^2 \cos^2\theta \qquad (26)
\end{aligned}
$$

Substituting these results in (21), the ACRB for the case of a ULA at the receiver is given by,

$$
\begin{aligned}
\mathrm{ACRB}(\theta) = \frac{6}{(N^2-1)L\pi^2\cos^2\theta} \Bigg[&\frac{M}{(M-1)\,N\mathrm{SNR}} + \\
&\frac{M^2}{(M-1)\,(M-2)\,(N\mathrm{SNR})^2} \Bigg] \quad (27)
\end{aligned}
$$

Let us investigate some special cases.

$\theta \to \pi/2$: Here, $\mathrm{ACRB}(\theta) \to \infty$, confirms that the direction cannot be estimated at endfire, since the array has a zero effective aperture (zero resolution).

$\theta = 0$: This is the best case for estimating the direction parameter. Indeed, at broadside the array has the largest effective aperture (best resolution).

$N = 1$: The bound is infinite. Indeed, a single omnidirectional antenna cannot measure the angle of arrival.

Numerical Results

In this section, numerical results are provided on the ACRB for a ULA with $N = 6$ elements at the receiver. Performance is parameterized by the number of transmitting antennas M. The transmit antennas are spaced sufficiently to achieve diversity.

Fig. 2 depicts the ACRB for various SNR's for large transmitting arrays with $M = 4$ and $M = 16$ transmitting elements, respectively. The CRB for the case of a Gaussian source without fading is depicted as well. Empirical results are represented through root mean square (RMS) errors of the MLE. It is well known that (for spatially and temporally white noise) the MLE for bearing estimation of a single source is given by a conventional beamformer. The MLE is the value of the angle of arrival that maximizes the output of the beamformer. That number of independent snapshots used in the estimation was $L = 80$.

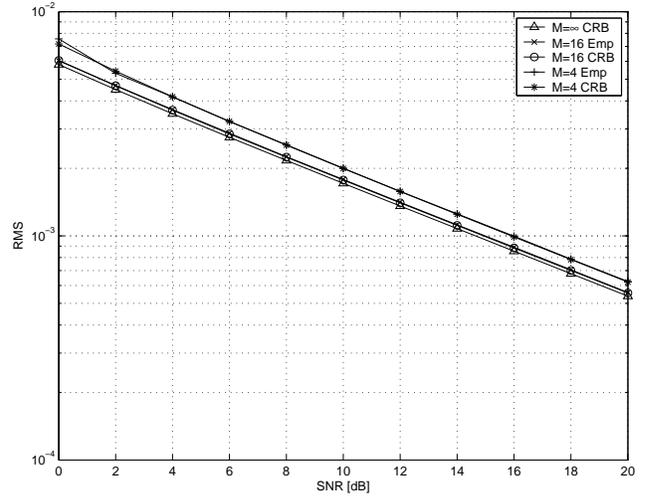It is evident from the figure that the ACRB values match the empirical results for both $M = 4$ and $M = 16$. When

Fig. 2. Average Cramer-Rao bound and empirical MSE for direction finding of a Gaussian source. Large transmit array.

a transmitting array with $M = 4$ elements is used, the fading loss is about 1.3 dB. This value is consistent with our analysis based on (22) predicts that the fading loss will be between 1.25 dB and 2.1 dB. When the array with $M = 16$ is used, the fading loss is negligible, as also predicted by our analysis.

In Fig. 3, the ACRB and the RMS error of the MLE are shown as functions of SNR for small transmit arrays with $M = 1$ and $M = 2$ elements, respectively. In addition, also shown is the CRB for the case of a Gaussian source without fading. For the case of $M = 2$ transmit antennas, we also plot a *Modified* ACRB (MACRB). The MACRB is defined as the lower bound of the CRB at high SNR, such that $\mathrm{SNR}^{-1} \gg \mathrm{SNR}^{-2}$, and terms containing the latter are neglected. From (16) we obtain

$$
\begin{aligned}
\mathrm{MCRB}(\theta|\boldsymbol{\alpha}) =\ &\frac{N}{2L} \frac{2M}{N\mathrm{SNR}\|\boldsymbol{\alpha}\|^2} \cdot \\
&\left(\left\| \dot{\mathbf{a}}(\theta) \right\|^2 - \left\| \dot{\mathbf{a}}^\dagger(\theta)\mathbf{a}(\theta) \right\|^2 / N \right)^{-1} (28)
\end{aligned}
$$

The MACRB is obtained by averaging (28) with respect to $\boldsymbol{\alpha}$. It is easy to see that the MACRB for the ULA at the receiver equals

$$
\mathrm{MACRB}(\theta) = \frac{6}{(N^2-1)L\pi^2\cos^2\theta} \frac{M}{(M-1)\,N\mathrm{SNR}} \quad (29)
$$

We can observe from the figure that if only one transmitting element is used, the fading loss is very large and it is about 15 dB. However, for large SNR, by adding an additional transmitting element, the fading loss decreases to about 2 dB. Moreover, when the SNR is large and (28) is tight, the MACRB fits the empirical results quite well.
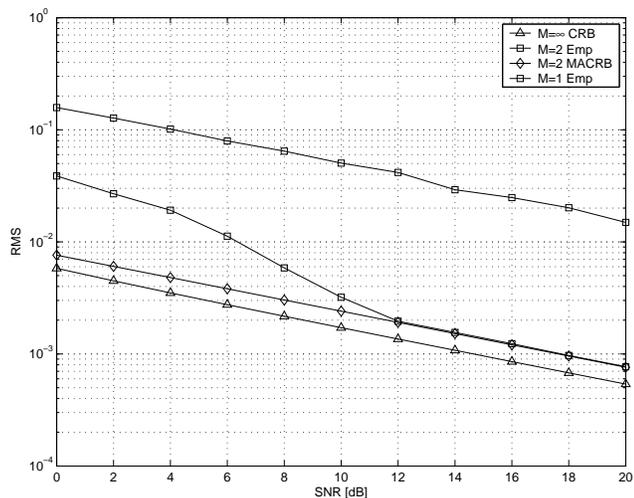
Fig. 3. Average Cramer-Rao bound and empirical MSE for direction finding of a Gaussian source. Small transmit array.

IV. Conclusions

In this paper, we introduce MIMO radar, a new concept in radar that capitalizes on the RCS scintillations with respect to the target aspect in order to improve the radar's performance. We introduced a generalized framework for the signal model that can accommodate conventional radars, beamformers, and MIMO radar. As demonstration of the potential advantages that MIMO radar can offer, we evaluated the Cramer-Rao bound for bearing estimation. We have shown that with a few transmit antennas that illuminate uncorrelated aspects of the target, the performance (in terms of Cramer-Rao bound) of MIMO radar approaches that of a steady target. This paper is meant only to introduce the MIMO radar concept. In subsequent work, we will continue to investigate this promising new approach to radar.

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