Advance Demand Information and Safety Capacity as a Hedge Against Demand and Capacity Uncertainty

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Abstract

To control a production-inventory system, a manager has to consider the variability in demand as well as variability in her production process. Both types of variability corrupt system performance and by alleviating either of them, the manager can improve the performance of the system. There has been a recent trend towards investing in better information systems to provide better advance demand information. Also, many firms have focused on having safety capacity (e.g., outsourcing or overtime) that they can rely on as needed to protect themselves against uncertainty in demand and production.

In this paper, we first address the tactical decision of how a firm decides on production-inventory-safety capacity levels when faced with production and demand uncertainty. We use a multi-period production-inventory model with backordering to fully characterize the structure of optimal policies. We explore the sensitivity of optimal policies and costs to parameters such as demand and production variability, service level, and utilization. We also analytically show that uncertainty in capacity may result in nonintuitive behavior, such as more variable capacity resulting in less inventory.

Using derived policy structure, through a computational study, we address the strategic decision of investing in better information or creating sources of safety capacity. Our study shows that reductions in costs are significant, with averages up to 30% for advance demand information, and up to 85% for outsourcing. Furthermore, conditions that make demand information more valuable tend to make safety capacity less valuable and vice versa and we identify when either will be more valuable. We also show that the benefits from both can exceed the sum of the benefits from either safety capacity or better information.

1 Introduction and Overview

1.1 Introduction

When manufacturers make production decisions, they need to consider the uncertainty in both demand and production capacity. Usually, safety inventory is used to protect firms against both sources of uncertainty. However, using inventory alone as a hedge against demand and capacity uncertainty can be expensive, especially when holding costs are high.

With the development of better information technology, manufacturers can collect and transmit more precise advance information on future demand. Electronic data interchange and Internet-based software effectively link customers' and manufacturers' computers directly and offer a medium to convey orders instantaneously, providing manufacturers with advance demand signals. Collaborative planning, forecasting and replenishment (CPFR), is becoming more common, especially in retailing and it enables supply chain partners throughout the whole chain to receive better information on demand. Retailers and manufacturers can also use price incentives to induce customers to place orders for future delivery, thus obtaining the advance demand information.

Information systems that enable manufacturers to obtain better advance demand information can require significant investment, yet only address one source of uncertainty. Manufacturers not only face demand uncertainty but also capacity uncertainty. This is because, in any plant, machines can go down; raw material deliveries may not be on time, or quality problems can reduce the effective capacity (for a general discussion of effective capacity and how it is influenced by variability in manufacturing systems, see Chapters 8-10 of Hopp and Spearman [21]). Thus, even if a manufacturer had perfect demand information, she still has to deal with uncertainties in production.

Manufacturers can deal with uncertainties in production by once again relying on inventory as a protection or by having extra sources of more expensive capacity (e.g., overtime, or a more reliable supplier from which some demand can be outsourced). Having such a source of safety capacity can also benefit manufacturers as it can protect against a demand surge or a line that is down causing a lot of backlogged orders. In fact, any variability in a production system is always buffered either with excess capacity or safety inventory or by making customers wait if neither of the first two buffering mechanisms are feasible (see p. 295 of Hopp and Spearman [21])).

In this paper, we focus on the safety capacity that is more expensive than regular capacity, which is often relied on by the manufacturers. As an example, we recently worked with a manufacturer of diesel engines, a supplier to Chrysler and many other commercial truck manufacturers. We found that the weekly production capacity of this firm's castings plant was highly random, due to downtime and low, variable quality. One of the options the company was considering was using a castings supplier as an emergency source. Since the engine plant uses a fairly standard number of castings; the contract with the emergency supplier would require delivery within a very short time. The emergency supplier was much more expensive, but in return could provide short deliveries and 6-sigma levels of quality. Thus, the firm was creating a form of more expensive "safety" (or "reactive") capacity.

A manufacturer that has decided to have a form of safety capacity still has the following problem to address: Given some level of advance demand information, how much should I produce this period in my regular plant and how much of the more expensive safety capacity will I need? In this context, a manufacturer also needs to decide how valuable advance demand information is, how much she should invest in trying to obtain it, and how valuable a source of safety capacity (such as outsourcing) is. We aim to answer these questions and also to understand how firms should prioritize whether it is more important to invest in advance demand information (e.g., work on implementing an information system whereby customers can provide demand forecasts) or work on creating more safety capacity for the firm (e.g., sign a contract with a local expensive subcontractor who, at a higher price, can deliver desired quantities on a short notice).

In this paper, we first analyze the structure of optimal production and safety capacity usage decisions made by a manufacturer facing uncertain demand and capacity who has some advance demand information. We show that the optimal decision can be characterized by a double-threshold policy. Having addressed the tactical decision of how to operate given the firm has advance demand information and the option to use safety capacity enables us to address the more strategic question of the conditions that make either of these more beneficial for a firm. That is; should a manager spend a lot of effort building better information systems or should she partner with some other contract manufacturers who can serve as a source of safety capacity? We show that the utilization of the regular capacity of the firm, the level of capacity and demand variability, and the service level that the firms provide all influence the value of safety capacity versus better information. Two of our findings are that 1) in general, conditions that make better demand information more valuable make the option to outsource less valuable (and vice versa) and we identify these conditions, 2) the combined effect of both options can be greater than the sum of the individual effects. We are thus able to provide managers with some guidance on how to think about their need for more safety capacity or better information (or both) as a hedge against uncertainty.

In the rest of this section, we review literature relevant to our problem. In Section 2, we develop a model that allows us to find the optimal policies in systems with outsourcing (in the remainder of the paper, we use "outsourcing" as an example of any safety capacity that is more expensive than regular capacity) and demand information for any fixed length of forecasting horizon. We show that the optimal policy is a double-threshold policy, where the manufacturer first outsources up to a certain level and then produces up to a higher level. We extend our discussion in Section 3 to explore how the optimal policy is affected by capacity variability. We show that although decreasing capacity variability always leads to lower costs, it can lead the outsourcing and production thresholds (inventory levels) to rise. In Section 4, using a comprehensive empirical study, we explore the effects of the system parameters (service level, demand and capacity variability, utilization, etc.) on the individual and joint benefits of information and outsourcing. The paper concludes in Section 5.

1.2 Literature Review

There is a tremendous literature on inventory and production management under demand uncertainty (see Chen [8], Porteus [31], and Zipkin [40] for comprehensive references). Although most of the literature assumes a general stochastic demand distribution, it does not assume variability in capacity or advance demand knowledge. We first review the work on capacity uncertainty which has received limited attention and then review the literature on advance demand information.

Production uncertainty has been modeled in two different ways. One approach has been heavily influenced by yield issues in electronics manufacturing and uses the concept of stochastically proportional yield or random yield, as defined in Henig and Gerchak [20]. The other approach regards the capacity in a given time interval as a random variable.

Random-yield models assume that a random fraction of a quantity ordered (or attempted to produce) is actually good (Henig and Gerchak). This is an appropriate model when the uncertainty is due to uncertain quality of individual items produced in a batch. Lee and Yano [26] consider a multi-echelon system with yield losses. Lee [25] and Gerchak, Wang, and Yano [15] consider the case when components are assembled and the suppliers have uncertain yields. Yano [38] allows for random lead times. Grosfeld-Nir, Gerchak, and He [16] take inspection of the possibly defective units into account. None of these papers has any advance demand information nor considers safety capacity. Yano and Lee [39] is an excellent review of random yield literature in the context of lot sizing.

Ciarallo *et al* [5] and Duenyas *et al* [10] follow the second approach (random capacity model) which is to assume that the number that can be produced in any given time interval is uncertain (but with known distribution). This is appropriate when modeling the effects of variability in processing times, machine downtime, etc. Ciarallo *at al* considers a simple stylized model and shows that the production policy is base-stock level. Duenyas *et al* consider a similar model, but allow for adding an alternative source of capacity but do not fully characterize the optimal policy. Also, in both of these papers, the firm does not have any advance demand information, and in Ciarallo, a form of safety capacity does not exist as well.

There have been some papers that explicitly model advance demand information. In most of these papers, advance demand information is usually modeled as a form of forecast updates, shared by parties in the supply chain. Hariharan and Zipkin [18] analyze a multi-echelon model with *demand lead time* and *supply lead time*. Their demand lead time essentially corresponds to advance demand information. They refine basic inventory models, propose several control policies, and show that these policies are slight variants of the policies for conventional systems. In the case of constant lead times, they show that a *base-stock policy* is optimal in many systems. The base-stock policies continue to be optimal in more general situations when the supply lead time is sequential stochastic and the demand lead time remains constant. However, the authors do not consider variability in production capacity, and they do not allow for errors in demand information (i.e., they assume perfect information).

Schwarz, Petruzzi, and Wee [33] consider one-stage system where customers place orders in advance (similar to Harriharan and Zipkin [18]), but customers may cancel some of them. Marklund [28] describes a different generalization of Hariharan and Zipkin [18] to the case of a distribution network and derives the total cost function for some special cases. Cachon and Fisher [4] model one supplier providing goods to multiple retailers. In their case advance demand information allows them to allocate orders better and through simulation they estimate the corresponding benefits. Chen [6] considers a serial multi-echelon system for one producer. He explores the value of POS (Point of Sale) data in the context of centralized and decentralized control. The information is limited to knowing about the orders in the lower stages of the supply chain. Both [6] and [4] find limited benefits from knowing POS data in their setting. (Note that our setting is different because we are considering a single echelon but with more extensive advance demand information as well as capacity uncertainty). Gallego and Özer [12] consider a periodic-review multi-stage system with one product and many classes of customers, each class providing a different level of advance demand information, and show optimality of a base-stock-like policy. In somewhat different settings, one stage and fixed ordering cost, Gallego and Özer [13] show optimality of (s, S) policies.

Gavirneni, Kapuscinski, and Tayur [14] consider a model where a manufacturer has limited but certain capacity and uses POS data for production decisions. The retailer places an order based on her (s, S) policy, which translates into uncertainty about timing of the orders and the quantity. The authors compare the benefits of sharing some information and all information (in every period) and show that in both cases the benefits of information decrease as capacity becomes tighter. Lee et al [24] consider very similar settings, except they assume no capacity constraint and allow for correlated demand across periods. They argue that the benefits of information increase when demand is positively correlated across time. Huang and Iravani [22] consider a capacitated supplier delivering goods to two retailers and examine under what conditions the benefits of POS data from one of two retailers are largest. Simchi-Levi and Zhao [35] analyze a setting similar to [14], but the retailer is constrained to ordering at fixed intervals of time and the authors divide this interval into shorter information sharing subintervals. Özer and Wei [29] study a similar model to [13], except they include the effect of limited capacity. Through a numerical study they evaluate the benefits of advance demand information as a function of capacity and other parameters. In all of these papers capacity is certain, and except for Gavirneni et al., the papers do not allow for a source of safety capacity. Consequently, none of these papers address the question when safety capacity versus advance demand information is more likely to be beneficial, the main focus of this paper.

Some of the literature is indirectly related to our questions. Even though some form of advance information is considered, it does not play a central role. Aviv and Federgruen [3] concentrate on a distribution network, where forecast is updated periodically using a Bayesian approach and allocation decisions are based on it. Chen [7] assumes multi-echelon setting, and models advance order information using an approach similar to that of Gallego and Özer [13]. However, the focus is on pricing – customers have different costs of waiting and prices are used to reveal their sensitivities to delay. He shows that production policy remains base stock.

In our paper, we use the Martingale Model of Forecast Evolution (MMFE) to study the value of advance demand information as compared to outsourcing. MMFE was originally developed by Heath and Jackson [19]. They assume additive forecast updates, which follow a *iid* Normal distribution, independent of the past information. We consider a slight generalization of their model. We do not assume Normal distributions and allow the forecast update to be any function of the last forecast. Gullu [17] considers a special case of Heath and Jackson [19]: for a single product with at most two forecast updates. Similarly, the forecasting model of Sethi *et al* [34] is also for single product and allows two updates. Their model allows, however, the demand to be a joint function of both updates. Toktay and Wein [37] propose an approximation of a production/inventory model that allows for advance demand information, and they apply Heath and Jackson's model [19] for a single product with *iid* (Normal) updates. The updating framework of Aviv [1] is described in terms of Kalman filter. The underlying assumptions are similar to those of Heath and Jackson for a single product and Normal distributions. Aviv allows the demand to depend in a *linear fashion* on all previous forecasts. Allowing dependence on previous forecasts is more general than our model, but linear dependence is more restrictive than our model. Once again, our use of MMFE is similar to most of these papers but our focus on the value of safety capacity versus advance demand information is unique, since except for Toktay and Wein, none of the above papers consider capacity uncertainty.

Toktay and Wein [37] is the only paper we are aware of that has elements of advance demand information and capacity uncertainty. In order to determine the optimal policy, capacity is assumed to be Normal and the model is approximated through rescaling the difference between demand and capacity (without changing values of demand and capacity) and translating it into a Brownian motion. The authors address the tactical problem of deciding how much to make using only one source of capacity, e.g., regular capacity, and develop an approximation for the optimal policy. Instead of using an approximation, we are able to characterize the structure of the optimal production policy for a system that has regular and safety capacity, (a feature Toktay and Wein do not consider) and use this optimal policy to provide managerial insights for when more safety capacity versus more advance demand information will be useful, a question they do not address. Furthermore, our focus on the sensitivity of optimal policies to changes in demand and production variance is also unique in the literature.

2 Model and Main Results

We consider a manufacturer satisfying stochastic customer demand and facing an uncertain capacity in multi-period settings. Any unsatisfied demand is backlogged. The manufacturer may receive advance demand information before making any decision, and she is allowed to use outsourcing. Later, in the computational part, we also consider cases when outsourcing is not available or prohibitively expensive.

We start by explaining how we model the advance demand information, the unreliable capacity, and outsourcing option. After introducing the time sequence and the notation, we present the formulation of the model with H-period advance demand information and outsourcing option.

In our model, a firm has initial knowledge of the general distribution of demand in every period, and may receive advance information for the demand within a certain forecast horizon. Suppose the forecast horizon covers H periods, then in any period s, the firm has demand forecasts for all the periods from s to $s + H$. The demand forecast evolution is modeled by MMFE, and we now describe how the advance demand information is updated.

We use the following notation for the demand information that the firm has in an arbitrary period s:

$D_{s,t} =$ advance information (forecast) for demand in period t, $s \le t \le s + H$;

$\epsilon_{s,t} = D_{s,t} - D_{s-1,t} =$ forecast update received for the demand in period t;

$$\epsilon_s = (\epsilon_{s,s}, \epsilon_{s,s+1}, \cdots, \epsilon_{s,s+H}) = \text{the information update vector collected for all the periods within forecast horizon;}$$

$$D_{s+H,s+H} = \epsilon_{s,s+H} + \epsilon_{s+1,s+H} + \cdots + \epsilon_{s+H,s+H} = \text{the actual demand in period } s+H;$$

$$\mathbf{D}_s = (D_{s-1,s}, D_{s-1,s+1}, \cdots, D_{s-1,s+H-1}) = \text{advance information relevant for decision making in period } s.$$

In period s, the firm receives its first forecast on demand in period $s+H$; which is a realization $D_{s,s+H}$ generated from distribution Z_{s+H} with pdf. q_{s+H} and cdf. Q_{s+H}. Thus, until the actual forecast is generated, the firm has knowledge only of the distribution Z_{s+H} for that period. In every period after s, e.g., $s+k$, the firm receives forecast updates. We assume that any forecast update $\epsilon_{s+k,s+H}$ (update received in period $s+k$ for period $s+H$ demand forecast) is a function of $D_{s+k-1,s+H}$, the previous forecast for period $s+H$. We thus have $\epsilon_{s+k,s+H} = \mathcal{E}_{s+k,s+H}(D_{s+k-1,s+H})$, and $Pr\{\epsilon_{s+k,s+H} < -D_{s+k-1,s+H}\} = 0$.

Because of variability in the manufacturing facility, the capacity is assumed to be random. Note that in reality all plants have random capacities due to such factors as unexpected outages, quality and yield problems etc., although the extent of randomness varies widely across industries and companies. We denote the capacity in period s by Y_s with *cdf* F_s and *pdf* f_s. Therefore, in period s, if the manufacturer aims to produce u_s units, the actual realization of the production quantity is minimum of u_s and Y_s, which we denote $u_s \wedge Y_s$.

Since the manufacturer has unreliable capacity, outsourcing provides her with a reliable source of additional capacity, although at a premium price. (We use the term outsourcing to refer in general to any form of acquiring extra safety capacity such as obtaining extra materials from other reliable sources, expediting, transporting from other locations, buying at a spot market, running overtime to meet desired production levels etc.) As our focus is on understanding when outsourcing and information are valuable as a function of the production and demand uncertainty that the manufacturer faces, we do not model an unreliable channel for outsourced goods. Clearly, if the manufacturer can only outsource production to a very unreliable supplier, this would not make outsourcing a very valuable option. Note that we assume that these outsourcing activities are more expensive than producing during regular time. (If outsourcing is cheaper than regular time production e.g., a supplier has economies of scale that the firm does not have, the firm may outsource all its production. This is not the case we consider here and in fact recent work by Parmigani [30] shows that even in those cases, firms may use a make and buy policy for strategic reasons.) Therefore, whenever the firm is making production decisions in each period, it simultaneously has to make a decision on how much to outsource that period.

We consider an N-period problem and the firm has H periods advance demand information. In any period s, the sequence of events is as follows:

1. At the beginning of period s, the manufacturer observes the starting inventory position x_s and advance demand information

$$\mathbf{D}_s = (D_{s-1,s}, D_{s-1,s+1}, \cdots, D_{s-1,s+H-1}),$$

which was updated in period $s-1$.

2. The decisions on outsourcing quantity v_s and production quantity u_s are jointly (and simultaneously) made.
3. At the end of the period s, the actual production quantity $u_s \wedge Y_s$ is realized and the outsourced units v_s are received.
4. ϵ_s is realized, i.e., the actual demand is realized and the advance demand information is updated to $\mathbf{D}_{s+1}$.

Our aim is to set the optimal outsourcing quantity v_s and target production quantity u_s simultaneously. Instead of dealing with u_s and v_s directly, we use $\underline{x}_s = x_s + v_s$ and $\bar{x}_s = x_s + v_s + u_s$ as decision variables. Notice that $\bar{x}_s$ is the targeted inventory position after desired production quantity and outsourcing.

Denote the starting state of period s as $I_s = (x_s, \mathbf{D}_s)$. Then the state evolves according to $I_{s+1} = (x_{s+1}, \mathbf{D}_{s+1})$:

$$\begin{aligned} x_{s+1} &= \underline{x}_s + (\bar{x}_s - \underline{x}_s) \wedge Y_s - [D_{s-1,s} + \mathcal{E}_{s,s}(D_{s-1,s})], \\ D_{s,t} &= D_{s-1,t} + \mathcal{E}_{s,t}(D_{s-1,t}), \qquad t = s, \cdots, s+H-1, \\ D_{s,s+H} &= Z_{s+H}. \end{aligned}$$

i.e., besides the evolution of advance demand information, note that the inventory in the next period is the sum of current period inventory plus the outsourced quantity ($\underline{x}_s$) plus the amount of units actually produced $(\bar{x}_s - \underline{x}_s) \wedge Y_k$, during regular production minus the realization of demand, $D_{s-1,s} + \mathcal{E}_{s,s}(D_{s-1,s})$.

We assume that the firm incurs holding and penalty costs. The holding and penalty cost function is assumed to be a general non-negative convex function $b_s(\cdot)$, with $\lim_{|\cdot| \to \infty} b_s(\cdot) = \infty$, charged according to the inventory position at the end of the period s. (E.g., $b_s(x) = h * x^+ + p * x^-$, would represent the typical linear holding and penalty costs.)

Our objective is to minimize the total production, outsourcing, holding, and penalty costs. We assume that the firm incurs linear production and outsourcing costs. Due to backlogging, without loss of generality, we can rescale production costs so that in-house production has 0 cost, while outsourced production costs $c_s > 0$ per unit in period s. For instance, for $s < N$, if before rescaling, the production cost is c_s^u, the outsourcing cost is $c_s^v (= c_s^u + c_s)$, and the general non-negative convex function for the ending inventory position

x_{s+1} is $\tilde{b}_s(x_{s+1})$, then after rescaling, we have the production cost as 0, the outsourcing cost is c_s, and $b_s(x_{s+1}) = \tilde{b}_s(x_{s+1}) + (c^u_s - \alpha_s c^u_{s+1})x_{s+1} + c^u_s[D_{s-1,s} + E\mathcal{E}_{s,s}(D_{s-1,s})]$, where α_s is a (positive) discount rate for period s. If $s = N$, then the rescaled convex function is $b_N(x_{N+1}) = \tilde{b}_N(x_{N+1}) + c^u_N x_{N+1} + c^u_N[D_{N-1,N} + E\mathcal{E}_{N,N}(D_{N-1,N})]$.

The expected one-period cost function can be expressed as $G_s(I_s, \underline{x}_s, \bar{x}_s)$:

$$G_s(I_s, \underline{x}_s, \bar{x}_s) = -c_s x_s + E[c_s \underline{x}_s + b_s(x_{s+1})].$$

$V_s(I_s, \underline{x}_s, \bar{x}_s)$ is the expected discounted total cost for periods s through N, if the optimal policy is used starting with period $s+1$, and quantities $\underline{x}_s$ and $\bar{x}_s$ are chosen in period s. (Recall that $I_s = (x_s, \mathbf{D}_s)$.) Thus, we have the (MDP_H) problem as follows:

$$\begin{aligned} J_s(I_s) &= \min_{\bar{x}_s \geq \underline{x}_s \geq x_s} V_s(I_s, \underline{x}_s, \bar{x}_s) \qquad (MDP_H) \\ &= \min_{\bar{x}_s \geq \underline{x}_s \geq x_s} G_s(I_s, \underline{x}_s, \bar{x}_s) + \alpha_s E J_{s+1}(I_{s+1}), \end{aligned}$$

and $J_{N+1} \equiv 0$.

We base our analysis on the first-order optimality conditions (KKT conditions). We can express the KKT conditions for $V_s(I_s, \underline{x}_s, \bar{x}_s)$ as follows:

$$\lambda^H_s(\underline{x}_s, \bar{x}_s) := \frac{\partial}{\partial \bar{x}_s} V_s(I_s, \underline{x}_s, \bar{x}_s) = \Big(1 - F_s(\bar{x}_s - \underline{x}_s)\Big)\zeta^H_s(\bar{x}_s) \geq 0 \tag{1}$$

$$\mu^H_s(\underline{x}_s, \bar{x}_s) := \frac{\partial}{\partial \underline{x}_s} V_s(I_s, \underline{x}_s, \bar{x}_s) = c_s + \lambda^H_s(\underline{x}_s, \bar{x}_s) + \int_0^{\bar{x}_s - \underline{x}_s} \zeta^H_s(\underline{x}_s + y) f_s(y)\mathrm{d}y \geq 0 \tag{2}$$

$$\bar{x}_s \geq \underline{x}_s, \qquad (\bar{x}_s - \underline{x}_s) \times \lambda^H_s(\underline{x}_s, \bar{x}_s) = 0 \tag{3}$$

$$\underline{x}_s \geq x_s, \qquad (\underline{x}_s - x_s) \times \mu^H_s(\underline{x}_s, \bar{x}_s) = 0, \tag{4}$$

with

$$\zeta^H_s(\bar{x}_s) := E[b'_s(\bar{x}_s - D_{s-1,s} - \mathcal{E}_{s,s}(D_{s-1,s})) + \alpha_s (J_{s+1})'_1(\bar{x}_s - D_{s-1,s} - \mathcal{E}_{s,s}(D_{s-1,s}), \mathbf{D}_{s+1})], \tag{5}$$

and we suppress the dependence of λ^H_s and μ^H_s on I_s. The direction of inequalities (1) and (2) follows from constraints on $\bar{x}_s$ and $\underline{x}_s$. Note that $V_s(x_s, \underline{x}_s, \bar{x}_s)$ is not jointly convex in $(\underline{x}_s, \bar{x}_s)$, implying that KKT conditions are necessary but not sufficient. We show that the optimal cost-to-go function $J_s(I_s)$ is, nevertheless, convex in x_s.

We next characterize the policy that is shown later to be optimal for a firm which has access to safety capacity (outsourcing option) and advance demand information:

Definition 1 *We define a* **double-threshold policy** *with outsourcing threshold of* $\mathcal{X}^0$ *and production threshold of* $\mathcal{X}^1$*, as follows:*

- *Produce nothing and outsource nothing, if* $x \geq \mathcal{X}^1$*;*
- *Produce* $\mathcal{X}^1 - x$ *and outsource nothing, if* $\mathcal{X}^1 > x \geq \mathcal{X}^0$*;*
- *Produce* $\mathcal{X}^1 - \mathcal{X}^0$ *and outsource* $\mathcal{X}^0 - x$*, if* $\mathcal{X}^0 > x$*.*

Note that production and outsourcing decisions are actually made simultaneously in the beginning of the period.

In our current setting, given $\mathbf{D}_s$, we use (5) to define $\mathcal{X}_s^0(\mathbf{D}_s)$ and $\mathcal{X}_s^1(\mathbf{D}_s)$ as follows:[1]

$$\mathcal{X}_s^1(\mathbf{D}_s) := \frac{\sup\{\bar{x}_s|\,\zeta_s^H(\bar{x}_s) < 0\} + \inf\{\bar{x}_s|\,\zeta_s^H(\bar{x}_s) > 0\}}{2}, \tag{6}$$

$$\mathcal{X}_s^0(\mathbf{D}_s) := \frac{\sup\{\underline{x}_s|\,\eta_s^H(\underline{x}_s) \leq -c_s\} + \inf\{\underline{x}_s|\,\eta_s^H(\underline{x}_s) \geq -c_s\}}{2}, \tag{7}$$

where $\eta_s^H(\underline{x}_s) := \int_0^{\mathcal{X}_s^1(\mathbf{D}_s)-\underline{x}_s} \zeta_s^H(\underline{x}_s + y) f_s(y)\mathrm{d}y$. Intuitively, $\mathcal{X}_s^1(\mathbf{D}_s)$ is the point at which $\zeta_s(\bar{x}_s)$ crosses 0. If $\zeta_s(\bar{x}_s)$ is non-continuous when it crosses 0 (corresponding to the case when V_s has multiple subgradients), we are free to choose either one of them and, therefore, we assume $\zeta_s(\mathcal{X}_s^1(\mathbf{D}_s)) = 0$. Similar logic also applies to $\mathcal{X}_s^0(\mathbf{D}_s)$ and $\eta_s(\underline{x}_s)$ such that $\eta_s(\mathcal{X}_s^0(\mathbf{D}_s)) = -c_s$. In the following Theorem refthm:twoinfor, we express the optimal policy in terms of $\mathcal{X}_s^0(\mathbf{D}_s)$ and $\mathcal{X}_s^1(\mathbf{D}_s)$.

Theorem 1 *The optimal value function for problem* (MDP_H), $J_s(x_s, \mathbf{D}_s)$, *is non-negative convex in* x_s, *and the optimal policy is a state-dependent double-threshold policy, with outsourcing and production thresholds* $\mathcal{X}_s^0(\mathbf{D}_s)$ *and* $\mathcal{X}_s^1(\mathbf{D}_s)$.

Proof: We prove the theorem by induction, i.e., for each period s, we show the following two facts:

1. The optimal structure: (a) $\mathcal{X}_s^0(\mathbf{D}_s) \leq \mathcal{X}_s^1(\mathbf{D}_s)$, and (b) the optimal decision variables are $\underline{x}_s^* = \mathcal{X}_s^0(\mathbf{D}_s) \vee x_s$ and $\bar{x}_s^* = \mathcal{X}_s^1(\mathbf{D}_s) \vee x_s$.

2. The inductional hypotheses for period s:

   $\mathcal{A}_s$. $J_s(I_s)$ is non-negative convex in x_s.

Clearly, $J_{N+1} \equiv 0$ satisfies $\mathcal{A}_{N+1}$. Suppose for period $s+1$, $\mathcal{A}_{s+1}$ holds. We show below that both points *1* and *2* hold for period s. We write $\mathcal{X}_k^0$ and $\mathcal{X}_k^1$ for $\mathcal{X}_k^0(\mathbf{D}_s)$ and $\mathcal{X}_k^1(\mathbf{D}_s)$.

1. The optimal structure

First note that, based on (6) and (7), $\mathcal{X}_s^0$ and $\mathcal{X}_s^1$ are well defined.

(a) $\mathcal{X}_s^0 \leq \mathcal{X}_s^1$.

$\mathcal{X}_s^1$ is defined in (6) by function $\zeta_s(\bar{x}_s)$, which has the following properties:

- $\zeta_s(\bar{x}_s)$ is non-decreasing in $\bar{x}_s$(from convexity of b_s and $\mathcal{A}_{s+1}$);

- $\lim_{\bar{x}_s \to -\infty} \zeta_s(\bar{x}_s) < 0$ and $\lim_{\bar{x}_s \to \infty} \zeta_s(\bar{x}_s) > 0$ (from $\lim_{|x| \to \infty} b_s(x) = \infty$ and $\mathcal{A}_{s+1}$).

[1] Assume $\inf \mathbb{R} = -\infty$, $\sup \mathbb{R} = \infty$, $\inf \emptyset = \infty$, and $\sup \emptyset = -\infty$.

The above, combined with the definition of $\mathcal{X}_s^1$, (6), give us:

- $-\infty < \mathcal{X}_s^1 < \infty$ and $\zeta_s(\mathcal{X}_s^1) = 0$. (8)

On the other hand, $\mathcal{X}_s^0$ is defined in (7) by the function $\eta_s(\underline{x}_s) = \int_0^{\mathcal{X}_s^1 - \underline{x}_s} \zeta_s(\underline{x}_s + y) f_s(y) \mathrm{d}y$. Clearly, $\eta_s(\mathcal{X}_s^1) = 0$, and due to monotonicity of ζ_s and (8), $\eta_s(\underline{x}_s)$ is non-decreasing in $\underline{x}_s$. Combined with the definition of $\mathcal{X}_s^0$, (7), we know that if $\lim_{\underline{x}_s \to -\infty} \eta_s(\underline{x}_s) \geq -c_s$, $\mathcal{X}_s^0 = -\infty$. If $\mathcal{X}_s^0 = -\infty$, then clearly $\mathcal{X}_s^0 < \mathcal{X}_s^1$; otherwise, from the definitions of $\mathcal{X}_s^0$ and $\mathcal{X}_s^1$, since $\eta_s(\mathcal{X}_s^0) = -c_s, \eta_s(\mathcal{X}_s^1) = 0$, and η_s is non-decreasing (but not necessarily continuous), we have $\mathcal{X}_s^0 \leq \mathcal{X}_s^1$.

(b) $\underline{x}_s^* = \mathcal{X}_s^0 \vee x_s$ and $\bar{x}_s^* = \mathcal{X}_s^1 \vee x_s$.

It is easy to verify that $(\underline{x}_s^*, \bar{x}_s^*)$ defined above satisfy KKT conditions (1-4). It suffices, therefore, to prove that for any other $(\underline{x}_s', \bar{x}_s') \neq (\underline{x}_s^*, \bar{x}_s^*)$, also satisfying (1-4), we have $V_s(I_s, \underline{x}_s', \bar{x}_s') \geq V_s(I_s, \underline{x}_s^*, \bar{x}_s^*)$.

First, consider the difference only in $\bar{x}_s'$. We show that $V_s(I_s, \underline{x}_s', \bar{x}_s') \geq V_s(I_s, \underline{x}_s', \bar{x}_s^*)$.

- If $\bar{x}_s' > \bar{x}_s^* (\geq \mathcal{X}_s^1)$. From the monotonicity of $\zeta_s(\bar{x}_s)$ and (8), we know that, for all $\bar{x}_s > \bar{x}_s^* \geq \mathcal{X}_s^1$, the first derivative of V_s with respect to $\bar{x}_s$, $\left(1 - F_s(\bar{x}_s - \underline{x}_s')\right) \zeta_s(\bar{x}_s) \geq 0$, which implies $V_s(I_s, \underline{x}_s', \bar{x}_s') \geq V_s(I_s, \underline{x}_s', \bar{x}_s^*)$.

- If $\bar{x}_s' < \bar{x}_s^*$, then, from definition of $\bar{x}_s^*$, $x_s \leq \bar{x}_s' < \mathcal{X}_s^1$, which, combined with (8) and monotonicity of $\zeta_s(\bar{x}_s)$, implies $\zeta_s(\bar{x}_s') \leq \zeta_s(\mathcal{X}_s^1) = 0$. If $\zeta_s(\bar{x}_s') = 0$, then $0 \leq \zeta_s(\bar{x}_s) \leq \zeta_s(\mathcal{X}_s^1) \leq 0$ for all $\bar{x}_s \in [\bar{x}_s', \mathcal{X}_s^1)$. If, on the other hand, $\zeta_s(\bar{x}_s') < 0$, the fact that $(\underline{x}_s', \bar{x}_s')$ satisfies (1), implies $1 = F_s(\bar{x}_s' - \underline{x}_s') \leq F_s(\bar{x}_s - \underline{x}_s') \leq 1$. Therefore, for both cases, $\left(1 - F_s(\bar{x}_s - \underline{x}_s')\right) \zeta_s(\bar{x}_s) = 0$ for all $\bar{x}_s \in [\bar{x}_s', \mathcal{X}_s^1)$, i.e., $V_s(I_s, \underline{x}_s', \bar{x}_s') = V_s(I_s, \underline{x}_s', \bar{x}_s^*)$.

Second, consider the difference in $\underline{x}_s$. We show that $V_s(I_s, \underline{x}_s', \bar{x}_s^*) \geq V_s(I_s, \underline{x}_s^*, \bar{x}_s^*)$.

- From (2) and the monotonicity of η_s, $\mu_s(\underline{x}_s, \bar{x}_s)$ is non-decreasing in $\underline{x}_s$. which implies that $V_s(I_s, \underline{x}_s, \bar{x}_s^*)$ is convex in $\underline{x}_s$ for given $\bar{x}_s^*$. Hence, $V_s(I_s, \underline{x}_s, \bar{x}_s^*)$ is minimized at $\underline{x}_s^*$, since (2) and (4) are satisfied.

This completes the proof of 1.

2. The inductional hypothesis

We prove the results by considering the following three cases for x_s: $x_s \leq \mathcal{X}_s^0$, $\mathcal{X}_s^0 < x_s \leq \mathcal{X}_s^1$, and $\mathcal{X}_s^1 < x_s$. Here, we only present the detailed proof for $\mathcal{X}_s^0 < x_s \leq \mathcal{X}_s^1$ (since the other cases use exactly the same logic).

If $\mathcal{X}_s^0 < x_s \le \mathcal{X}_s^1$, then $\underline{x}_s^* = x_s$ and $\bar{x}_s^* = \mathcal{X}_s^1$, so

$$\begin{aligned} J_s(x_s, \mathbf{D}_s) = E\Big[& b_s(x_s + (\mathcal{X}_s^1 - x_s) \wedge Y_s - D_{s-1,s} - \mathcal{E}_{s,s}(D_{s-1,s})) \\ & + \alpha_s J_{s+1}(x_s + (\mathcal{X}_s^1 - x_s) \wedge Y_s - D_{s-1,s} - \mathcal{E}_{s,s}(D_{s-1,s}), \mathbf{D}_{s+1})\Big]. \end{aligned}$$

Denote y as a realization of Y_s, then $(J_s)_1'(x_s, \mathbf{D}_s) = \int_0^{\mathcal{X}_s^1 - x_s} \zeta_s(x_s + y) f_s(y) \mathrm{dy}$. Using definition of ζ_s and (8), $(J_s)_{11}''(x_s, \mathbf{D}_s) = \int_0^{\mathcal{X}_s^1 - x_s} \zeta_s'(x_s + y) f_s(y) \mathrm{dy} \ge 0$, where inequality follows from monotonicity of ζ_s. This proves $\mathcal{A}_s$.

For the other two cases, notice that if $x_s \le \mathcal{X}_s^0$, then $\underline{x}_s^* = \mathcal{X}_s^0$ and $\bar{x}_s^* = \mathcal{X}_s^1$; if $\mathcal{X}_s^1 < x_s$, then $\underline{x}_s^* = \bar{x}_s^* = x_s$. ■

The double-threshold structure of the optimal policy is intuitively driven by the existence of outsourcing option, which serves as a protection for the manufacturer against her capacity unreliability.

Theorem 1 is a generalization of results of Ciarallo *et al* [5] by including the interdependence among outsourcing, advance demand information, and capacity uncertainty (as compared to only capacity uncertainty in [5]). The underlying problem identified by [5], non-convexity of cost functions, obviously continues to exist in our setting. We model a larger scope of decisions here (production and outsourcing) in the existence of advance demand information (not modeled in [5]) and we do not impose any assumptions on the continuity of demand or capacity distributions or linearity of cost function (all of them implicitly or explicitly assumed in [5]).

We also note that our result differs from that of Gallego and Özer [13], where, with zero set-up costs, they write that "information *beyond* the protection period does not affect the order-up-to level when we assume stationary costs and demand distribution." In [13], with instantaneous replenishment, a decision-maker could ignore all advance demand information except for the current period. In our model, even in stationary setting, we still need to take all known information about demand into account and set production policy accordingly, since we cannot produce the desired quantity we would like with certainty.

Thus, our analysis indicates how production variability makes it more difficult to plan production levels and in turn causes a requirement of better information about future demand. Hence, better demand information is not only needed when demand variability is a problem but also when production variability is an issue.

2.1 Extension to Markov-Modulated Demand

We can extend our model and include dependencies between periods using a Markov-modulated demand process. This captures situations where next periods' demand distributions are related to the current period's demand if not to past periods' demands. Markov-modulated processes have been used in a number of papers (see, e.g., Song and Zipkin [36] and an excellent list of references therein) and in multiple book chapters (see, e.g., Kapuscinski and Tayur [23], p. 16).

We assume that the manufacturer has H-period advance demand information and can divide the states of the world and the corresponding demand into classes such as, e.g., low, medium, or high, with the associated demand distributions. Assume that the set of demand classes, S, is finite, and p_{ij}, for $i, j \in S$, is the transition probability from i to j (given that demand in the current period is in class i, the probability that in the following period, it will be in class j). Consider a finite N-period problem. At the beginning of period s, the manufacturer receives (i) the accurate indicators of the demand type i_s for period s and j_{s+1} for period $s+1$, and (ii) not-necessarily-accurate information of the possible demand realization $D_{s-1,s}$ for period s, which has a random update, $\mathcal{E}^{i_s}_{s,s}(D_{s-1,s})$, with mean of 0, as well as the information $D_{s-1,s+k}$ for period $s+k$, $k = 1, , H-1$, which is subject to a random revision $\mathcal{E}^{j_{s+1}}_{s,s+k}(D_{s-1,s+k})$.

All the other assumptions about capacity, demand information, and cost coefficients remain unchanged. We have the following:

Theorem 2 *Consider a multi-period problem with H-period advance-demand information and a pair of demand class and demand realization with Markov transitions between pairs of states. The optimal policy is a state-dependent double-threshold policy, $\mathcal{X}^0_s(i_s, j_{s+1}, \mathbf{D}_s) \leq \mathcal{X}^1_s(i_s, j_{s+1}, \mathbf{D}_s)$. Furthermore, if the update for the demand in period s depends only on the demand type i_s (but is independent of $D_{s-1,s}$), then both $\mathcal{X}^0_s - D_{s-1,s}$ and $\mathcal{X}^1_s - D_{s-1,s}$ are independent of $D_{s-1,s}$.*

Proof: We omit the proof due to space considerations. ■

2.2 The Properties of Double-Threshold Policy

We now concentrate on the properties and sensitivity to problem parameters of the double-threshold policy, and optimal costs. Specifically, we describe: (1) the "consistency" of the current-period production threshold, (2) the monotonicity of the thresholds with respect to advance demand information, and (3) the effects of the outsourcing capacity on the thresholds.

Our first result states that although the current-period production threshold depends on future periods' capacity distributions, it is independent of the current-period capacity distribution. This result may appear to be surprising at first sight but to see why it is correct, recall that the production threshold is where the firm would like its inventory level to be before facing demand, and therefore the target level remains the same regardless of what the firm may or may not be able to achieve this period. Increased variability in capacity for future periods, however, may require the firm to build more inventory now to protect itself and therefore the current-period threshold will be affected by future periods' capacity distributions.

Proposition 1 *The optimal production threshold, $\mathcal{X}_s^1(\mathbf{D}_s)$, is independent of the capacity distribution and the availability (or unavailability) of outsourcing option in the current period s. (Future-periods' capacity distributions and the availability of outsourcing option obviously influence the threshold.)*

The following results establish the monotonicity of the double-thresholds with respect to advance demand information. The MMFE model allows the manufacturer's information not to be completely accurate. Here, we also assume that a higher demand level, predicted in advance, results in a stochastically higher demand in realization. We refer to advance demand information that has this property as *proper*.

Definition 2 *Advance demand information $\mathbf{D}$ is* proper*, if the coordinate of the induced random variable $\mathbf{D} + \mathcal{E}_{\mathbf{D}}$ is* stochastically increasing *in the corresponding coordinate of $\mathbf{D}$, where $\mathcal{E}_{\mathbf{D}}$ is $\mathbf{D}$'s associated random error or revision. I.e., $D_{s-1,t} + \mathcal{E}_{s,t}(D_{s-1,t})$ is stochastically increasing in $D_{s-1,t}$.*

Based on the requirement that the advance demand information is proper (which we believe is a very realistic and mild requirement), we show the following two properties of the optimal double-threshold policy:

Proposition 2 *If, for all periods s, the H-period advance demand information $\mathbf{D}_s$ is proper, and $\mathcal{E}_s$ are independent, then both $\mathcal{X}_s^0(\mathbf{D}_s)$ and $\mathcal{X}_s^1(\mathbf{D}_s)$ are non-decreasing functions of $D_{s-1,t}$, $t = s, s+1, \cdots, s+H-1$.*

Proof: See Appendix A. ■

Basically, Proposition 2 establishes a sufficient condition for thresholds to increase when forecasts indicate higher demand. Without the "properness" defined in Definition 2, such monotonicity cannot be guaranteed.

As already mentioned, outsourcing gives the manufacturer a protection against unreliable production capacity. It is obvious that the availability (or unavailability) of outsourcing should not change the structure of the optimal production policy. Below we show formally that a bound on outsourcing does not change the structure of the policy, but affects both the production and outsourcing thresholds in a monotonic way.

Proposition 3 *Assume that in any given period s, outsourcing is limited to at most $\bar{v}$ units. The optimal policy is a capacity-constrained double-threshold policy, where the outsourcing quantity is limited by V, i.e., in any period s, $v_s^* = \bar{v} \wedge \left(\mathcal{X}_s^0(\bar{v}) - x_s\right)^+$, and the production threshold is $\mathcal{X}_s^1(\bar{v})$. Furthermore, $\mathcal{X}_s^0(\bar{v})$ and $\mathcal{X}_s^1(\bar{v})$ are non-increasing in $\bar{v}$.*

Proof: The proof is based on ordering of derivatives and is omitted due to space considerations. ■

Proposition 3 shows how outsourcing capacity can affect the production quantities. Clearly, the production capacity and its uncertainty can also significantly affect the thresholds and we explore that relationship in Section 3.

3 Sensitivity of Policy to Capacity Parameters

In Section 2, we characterized the optimal policy structure, which is a double-threshold policy. Here we examine how outsourcing and production thresholds change when capacity is changed. We note that previous literature has not addressed the issue of capacity limitation and its variability. Therefore, it is silent on whether higher or less variable capacity results in less inventory. Simple intuition would predict that, as capacity variability decreases, the firm needs to hold less inventory for future periods and, therefore, both the outsourcing and production thresholds go down. As we analytically show below, this is not always the case.

To describe the changes in randomness of capacity, we compare two facilities, **A** and **B**, that differ in mean or variability of capacity, one at a time, by using *stochastically larger* and *stochastically more variable* capacity.

3.1 Stochastically Larger Capacity

Capacity of production facility **A** is *stochastically larger* than of facility **B**, if the corresponding capacity random variables $Y^A \geq_{st} Y^B$, i.e., for any increasing function $\bar{G}$, $E[\bar{G}(Y^A)] \geq E[\bar{G}(Y^B)]$. Notice that, if the manufacturer's desired production quantity is u units, then he is more likely to fulfill his production plan in facility **A**, i.e., $1 - F^A(u) \geq 1 - F^B(u)$. Notice also that, stochastically larger capacity often leads to a lower utilization under the same demand. Therefore, it is intuitive that facility **A**, compared with facility **B**, has lower costs and lower inventory levels.

Consider an N-period problem. In any period s, with starting state $I_s = (x_s, \mathbf{D}_s)$, the optimal cost for system i (i =A, B) is denoted by $J_s^i(I_s)$, and the corresponding outsourcing and production thresholds are $\mathcal{X}_s^{0i}(\mathbf{D}_s)$ and $\mathcal{X}_s^{1i}(\mathbf{D}_s)$, respectively. We have the following proposition:

Proposition 4 *If facility* **A** *has stochastically larger capacity than facility* **B**, *then*

$$J_s^A(I_s) \leq J_s^B(I_s),\ \mathcal{X}_s^{0A}(\mathbf{D}_s) \leq \mathcal{X}_s^{0B}(\mathbf{D}_s),\ \textit{and}\ \ \mathcal{X}_s^{1A}(\mathbf{D}_s) \leq \mathcal{X}_s^{1B}(\mathbf{D}_s).$$

Proof: See Appendix B. ■

Proposition 4 verifies our intuition that stochastically larger capacity results in lower costs and lower thresholds (and the associated lower inventory levels). However, as the next subsection demonstrates, the same intuition does not automatically carry over to stochastically less variable capacity.

3.2 Stochastically More Variable Capacity

A production facility **A** is *stochastically more variable* than facility **B**, if the corresponding capacity random variables $Y^A \geq_v Y^B$, i.e., for any convex function $\bar{G}$, $E[\bar{G}(Y^A)] \geq E[\bar{G}(Y^B)]$. When, compared with facility **B**, facility **A** has more variable capacity, we conjecture that the manufacturer gains from upgrading facility **A** to **B**.

We first show that higher variability in capacity always results in higher expected costs. However, there is no immediate ordering of the thresholds; this requires additional conditions, as we show in Propositions 6 and 7.

Proposition 5 *If facility* **A** *is stochastically more variable than facility* **B**, *then* $J_s^A(I_s) \geq J_s^B(I_s)$.

Proof: See Appendix C. ■

Consider now the ordering of thresholds. We show that the interactions between capacity variability, the thresholds, and other parameters are somewhat more complex and depend on the distribution of demand. Since we are showing that these thresholds depend on how demand is distributed, here we focus on the case with no advance demand information (otherwise, the results would also depend on the distribution of forecast updates).

We first define three types of demand distribution: *decreasing demand*, *increasing demand*, and *short-tailed concave demand*.

Definition 3 *We call a demand* Z_s, *with* pdf q_s *and* cdf Q_s
(a) **decreasing** *demand if* $q_s(z)$ *is decreasing in* z,
(b) **increasing** *demand if* $q_s(z)$ *is increasing in* z,
(c) **short-tailed unimodal** *demand if* $q_s(z)$ *is unimodal in* z, *and* $Q_s(\arg\max q_s(z)) > \frac{1}{2}$.

For instance, geometrically distributed demand is a decreasing demand. All unimodal distributions (e.g., gamma, normal, or log-normal) truncated below their maximum will result in increasing demand distributions. The most straightforward short-tailed unimodal demand distributions are triangle distributions with "steeper" right-hand-side slope. The following propositions show the dependence of thresholds on type of the distribution:

Proposition 6 *Consider a manufacturer who incurs linear holding cost h and penalty cost p, and faces a decreasing demand. With stochastically more variable capacity, the thresholds are increased.*

Proof: See Appendix D. ■

In the case when the manufacturer faces either an increasing or a short-tailed concave demand, there are no definite trends in the thresholds. The following single-period scenario is a case where the ordering of outsourcing thresholds depends on manufacturer's cost structure.

Definition 4 *Let F^i, for $i = A, B$, be the* cdf *of the corresponding underlying random variable Y^i. If*

- $\forall\, y \geq a,\ 1 - F^A(y) = 1 - F^B(y)$,
- $\forall\, 0 \leq y < a,\ \displaystyle\int_y^\infty (1 - F^A(x))\mathrm{dx} \geq \int_y^\infty (1 - F^B(x))\mathrm{dx}$.

then we define Facility A to be **lower more variable** *than facility B.*

Proposition 7 *Consider a one-period problem, where the manufacturer faces an increasing demand or a short-tailed unimodal demand. Let the holding cost be h, the penalty cost for each backordered unit be p, and the service level $p/(p+h)$ satisfy $\frac{1}{2} < \frac{p}{p+h} < Q(\arg\max q(z))$. Consider two facilities* **A** *and* **B**, *which have the same mean capacity, with facility* **A** *lower more variable than facility* **B**. *Then there exists $c_0 > 0$, such that for outsourcing costs, $c \geq c_0$, the outsourcing thresholds are ordered according to $\mathcal{X}^{0A} \leq \mathcal{X}^{0B}$, while the production thresholds equal $\mathcal{X}^{1A} = \mathcal{X}^{1B} = Q^{-1}(p/p + h)$.*

Proof: See Appendix E. ■

Proposition 7 demonstrates, how a reduction in inventory thresholds can occur with more variable capacity. It implies that a more variable capacity does not necessarily result in a higher outsourcing quantity. The equality of the two production thresholds is due to the fact that we are only considering a one-period problem. In a two-period problem, we can show that the production threshold in the first period is not necessarily higher for facility **A** than for facility **B**. Therefore, we conclude that firms should expect cost reductions from their variability reduction efforts, but although overall costs may decrease, inventory levels and the associated inventory costs may in some cases increase.

4 Computational Results

In the previous sections, we characterized both the structure and the properties of optimal production and outsourcing policies in the presence of advance demand information. Firms use advance demand information and the outsourcing option as mechanisms for reducing uncertainty. This, therefore, raises the question of under what conditions either approach is effective. In this section, through a computational study, we focus on how the environment in which the firm operates influences the benefits gained by using either approach. Specifically, we look at capacity variability, demand variability, utilization, and service level. In all cases we follow exactly the same sequence of events, i.e., demand is realized after production and outsourcing decisions are made and after capacity realization. Our benchmark is a case where neither advance demand information nor outsourcing option is available.

For the purpose of the study, advance demand information consists of one-period accurate advance demand information, i.e., the advance information has no error. (In the study we use the fact that the no-error case may be interpreted as knowing demand before the production and outsourcing decisions are taken.) After calculating the costs for our benchmark case, we estimate the value of the accurate advance demand information in absence of the outsourcing option and compare it to the benchmark. Then, we estimate the value of outsourcing by considering a symmetric situation: the outsourcing option is allowed but there is no advance demand information available. Finally, we estimate the value of the joint use of the accurate current-period demand information and the outsourcing option.

We solve the dynamic programming formulation (MDP) with backward induction using value iteration to obtain the results. In each example, we solve a 20-period horizon problem with initial inventory of 0 (we verified that extending the time horizon does not change any of the conclusions). The benefits are measured as a relative value

$$\frac{\text{cost(Benchmark)} - \text{cost(Considered Approach)}}{\text{cost(Benchmark)}}.$$

To cover a broad range of each of the parameters, we consider five values for each of them, ranging from very small through intermediate to very large. Both the capacity and the demand are assumed to be integer random variables starting from 0, with non-negative mass for each value in the range of the random variable. The shape is assumed to be triangular (we use inversed triangle to achieve hige standard deviations). We keep the mean demand fixed at 18, while the considered standard deviations are 1.8, 5.4, 10.8, 16.2, or 21.6, resulting in the coefficients of variability of capacity equal to 0.1, 0.3, 0.6, 0.9, and 1.2. (To generate the triangular distribution that has the desired mean and desired coefficient of variability, in each case, we assign appropriate probability mass to each of the values between 0 and 68). For each of the demand distributions, we adjust the capacity distribution to achieve the desired target system utilizations and capacity variabilities. The target utilization ranges from 0.6, 0.8, 0.9, 1.0 to 1.2, while the capacity variability is characterized by the same set of coefficients of variability as for the demand distribution. (The capacity distribution is also triangular taking values between 0 and 114. The mean is determined by the utilization and the mass values are determined to achieve the desired mean and coefficient of variability). Service level is defined as penalty cost/(penalty cost + holding cost). The values we use for service level are: 0.5, 0.8, 0.9, 0.95, and 0.99. We fix the holding cost at 0.2 and change the penalty cost to achieve a specific service level. The outsourcing cost is set at 0.2, 0.5, and 0.8 times the penalty cost.

For every combination of the considered parameters (i.e, capacity variability, demand variability, utilization, and service level), and for each of the considered scenarios, we compute the optimal costs, which results in a total of 7500 experiments in this computational study. All the results below present averages across all of those parameters.

4.1 Benefits of Advance Demand Information

As one might expect, advance demand information always benefits the manufacturer. In our setting, where service levels are 50% or higher, it manifests itself through decreased order-up-to levels, or equivalently smaller safety stocks.

Increased capacity variability, utilization, and a higher service level all have similar effects on the value of advance demand information – the value of information is decreasing in any of them. Therefore, we consider them together. The effect of the demand variability, however, is more complicated and, thus, is described separately. The benefits obviously vary across ranges of parameters studied. We typically see the averages of benefits in the range between 0% and 35%.

1. Effects of capacity variability, utilization, and service level.

Figure 1 illustrates the effects of capacity variability and of the service level. Each line represents the average value of the information across the different demand variabilities and utilizations. As capacity variability increases, the value of information decreases; as the service level is increased, the value of information is also reduced. For lower service levels, e.g., around 50%, the benefits span the range of 0-35%. However, as the service level increases, the range of benefits shrinks. The two behavioral patterns for capacity variability and service level can actually be explained using similar logic. Advance demand information allows the firm to lower the (average) order-up-to levels, which results in cost reductions. One-period demand information cannot, however, help with the increasing pressure the manufacturer faces when any of the above two parameters, capacity variability or service level, increases. (That is, even if the firm knows the demand for this period, it will still keep a large safety stock as it is uncertain about the demands for future periods

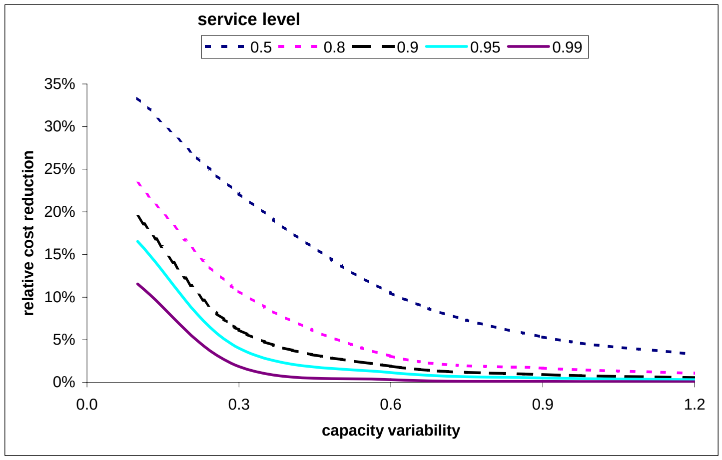

Figure 1: Effects of capacity variability and service level on the value of information.

or its ability to keep up with demand in the future given its capacity uncertainty and high utilizations). Consequently, the order-up-to levels are driven up by either higher capacity variability or service levels. At the same time, however, the absolute difference between the threshold up-to levels for the cases with and without demand information is typically bounded by mean demand for one period. Thus, as variability in capacity increases, order-up-to levels increase both with and without advance demand information but the differences of the order-up-to levels remain fairly constant, resulting in decreasing percentage benefits from information. For instance, for the highest line in Figure 1, with a service level of 0.5, when both demand variability and utilization are set to 0.6, the differences in up-to quantities remain between 18 or 25. When the service level is 0.9, as shown by the third line, for the same demand variability and utilization as above, the difference between the thresholds decreases from 30 down to 19 as capacity variability increases. The same behavior can be observed for the service level – the differences in up-to levels (between the no-information and information cases) are bounded and the information benefits become smaller as service levels increase. Note that even though we compare the benefits of 1-period demand information to no information (comparisons with the case with more periods' information becomes time consuming), the same logic applies to any constant number of periods worth of demand information.

Figure 2 illustrates the effect of utilization on the average value of information. Each point in Figure 2 shows the value of information for the given level of demand variability and utilization level (averaged over all 25 cases for capacity variability and service level). As expected, lower utilization translates into higher **relative** information benefits. For low utilizations, e.g., 60%, the relative costs reductions reach 30%, while for high utilizations there are noticeably smaller (e.g., for 90% they go only up to 6.6%). The drivers of this behavior are exactly the same as for increasing capacity variability and service level above. (Note our emphasis on "relative" benefits as in lower utilizations, costs are lower and therefore a high percentage savings of a low base cost can also be low in absolute terms). Note

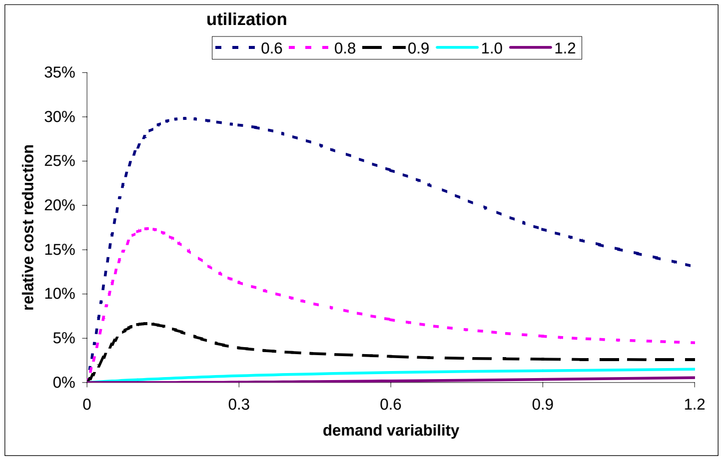

Figure 2: Effects of utilization and demand variability on the value of information.

that these results are similar in their spirit to the computational results in [14] and [37], who also notice decreasing effect of information (although defined differently than here) as a function of capacity.

2. Effects of demand variability.

Figure 2 also presents the effect of demand variability on value of information. The value of information is increasing first and decreasing later, with the highest value achieved at some intermediate value of demand variability. Its pseudo-concave shape, as shown in Figure 2 is maintained across all levels of capacity variability, service level, and utilization, with the maximum obviously dependent on the combination of these three factors.

When the demand variability is low, such as 0.1, the system acts almost like a deterministic one and the information is marginally beneficial. On the other hand, when the demand is highly uncertain, current-period demand information is valuable in absolute terms. However, since demand information only reveals the demand for one period, as demand uncertainty increases, the firm is once again forced to keep larger safety stocks. Furthermore, the differences in the threshold levels between the case with and without information remain fairly constant. Thus, when demand variability is really high, the value of demand information also decreases in "relative" terms. We note that our conclusions would remain the same even if the firm obtained advance demand information for more than one period as long as the demand information is for a limited number of periods (i.e., clearly, demand information would be very valuable in relative and absolute terms if it was available for all periods thus making demand deterministic).

4.2 Benefits of Outsourcing

Similar to the case where advance demand information is available, having an option to outsource part of the demand can reduce the manufacturer's costs by allowing for lower inventory levels and, thus, decreased holding costs. In the study we observe a wide range of benefits – relative cost reductions are usually between 20% and 85%. What is really interesting is that, in general, the effects of the system parameters on the value of outsourcing option have the opposite trends to that for the value of demand information. That is, the conditions that tend to make outsourcing more valuable tend to make demand information less valuable and vice versa. For instance, the higher the capacity variability, the utilization, or the service level, the larger the relative gains from outsourcing but lower benefits from advance demand information.

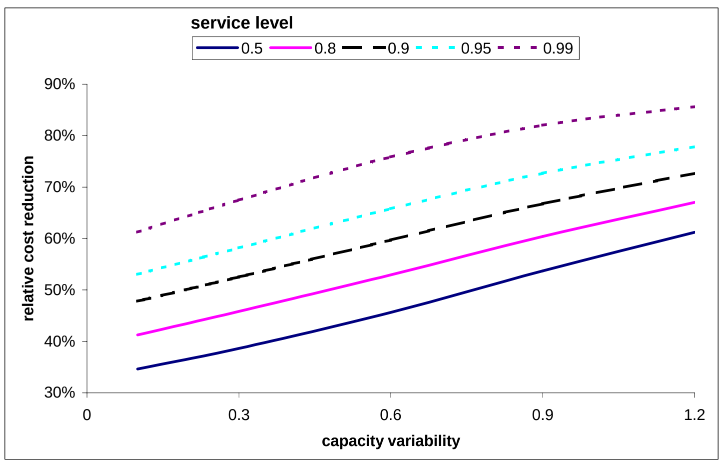

Figure 3: Effects of capacity variability and service level on the value of outsourcing.

1. The effects of the capacity variability, the service level, and the utilization

In Figure 3, we present average percentage cost reduction as a function of service level and capacity variability. (Each point in the graph represents an average percentage reduction over all 75 cases of demand variability, utilization, and outsourcing cost). The figure clearly shows that the benefits from outsourcing increase as the capacity becomes more variable – a behavior one might expect. At the same time, the higher the service level the firm aims to achieve, the higher the value of outsourcing. For high service level (99%) we see reductions of 60% to 85%, while for lower service levels (50%) the reductions typically do not exceed 65%. The higher service levels do require dramatically increased safety stocks when no outsourcing is available, but in the presence of outsourcing (especially when outsourcing costs are relatively small), this increase in the safety stocks is mild. Thus, the option that the firm has to outsource when needed is becoming more valuable as the firm aims at providing its customers a higher service level.

Figure 4 shows that the effects of utilization are similar – outsourcing is more beneficial for higher utilizations. Intuitively, capacity variability, service level, and utilization, all have similar effects on the value of outsourcing. Increasing any of them translates into a higher pressure on capacity. The outsourcing option, however, can directly and significantly decrease this pressure and therefore becomes valuable.

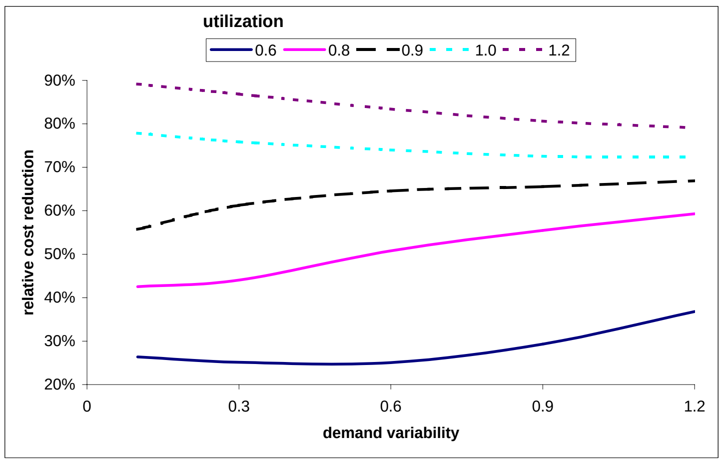

Figure 4: Effects of utilization and demand variability on the value of outsourcing

2. The effects of the demand variability.

The effects of demand variability on the value of outsourcing are more complicated. For high values of utilization, e.g., 1.2 and 1, the value of outsourcing tends to be decreasing and for smaller values, e.g., 0.6, 0.8, and even 0.9, it tends to be increasing in demand variability – see Figure 4. The initial part of the curve at a utilization level of 0.6 is, however, decreasing. Also, it is possible to find some individual cases (as opposed to the average behavior presented here), where some minor non-monotonicity takes place.

This phenomenum can be explained as follows: When the utilization level is very high, it is very difficult to meet the demand, thus necessitating the system to carry a lot of inventory (threshold levels are very high). Although outsourcing helps, as demand variability increases, safety stocks increase even further and the relative decrease in cost due to outsourcing becomes smaller. That is, at high levels of utilization, the cost increases due to increases in safety stocks is much more significant than any decrease that one could obtain from outsourcing. On the other hand, in a low or moderate utilization environment where demand variability is low (as seen when utilization is 0.6), outsourcing does not have much value. When demand variability becomes higher, more safety stocks will be carried again, but also the outsourcing threshold increases. Therefore, as demand variability increases, outsourcing becomes a more useful (and cost saving) option.

4.3 Joint Benefits of Information and Outsourcing

So far, we have only considered the benefits of value of information or outsourcing individually. However, if the manufacturer has access to better demand information as well outsourcing, the benefits from the combined use of better information and this source of safety capacity can sometimes result in much larger savings than the sum of the savings from using either resource. Also, whether outsourcing or better information will be most useful individually depends on the particular environment that the firm operates in.

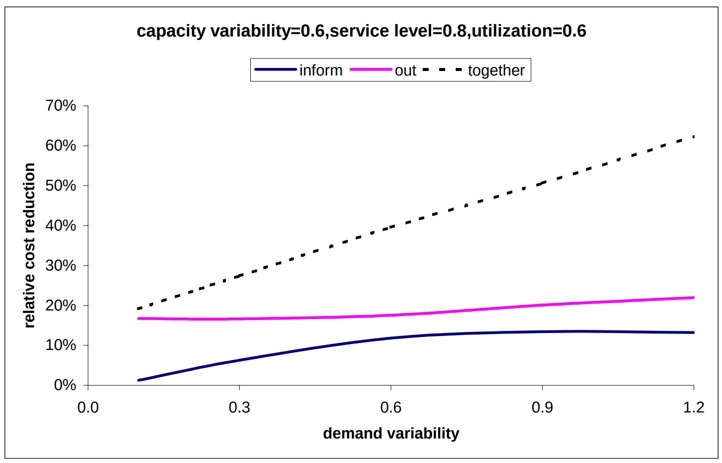

Figure 5: Joint value of information and outsourcing (By demand variability)

In Figure 5, we demonstrate the cost savings obtained by using outsourcing or demand information individually and jointly. In this example, the rather interesting fact is that as demand variability increases, the reduction of costs obtained by using both approaches at the same time far exceeds the sum of the savings from using either approach. This has a fairly simple explanation. In this environment, the system is pressured to keep higher inventory levels with increased demand uncertainty. The advance demand information helps in the current period (or in general, for a fixed number of periods), but does not change the uncertainty in future periods. To protect against future stockouts, the manufacturer, therefore, still needs to build a fairly high level of safety stock. With additional outsourcing option (as long as outsourcing costs are fairly reasonable), however, the manufacturer can cut down the safety stock significantly as it now has very good information on the current period, and therefore does not need to carry large inventories to meet current demand, **and** because outsourcing is available, she does not need to carry large inventories for spikes in the future either.

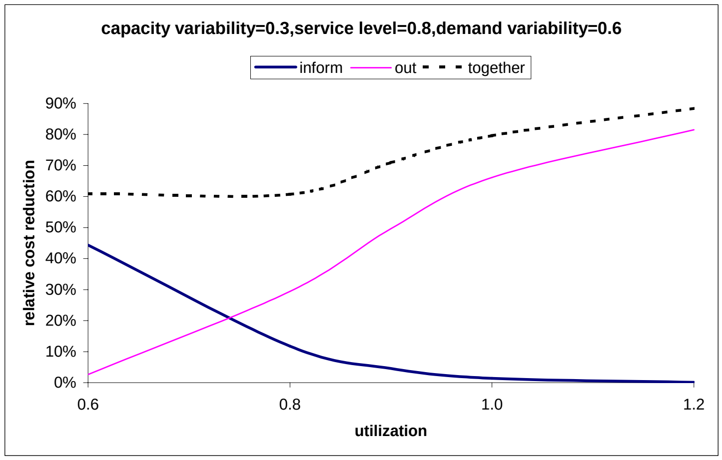

Figure 6: Joint value of information and outsourcing (By utilization)

Figure 6 provides another example of how the combined benefits from both approaches can exceed the sum of the benefits form each. The example illustrates the benefits obtained from either (or both) approaches as a function of system utilization. The system under consideration in Figure 6 has low capacity variability and moderate demand variability. The main pressure it faces when the utilization is high is clearly due to the lack of capacity. Therefore, in high utilization states, outsourcing has a very large benefit. When utilization is low, capacity availability is no longer a challenge but decreasing the safety stock by reducing demand uncertainty can make a significant difference and, therefore, demand information has more value than outsourcing. The interesting case is under medium utilization, where once again the combined benefits of both information and outsourcing far exceed the sum of the benefits from each approach.

We have, therefore, shown that whether firms should give serious consideration to ensuring outsourcing is an option or investing in better information systems and working with customers to get better demand information depends on the environment that they operate in now. Demand and production variability, utilization, and costs of outsourcing as well as the desired service level have a key influence on which factor will have more impact on costs. Furthermore, the benefits of using both approaches simultaneously can exceed the sum of the benefits from each approach.

5 Conclusions and Further Research

We have developed a general multi-period production-inventory model with demand and capacity uncertainty to consider the conditions under which it is best for a firm to invest in better demand information versus in sources of extra capacity. We were able to characterize the structure of optimal production and inventory policies under these two types of uncertainty. We were also able to show that in many situations, one or the other of these two mechanisms to protect against uncertainty have larger impact on reducing a firm's costs, although in some situations, investing in both can lead to large cost savings.

Our model has addressed these issues only in the context of a single manufacturer serving a stochastic demand. Recent research has focused on the impact of better information sharing in multi-echelon supply chains. Further research should extend our analysis to multi-echelon supply chains and characterize the benefits of safety capacity or advance demand information at different echelons.

A Proof of Proposition 2

Proof: It suffices to prove that $\zeta_s^H(\bar{x}_s)$ and $\eta_s^H(\underline{x}_s)$ are non-increasing in $D_{s-1,s-1+k}$, $k = 1, \cdots, H$. Denote $\widehat{\mathcal{E}}_{s,s-1+k}(D_{s-1,s-1+k}) = D_{s-1,s-1+k} + \mathcal{E}_{s,s-1+k}(D_{s-1,s-1+k})$. Since $\mathbf{D}_s$ is proper for all periods s, it follows that $\widehat{\mathcal{E}}_{s,s-1+k}(D_{s-1,s-1+k})$ is stochastically increasing in $D_{s-1,s-1+k}$ for all s's.

Denote $(J_s)''_{1,k+1}(x_s, \mathbf{D}_s) = \dfrac{\partial}{\partial D_{s-1,s-1+k}}\Big[\dfrac{\partial}{\partial x_s} J_s(x_s, \mathbf{D}_s)\Big]$, then the proof of the proposition is based on the inductional hypothesis for period $s+1$: $\underline{(J_{s+1})''_{1,k+1}(x_{s+1}, \mathbf{D}_{s+1}) \leq 0,}$ $\underline{k = 1, \cdots, H}$, which obviously holds for ending period $(s+1 =)N+1$. We can consequently prove the following two facts (a) and (b), and finally prove the inductional hypothesis for period s in (c):

(a) $\zeta_s^H(\bar{x}_s)$ *and* $\eta_s^H(\underline{x}_s)$ *are non-increasing in* $D_{s-1,s}$.

Note that $D_{s-1,s}$ is a parameter influencing ζ_s^H and $\mathcal{X}_s^1(\mathbf{D}_s)$ in definition of η_s^H. Since $\widehat{\mathcal{E}}_{s,s}(D_{s-1,s})$ is stochastically increasing in $D_{s-1,s}$, from definition of ζ_s^H, convexity of b_s, and Theorem 1, we have:

- $E[b'_s(\bar{x}_s - D_{s-1,s} - \mathcal{E}_{s,s}(D_{s-1,s}))]$ is non-increasing in $D_{s-1,s}$;
- $\tilde{g}(\epsilon) := E(J_{s+1})'_1(\bar{x}_s - \epsilon, \mathbf{D}_{s+1})$ is non-increasing in ϵ, and

$$\begin{aligned} E[\tilde{g}(\widehat{\mathcal{E}}_{s,s}(D_{s-1,s}))] &= E\tilde{g}(D_{s-1,s} + \mathcal{E}_{s,s}(D_{s-1,s})) \\ &= E(J_{s+1})'_1(\bar{x}_s - D_{s-1,s} - \mathcal{E}_{s,s}(D_{s-1,s}), \mathbf{D}_{s+1}); \end{aligned}$$

- for any $D'_{s-1,s} \geq D_{s-1,s}$, since $\widehat{\mathcal{E}}_{s,s}(D'_{s-1,s}) \geq_{st} \widehat{\mathcal{E}}_{s,s}(D_{s-1,s})$, therefore $E[\tilde{g}(\widehat{\mathcal{E}}_{s,s}(D'_{s-1,s}))] \leq E[\tilde{g}(\widehat{\mathcal{E}}_{s,s}(D_{s-1,s}))]$.

Hence, $\zeta_s^H(\bar{x}_s)$ is non-increasing in $D_{s-1,s}$ and $\mathcal{X}_s^1(\mathbf{D}_s)$ is non-decreasing in $D_{s-1,s}$. Due to the opposite monotonicities of $\zeta_s^H(\bar{x}_s)$ and $\mathcal{X}_s^1(\mathbf{D}_s)$, from definition of η_s^H, as well as $\zeta_s^H(\mathcal{X}_s^0(\mathbf{D}_s)) = 0$, we conclude that $\eta_s^H(\underline{x}_s)$ is also non-increasing in $D_{s-1,s}$, therefore, $\mathcal{X}_s^0(\mathbf{D}_s)$ is non-decreasing in $D_{s-1,s}$.

(b) *$\zeta_s^H(\bar{x}_s)$ and $\eta_s^H(\underline{x}_s)$ are non-increasing in $D_{s-1,s-1+k}$, $k = 2, \cdots, H$.*

Note that $\widehat{\mathcal{E}}_{s,s-1+k}(D_{s-1,s-1+k})$ is stochastically increasing in $D_{s-1,s-1+k}$ for all s's. From definition of ζ_s^H and $(J_{s+1})''_{1k}(x_{s+1}, \mathbf{D}_{s+1}) \leq 0$, it follows that:

- $\bar{g}^{k-1}(\epsilon) := E(J_{s+1})'_1(\bar{x}_s - D_{s-1,s} - \mathcal{E}_{s,s}(D_{s-1,s}), \mathbf{D}_{s+1}^{-(k-1)}, \epsilon, \mathbf{D}_{s+1}^{+(k-1)})$ is non-increasing in ϵ, where $\mathbf{D}_{s+1}^{-(k-1)} = (D_{s,s+1}, \cdots, D_{s,s+k-2})$, $\mathbf{D}_{s+1}^{+(k-1)} = (D_{s,s+k}, \cdots, D_{s,s+H})$, and

$$\begin{aligned} E[\bar{g}^{k-1}(\widehat{\mathcal{E}}_{s,s-1+k}(D_{s-1,s-1+k}))] &= E\bar{g}^{k-1}(D_{s-1,s-1+k} + \mathcal{E}_{s,s+k}(D_{s-1,s-1+k})) \\ &= E(J_{s+1})'_1(\bar{x}_s - D_{s-1,s} - \mathcal{E}_{s,s}(D_{s-1,s}), \mathbf{D}_{s+1}); \end{aligned}$$

- for any $D'_{s-1,s-1+k} \geq D_{s-1,s-1+k}$, since $\widehat{\mathcal{E}}_{s,s-1+k}(D'_{s-1,s-1+k}) \geq_{st} \widehat{\mathcal{E}}_{s,s-1+k}(D_{s-1,s-1+k})$, therefore $E[\bar{g}^{k-1}(\widehat{\mathcal{E}}_{s,s-1+k}(D'_{s-1,s-1+k}))] \leq E[\bar{g}^{k-1}(\widehat{\mathcal{E}}_{s,s-1+k}(D_{s-1,s-1+k}))]$.

Also, $E[b'_s(\bar{x}_s - D_{s-1,s} - \mathcal{E}_{s,s}(D_{s-1,s}))]$ is obviously independent of $D_{s-1,s-1+k}$. Hence, $\zeta_s^H(\bar{x}_s)$ is non-increasing in $D_{s-1,s-1+k}$, and $\mathcal{X}_s^1(\mathbf{D}_s)$ is non-decreasing in $D_{s-1,s-1+k}$. Using the same logic as in the proof of part (a), we also conclude that $\eta_s^H(\underline{x}_s)$ is non-increasing in $D_{s-1,s-1+k}$. Therefore, $\mathcal{X}_s^0(\mathbf{D}_s)$ is non-decreasing in $D_{s-1,s-1+k}$.

(c) Now we prove the inductional hypothesis for period s, i.e., $(J_s)''_{1,k+1}(x_s, \mathbf{D}_s) \leq 0$, for $k = 1, \cdots, H$.

This can be proved by considering three cases for x_s: $x_s \leq \mathcal{X}_s^0(\mathbf{D}_s)$, $\mathcal{X}_s^0(\mathbf{D}_s) < x_s \leq \mathcal{X}_s^1(\mathbf{D}_s)$, and $\mathcal{X}_s^1(\mathbf{D}_s) < x_s$, to derive the formulation of $(J_s)'_1(x_s, \mathbf{D}_s)$. But here we only provide the detailed proof for $\mathcal{X}_s^0(\mathbf{D}_s) < x_s \leq \mathcal{X}_s^1(\mathbf{D}_s)$, where $\underline{x}_s^* = x_s$ and $\bar{x}_s^* = \mathcal{X}_s^1(\mathbf{D}_s)$. Then,

$$(J_s)'_1(x_s, \mathbf{D}_s) = \int_0^{\mathcal{X}_s^1(\mathbf{D}_s) - x_s} \zeta_s^H(x_s + y) f_s(y) \mathrm{d}y.$$

<u>For $k = 1$</u>: Since $\widehat{\mathcal{E}}_{s,s}(D_{s-1,s})$ is stochastically increasing in $D_{s-1,s}$, by the convexity of b_s, and Theorem 1, the following hold:

- $E[b'_s(x_s + y - D_{s-1,s} - \mathcal{E}_{s,s}(D_{s-1,s}))]$ is non-increasing in $D_{s-1,s}$ (as proof in part (a));
- $\hat{g}(\epsilon) := E(J_{s+1})'_1(x_s + y - \epsilon, \mathbf{D}_{s+1})$ is non-increasing in ϵ, and

$$\begin{aligned} E[\hat{g}(\widehat{\mathcal{E}}_{s,s}(D_{s-1,s}))] &= E\hat{g}(D_{s-1,s} + \mathcal{E}_{s,s}(D_{s-1,s})) \\ &= E(J_{s+1})'_1(x_{s+1}, \mathbf{D}_{s+1}); \end{aligned}$$

- for any $D'_{s-1,s} \geq D_{s-1,s}$, since since $\widehat{\mathcal{E}}_{s,s}(D'_{s-1,s}) \geq_{st} \widehat{\mathcal{E}}_{s,s}(D_{s-1,s})$, we have $E[\hat{g}(\widehat{\mathcal{E}}_{s,s}(D_{s-1,s}))] \leq E[\hat{g}(\widehat{\mathcal{E}}_{s,s}(D_{s-1,s}))]$.

Combined with $\zeta_s^H(\mathcal{X}_s^0(\mathbf{D}_s)) = 0$, the above three facts imply that $(J_s)'_1(x_s, \mathbf{D}_s)$ is non-increasing in $D_{s-1,s}$.
For $k = 2, \cdots, H$: Since b_s is independent of $D_{s-1,s-1+k}$, by the similar logic in (b), we can immediately get the conclusion from $(J_{s+1})''_{1k}(x_{s+1}, \mathbf{D}_{s+1}) \leq 0$. ■

B Proof of Proposition 4

Proof: Let

$$\begin{aligned}
(\zeta_s^H)^i(\bar{x}_s) &= E[b'_s(\bar{x}_s - D_{s-1,s} - \mathcal{E}_{s,s}(D_{s-1,s})) \\
&\quad + \alpha_s (J_{s+1}^i)'_1(\bar{x}_s - D_{s-1,s} - \mathcal{E}_{s,s}(D_{s-1,s}), \mathbf{D}_{s+1})] \\
(\eta_s^H)^{ij}(\underline{x}_s) &= \int_0^{\mathcal{X}_s^{1j}(\mathbf{D}_s) - \underline{x}_s} (\zeta_s^H)^j(\underline{x}_s + y) f_s^i(y)\mathrm{d}y, \text{ and } i, j = \text{A,B},
\end{aligned}$$

with $(\eta_s^H)^{ij}$ corresponding to an intermediate construct, where instead of using parameters of system j throughout, the distribution of capacity in the current period alone is replaced by that of i.

We prove, by induction in s, the following four points:

$\hat{a}_s$. $\mathcal{X}_s^{1A} \leq \mathcal{X}_s^{1B}$;

$\hat{b}_s$. $\mathcal{X}_s^{0A} \leq \mathcal{X}_s^{0B}$;

$\hat{c}_s$. $(J_s^A)'_1(I_s) \geq (J_s^B)'_1(I_s)$, which implies, $(\zeta_{s-1}^H)^A(\bar{x}_{s-1}) \geq (\zeta_{s-1}^H)^B(\bar{x}_{s-1})$;

$\hat{d}_s$. $J_s^A(I_s) \leq J_s^B(I_s)$.

It is trivial to show that these conditions hold for $s = N + 1$.

Notice that $\hat{a}_s$ follows immediately from $\hat{c}_{s+1}$. Also, $\hat{d}_s$ follows directly from $\hat{d}_{s+1}$. To prove $\hat{b}_s$, we need to define the following function of y:

$$\theta_s^i(x)(y) = \begin{cases} (\zeta_s^H)^i(x+y) & \text{if } y < \mathcal{X}_s^{1i} - x \\ 0 & \text{if } y \geq \mathcal{X}_s^{1i} - x \end{cases}$$

It is easy to verify that $\theta_s^i(x)(y) = \min((\zeta_s^H)^i(x+y), 0)$ (implying $\theta_s^i(x)(y) \leq 0$) and that $\theta_s^i(x)(y)$ is increasing in y. From $\hat{c}_{s+1}$, $\theta_s^A(x)(y) \geq \theta_s^B(x)(y)$. Definition of $(\eta_s^H)^{ii}(\underline{x}_s)$ implies that $(\eta_s^H)^{ii}(\underline{x}_s) = E[\theta_s^i(\underline{x}_s)(Y^i)]$. Due to monotonicity of $\theta_s^i(\underline{x}_s)(y)$ in y and $Y^A \geq_{st} Y^B$, we have

$$(\eta_s^H)^{AA}(\underline{x}_s) = E[\theta_s^A(\underline{x}_s)(Y^A)] \geq E[\theta_s^A(\underline{x}_s)(Y^B)] \geq E[\theta_s^B(\underline{x}_s)(Y^B)] = (\eta_s^H)^{BB}(\underline{x}_s).$$

From definition of $\mathcal{X}_s^{0i}$, $(\eta_s^H)^{ii}(\mathcal{X}_s^{0i}) = -c_s$. Thus, we have $\mathcal{X}_s^{0A} \leq \mathcal{X}_s^{0B}$.

To prove $\hat{c}_s$, note that $J_s^i(x_s, \mathbf{D}_s)$ is convex in x_s, and reaches its minimum at $\mathcal{X}_s^{1i}$. Based on $\hat{a}_s$ and $\hat{b}_s$, we consider the following cases (some of the intervals may be empty) and show that inequality holds for each of them:

- $x_s \leq \mathcal{X}_s^{0B}$: $(J_k^A)'_1(x_s, \mathbf{D}_s) \geq -c_s = (J_s^B)'_1(x_s, \mathbf{D}_s)$;
- $\mathcal{X}_s^{0B} < x_s \leq \mathcal{X}_s^{1A}$: Since $(J_k^i)'_1(x_s, \mathbf{D}_s) = \int_0^{\mathcal{X}_s^{1i}-x_s} (\zeta_s^H)^i(x_s+y) f_s^i(y)\mathrm{d}y = E_{Y_s^i}[\theta_s^i(x_s)(Y_s^i)]$,

  $(J_s^A)'_1(x_s, \mathbf{D}_s) = E[\theta_s^A(x_s)(Y_s^A)] \geq E[\theta_s^A(x_s)(Y_s^B)] \geq E[\theta_s^B(x_s)(Y_s^B)] = (H_s^B)'_1(x_s, \mathbf{D}_s)$;
- $\mathcal{X}_s^{1A} < x_s \leq \mathcal{X}_s^{1B}$: $(J_s^A)'_1(x_s, \mathbf{D}_s) \geq 0 \geq (J_s^B)'_1(x_s, \mathbf{D}_s)$;
- $\mathcal{X}_s^{1B} < x_s$: $(J_s^i)'_1(x_s, \mathbf{D}_s) = E[b'_s(x_s - D_{s-1,s} - \mathcal{E}_{s,s}(D_{s-1,s})) + \alpha_s(J_{s+1}^i)'_1(x_{s+1}, \mathbf{D}_{s+1})]$, using $\hat{c}_{s+1}$, we get $(J_s^A)'_1(x_s, \mathbf{D}_s) \geq (J_s^B)'_1(x_s, \mathbf{D}_s)$. ■

C Proof of Proposition 5

Proof: We prove by induction that $J_s^A(x_s, \mathbf{D}_s) \geq J_s^B(x_s, \mathbf{D}_s)$. The statement holds trivially for the final period $N+1$. Suppose it holds for period $s+1$. For $i, j = A, B$, let us define

$$\begin{aligned}
hh_s^i(I_s, \underline{x}_s, \bar{x}_s) &= E[b_s(\bar{x}_s - D_{s-1,s} - \mathcal{E}_{s,s}(D_{s-1,s})) \\
&\quad + \alpha_s(J_{s+1})_1(\bar{x}_s - D_{s-1,s} - \mathcal{E}_{s,s}(D_{s-1,s}), \mathbf{D}_{s+1})] \\
\bar{V}_s^{ij}(I_s, \underline{x}_s, \bar{x}_s) &= E_{Y_s^i}[hh_s^j(I_s, \underline{x}_s, (\underline{x}_s + (\bar{x}_s - \underline{x}_s) \wedge Y_s^i))] \\
V_s^{ij}(I_s, \underline{x}_s, \bar{x}_s) &= c_s(\underline{x}_s - x_s) + \bar{V}_s^{ij}(I_s, \underline{x}_s, \bar{x}_s)
\end{aligned}$$

$\bar{V}_s^{ij}(I_s, \underline{x}_s, \bar{x}_s)$ and $V_s^{ij}(I_s, \underline{x}_s, \bar{x}_s)$ represent the scenario where the distribution of capacity in the current period, s, is i, while in periods $s+1$ through $N+1$ is j, and, for $\bar{V}_s^{ij}(I_s, \underline{x}_s, \bar{x}_s)$, outsourcing is not allowed in the current period s. Then,

$$\begin{aligned}
\bar{J}_s^i(I_s, \underline{x}_s) &:= \min_{\bar{x}_s \geq \underline{x}_s} \bar{V}_s^{ii}(I_s, \underline{x}_s, \bar{x}_s) \\
J_s^i(I_s) &= \min_{\bar{x}_s \geq \underline{x}_s \geq x_s} V_s^{ii}(I_s, \underline{x}_s, \bar{x}_s) = \min_{\underline{x}_s \geq x_s} \{c_s(\underline{x}_s - x_s) + \bar{J}_s^i(I_s, \underline{x}_s)\} \qquad (9)
\end{aligned}$$

and $J_{N+1}^i \equiv 0$. Clearly the total cost can be viewed as an effect of outsourcing followed by production and, based on (9) above, it is sufficient to show that $\bar{J}_s^A(I_s, \underline{x}_s) \geq \bar{J}_s^B(I_s, \underline{x}_s)$, or equivalently to consider the scenario when outsourcing is not allowed in the current period.

From inductional assumption, clearly $hh_s^A(I_s, \underline{x}_s, \bar{x}_s) \geq hh_s^B(I_s, \underline{x}_s, \bar{x}_s)$. Suppose $\mathcal{X}_s^{1ij}$ is the optimal production threshold for $\bar{V}_s^{ij}(I_s, \underline{x}_s, \bar{x}_s)$. Then $\mathcal{X}_s^{1i} = \mathcal{X}_s^{1ii}$. Based on Proposition

1, we clearly have $\mathcal{X}_s^{1ji} = \mathcal{X}_s^{1i}$. Considering the total cost as a function of capacity realization y:

$$\psi_s^{\underline{x}_s,i}(y) = \begin{cases} E[b_s(\underline{x}_s + y - D_{s-1,s} - \mathcal{E}_{s,s}(D_{s-1,s})) \\ \quad +\alpha_s(J_{s+1})_1(\underline{x}_s + y - D_{s-1,s} - \mathcal{E}_{s,s}(D_{s-1,s}), \mathbf{D}_{s+1})] & \text{if } y < \mathcal{X}_s^{1i} - \underline{x}_s \\ E[b_s(\mathcal{X}_s^{1i}(\mathbf{D}_s) - D_{s-1,s} - \mathcal{E}_{s,s}(D_{s-1,s})) \\ \quad +\alpha_s(J_{s+1})_1(\mathcal{X}_s^{1i}(\mathbf{D}_s) - D_{s-1,s} - \mathcal{E}_{s,s}(D_{s-1,s}), \mathbf{D}_{s+1})] & \text{if } y \geq \mathcal{X}_s^{1i} - \underline{x}_s \end{cases}$$

we get the following

- Since $\mathcal{X}_s^{1i} = \mathcal{X}_s^{1ji}$, given any value of $\underline{x}_s$, we have

$$\bar{J}_s^i(I_s, \underline{x}_s) = E\Big[\psi_s^{\underline{x}_s,i}(Y^i)\Big] \quad \text{and} \quad \min_{\{\bar{x}_s | \bar{x}_s \geq \underline{x}_s\}} \bar{V}_s^{ji}(I_s, \underline{x}_s, \bar{x}_s) = E\Big[\psi_s^{\underline{x}_s,i}(Y^j)\Big]$$

- $\psi_s^{\underline{x}_s,i}(y)$ is a decreasing convex function of y

Now, for period s: $Y^A \geq_v Y^B \Rightarrow \bar{J}_s^A(I_s, \underline{x}_s) \geq E\Big[\psi_s^{\underline{x}_s,A}(Y^B)\Big]$. On the other hand, as argued above, $hh_s^A(I_s, \underline{x}_s, \bar{x}_s) \geq hh_s^B(I_s, \underline{x}_s, \bar{x}_s)$, which implies $\bar{V}_s^{BA}(I_s, \underline{x}_s, \bar{x}_s) \geq \bar{V}_s^{BB}(I_s, \underline{x}_s, \bar{x}_s)$. Therefore, we have $E\Big[\psi_s^{\underline{x}_s,A}(Y^B)\Big] \geq \bar{J}_s^B(I_s, \underline{x}_s)$, which implies $\bar{J}_s^A(I_s, \underline{x}_s) \geq \bar{J}_s^B(I_s, \underline{x}_s)$, for any $\underline{x}_s$, which we needed to prove. ■

D Proof of Proposition 6

Proof: Since $b_s(x) = E[h(x - Z_s)^+ + p(x - Z_s)^-]$, recall that *c.d.f.* of Z_s is Q_s. Decreasing demand implies that $Eb_s'(x) = -p + (h + p)Q_s(x)$ is concave for all the periods s. We first prove, by induction, that $(J_s)_1'(x_s)$ is concave in x_s.

The following three cases (for x_s) can take place: $x_s \leq \mathcal{X}_s^0$, $\mathcal{X}_s^0 < x_s \leq \mathcal{X}_s^1$, and $\mathcal{X}_s^1 < x_s$. Since they are similar, we provide the detailed proof only for $\mathcal{X}_s^0 < x_s \leq \mathcal{X}_s^1$, where $\underline{x}_s^* = x_s$ and $\bar{x}_s^* = \mathcal{X}_s^1$. The result obviously holds for $N+1$. Suppose it also holds for $s+1$. Then

$$(J_s)_{111}'''(x_s) = \int_0^{\mathcal{X}_s^1 - x_s} \zeta_s''(x_s + y) f_s(y) \mathrm{d}y - \zeta_s'(\mathcal{X}_s^1) f_s(\mathcal{X}_s^1 - x_s) \leq 0.$$

The inequality follows form non-negativity of ζ_s' (based on convexity of b_s and of J_{s+1}, i.e., Theorem 1) and negativity of ζ_s'' (based on concavity of Eb_s' and of $(J_{s+1})_1'$, which is the inductional assumption).

Then, based on monotonicity and concavity of ζ_s, we have that:

$$\theta_s^i(x)(y) = \begin{cases} \zeta_s^i(x + y) & \text{if } y < \mathcal{X}_s^{1i} - x \\ 0 & \text{if } y > \mathcal{X}_s^{1i} - x \end{cases}$$

is increasing and concave in y. Concavity of $\theta_s^i(x)(y)$, combined with stochastically more variable property of production, implies that $\eta_s^{AA} \leq \eta_s^{BB}$, where $\eta_s^{ii}(\underline{x}_s) = E[\theta_s^i(\underline{x}_s)(Y^i)]$, $i = A, B$. Formally, by induction that uses the logic of Appendix B, the following hold for any period s:

a'_s. $\mathcal{X}_s^{1A} \geq \mathcal{X}_s^{1B}$;

b'_s. $\mathcal{X}_s^{0A} \geq \mathcal{X}_s^{0B}$;

c'_s. $(J_s^A)'_1(x_s) \leq (J_s^B)'_1(x_s)$, consequently, $\zeta_{s-1}^A(\bar{x}_{s-1}) \leq \zeta_{s-1}^B(\bar{x}_{s-1})$.

which justify Proposition 6. ■

E Proof of Proposition 7

Proof: Using the Newsvendor logic, the production thresholds in both facilities are $\mathcal{X}^1 := \mathcal{X}^{1A} = \mathcal{X}^{1B} = Q^{-1}(p/p+h) > 0$.

For $i = A, B$, let $\gamma^i(x) := \int_0^{\mathcal{X}^1 - x}[-p + (h+p)Q(x+y)]f^i(y)\mathrm{dy}$. Clearly, $\gamma^i(\mathcal{X}^1) = 0$, and $\gamma^i(x)$ is non-decreasing in x.

Since **A** is lower more variable than **B**, there exists $a > 0$, such that for all $y \geq a$, $1 - F^A(y) = 1 - F^B(y)$, that is, for all $y \geq a$, $f^A(y) = f^B(y)$. Hence, for $x \leq \mathcal{X}^1 - a$.

$$\gamma^A(x) - \gamma^B(x) = \int_0^a [-p + (h+p)Q(x+y)]f^A(y)\mathrm{dy} - \int_0^a [-p + (h+p)Q(x+y)]f^B(y)\mathrm{dy}$$

Denote $$\phi^a(x)(y) := \begin{cases} -p + (h+p)Q(x+y) & \text{if } 0 \leq y < a \\ -p + (h+p)Q(x+a) + (h+p)q(x+a)(y-a) & \text{if } y \geq a \end{cases}$$

The assumption on the service level implies that for all $x \leq \mathcal{X}^1$, $q'(x) \geq 0$. Thus $\phi^a(x)(y)$ is non-decreasing and convex in y. $Y^A \geq_v Y^B$ implies that, for all $x \leq \mathcal{X}^1 - a$,

$$\gamma^A(x) - \gamma^B(x) = \mathrm{E}[\phi^a(x)(Y^A)] - \mathrm{E}[\phi^a(x)(Y^B)] \geq 0 \tag{10}$$

Notice that in one period, $\mathcal{X}^{0i}$ is defined by $\gamma^i(\mathcal{X}^{0i}) = -c$. Let $c_0 = -\gamma^B(\mathcal{X}^1 - a)$. Then, if $c \geq c_0$, we have $\gamma^A(\mathcal{X}^1 - a) \geq \gamma^B(\mathcal{X}^1 - a) \geq -c$. Hence, there exists $\mathcal{X}^{0i} < \mathcal{X}^1 - a$, such that $\gamma^i(\mathcal{X}^{0i}) = -c$ (i =A,B), and by (10), $\mathcal{X}^{0A} \leq \mathcal{X}^{0B}$. ■

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