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TECHNICAL NOTE

Lost-Sales Problems with Stochastic Lead Times: Convexity Results for Base-Stock Policies

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We consider a single-location inventory system with periodic review and stochastic demand. It places replenishment orders to raise the inventory position—that is, inventory on hand plus inventory in transit—to exactly S at the beginning of every period. The lead time associated with each of these orders is random. However, the lead-time process is such that these orders do not cross. Demand that cannot be met with inventory available on hand is lost permanently. We state and prove some sample-path properties of lost sales, inventory on hand at the end of a period, and inventory position at the end of a period as functions of S. The main result is the convexity of the expected discounted sum of holding and lost-sales costs as a function of S. This result justifies the use of common search procedures or linear programming methods to determine optimal base-stock levels for inventory systems with lost sales and stochastic lead times. It should be noted that the class of base-stock policies is suboptimal for such systems, and we are primarily interested in them because of their widespread use.

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1. Introduction

We consider a single-location inventory system with periodic review and stochastic demand. It places replenishment orders to raise the inventory position, that is, inventory on hand plus inventory in transit, to exactly S at the beginning of every period. The lead time associated with each of these orders is random. However, the lead-time process is such that these orders do not cross. Demand that cannot be met with inventory available on hand is lost permanently.

First, we establish some elementary sample-path properties of lost sales and end-of-period inventories as functions of S. Subsequently, we show that a discounted sum of lost sales over a finite horizon is convex in S for every sample path of demands and lead times. We provide a counterexample to show that this result is not true when orders cross. We also establish the convexity result for the discounted sum of end-of-period inventory position for every sample path of demands and lead times. Consequently, for a cost model with holding costs that are linear in the inventory position at the end of the period and lost-sales costs that are linear in the lost sales incurred, the discounted-cost function is easily seen to be convex. This cost model is appropriate when capital costs are large compared to other inventory holding costs, and payment due dates are based on order placement dates rather than order arrival dates. We provide an example to show that if holding costs are charged on the inventory on hand at the end of each period, then the discounted-cost function need not be convex along every sample path.

With an additional assumption about lead times (see Kaplan 1970), we show that the expected discounted sum of end-of-period inventories on hand over a finite horizon is convex in S. With this assumption, we establish the convexity of the expected discounted sum of holding and lost-sales costs for cost models that include two holding terms—the first one proportional to the inventory position and the second one proportional to on-hand inventory. This result was first established by Downs et al. (2001) for the case of deterministic lead times when holding costs are charged only on the inventory on hand. This result justifies the use of common search procedures or linear programming methods to determine optimal base-stock levels for inventory systems with lost sales and stochastic lead times.

We assume that there are no fixed ordering costs in this environment. Though order-up-to-S policies are optimal for systems without fixed costs when excess demand is backordered (for example, see Erhardt 1984 for such a result with backorders and stochastic lead times), these policies are known to be suboptimal for lost-sales problems with positive lead times. However, they are very commonly used in practice in lost-sales environments, and this serves to motivate our work.

2. Motivation and Literature Survey

Karlin and Scarf (1958) considered an inventory model with lost sales and a lead time of one period. They assumed demands with positive continuous density functions, a fixed lead time of one period, linear and proportional purchase and lost-sales costs, and convex, increasing holding costs. They prove that base-stock policies are not optimal for these systems and also establish some elementary properties of the optimal ordering policy. They also state that though order-up-to-S policies are not optimal for this situation, there are examples of their use in both military and industrial areas in lost-sales problems. Their paper also has a brief discussion on finding the optimal S when demand is exponentially distributed.

With the added assumption that the holding costs are linear, Morton (1969) extended the result of Karlin and Scarf (1958) on the properties of the optimal order quantity to the case where the lead time is constant and equal to some nonnegative and constant integer. He also developed tight upper and lower bounds on the optimal ordering policy and used these bounds in heuristics. Morton (1971) proposed myopic policies as approximate solutions to this problem as well as a larger class of proportional cost problems, and presented computational results.

Nahmias (1979) studied the periodic-review lost-sales problem with set-up costs, random lead times without order crossing and partial backordering, and proposed myopic heuristics. In addition, he used a simulator to find the best value of S among the order-up-to-S or base-stock policies using Fibonacci search for problems with no set-up costs. He observed that the response surface as a function of S was convex for all of the models considered. Donselaar et al. (1996) derived heuristics for finding the optimal S and compared that order-up-to-S policy with a different policy that they proposed.

Downs et al. (2001) considered order-up-to-S policies for a lost-sales problem with deterministic lead times and proved the convexity of the average cost function with respect to S. They derived nonparametric estimates of the costs that they used in a linear-programming-based policy to determine the optimal order-up-to levels for multiple products in the presence of resource constraints. This policy is computationally simple and can be used even in the absence of the specification of the demand distribution. Karush (1957) analyzed a continuous-time inventory system with Poisson demands and independent and identically distributed lead times and demonstrated the convexity of the steady-state rate of lost sales as a function of the order-up-to level.

Healy (1992) considered (s, S) policies for an inventory problem with backorders and showed that for a fixed value of $S - s$, the n-period cost function is convex in S for every sample path of demands. Fu and Healy (1997) used this convexity result to determine the optimal (s, S) pair for a given sample path of demands. Glasserman and Tayur (1995) used perturbation analysis to determine base-stock levels in capacitated multiechelon inventory systems with backorders.

Agrawal and Smith (1996) developed a parameter-estimation methodology to estimate parameters for negative binomial demand distributions in the presence of unobservable lost sales. Ketzenberg et al. (2000) and Metters (1997; 1998a, b) are some papers that developed heuristic inventory policies for systems with lost sales.

Hill (1999) studied a continuous-review inventory problem with constant lead times, Poisson demand, and lost sales. He proved that $(S-1, S)$ policies can never be optimal if S is greater than one unit. Johansen (2001) considered the same problem in a periodic-review environment and proposed a policy called the modified base-stock policy, which has two parameters S and t. S is the base-stock level and t is the minimum number of periods between two successive replenishment orders.

Kaplan (1970), Erhardt (1984), and Zipkin (1986) are some of the papers in the literature that studied inventory problems with stochastic, noncrossing lead times and backorders.

To our knowledge, there is no existing analytical work in the discrete-time inventory control literature with lost sales and stochastic lead times.

Though order-up-to-S policies are suboptimal for lost-sales problems, there are two reasons for working with them. One is the simplicity and widespread use of these models. The second one is that it seems to be impossible to analytically infer the structure of optimal policies in periodic-review, stochastic lead time, lost-sales models.

3. Problem Definition

Throughout this paper we use the terms "increasing" and "decreasing" in the weak sense. We begin with some notation. n, t, and k are indices for time periods, where $0 \leqslant n, t, k \leqslant N$. N denotes the length of the horizon. D_n is the demand that occurs in period n. x_n is the amount of inventory on hand after receiving the shipments that arrive in period n. q_n is the size of the order placed in period n. $L(n)$ is the lead time of that order. l_n is the amount of sales lost in period n. i_n is the amount of inventory on hand at the end of the period. π_n is the inventory position (inventory on hand plus in transit) at the end of the period. λ_n is the period index of the latest shipment that arrived in or before period n. Thus, $\lambda_n = \max\{t \colon t + L(t) \leqslant n\}$. Assume that the stochastic process $L = (L(0), L(1), \ldots, L(N))$ is such that:

Assumption 1. *Orders do not cross, i.e., $n + L(n)$ is increasing in n.*

Next, we describe the sequence of events in period n $(0 \leqslant n \leqslant N)$ when a base-stock policy with parameter S (also called an order-up-to-S policy) is used.

1. Place an order of size q_n to raise the inventory position, that is, on-hand inventory plus inventory in transit, to S.
2. Receive all shipments that arrive in period n, bringing the on-hand stock to its new value x_n. Because an order-up-to-S policy is being followed, it is easy to see that

$$x_n = S - \sum_{\lambda_n < t \leqslant n} q_t.$$

3. Observe demand D_n.
4. Satisfy as much demand as possible. Let y_n denote the amount of demand satisfied in period n. Note that $y_n = \min(x_n, D_n)$. That is, the inventory on hand drops from x_n by y_n to $(x_n - D_n)^+$ and the inventory position drops to $S - y_n$. The amount of sales lost is given by $l_n = (D_n - x_n)^+ = D_n - y_n$. That is,

$$i_n = x_n - y_n \quad \text{and} \quad \pi_n = S - y_n.$$

Because the inventory position falls by y_n in period n, the amount ordered in period $n+1$ is given by $q_{n+1} = y_n$.

We assume that the system starts with inventory on hand of size S, and none on order. Consequently, $x_0 = S$ and $q_0 = 0$. We also define λ_0 to be zero.

We consider a cost model where a holding cost of h_1 is charged for the inventory position (inventory on hand and in transit) at the end of each period, a storage cost of h_2 is charged for every unit of inventory on hand at the end of each period, and a lost-sales cost of p is charged on every unit of sale lost in each period. There is a discount factor of α $(0 < \alpha \leqslant 1)$. Note that there is no discounting if $\alpha = 1$. It can be shown that if purchase costs are linear, the unit purchase cost can be assumed to be zero without loss of generality by assuming complete salvage at the end of the horizon (see Janakiraman and Muckstadt 2004 for the proof of a similar result for more general systems). For a given realization of lead times and demands, and a given order-up-to level S, the discounted cost is

$$\sum_{n=0}^{N} \alpha^n (h_1 \cdot \pi_n + h_2 \cdot i_n + p \cdot l_n). \tag{3.1}$$

Using relations derived earlier, we can rewrite this cost as

$$\sum_{n=0}^{N} \alpha^n (h_1 \cdot (S - y_n) + h_2 \cdot (x_n - y_n) + p \cdot (D_n - y_n)). \tag{3.2}$$

The cost of capital is captured by h_1 if payment due dates are based on order placement dates, or by h_2 if due dates are based on order arrival dates. Costs related to the physical storage of inventory are captured by h_2. In highly industrialized countries, in make-to-stock inventory systems, payment is usually due when the inventory arrives, not when orders are placed. Consequently, h_1 is often zero. However, third-world importers often face a different situation. Consider two Costa Rican companies. One of them imports fine liquors from the United States and sells them in Costa Rica. The other imports galvanized wire from the United States and uses it to make barbed wire for local needs. These companies buy relatively small quantities and serve a limited market. Consequently, they are obliged to purchase inventory in the country of origin and ship it themselves. In these settings, h_1 is often the dominant holding-cost term.

In the following section, we will state and prove properties of the behavior of this system as S varies. For this reason, we will henceforth use the more descriptive notation $x_n(S)$, $l_n(S)$, $q_n(S)$, $i_n(S)$, and $\pi_n(S)$.

4. Analytical Results

This section contains our analytical results. In §4.1, we derive sample-path results by limiting attention to a given realization of the demands D_n and lead times $L(n)$, $0 \leqslant n \leqslant N$. We establish elementary properties of (i) the inventory on hand at the end of a period, (ii) the inventory position at the end of a period, and (iii) the lost sales incurred in a period, as functions of S. Furthermore, we show that both the discounted sum of lost sales and the discounted sum of inventory position are convex functions of S. The convexity of these two functions implies that the discounted sum defined in (3.1) is a convex function of S when h_2 is zero. Because expectations of convex functions are convex, this establishes the convexity of the expected cost when $h_2 = 0$. We provide an example showing that the discounted sum of lost sales can fail to be convex when lead times are allowed to cross. In addition, we present an example to show that the discounted cost in (3.1) can fail to be convex when h_2 is not zero, meaning that the assumptions in §4.1 are too weak to establish convexity when $h_2 > 0$.

In §4.2, we address the case $h_2 > 0$. We restrict ourselves to a class of lead-time processes first proposed by Kaplan (1970). With this restriction, we show that the expected discounted sum of inventory on hand at the end of each period is convex in S, where the expectation is taken over the lead-time process, for any realization of demands. Our main theorem, that the expected value of the discounted sum defined in (3.1) is a convex function of S, follows immediately.

4.1. Sample-Path Results

The results in this section are derived for a given sample path of lead times and demands. Many of the sample-path results directly imply corresponding results relating to the expectations. We start by stating a lemma that describes properties of the functions $x_n(S)$, $q_n(S)$, $i_n(S)$, and $\pi_n(S)$.

Lemma 1. *Consider a given sample path of demands and lead times. For all $n \in \{0, 1, \ldots, N\}$, $x_n(S)$, $q_n(S)$, $i_n(S)$, and $\pi_n(S)$ are continuous and increasing. Furthermore, they are all differentiable except at a finite number of points. Their derivatives (wherever they exist) belong to the set $\{0, 1\}$.*

Proof. The proof is by induction and is straightforward, based on the following relations:

$$q_{n+1}(S) = y_n(S) = \min(x_n(S), D_n),$$

$$x_{n+1}(S) = (x_n(S) - D_n)^+ + \sum_{\{t\,:\, t+L(t)=n+1\}} q_t(S)$$
$$= S - \sum_{\lambda_{n+1} < t \leqslant n+1} q_t(S), \tag{4.3}$$

$$i_n(S) = (x_n(S) - D_n)^+ = x_n(S) - y_n(S), \quad \text{and} \tag{4.4}$$

$$\pi_n(S) = S - y_n(S). \tag{4.5}$$

In the induction proof, the first parts of (4.3) and (4.4) establish that $x'_{n+1}(S)$ and $i'_n(S)$ are nonnegative integers, respectively. The second parts of the same equations show that $x'_{n+1}(S) \leqslant 1$ and $i'_n(S) \leqslant 1$. (4.5) shows that $\pi'_n(S) \in \{0, 1\}$. □

The following statement about $l_n(S)$, the nth period lost-sales function, is a direct corollary to this lemma and is based on the relation

$$l_n(S) = D_n - \min(x_n(S), D_n) = D_n - y_n(S). \tag{4.6}$$

Corollary 2. *Consider a given sample path of demands and lead times. For all $n \in \{0, 1, \ldots, N\}$, $l_n(S)$ is continuous and decreasing. Further, it is differentiable everywhere except at a finite number of points. Their derivatives (wherever they exist) belong to the set $\{-1, 0\}$.*

We see from Lemma 1 that Figure 1 shows a typical plot of $q_n(S)$ versus S. Typical plots of $x_n(S)$, $\pi_n(S)$, and $i_n(S)$ are similar. Clearly, these functions are neither convex nor concave. However, we will now establish the concavity of the discounted sum of the $y_n(S)$ and $q_n(S)$ functions. The convexity of the discounted sum of lost sales and of the inventory position follows immediately. The importance of these results can be seen by examining the discounted-cost function defined in the previous section.

Lemma 3. *Consider a given sample path, i.e., a specific realization of all lead times and demands. For all $m \in \{0, \ldots, N+1\}$ and for all nonnegative and decreasing sequences $\{\beta_n\}$,*

$$W_m^{\{\beta_n\}}(S) \overset{\text{def}}{=} \sum_{n=0}^{m} \beta_n \cdot q_n(S)$$
$$= \sum_{n=1}^{m} \beta_n \cdot y_{n-1}(S) \quad \textit{is concave in } S.$$

(When $\beta_n = \alpha^{n-1}$ for all n, $W_m^{\{\beta_n\}}(S)$ is the discounted sum of the sales in periods $0, 1, \ldots, m-1$.)

Proof. The proof is by induction. The statement is trivially true for $m = 0$ because $W_0^{\{\beta_n\}}(S) = \beta_0 \cdot q_0(S) = 0\ \forall\, S$. Assume that the statement is true for some $m \geqslant 0$. We will now prove the statement for $m+1$.

$$W_{m+1}^{\{\beta_n\}}(S) = \sum_{n=1}^{m} \beta_n \cdot q_n(S) + \beta_{m+1} \cdot q_{m+1}(S)$$
$$= \sum_{n=1}^{m} \beta_n \cdot q_n(S) + \beta_{m+1} \cdot \min(x_m(S), D_m)$$
$$= \sum_{n=1}^{m} \beta_n \cdot q_n(S) + \beta_{m+1} \cdot \min\Bigl(S - \sum_{\lambda_m < t \leqslant m} q_t(S), D_m\Bigr)$$
$$= \min\Bigl(\beta_{m+1} \cdot S + \sum_{n=1}^{m} \beta'_n \cdot q_n(S),\ \beta_{m+1} \cdot D_m + \sum_{n=1}^{m} \beta_n \cdot q_n(S)\Bigr),$$

where $\beta'_n = \beta_n$ if $n \leqslant \lambda_m$ and $\beta'_n = \beta_n - \beta_{m+1}$ if $\lambda_m < n \leqslant m$. Therefore, we can write $W_{m+1}^{\{\beta_n\}}(S)$ as

$$W_{m+1}^{\{\beta_n\}}(S) = \min\bigl(\beta_{m+1} \cdot S + W_m^{\{\beta'_n\}}(S), \beta_{m+1} \cdot D_m + W_m^{\{\beta_n\}}(S)\bigr).$$

It is easy to verify that $\{\beta'_n\}$ is a nonnegative, decreasing sequence. Therefore, $W_m^{\{\beta'_n\}}(S)$ is concave by induction, so $W_{m+1}^{\{\beta_n\}}(S)$ is a minimum of two concave functions and is, hence, concave. □

An alternate proof of this lemma can be found in Janakiraman and Roundy (2002). (This alternate proof technique was suggested by a referee.)

We now show that the discounted sum of lost sales and the discounted sum of end-of-period inventory position are convex functions of S. The proof is a direct corollary to Lemma 3 and is based on (4.5) and (4.6).

Corollary 4. *Consider a given sample path, i.e., a specific realization of all lead times and demands. For all $m \in \{0, \ldots, N\}$ and for all nonnegative and decreasing sequences $\{\beta_n\}$, $\sum_{n=0}^{m} \beta_n \cdot l_n(S)$ and $\sum_{n=0}^{m} \beta_n \cdot \pi_n(S)$ are convex in S. (When $\beta_n = \alpha^n$ for all n, $\sum_{n=0}^{m} \beta_n \cdot l_n(S)$ and $\sum_{n=0}^{m} \beta_n \cdot \pi_n(S)$ are the discounted sums of $l_n(S)$ and $\pi_n(S)$, respectively.)*

The first of our main results is the convexity of the discounted-cost function in (3.1) when $h_2 = 0$, a direct consequence of Corollary 4.

Figure 1. Plot of $q_n(S)$.

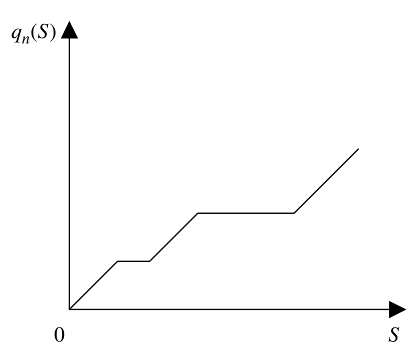


Table 1. Calculations for Example 1.

n	0	1	2	3	4
x_n	S	$(S-1)^+$	$(S-2)^+$	$(S-3)^+ + \min(S,1)$	$(S-3)^+ + \min(S,1)$
q_n	0	$\min(S,1)$	$\min((S-1)^+,1)$	$\min((S-2)^+,1)$	0
i_n	$(S-1)^+$	$(S-2)^+$	$(S-3)^+$	$(S-3)^+ + \min(S,1)$	$(S-3)^+ + \min(S,1)$
l_n	$(1-S)^+$	$\min((2-S)^+,1)$	$\min((3-S)^+,1)$	0	0

Theorem 5. *For a given sample path of demands and lead times,*

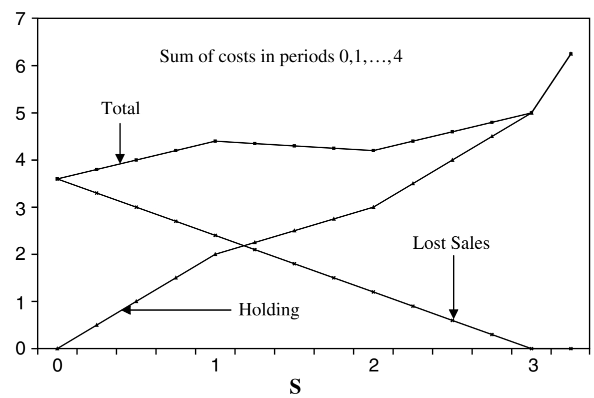
$$\sum_{n=0}^{N} \alpha^n (h_1 \cdot \pi_n(S) + p \cdot l_n(S)) \quad \textit{is convex in } S.$$

The natural next step is to investigate whether the discounted sum of end-of-period inventory on hand is convex or not. Convexity of this function would imply the convexity of the discounted cost in (3.1) for all values of h_1, h_2, and p. Unfortunately, this is not the case, as is shown by the following example.

Example 1. Consider a problem where $N = 4$. Assume that $\beta_n = 1\ \forall n \in \{0, 1, \ldots, 4\}$. The amounts of inventory on hand and in the pipeline at the start of period 0 are S and 0, respectively. Assume that $L(1) = 2$, $L(2) = L(3) = L(4) = 3$ (so orders do not cross), and that $D_n = 1$, $n \in \{0, 1, 2\}$, and $D_n = 0$, $n \in \{3, 4\}$. An order-up-to-S policy is followed. Under these assumptions, the variables that describe the system in the different periods are calculated and given in Table 1. Note that the following functions have discontinuities in their derivatives at $S = 1$: $i_0(S), i_3(S), i_4(S), l_0(S), l_1(S)$. Let $h_1 = 0$, $h_2 = 1$, and $p = 1.2$. The total sum of all costs in all periods, the sum of holding costs in all periods, and the sum of lost-sales costs in all periods are plotted in Figure 2. The holding-cost function and the total-cost function are locally concave at $S = 1$. □

Next, we provide an example to show that Lemma 3, Corollary 4, and consequently Theorem 5, can fail when the lead-time process is such that orders can cross.

Figure 2. Plots of costs.



Example 2. Consider a problem where $N = 3$. Assume that $\beta_n = 1\ \forall n \in \{0, 1, 2, 3\}$. The system starts with inventory on hand of size S and none on order. Assume that $L(1) \geqslant 2$, $L(2) = 0$, and that for some $\epsilon > 0$, $-\epsilon \leqslant S - D_0 \leqslant \epsilon$ and $D_k > \epsilon$, $k \in \{1, 2, 3\}$. An order-up-to-S policy is followed. Under these assumptions, the variables that describe the system in the different periods are calculated and given in Table 2. (The values of the "?"s in the table depend on $L(1)$ and $L(3)$.) We see that $\sum_{n=0}^{3} \beta_n \cdot q_n(S) = \sum_{n=0}^{2} \beta_{n+1} \cdot (D_n - l_n(S))$ is equal to $S + (S - D_0)^+$. This is clearly not concave for S in a small neighborhood of D_0. □

4.2. The Kaplan Lead-Time Model

In this section, we assume that our lead-time process belongs to the family introduced by Kaplan (1970). As we will see, this assumption will let us extend Theorem 5 to include holding costs charged on the inventory on hand. We state the "Kaplan assumption" in a form different from the original. The form of the assumption we use is very similar to the one used by Nahmias (1979) and Erhardt (1984).

Assumption 2. $\lambda_n = \max(\lambda_{n-1}, n - \rho_n)\ \forall n$ *where* (ρ_n) *is a sequence of i.i.d., nonnegative, integer-valued random variables.*

Equivalently, the orders received in period n are the orders that are outstanding at the beginning of period n and were placed in period $n - \rho_n$ or earlier. Observe that this assumption implies that the lead times $\{L(n)\}$ are identically distributed. In fact, $P(L(n) = 0) = P(\rho_n = 0)$ and

$$\forall k \geqslant 1, \quad P(L(n) = k) = P(\rho_n > 0) \cdot P(\rho_{n+1} > 1) \cdot \ldots \cdot P(\rho_{n+k-1} > k-1) \cdot P(\rho_{n+k} \leqslant k).$$

(This distribution is derived on page 123 of Erhardt 1984.)

It should be noted that this model for stochastic lead times does not allow arbitrary distributions of the lead time. For example, Zipkin (1986, p. 766) shows that this model implies that the failure rate of the lead-time distribution, $P(L(n) = k)/P(L(n) \geqslant k)$, is nondecreasing and converges to one as k approaches ∞.

We are now ready to prove the convexity of a discounted sum of the expected amounts of inventory on hand at the start of each period. We use the notation $\rho = (\rho_n, n = 0, 1, \ldots, N)$. We use $x_n(S, \rho)$, $q_n(S, \rho)$, $l_n(S, \rho)$, $i_n(S, \rho)$, $\pi_n(S, \rho)$, $\lambda_n(\rho)$, and $L(n, \rho)$ to make the dependence of these quantities on ρ explicit.

Table 2. Calculations for Example 2.

Period n	0	1	2	3
q_n	0	$S-(S-D_0)^+$	$(S-D_0)^+$	$(S-D_0)^+$
x_n	S	$(S-D_0)^+$	$(S-D_0)^+$	?
Lost sales in period n	$(D_0-S)^+$	$D_1-(S-D_0)^+$	$D_2-(S-D_0)^+$	?

LEMMA 6. *Under Assumption 2, for every realization of demands, the function $\sum_{n=0}^m \beta_n \cdot E_\rho[x_n(S,\rho)]$ is convex for all $m \in \{0,1,\ldots,N\}$ for every nonnegative and decreasing sequence $\{\beta_n\}$, where E_ρ denotes the expectation operator with respect to the random vector ρ.*

PROOF. Recall that

$$x_n(S,\rho) = S - \sum_{\lambda_n(\rho)<t\leqslant n} q_t(S,\rho).$$

Consequently, we have

$$\sum_{n=0}^m \beta_n \cdot x_n(S,\rho) = \sum_{n=0}^m \beta_n\left(S - \sum_{\lambda_n(\rho)<t\leqslant n} q_t(S,\rho)\right).$$

By interchanging the order of summation and taking the expectation, it can be verified that

$$E_\rho\left[\sum_{n=0}^m \beta_n \cdot x_n(S,\rho)\right] = \left(\sum_{n=0}^m \beta_n\right)S - \sum_{n=0}^m E_\rho\left[\left(\sum_{t=n}^{\min(m,\,n+L(n,\rho)-1)} \beta_t\right) q_n(S,\rho)\right].$$

(The interchange of summation operators can be seen intuitively using the following argument. $\sum_{\lambda_n(\rho)<t\leqslant n} q_t(S,\rho)$ is the amount of inventory on order in period n and $\sum_{n=0}^m \sum_{\lambda_n(\rho)<t\leqslant n} q_t(S,\rho)$ is the sum of the amounts of inventory on order in all the periods. This sum can also be computed by tracking the time periods in which any given order is outstanding. Note that $n, n+1, \ldots, \min(m, n+L(n,\rho)-1)$ are the periods when $q_n(S,\rho)$ is an outstanding order.)

Also note that for fixed demands, the random variable $q_n(S,\rho)$ is a deterministic function of $\rho_1, \rho_2, \ldots, \rho_{n-1}$ because $q_n(S,\rho)$ equals $\min(x_{n-1}(S,\rho), D_{n-1})$ and $x_{n-1}(S,\rho)$ is completely determined by the set of orders that arrive in or before period $n-1$. Furthermore, note that $L(n,\rho)$ is a function of $(\rho_n, \rho_{n+1}, \ldots)$, which is probabilistically independent of $(\rho_1, \rho_2, \ldots, \rho_{n-1})$. Consequently, we can write

$$E_\rho\left[\sum_{n=0}^m \beta_n \cdot x_n(S,\rho)\right] = \left(\sum_{n=0}^m \beta_n\right)S - \sum_{n=0}^m E_\rho\left(\sum_{t=n}^{\min(m,\,n+L(n,\rho)-1)} \beta_t\right) E_\rho[q_n(S,\rho)].$$

Using the facts that $\{\beta_n\}$ is a decreasing sequence and the distribution of $L(n)$ is identical for all n, it can be verified that $E_\rho(\sum_{t=n}^{\min(m,\,n+L(n,\rho)-1)} \beta_t)$ is a decreasing sequence in n. Therefore, $E_\rho[\sum_{n=0}^m \beta_n \cdot x_n(S,\rho)]$ is convex in S by Lemma 3. □

Using the fact that $i_n(S,\rho) = x_n(S,\rho) - y_n(S,\rho)$ along with Lemmas 3 and 6, we can now state an identical convexity result for the discounted sum of end-of-period inventories on hand.

COROLLARY 7. *Under Assumption 2, for every realization of demands, the function $\sum_{n=0}^m \beta_n \cdot E_\rho[i_n(S,\rho)]$ is convex for all $m \in \{0,1,\ldots,N\}$ for every nonnegative and decreasing sequence $\{\beta_n\}$, where E_ρ denotes the expectation operator with respect to the random vector ρ.*

We now state the second of our main results.

THEOREM 8. *Under Assumption 2, for every realization of demands and every set of nonnegative cost parameters (h_1, h_2, p), the function*

$$E_\rho\left[\sum_{n=0}^N \alpha^n (h_1 \cdot \pi_n(S,\rho) + h_2 \cdot i_n(S,\rho) + p \cdot l_n(S,\rho))\right]$$

is convex.

PROOF. A direct consequence of Theorem 5 and Corollary 7. □

REMARKS. By examining the proof of Lemma 6 carefully, one can observe that the proof holds even when the ρ_ns are not identically distributed. The proof requires only the independence of ρ_ns and the property that the lead times are stochastically decreasing; that is, $P(L(n) \leqslant k)$ is increasing in n for all k. Similarly, the proofs of Theorems 5 and 8 hold even when the cost parameters are nonstationary under certain conditions. More importantly, every result proved in this paper so far is true for any sample path of demands. Consequently, these results are valid for nonstationary and/or correlated demand processes as long as the demand process is independent of the lead-time process and the inventory policy. Although each one of the cases mentioned above leads to an easy generalization of our results, it is unnatural to assume that one would use stationary order-up-to levels in these situations.

Our convexity results easily extend to the infinite-horizon case as we show in the next section.

5. Infinite-Horizon Results

In this section, we use $\pi_n(S)$, $i_n(S)$, and $l_n(S)$ to denote unconditional random variables and $\pi_n(S,\rho,D)$, $i_n(S,\rho,D)$, and $l_n(S,\rho,D)$ to denote the respective values for a given realization $(\rho, D) = ((\rho_n, D_n): n \geqslant 0)$.

First, we show that Theorem 5 holds for an infinite-horizon, discounted-cost model.

THEOREM 9. *Let $\alpha < 1$. For every set of nonnegative cost parameters (h_1, h_2, p) and every sample path of demands and lead times, the following possibly infinite-valued function is convex*:

$$\sum_{n=0}^{\infty} \alpha^n (h_1 \cdot \pi_n(S, \rho, D) + p \cdot l_n(S, \rho, D)) \quad \text{is convex in } S.$$

If the demand D_n in each period n is bounded above by M, then this function is finite valued. Similarly, for every realization of lead times, the following possibly infinite-valued function is convex:

$$E_D\left[\sum_{n=0}^{\infty} \alpha^n (h_1 \cdot \pi_n(S, \rho, D) + p \cdot l_n(S, \rho, D))\right] \quad \text{is convex in } S,$$

where E_D denotes the expectation with respect to the demand process $(D_n\colon n \geqslant 0)$. If the expected demand in each period n, $E(D_n)$, is bounded above by M, then this function is finite valued.

PROOF. We sketch the proof for the second part of this theorem. The proof for the first part is similar, but simpler. For any sample path of lead times and any S, the expectation $E_D[\sum_{n=0}^{N} \alpha^n (h_1 \cdot \pi_n(S, \rho, D) + p \cdot l_n(S, \rho, D))]$ exists and is increasing in N. By the monotone convergence theorem, the (possibly infinite) limit as $N \to \infty$ exists, and we can interchange the expectation and the limit. Using Theorem 5 and the fact that the limits of convex functions are convex, we can see that $E_D[\sum_{n=0}^{\infty} \alpha^n (h_1 \cdot \pi_n(S, \rho, D) + p \cdot l_n(S, \rho, D))]$ exists and is convex. If $E(D_n) < M \ \forall n$, then this function is bounded above by $\max(h_1 \cdot S, p \cdot M)/(1-\alpha) < \infty$. □

Next, we extend Theorem 8 to the infinite-horizon discounted case in the same way. We omit the proof.

THEOREM 10. *Let $\alpha < 1$. Under Assumption 2, for every set of nonnegative cost parameters (h_1, h_2, p) and every sample path of demands, the following possibly infinite-valued function is convex*:

$$E_\rho\left[\sum_{n=0}^{\infty} \alpha^n (h_1 \cdot \pi_n(S, \rho, D) + h_2 \cdot i_n(S, \rho, D) + p \cdot l_n(S, \rho, D))\right].$$

If D_n is bounded above by $M \ \forall n$, this function is finite valued. The following possibly infinite-valued function is also convex:

$$E_{\rho, D}\left[\sum_{n=0}^{\infty} \alpha^n (h_1 \cdot \pi_n(S, \rho, D) + h_2 \cdot i_n(S, \rho, D) + p \cdot l_n(S, \rho, D))\right],$$

where $E_{\rho, D}$ is the expectation operator over the random sequence $((\rho_n, D_n)\colon n \geqslant 0)$. If $E(D_n)$ is bounded above by $M \ \forall n$, this function is finite valued.

Next, we extend these results to the long-run average-cost model. For these results, we make three additional assumptions.

ASSUMPTION 3. *First, we assume that $((\rho_n, D_n)\colon n \geqslant 0)$ is an i.i.d. sequence of random vectors. Second, we assume that $P(D_n = 0) > 0 \ \forall n$. Third, we assume that the demand in any period is an integer.*

These assumptions are stronger than necessary, but they facilitate easy proofs. Our arguments for the average-cost equivalents of Theorems 9 and 10 are the following.

Assume that the inventory on hand at the start of period 0 is S, and there is no pipeline inventory in period 0. Let $v_n(S)$ represent the random vector of inventory amounts in different stages of the pipeline, including the inventory on hand, at the beginning of period n. For example, if the maximum possible lead time is three periods, $v_n(S)$ is a three-dimensional vector. Let us define the state space of the Markov chain $(v_n(S)\colon n \geqslant 0)$ as all possible vectors $v_n(S)$ accessible from the state at which we start period 0 (S on hand and none in the pipeline). Because we have assumed that the probability of zero demand occurring in a period is strictly positive, we can show that the state where $v_n(S)$ is such that inventory on hand is S, and such that the pipeline has no inventory is a positive recurrent state of the Markov chain. This is because a sufficiently long sequence of zero demands will get the system back to this state. Now, we can observe that $(v_n(S)\colon n \geqslant 0)$ is an irreducible and positive recurrent Markov chain. Consequently, it has a stationary distribution, represented by the random variable $v_\infty(S)$. This also ensures the existence of stationary distributions for $i_n(S)$, $\pi_n(S)$, and $l_n(S)$. Furthermore, using Proposition 2.12.4 from Resnick (1992), we can see that the long-run average cost converges to the stationary expected one-period cost almost surely, that is, the expected cost incurred in a period, say n, where $v_n(S)$ has the same distribution as $v_\infty(S)$.

THEOREM 11. *Under Assumption 3, for every set of nonnegative cost parameters (h_1, h_2, p) and almost every sample path of demands and lead times,*

$$\lim_{N\to\infty}\left[\sum_{n=0}^{N} (h_1 \cdot \pi_n(S, \rho, D) + p \cdot l_n(S, \rho, D))/N\right] = (h_1 \cdot E[\pi_\infty(S)] + p \cdot E[l_\infty(S)]),$$

where $\pi_\infty(S)$ and $l_\infty(S)$ are random variables whose distributions are the stationary distributions of the random variables $\pi_n(S)$ and $l_n(S)$, respectively. Furthermore, this long-run average cost is convex in S.

PROOF. The arguments preceding the theorem establish the equation and the existence of the quantities in the equation. The convexity of the long-run average cost is a direct consequence of the existence of these distributions and the finite-horizon result, that is, Theorem 5. □

Similarly, Theorem 10 can be extended to the average-cost case. We state the result for the sake of completeness.

THEOREM 12. *Under Assumptions* 2 *and* 3, *for every set of nonnegative cost parameters* (h_1, h_2, p),

$$E_{\rho, D}\left[\lim_{N\to\infty}\sum_{n=0}^{N}(h_1\cdot\pi_n(S,\rho,D) + h_2\cdot i_n(S,\rho,D)+p\cdot l_n(S,\rho,D))/N\right] = (h_1\cdot E[\pi_\infty(S)]+h_2\cdot E[i_\infty(S)]+p\cdot E[l_\infty(S)]),$$

where $\pi_\infty(S)$, $i_\infty(S)$, *and* $l_\infty(S)$ *are random variables whose distributions are the stationary distributions of the random variables* $\pi_n(S)$, $i_n(S)$, *and* $l_n(S)$, *respectively. Furthermore, this long-run average cost is convex in* S.

This concludes our discussion of infinite-horizon models.

Note that all the results we have developed in this paper assume that we start period 0 in a very specific way; that is, the inventory on hand is S and there is no inventory in the pipeline. A natural question is whether these results are valid if this assumption is violated. We address this issue in the appendix.

6. Conclusion

Some sample-path properties of a discrete-time inventory system with lost sales, with stochastic lead times such that orders do not cross, and operating under a base-stock policy or order-up-to-S policy have been presented. Using these results, we derive the convexity of the expected discounted sum of holding and lost-sales costs in the planning horizon with respect to the order-up-to parameter S. The proof of this result requires an additional assumption about the lead-time process. This result is an extension of the result shown by Downs et al. (2001) for systems where lead times are deterministic and holding costs are charged only on inventory on hand. The convexity result can be exploited (as is done in Downs et al. 2001) to compute optimal order-up-to policies (optimal within that class of policies) for multiple products with a budget constraint. In addition, it justifies the use of common search techniques or infinitesimal perturbation analysis for determining optimal order-up-to levels. The convexity result has also been extended to the infinite-horizon, discounted-cost, and average-cost models.

To our knowledge, this paper is the first analytical work on discrete-time inventory models with lost sales and stochastic lead times. It is hoped that this work creates more interest in analytical results for lost-sales problems.

Appendix. Alternate Assumption about the Starting State

We assume that the starting state is a fixed vector independent of S. We state this assumption next.

ASSUMPTION 4. *Let* v_0 *represent the vector of inventory levels at different stages of the pipeline, including inventory on hand, at the start of period* 0 *before placing an order in period* 0. v_0 *is independent of* S. *The dimension of* v_0 *is the maximum possible lead time.*

Let $x_0(S)$ and $\pi_0(S)$ represent the inventory on hand and the total pipeline inventory at the start of period 0 after placing an order and receiving shipments due in period 0, when an order-up-to-S policy is used. Consequently, $x_0(S)+\pi_0(S)\geqslant S$.

Lemma 1 and Corollary 2 can easily be verified with the alternate assumption about the starting state. Unfortunately, the discounted sum of lost-sales costs over a finite or infinite horizon need not be convex in S with this assumption. To see this, consider an example where v_0 is such that the pipeline is empty and $x_0 = S_0+\epsilon$. Consider three order-up-to policies with the following order-up-to levels: S_0, $S_0+\epsilon$, and $S_0+2\epsilon$. The first two policies order nothing in period 0, but the third system orders ϵ. Assume $d_0 = S_0+\epsilon$, $L_0 = 1$, $L_1\geqslant 1$, and $d_1=\epsilon$, so $l_0(S_0) = l_0(S_0+\epsilon) = l_0(S_0+2\epsilon) = 0$, $l_1(S_0) = l_1(S_0+\epsilon)=\epsilon$, and $l_1(S_0+2\epsilon) = 0$. Therefore, $l_0(S)+l_1(S)$ is not a convex function. However, we argue next that the infinite-horizon results for the undiscounted model continue to hold.

Let us now assume that there is a positive probability of nonzero demand occurring in a period, in addition to Assumption 3. We now claim that Theorems 11 and 12 continue to hold with the alternate assumption about the starting state. This is easy to see because it is clear that the time to reach a state *in which the inventory on hand is S and none is on order* is finite, with probability one. Consequently, the contribution of the periods before the period in which this state is reached to the long-run average cost is zero, with probability one.

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References

Agrawal, N., S. Smith. 1996. Estimating negative binomial demand for retail inventory management with unobservable lost sales. *Naval Res. Logist.* **43**(6) 839–861.

Donselaar, K., T. Kok, W. Rutten. 1996. Two replenishment strategies for the lost sales inventory model: A comparison. *Internat. J. Production Econom.* **46–47** 285–295.

Downs, B., R. Metters, J. Semple. 2001. Managing inventory with multiple products, lags in delivery, resource constraints and lost sales: A mathematical programming approach. *Management Sci.* **47**(3) 464–479.

Erhardt, R. 1984. (s, S) policies for a dynamic inventory model with stochastic lead times. *Oper. Res.* **32**(1) 121–132.

Fu, M., K. Healy. 1997. Techniques for optimization via simulation: An experimental study on an (S, S) inventory system. *IIE Trans.* **29** 191–199.

Glasserman, P., S. Tayur. 1995. Sensitivity analysis for base-stock levels in multiechelon production-inventory systems. *Management Sci.* **41**(2) 263–281.

Healy, K. 1992. Optimizing stochastic systems: A retrospective/ deterministic approach. Ph.D. dissertation, School of Operations Research and Industrial Engineering, Cornell University, Ithaca, NY.

Hill, R. 1999. On the suboptimality of $(S-1, S)$ lost sales inventory policies. *Internat. J. Production Econom.* **59** 387–393.

Janakiraman, G., J. Muckstadt. 2004. Inventory control in directed networks: A note on linear costs. *Oper. Res.* **52**(3) 491–495.

Janakiraman, G., R. Roundy. 2002. An alternate proof technique for a key result in "Lost-sales problems with stochastic lead times: Convexity results for base-stock policies." Technical report 1336, School of Operations Research and Industrial Engineering, Cornell University, Ithaca, NY.

Johansen, S. 2001. Pure and modified base-stock policies for the lost sales inventory system with negligible set-up costs and constant lead times. *Internat. J. Production Econom.* **71** 391–399.

Kaplan, R. 1970. A dynamic inventory model with stochastic lead times. *Management Sci.* **16**(7) 491–507.

Karlin, S., H. Scarf. 1958. Inventory models of the Arrow-Harris-Marschak type with time lag. *Studies in the Mathematical Theory of Inventory and Production*. Stanford University Press, Stanford, CA, 155–178.

Karush, W. 1957. A queuing model for an inventory problem. *Oper. Res.* **5**(5) 693–703.

Ketzenberg, M., R. Metters, V. Vargas. 2000. Inventory policy for dense retail outlets. *J. Oper. Management* **18**(3) 303–316.

Metters, R. 1997. Production planning with stochastic seasonal demand and capacitated production. *IIE Trans.* **29**(11) 1017–1029.

Metters, R. 1998a. General rules for production planning with seasonal demand. *Internat. J. Production Res.* **36**(5) 1387–1399.

Metters, R. 1998b. Producing multiple products with stochastic seasonal demand and capacity limits. *J. Oper. Res. Soc.* **49**(3) 263–272.

Morton, T. 1969. Bounds on the solution of the lagged optimal inventory equation with no demand backlogging and proportional costs. *SIAM Rev.* **11**(4) 572–596.

Morton, T. 1971. The near-myopic nature of the lagged-proportional-cost inventory problem with lost sales. *Oper. Res.* **19**(7) 1708–1716.

Nahmias, S. 1979. Simple approximations for a variety of dynamic lead-time lost-sales inventory models. *Oper. Res.* **27**(5) 904–924.

Resnick, S. 1992. *Adventures in Stochastic Processes*. Birkhauser, Boston, MA.

Zipkin, P. 1986. Stochastic leadtimes in continuous-time inventory models. *Naval Res. Logist. Quart.* **33** 763–774.