

New Policies for the Stochastic Inventory Control Problem with Two Supply Sources

Anshul Sheopuri, Ganesh Janakiraman, Sridhar Seshadri ¹

August 2007

Subject Classification: Inventory/Production: Lead Times, Dual Sourcing, Heuristics

Abstract

We study an inventory system under periodic review in the presence of two suppliers (or delivery modes). The emergency supplier has a shorter lead-time than the regular supplier, but the unit price he offers is higher. Excess demand is backlogged. We show that the classical “Lost Sales inventory problem” is a special case of this problem. Then, we generalize the recently studied class of Dual Index policies (Veeraraghavan and Scheller-Wolf (2007)) by proposing two classes of policies. The first class consists of policies that have an order-up-to structure for the emergency supplier. We provide analytical results that are useful for determining optimal or near-optimal policies within this class. This analysis and the policies that we propose leverage the connections we make between our problem and the lost sales problem. The second class consists of policies that have an order-up-to structure for the combined orders of the two suppliers. Here, we derive bounds on the optimal order quantity from the emergency supplier, in any period, and use these bounds for finding effective policies within this class. Finally, we undertake an elaborate computational investigation to compare the performance of the policies we propose with that of Dual Index policies. One of our policies provides an average cost-saving of 1.1 % over the Best Dual Index policy and has the same computational requirements. Another policy that we propose has a cost performance similar to the Best Dual Index policy but its computational requirements are lower.

¹IOMS-OM Group, Stern School of Business, New York University, 44 W. 4th Street, Rooms 8-71/73/154, New York, NY 10012-1126.

1 Introduction

In this paper, we study the optimal inventory control problem for a periodically reviewed, single stage system facing stochastic demand when inventory can be replenished from two suppliers with different lead times and unit costs. Our study is motivated by the practice of some firms to procure components through multiple supply sources that have different delivery characteristics (see Beyer and Ward (2000), Rao et al. (2000) and Threatte and Graves (2002) for examples). We assume there are no fixed costs for ordering from either supplier and that demands in different periods are independently and identically distributed. Demand in excess of available inventory in any period is back-ordered. Linear holding and shortage costs are charged at the end of each period. We consider the problem of minimizing the expected sum of procurement, holding and shortage costs over multiple periods. We use “Dual Sourcing” to refer to this problem/system.

We show that the classical “Lost Sales inventory problem” (studied by Karlin and Scarf (1958), Morton (1969) and Zipkin (2006)) is a special case of the dual sourcing problem. Then, we generalize the recently studied class of Dual Index policies (Veeraraghavan and Scheller-Wolf (2007)) by proposing two classes of policies. The first class consists of policies that have an order-up-to structure for the emergency supplier. We provide analytical results that are useful for determining optimal or near-optimal policies within this class. This analysis and the policies that we propose leverage the connections we make between our problem and the lost sales problem. The second class consists of policies that have an order-up-to structure for the combined orders of the two suppliers. For this class of policies, we derive bounds on the optimal order quantity from the emergency supplier, in any period, and use these bounds for finding effective policies within this class. Finally, we undertake an elaborate computational investigation to compare the performance of the policies we propose with Dual Index policies. One of our policies provides a cost-saving of 1.1 % over the Best Dual Index policy and has the same computational requirements. Another policy that we propose has a cost performance similar to the Best Dual Index policy but its computational

requirements are lower.

The remainder of this section is organized as follows. We present the mathematical statement of our problem in Section 1.1. In Section 1.2, we describe the existing literature on the dual sourcing problem. We summarize our contributions relative to the state of the art on this problem in Section 1.3.

1.1 Notation and Problem Definition

Let c^R (l^R) and c^E (l^E) denote the unit procurement costs (lead times) from the first and second suppliers, respectively. The lead time l^R is any non-negative integer and the lead time l^E is any non-negative integer or -1 (we will explain what we mean by a lead time of -1 later). We assume that

$$l^R > l^E \text{ and } c^R < c^E .$$

That is, the second supplier has a shorter lead time but charges a premium of $(c^E - c^R)$ above the first supplier's unit price for the faster delivery. Notice that if the assumption does not hold, it is optimal to procure exclusively from one of the two suppliers and the problem is then just a standard inventory control problem with a single supplier, which is trivial to solve under the assumptions we make on the demand process. So, from now on, we will refer to the first supplier as “R” (standing for “regular”) and the second supplier as “E” (standing for “expedited”).

The system is reviewed periodically. We use $t \in \{1, 2, \dots\}$ as the period index. The demand in period t is D_t , which is random. We assume that $\{D_t\}$ is a sequence of i.i.d. random variables with cumulative distribution function Φ and a finite mean, μ . We use D to denote a generic random variable with this distribution.

The net-inventory, i.e., the amount of inventory on hand minus the amount on backorder, at the beginning of period t , is denoted by w_t . Let q_t^R and q_t^E denote the replenishment orders

placed in period t on R and E, respectively. The holding and shortage cost parameters are h and b , respectively.

The sequence of events in period t is as follows, when $l^E \geq 0$:

1. The replenishment quantities q_t^R and q_t^E are decided using the information on the state of the system in that period, namely, $w_t, (q_{t-l^R}^R, \dots, q_{t-1}^R)$ and $(q_{t-l^E}^E, \dots, q_{t-1}^E)$, .
2. Inventory is replenished by the deliveries that are due in period t , namely $q_{t-l^R}^R$ and $q_{t-l^E}^E$.
3. The demand D_t is realized.

The mismatch between supply and demand in period t is given by the quantity $w_t + q_{t-l^R}^R + q_{t-l^E}^E - D_t$. The cost incurred in period t is, therefore,

$$c^R \cdot q_t^R + c^E \cdot q_t^E + h \cdot (w_t + q_{t-l^R}^R + q_{t-l^E}^E - D_t)^+ + b \cdot (D_t - w_t - q_{t-l^R}^R - q_{t-l^E}^E)^+ .$$

When l^E is -1 , the only change to the sequence of events is that the decision q_t^E is taken after D_t is realized, and this order is delivered instantly and the cost incurred in period t is

$$c^R \cdot q_t^R + c^E \cdot q_t^E + h \cdot (w_t + q_{t-l^R}^R + q_t^E - D_t)^+ + b \cdot (D_t - w_t - q_{t-l^R}^R - q_t^E)^+ .$$

We denote this dual sourcing system by $\mathcal{DS}(h, b, c^E, D, l^E, l^R)$. An admissible policy is defined to be a rule for placing orders using the historical information as defined in 1. Let Π denote the set of admissible policies and let π denote any element of Π . Let Π^R and Π^E denote the classes of policies that source exclusively from R (i.e., $q_t^E = 0$ for every t) and exclusively from E, respectively.

Each of our analytical results holds for one or more of the following performance measures: (i) finite horizon expected total cost, (ii) infinite horizon expected discounted cost, and (iii) infinite horizon expected average cost. We will refer to the problem of minimizing (i) as the finite horizon problem, (ii) as the infinite horizon discounted cost problem, and

(iii) as the infinite horizon average cost problem.

Notice that any policy with a finite long run average cost will incur a procurement cost of at least $c^R \cdot \mu$ per period. In fact, it is easy to show that the optimal procurement policy is independent of c^R and c^E for a given value of the premium, $(c^E - c^R)$.

Consequently, we will assume without loss of generality that $c^R = 0$.

Let $\delta = l^R - l^E$. It is easy to show that if $c^E \geq b \cdot \delta$, the class Π^R contains an optimal policy and the problem reduces to the standard single supplier inventory problem. This is because the maximum benefit we can possibly derive by procuring one unit from E, instead of R, is the backorder cost incurred on a unit over the difference in the lead times. To avoid this trivial possibility, we will assume throughout that

$$c^E < b \cdot \delta .$$

Before we proceed, we make an elementary observation about the minimal information required to make optimal ordering decisions in this system. The proof follows standard arguments in inventory theory and is, consequently, omitted.

We use the following convention throughout this paper. Any summation of the form \sum_a^b is zero if $a > b$. Any vector $(a, a + 1, \dots, b)$ is the null vector if $a > b$.

Define the *expedited inventory position* at the beginning of period t , IP_t^E , as follows:
(i) $IP_t^E = w_t + \sum_{t-l^E}^{t-1} q_u^E + \sum_{t-l^R}^{t-l^R+l^E} q_u^R$, when $l^E \geq 0$, (ii) $w_t + q_{t-l^R}^R$ when $l^E = -1$ and $l^R \geq 1$, and, (iii) w_t when $l^E = -1$ and $l^R = 0$.

Lemma 1.1. *Consider the finite horizon problem or the infinite horizon (average cost or discounted cost) problem for the dual sourcing system with $l^E \geq 0$. It is sufficient to consider policies in which q_t^R and q_t^E depend on the vector $(w_t, q_{t-1}^E, q_{t-2}^E, \dots, q_{t-l^E}^E, q_{t-1}^R, \dots, q_{t-l^R}^R)$ only*

through the vector $(IP_t^E, q_{t-1}^R, q_{t-2}^R, \dots, q_{t-l^R+l^E+1}^R)$. Thus, the state space of the corresponding dynamic program is $(l^R - l^E)$ dimensional.

This result follows from the observations that the costs in periods $\{t, t+1, \dots, t+l^E-1\}$ are *sunk costs*, as seen from period t , and the state of the system in any period after period $t+l^E-1$ depends on the state in period t only through the “compressed” state vector, $(IP_t^E, q_{t-1}^R, q_{t-2}^R, \dots, q_{t-l^R+l^E+1}^R)$.

Note that, for the finite horizon problem or the infinite horizon (average cost or discounted cost) problem for the dual sourcing system with $l^E = -1$, it is sufficient to consider policies in which q_t^R and q_t^E depend on the vector $(w_t, q_{t-1}^R, \dots, q_{t-l^R}^R)$ only through the vector $(w_t + q_{t-l^R}^R, q_{t-1}^R, q_{t-2}^R, \dots, q_{t-l^R+1}^R)$, if $l^R \geq 1$, or (w_t) , if $l^R = 0$. Thus, the state space of the corresponding dynamic program is l^R dimensional, if $l^R \geq 1$, or 1 dimensional, if $l^R = 0$.

1.2 Related Literature

In this section, we review the literature on the dual sourcing problem in chronological order.

Daniel (1962) and Neuts (1964) show the optimality of order-up-to policies for the case when $l^E = 0$ and $l^R = 1$. Fukuda (1964) extends the result to the case when $l^E = L$ and $l^R = L+1$, $L \in \{0, 1, \dots\}$. Let us briefly provide our own informal derivation of this result. Notice that, under the lead time assumptions made in the papers mentioned above, *orders do not cross*, i.e., any order placed in period t arrives no later than any order placed in period t' if $t' > t$. This property is critical in establishing the optimality of order-up-to policies. The following is one such argument. Consider splitting each ordering epoch into two epochs, the first of which is when the order from E is placed and the second is when the order from R is placed. Imagine that there is a demand of zero between these two epochs. Now, relative to the new epochs, it can be verified that the system is just a single supplier system in which unit costs alternate between c^R and c^E and the pair of holding and shortage cost parameters alternate between $(0, 0)$ and (h, b) . Optimality of order-up-to policies

now follows immediately using standard arguments. Moreover, the order-up-to levels can be easily derived using the Newsvendor formula (with appropriate adjustment of parameters) for E and a simple one-dimensional convex minimization yields the order-up-to level for R. To conclude our discussion on the special case of a one period difference in the lead times of the two suppliers, we can say that this case is now as well understood as the basic inventory problem with a single supplier.

Whittemore and Saunders (1977) study the dual sourcing problem when the difference between l^R and l^E is arbitrary and the holding and shortage cost functions are allowed to be non-linear. They derive sufficient conditions under which only one supplier is used in the infinite horizon discounted cost problem. Their conditions are generalizations of the conditions discussed in Section 1.1, namely, $c^E \leq c^R$ implies that it is optimal to exclusively use E and $c^E > c^R + b \cdot \delta$ implies that it is optimal to exclusively use R.

During the period between this early and elementary theoretical work to now, several authors have proposed heuristic policies for this problem (see, for example, Chiang and Gutierrez (1995) and Tagaras and Vlachos (2001)). Please see Veeraraghavan and Scheller-Wolf (2007) for a discussion of these papers and additional references.

Recently, there has been a renewed interest in studying this problem from an analytical/theoretical perspective and to use the resulting analysis to guide the development of heuristics. We discuss this recent work next.

Feng et al. (2006) examine the question of whether/when order-up-to policies are optimal for systems with 3 or more suppliers whose lead times are consecutive integers. Our dual sourcing system can be modeled using a system with $(l^R - l^E + 1)$ suppliers whose lead times are consecutive integers between l^E and l^R . Their main result is that even for a system with 3 such suppliers, order-up-to policies need not be optimal. They show this by constructing a

three period example with non-stationary demand distributions and non-stationary holding and shortage costs; their analysis is elaborate and requires studying several cases.

Veeraraghavan and Scheller-Wolf (2007) propose a class of policies called “Dual Index” policies for the dual sourcing problem. In this class of policies, there are two order-up-to levels, S^R and S^E , for R and E, respectively. At the beginning of a period, if the inventory position of E, IP_t^E , is below S^E , the difference is ordered from E. Next, the inventory position of R (defined as net-inventory plus receipts due from both R and E within the next l^R periods) is raised to S^R by ordering the appropriate quantity from R. This policy is both easy to use and understand, and, it captures information on the state of the supply chain using the two inventory positions. The authors show the following results on the joint optimization of S^E and S^R within this class. Let $\Delta = S^R - S^E$. For a given Δ , the optimal S^E can be found using the Newsvendor formula applied on the convolution of demand over $(l^E + 1)$ periods and a random variable, called the *overshoot*, whose distribution is independent of S^E and can be determined by simulation. The optimal Δ is found through a one-dimensional search by evaluating the cost of the system for different choices of Δ . They show that the cost of the optimal Dual Index policy is within 2-3 % of the optimal cost computed by solving the dynamic program over a small set of examples. Their work allows the possibility of incorporating capacity constraints on the orders placed with R and E.

Scheller-Wolf et al. (2005) propose a class of policies called “Single Index” policies for the dual sourcing problem. These policies are identical to Dual Index policies except that the order from E is also placed using the inventory position of R. So, the only information this policy uses about the state of the supply chain is this inventory position; in that sense, it is even simpler to use than Dual Index policies. The authors propose the same procedure as Veeraraghavan and Scheller-Wolf (2007) for finding the optimal policy within this class. However, they highlight one computational benefit. The distribution used in the Newsvendor formula for computing the optimal S^E for a given Δ involves a convolution which can

be evaluated without simulation by writing it as a linear combination of shifted distribution functions of sums of demands and residual demands. In their experiments, the performance of these policies is, on average, 6.4 % worse than the Best Dual Index policy. This, as well as, many of the numerical experiments attest to the superior performance of the Best Dual Index policies.

A related stream of literature studies systems in which there is an option to expedite orders after they have been placed, for example Groenevelt and Rudi (2003), Jain et al. (2007), Lawson and Porteus (2000) and Muharremoglu and Tsitsiklis (2003).

1.3 Our Results and Positioning in the Literature

We now summarize the main results of the paper. In Section 2, we show that the lost sales problem studied by Karlin and Scarf (1958) is a special case of the dual sourcing problem. A famous result of that paper is that order-up-to policies are not optimal for the lost sales problem. Thus, the non-optimality of order-up-to policies for the dual sourcing problem is easily established using these two results.

In Section 3, we assume that we use an order-up-to policy with target S^E for ordering from E. Then, we use the connection between the lost sales problem and the dual sourcing problem to guide the development of good policies for the latter. In particular, we argue that orders placed from E can be interpreted as “lost sales” for R. We prove that the optimal cost for the dual sourcing system (within the class of policies that use an order-up-to rule for E) is bounded above by the sum of the optimal cost of a lost sales system with a lead time of $(l^R - l^E - 1)$ periods and the optimal cost of a backorder system with a lead time of l^E periods. This leads us to consider policies for R that are known to work well for lost sales systems. Moreover, we show that for any *such policy* for R, computing the optimal S^E is as easy as it is within the class of Dual Index policies. Our results are a generalization of

the separability results of Veeraraghavan and Scheller-Wolf (2007).

In Section 4, we assume that we use an order-up-to policy for the combined orders of the two suppliers. To be precise, we assume that the total inventory position in the system is raised to the same level S^R in each period. Therefore, given a choice of S^R , the remaining question is how should the total order quantity in a period be allocated between E and R. We derive upper and lower bounds on the order quantity from E (equivalently, from R) in each period. We propose using these bounds as heuristics for the order allocation decision.

In Section 5, we present our numerical investigation. We present our conclusions in Section 6.

2 Connection to the Lost Sales Problem

In this section, we prove that the dual sourcing problem is a generalization of the lost sales problem and use this to show that order-up-to policies are not optimal, in general, in dual sourcing problems.

Consider the following dual sourcing system. The lead times of the suppliers are $l^E = -1$ and $l^R = \tau$, respectively. Let $c^E = p$. *Consider the infinite horizon, optimal control problem with a discount factor $\alpha < 1$.* Recall the sequence of events in a period, for the case where $l^E = -1$: first, the order from R due in that period is received; second, the demand is observed; third, an order is placed on E and received immediately; demand is met with available inventory to the extent possible and any unmet demand is backordered.

We first establish an upper bound on the inventory position when we follow an optimal policy in this system. The starting state $(w_1, q_t^R : t = -\tau + 1, -\tau + 2, \dots, 0)$ is given. We

need the following definitions first. Let $N(y) = \min\{t : D[1, t] \geq y\}$ and let

$$\overline{M} = \min \left\{ y : h \cdot \alpha^\tau \cdot E \left[\frac{1 - \alpha^{(N(y) - \tau)^+}}{1 - \alpha} \right] \geq p \right\} .$$

That is, $N(y)$ is the random variable for the number of periods required for the system to observe a cumulative demand of y units. \overline{M} is guaranteed to exist if

$$h \cdot \frac{\alpha^\tau}{1 - \alpha} \geq p.$$

Intuitively, \overline{M} is a large enough inventory position that ordering an extra unit beyond \overline{M} and incurring the holding cost on it is inferior to waiting until the demand for that unit materializes and sourcing that unit from E. This leads to the following result.

Lemma 2.1. *Assume $\sum_{u=-\tau+1}^0 q_u^R + w_1 \leq \overline{M}$. Then, under any optimal policy, the inventory position in period t , $\sum_{u=t-\tau}^{t-1} q_u^R + w_t$ is bounded above by \overline{M} for every t .*

We now make an assumption on the cost parameters.

Assumption 2.1. *The following condition holds:*

$$b > p + h \cdot \alpha^\tau \cdot E \left[\frac{1 - \alpha^{(N(\overline{M}) - \tau)^+}}{1 - \alpha} \right] .$$

This condition ensures that whenever the inventory position is bounded above by \overline{M} , it is optimal to clear any backorders that exist at the instant of time when orders from E are placed since the cost of backordering a unit of demand by a period exceeds the maximum possible savings in holding costs if we choose the option of satisfying this demand later using a unit of inventory in the pipeline. Also, observe that it is never optimal to order a unit from E when we know it will not be used in that same period. These properties are formally stated next. We omit their formal proofs since they follow the discussion above.

Lemma 2.2. *Suppose Assumption 2.1 holds. Also, assume $\sum_{u=-\tau+1}^0 q_u^R + w_1 \leq \overline{M}$. Then, in every period, t , any optimal policy will order from E exactly the amount on backorder at that instant, that is,*

$$q_t^E = \max(0, -[w_t + q_{t-\tau}^R]) .$$

Notice that the optimal ordering policy uses an order-up-to zero policy for E.

In the following lemma, we show that the optimal cost of this dual sourcing system equals the optimal cost of a specific lost sales inventory system. For conciseness, we introduce the notation $\mathcal{L}(h, p, D, \tau)$ to refer to a single supplier inventory system in which excess demands are lost, the holding cost parameter is h , the penalty cost per unit of lost sales is p , the lead time is τ periods and D represents the demand distribution. The state vector at the beginning of a period in $\mathcal{L}(h, p, D, \tau)$ is τ -dimensional, for $\tau \geq 1$; it consists of the inventory on hand and the vector of outstanding orders. Also, note that the state vector at the beginning of a period in $\mathcal{DS}(h, b, c^E = p, D, l^E = -1, l^R = \tau)$ is also τ -dimensional, for $\tau \geq 1$; it consists of the net-inventory and the vector of outstanding orders from R. If $\tau = 0$, the state vectors in both systems are one-dimensional, consisting only of the on-hand inventory in the former system and the net-inventory in the latter system.

Lemma 2.3. *Suppose (i) Assumption 2.1 holds, (ii) the period 1 starting inventory position, $\sum_{u=-\tau+1}^0 q_u^R + w_1$ is less than \bar{M} , and (iii) the state vectors in $\mathcal{L}(h, p, D, \tau)$ and $\mathcal{DS}(h, b, c^E = p, D, l^E = -1, l^R = \tau)$ are identical at the beginning of period 1. Then, for any given discount factor α , the infinite horizon optimal costs for $\mathcal{DS}(h, b, c^E = p, D, l^E = -1, l^R = \tau)$ and $\mathcal{L}(h, p, D, \tau)$ are equal.*

PROOF: Let q_t denote the order quantity in the lost sales system. By assumption, the state vectors in the two systems are identical in period t . Assume that q_t^R in the dual sourcing system equals q_t in the lost sales system. Using Lemma 2.2, notice that the order placed from E in the dual sourcing system is exactly the amount of lost sales in the other system. Observe that, with probability one, the costs in the two systems are identical in period t and the state vectors in the two systems are identical in period $t + 1$. Thus, for every policy in $\mathcal{L}(h, p, D, \tau)$, we can construct a policy in $\mathcal{DS}(h, b, c^E = p, D, l^E = -1, l^R = \tau)$ with the same costs in each period, and vice-versa. The desired result is a direct consequence of this observation. \square

The following theorem establishing the lost sales problem as a special case of the dual sourcing problem follows directly from the proof of Lemma 2.3.

Theorem 2.1. *Consider a lost sales inventory problem, $\mathcal{L}(h, p, D, \tau)$, and a dual sourcing problem, $\mathcal{DS}(h, b, c^E, D, l^E, l^R)$, in which $c^E = p$, $l^R = \tau$, $l^E = -1$ and (b, h, p) satisfies Assumption 2.1. The optimal policies in these two systems are identical, in the following sense. Assume that the state vectors in $\mathcal{L}(h, p, D, \tau)$ and $\mathcal{DS}(h, b, c^E, D, l^R, l^E)$ are identical at the beginning of some period t . Any optimal choice of q_t^R in $\mathcal{DS}(h, b, c^E, D, l^E, l^R)$ is also an optimal order quantity in $\mathcal{L}(h, p, D, \tau)$.*

Now, we use a result due to Karlin and Scarf (1958) for lost sales systems to establish that order-up-to policies are not optimal, in general, for dual sourcing systems.

Theorem 2.2. *Consider $\mathcal{DS}(h, b, c^E, D, l^E, l^R)$, when $c^E = p$, $l^R = 1$, $l^E = -1$ for any (b, h, p) satisfying Assumption 2.1. Let $0 < l \leq r < \infty$. Assume $P\{D \in [l, r]\} = 1$. The class of order-up-to policies (for ordering from R) does not contain an optimal policy for this system.*

PROOF: Let us assume that the period 1 starting inventory position, $q_0^R + w_1$ is less than \overline{M}). From Lemma 2.2, we know that the uniquely optimal ordering decision from E in any period t is given by

$$q_t^E = \max(0, -[w_t + q_{t-1}^R]) .$$

We can now prove the desired result by contradiction. Assume there exists an optimal policy that uses an order-up-to policy for deciding q_t^R , in each period t . We know from Theorem 2.1 that this same policy is optimal for $\mathcal{L}(h, p, D, 1)$. Theorem 2 of Karlin and Scarf (1958) establishes that order-up-to policies are not optimal for $\mathcal{L}(h, p, D, 1)$ if it is necessary to order a positive amount when the stock is small, and unprofitable to order a positive amount when the stock is large. Since $P\{D \in [l, r]\} = 1$, for $0 < l \leq r < \infty$, the above condition is satisfied. Thus, we have a contradiction. \square

Even though we have a negative result regarding the optimal policy, the connection to the lost sales systems can be exploited to develop good heuristics for the dual sourcing problem as shown in the next section.

3 Policies with an Order-up-to Structure for the Emergency Supplier

In Lemma 2.2, we showed, under some assumptions, that it is optimal to follow an order-up-to zero policy for orders from E, in dual sourcing systems in which l^E is -1 . Also, for systems in which the difference between l^R and l^E is one period, it is known that an order-up-to policy is optimal for E (and also for R). Moreover, the Dual Index policies of Veeraraghavan and Scheller-Wolf (2007) are order-up-to policies for E (and in fact, for R also) and have been shown to perform well in their computational experiments. We now explore the more general class of policies that use the order-up-to rule for E and any admissible ordering rule for R. These policies are a generalization of the Dual Index policy but almost as easy to compute.

We first make a simple observation, along the lines of Lemma 1.1, about the information requirement for the optimal ordering policy from R, given that E uses an order-up-to policy with a given target level, S^E . Notice that orders are placed from both suppliers, and therefore, there might be some periods in which the expedited inventory position, IP_t^E , exceeds S^E even if the state of the system in period 1 is carefully chosen. That is, there can be an *overshoot*, which we define formally first before stating the lemma.

$$OV_t = (IP_t^E - S^E)^+.$$

(Veeraraghavan and Scheller-Wolf (2007) introduced this notion of overshoot in the context of Dual Index policies.)

Lemma 3.1. *Consider the infinite horizon (discounted cost or average cost) problem for the dual sourcing system with $l^E \geq 0$. Assume that we follow an order-up-to S^E policy from E,*

i.e., $q_t^E = (S^E - IP_t^E)^+$ for every t . Then, for a given S^E , it is sufficient to consider policies in which q_t^R depends on the vector $(w_t, q_{t-1}^E, \dots, q_{t-l^E}^E, q_{t-1}^R, \dots, q_{t-l^R}^R)$ only through the vector $(OV_t, q_{t-1}^R, \dots, q_{t-l^R+l^E+1}^R)$.

The proof of this lemma is straight forward and, hence, omitted. The intuition for the result is similar to the explanation provided after Lemma 1.1. As a result of this lemma, in this section, we will restrict attention to order-up-to policies for E and policies that use only the information $(OV_t, q_{t-1}^R, \dots, q_{t-l^R+l^E+1}^R)$ for R. Let us use π_R to refer to any such ordering policy from R and $q_{\pi_R}(\cdot)$ to denote the order quantity as a function of this compressed information vector. We now state a technical result that is useful for finding optimal/good policies within this class.

Lemma 3.2. *Assume $OV_1 = 0$. Consider a given stationary ordering policy from R, π_R . For any sample path of demands (D_1, \dots, D_t) , the overshoot in period t , OV_t , does not depend on the choice of S^E . The same result also applies to q_t^R , the quantity ordered from R.*

It should be noted that Veeraraghavan and Scheller-Wolf (2007) show the same result *within the class of Dual Index policies*, which is a subset of our class of policies. The proof of Lemma 3.2 is inductive and straight forward, relying mainly on the following recursions. We omit it, in the interest of space.

$$OV_{t+1} = \max(0, OV_t - D_t + q_{t+1-l^R+l^E}^R) . \quad (3.1)$$

$$q_{t+1}^R = q_{\pi_R}(OV_{t+1}, q_t^R, \dots, q_{t+2-l^R+l^E}^R) . \quad (3.2)$$

Also, notice that the order quantity from E in period $t+1$ is given by the expression

$$q_{t+1}^E = \max(0, D_t - (OV_t + q_{t+1-l^R+l^E}^R)) . \quad (3.3)$$

Let $Y_t = OV_t + q_{t+1-l^R+l^E}^R$. So, OV_{t+1} can also be written as

$$\max(0, Y_t - D_t) .$$

Also, observe that the evolution of $\{Y_t\}$ is described by the recursion

$$Y_{t+1} = \max(0, Y_t - D_t) + q_{t+2-l^R+l^E}^R . \quad (3.4)$$

Now, consider a lost sales inventory system with i.i.d. demands with the same distribution as D and with a lead time of $(l^R - l^E - 1)$ periods. The evolution of the inventory on-hand at the beginning of a period (after receiving the delivery for that period) and the inventory on-hand at the end of the period in this system are exactly the same as the processes $\{Y_t\}$ and $\{OV_{t+1}\}$ under the ordering policy π^R .

Let us now assume that the processes $\{OV_t\}$ and $\{Y_t\}$ converge to stationary distributions represented by the random variables OV_∞ and Y_∞ , respectively. Notice that both these stationary random variables are also independent of the choice of S^E .

We will now provide an expression for the long run average cost for a policy in this class, i.e., the cost for a given S^E and π^R . We denote this quantity by C_{S^E, π^R} .

$$\begin{aligned} C_{S^E, \pi^R} &= c^E \cdot E[\max(0, D - Y_\infty)] + h \cdot E[(S^E + OV_\infty - D[1, l^E + 1])^+] \\ &\quad + b \cdot E[(D[1, l^E + 1] - S^E - OV_\infty)^+] . \end{aligned} \quad (3.5)$$

We will find the following alternate expression useful.

$$\begin{aligned} C_{S^E, \pi^R} &= c^E \cdot E[\max(0, D - Y_\infty)] + h \cdot E[(S^E + \max(0, Y_\infty - D) - D[1, l^E + 1])^+] \\ &\quad + b \cdot E[(D[1, l^E + 1] - S^E - \max(0, Y_\infty - D))^+] . \end{aligned} \quad (3.6)$$

We now derive a formula for the optimal S^E for a given π^R .

Lemma 3.3. *For a given ordering policy from R , π^R , the optimal value of S^E for the infinite horizon average cost problem is the solution to*

$$P(S^E \geq D[1, l^E + 1] - OV_\infty) = b/(b + h) .$$

This result follows directly from recognizing that C_{S^E, π^R} is a convex function of S^E and by solving for the first order condition with respect to S^E .

Let us now turn to the issue of finding an optimal π^R for a given S^E . Recall our observation earlier that the $\{Y_t\}$ process corresponds to the on-hand inventory in a lost sales inventory system with a lead time of $l^R - l^E - 1$ periods. Similarly, Y_∞ is the stationary distribution of the on hand inventory. Notice from (3.6) that the cost C_{S^E, π^R} , for a fixed S^E , is a non-linear (more importantly, not necessarily convex) function of Y_∞ . As a result, determining the optimal π^R is equivalent to finding the optimal policy in a lost sales system with a non-linear holding and shortage cost function. To our knowledge, the literature on the lost sales problem focuses on the case of linear holding and shortage cost functions. Given that solution techniques for the lost sales problem are quite limited even for this case, it seems highly unlikely that an efficient method can be developed to solve this problem with more general cost functions.

We propose a method to resolve this difficulty in a way in which the existing techniques for lost sales problems can be leveraged for finding good policies for the dual sourcing problem. We develop an upper bound on the right hand side of (3.6). It gives a neat decoupling of the dynamics of the inventory system that is of independent interest.

Lemma 3.4.

$$\begin{aligned} C_{S^E, \pi^R} \leq & h \cdot E[(Y_\infty - D)^+] + c^E \cdot E[(D - Y_\infty)^+] \\ & + h \cdot E[(S^E - D[1, l^E + 1])^+] + b \cdot E[(D[1, l^E + 1] - S^E)^+] . \end{aligned} \quad (3.7)$$

PROOF: The proof follows from the fact that

$$E[(S^E + \max(0, Y_\infty - D) - D[1, l^E + 1])^+] \leq E[(Y_\infty - D)^+] + E[(S^E - D[1, l^E + 1])^+].$$

□

Let $\mathcal{B}(h, b, D, \tau)$ be a single stage backordering system with a lead time of τ , holding and backordering cost parameters h and b , and demand distribution D . From Lemma 3.4, the cost of the dual sourcing system, $\mathcal{DS}(h, b, c^E, D, l^E, l^R)$ which orders-up-to S^E for E and follows

π_R for R is bounded above by the sum of the cost of the backordering system, $\mathcal{B}(h, b, D, l^E)$ with an order-up-to S^E policy and the cost of the lost sales system, $\mathcal{L}(h, c^E, D, l^R - l^E - 1)$ operating under the policy π_R .

We can now use Lemma 3.4 to write a bound on the optimal cost of a dual sourcing system in terms of the optimal costs of a lost sales system and a backorder system. Let $C^*(\mathcal{B}(h, b, D, \tau))$, $C^*(\mathcal{L}(h, p, D, \tau))$, and $C^*(\mathcal{DS}(h, b, c^E, D, l^E, l^R))$ denote the optimal infinite horizon average costs of $\mathcal{B}(h, b, D, \tau)$, $\mathcal{L}(h, p, D, \tau)$ and $\mathcal{DS}(h, b, c^E, D, l^E, l^R)$ respectively.

Theorem 3.1. *The optimal infinite horizon average cost for $\mathcal{DS}(h, b, c^E, D, l^E, l^R)$ is bounded above by the sum of the optimal infinite horizon average costs of $\mathcal{L}(h, c^E, D, l^R - l^E - 1)$ and $\mathcal{B}(h, b, D, l^E)$, i.e.,*

$$C^*(\mathcal{DS}(h, b, c^E, D, l^E, l^R)) \leq C^*(\mathcal{L}(h, c^E, D, l^R - l^E - 1)) + C^*(\mathcal{B}(h, b, D, l^E)) .$$

PROOF: Let us choose S^E and π^R to be the optimal order-up-to level for $\mathcal{B}(h, b, D, l^E)$ and the optimal ordering policy for $\mathcal{L}(h, c^E, D, l^R - l^E - 1)$, respectively. The right side of the inequality in Lemma 3.4 now equals the right side of the desired inequality of this theorem. Since $C^*(\mathcal{DS}(h, b, c^E, D, l^E, l^R))$ is, by definition, the optimal cost in \mathcal{DS} , it is smaller than C_{S^E, π^R} , the left side of Lemma 3.4. The result follows from the preceding two statements. \square

In practice, we can use the results of this theorem to construct a heuristic for the dual sourcing problem where both E and R follow an order-up-to rule. The order-up-to levels are computed by finding the optimal order-up-to levels for both the backordering system, $\mathcal{B}(h, b, D, l^E)$ and the lost sales system, $\mathcal{L}(h, c^E, D, l^R - l^E - 1)$. The optimal order-up-to level is easy to compute since the cost is known to be a convex function of the order-up-to level; see Janakiraman and Roundy (2004) and Downs et al. (2001) for details. Observe that this heuristic belongs to the class of Dual Index policies, and is, in general, sub-optimal within the class. Thus, it can never beat the best Dual Index policy. On the other hand, the order-up-to levels can be computed trivially. One can do much better (if the aim is to

improve upon the best Dual Index policy) than use this heuristic as discussed below.

The result in Theorem 3.1 motivates us to adopt policies that have been shown to work well for the lost sales problem as the ordering rule for R. (The rule for E is retained as order-up-to.) One such heuristic for R is the Vector Base-stock policy. To our knowledge, this name was given by Zipkin (2006) although it was proposed by Morton (1971) and Nahmias (1979) – we will use the same name to refer to our policy. We refer the readers to the papers mentioned above for a discussion on why these policies are intuitive for the lost sales problem. For lost sales systems, in the experiments performed by Zipkin (2006), this policy outperforms the Best Base-stock policy in 30 out of 32 experiments and, on an average, has a cost saving of 1.76 % over the Best Base-stock policy. The policy has to be adapted to the dual sourcing problem, as described next. The implementation involves the use of two parameters, θ and S^E .

Let E use an order-up-to policy with a target level of S^E . We use θ to denote a parameter for describing our policy for R. Let

$$q_t^R = \min(F_1^{-1}(\theta), (F_2^{-1}(\theta) - q_{t-1}^R)^+, (F_3^{-1}(\theta) - q_{t-1}^R - q_{t-2}^R)^+, \dots, (F_{l^R+l^E}^{-1}(\theta) - \sum_{u=t-l^R+l^E+1}^{t-1} q_u^R - OV_t)^+),$$

where $F_u(\cdot)$ is the cumulative distribution function of $D[1, u]$.

We propose two heuristic policies, which we call the Best Vector Base-stock policy and the Standard Vector Base-stock policy. In the *Best Vector Base-stock policy*, we optimize over the entire parameter space $\{(\theta, S^E) : 0 \leq \theta \leq 1\}$. For a given value of θ , the optimal S^E may be obtained by the Newsvendor formula in Lemma 3.3. Recall that the computation of the Best Dual Index policy is similar: for a given value of Δ , the optimal S^E may be obtained by the Newsvendor formula in Lemma 3.3. Consequently, the computational complexity of finding the best policy within this class is the same as obtaining the Best Dual Index policy.

In the *Standard Vector Base-stock policy*, we choose the parameter θ as $c^E/(c^E + h)$ (for the Standard Vector Base-stock policy for the lost sales problem, $\mathcal{L}(h, p, D, \tau)$, Zipkin (2006) suggests that we choose $\theta = p/(p + h)$) and optimize over S^E only. The interesting feature of this policy is that it requires *no search*. In fact, the best S^E can be easily obtained by the Newsvendor formula in Lemma 3.3.

Theorem 3.1 also motivates the following easily computable Dual Index policy, which we refer to as the *Standard Dual Index policy*. Let R order-up-to the optimal order-up-to level of $\mathcal{L}(h, c^E, D, l^R - l^E - 1)$ within the class of order-up-to policies using the inventory position $\sum_{u=t-l^R+l^E}^{t-1} q_t^R + OV_t$. We determine the order up to level S^E for the emergency supplier using Lemma 3.3.

We next propose another policy class, based on Theorem 3.1. It too uses the structure of the optimal policy of the lost sales problem. We refer to this heuristic as the *Weighted Dual Index policy*. This policy is also order-up-to for E. Thus,

$$q_t^E = (S^E - IP_t^E)^+.$$

To develop an ordering policy from the regular supplier, we note that Morton (1969) shows that, for the lost sales problem, the more recent an outstanding order is, the more sensitive the optimal order quantity is to that order. Motivated by this observation, we generalize the notion of inventory position to a weighted sum of the outstanding orders. For computational ease, we restrict ourselves to weighted inventory positions, where the weights depend on a single parameter only. Thus, we assume that the regular supplier orders q_t^R to raise the weighted inventory position

$$\beta^{l^R-l^E-1} \cdot OV_t + \beta^{l^R-l^E-2} \cdot q_{t-l^R+l^E+1}^R + \cdots + \beta \cdot q_{t-2}^R + q_{t-1}^R$$

to Δ , with $0 \leq \beta \leq 1$. Note that this policy class contains the Dual Index policy. By constraining the value of β to a fixed number of points, say n , the computational complexity is increased by no more than a factor of n . For details on the implementation of this policy,

please see Section 5.

Finally, we observe that policies developed for the lost sales problem in the future can be routinely tested for the dual sourcing problem.

Some of the results in this section can be extended to the capacitated case (please see Appendix A for details).

4 Policies with an Order-up-to Structure for the Combined Orders of the Two Suppliers

In this section, we restrict attention to policies that have an order-up-to structure for the combined orders of the two suppliers. That is, there is a target level S^R such that the total inventory position in the system at the beginning of each period, after ordering, is S^R . Notice that the optimal policy has this structure in systems in which $l^R - l^E$ is equal to 1.

Assuming that the total inventory position in the system at the beginning of period 1 is S^R , these policies have the following property that is characteristic of order-up-to policies: the total quantity ordered in period t is the demand in period $t - 1$, i.e.,

$$q_t^R + q_t^E = D_{t-1} \quad \forall t \geq 2 .$$

Therefore, for any given choice of S^R , the only remaining policy decision is how to allocate the total order quantity of D_{t-1} units in period t between the two suppliers. Notice that the allocation decision made in period t *directly* impacts the holding and shortage costs only in the interval $[t + l^E, t + l^R - 1]$. We use this observation to derive bounds on the optimal order quantity from E, as a function of the total order quantity which is known at the beginning of a period. We need some preliminary definitions first. Recall that the expedited inventory

position at the beginning of period t , IP_t^E , is defined as

$$IP_t^E = w_t + \sum_{u=t-l^E}^{t-1} q_u^E + \sum_{u=t-l^R}^{t-l^R+l^E} q_u^R.$$

Let

$$G^E(y) = h \cdot E[y - \mathcal{D}^E]^+ + b \cdot E[\mathcal{D}^E - y]^+, \text{ where}$$

\mathcal{D}^E is the $(l^E + 1)$ -fold convolution of the demand distribution. We denote the holding cost term in the definition of $G^E(y)$ by $H^E(y)$, i.e.,

$$H^E(y) = h \cdot E[y - \mathcal{D}^E]^+.$$

The following notation will be useful. Let

$$q^R[u, v] = \sum_{t=u}^v q_t^R \text{ and } D[u, v] = \sum_{t=u}^{v-1} D_t.$$

Let

$$\begin{aligned} & g^1(IP_t^E, q_{t-1}^R, \dots, q_{t-l^R+l^E+1}^R, q) \\ &= c^E \cdot q + E \left[\sum_{u=t}^{t+l^R-l^E-1} G^E(IP_t^E + q + q^R[t+1-l^R+l^E, u-l^R+l^E] - D[t, u]) \right], \\ & g^2(IP_t^E, q_{t-1}^R, \dots, q_{t-l^R+l^E+1}^R, q) \\ &= c^E \cdot q + E \left[\sum_{u=t}^{t+l^R-l^E-1} G^E(IP_t^E + q + q^R[t+1-l^R+l^E, u-l^R+l^E]) \right] \text{ and} \\ & g^3(IP_t^E, q_{t-1}^R, \dots, q_{t-l^R+l^E+1}^R, q) \\ &= c^E \cdot q + G^E(IP_t^E + q) \\ &+ E \left[\sum_{u=t+1}^{t+l^R-l^E-1} H^E(IP_t^E + q + q^R[t+1-l^R+l^E, u-l^R+l^E] - D[t, u]) \right]. \end{aligned}$$

For a given period t and a given vector $(IP_t^E, q_{t-1}^R, \dots, q_{t-l^R+l^E+1}^R)$, let

$$\begin{aligned} q^1(Q) &= \arg \min_q g^1(IP_t^E, q_{t-1}^R, \dots, q_{t-l^R+l^E+1}^R, q) \\ &\text{s.t. } 0 \leq q \leq Q. \end{aligned}$$

The quantities $q^2(Q)$ and $q^3(Q)$ are defined similarly by replacing g^1 with g^2 and g^3 , respectively.

Theorem 4.1. *Consider a given period t and a given vector $(IP_t^E, q_{t-1}^R, \dots, q_{t-l^R+l^E+1}^R)$. Assume that the total order quantity in this period, $q_t^R + q_t^E$, is constrained to be Q . Then, for any finite horizon problem with a horizon length greater than $t + l^R - 1$ periods, there exists an optimal choice of q_t^E which is bounded above by $q^1(Q)$.*

Proof: We first show that $q^1(Q)$ is an upper bound on the optimal choice of q_t^E . Let us consider any policy π that orders a feasible quantity $q_t^{E,\pi}$ from E, where $q_t^{E,\pi} > q^1(Q)$. The orders placed by π in all periods will be denoted by the superscript π . Let us consider the following alternate policy, $\tilde{\pi}$, whose orders are denoted using the superscript $\tilde{\pi}$. For all $u \neq t$, $q_u^{E,\tilde{\pi}} = q_u^{E,\pi}$ and $q_u^{R,\tilde{\pi}} = q_u^{R,\pi}$. In period t , let

$$q_t^{E,\tilde{\pi}} = q^1(Q) \text{ and } q_t^{R,\tilde{\pi}} = Q - q^1(Q).$$

By construction, the only difference in the costs incurred by π and $\tilde{\pi}$ are the procurement costs in period t and the holding/shortage costs in periods $[t + l^E, t + l^R - 1]$, since the two policies *couple* in period $t + l^E$. Let us compute, $DIFF$, the expected cost of π over $[t, T]$ minus the expected cost of $\tilde{\pi}$ over $[t, T]$, for any $T \geq t + l^E - 1$.

$$\begin{aligned} DIFF = & c^E \cdot (q_t^{E,\pi} - q^1(Q)) \\ & + E \left[\sum_{u=t}^{t+l^R-l^E-1} G^E(IP_t^E + q_t^{E,\pi} + q^{E,\pi}[t+1, u] + q^R[t+1-l^R+l^E, u-l^R+l^E] - D[t, u]) \right] \\ & - E \left[\sum_{u=t}^{t+l^R-l^E-1} G^E(IP_t^E + q^1(Q) + q^{E,\tilde{\pi}}[t+1, u] + q^R[t+1-l^R+l^E, u-l^R+l^E] - D[t, u]) \right]. \end{aligned}$$

By our choice of $\tilde{\pi}$, we have $q^{E,\pi}[t+1, u] = q^{E,\tilde{\pi}}[t+1, u] \geq 0$. Combining this fact with the convexity of G^E , we get

$$\begin{aligned} DIFF \geq & c^E \cdot (q_t^{E,\pi} - q^1(Q)) \\ & + E \left[\sum_{u=t}^{t+l^R-l^E-1} G^E(IP_t^E + q_t^{E,\pi} + q^R[t+1-l^R+l^E, u-l^R+l^E] - D[t, u]) \right] \\ & - E \left[\sum_{u=t}^{t+l^R-l^E-1} G^E(IP_t^E + q^1(Q) + q^R[t+1-l^R+l^E, u-l^R+l^E] - D[t, u]) \right]. \end{aligned}$$

The function, $c^E \cdot q + E \left[\sum_{u=t}^{t+l^R-l^E-1} G^E(IP_t^E + q + q^R[t+1-l^R+l^E, u-l^R+l^E] - D[t, u]) \right]$ is convex in q and is minimized at $q^1(Q)$, which is smaller than $q_t^{E,\pi}$. This fact, combined with the previous inequality, implies that $DIFF \geq 0$. Therefore, for $T \geq t + l^R - 1$, the policy $\tilde{\pi}$ incurs a cost that is no larger than the cost incurred by π . This proves the existence of an optimal choice of q_t^E that is smaller than $q^1(Q)$. \square

Theorem 4.2. *Assume that the combined order quantity in any period u is equal to the demand in period $u-1$. Consider a given period t and a given vector $(IP_t^E, q_{t-1}^R, \dots, q_{t-l^R+l^E+1}^R)$. Assume that the total order quantity in this period, $q_t^R + q_t^E$, is constrained to be Q . Then, for any finite horizon problem with a horizon length greater than $t + l^R - 1$ periods, there exists an optimal choice of q_t^E which is bounded below by $q^2(Q)$.*

Proof: The proof is similar to the proof of Theorem 4.1. We will again refer to two policies denoted by π and $\tilde{\pi}$ and denote the difference in their costs by $DIFF$. Consider any policy π (within the class of policies that, in any period u , order a combined amount equal to D_{u-1}) that uses some $q_t^{E,\pi}$ in $[0, q^2(Q))$. We define $\tilde{\pi}$ as a policy that uses the same order sizes as π in all periods except t . In period t , let

$$q_t^{E,\tilde{\pi}} = q^2(Q) \text{ and } q_t^{R,\tilde{\pi}} = Q - q^2(Q).$$

Now, the difference in the costs of π and $\tilde{\pi}$ is

$$\begin{aligned}
DIFF = & \\
& c^E \cdot (q_t^{E,\pi} - q^2(Q)) \\
& + E \left[\sum_{u=t}^{t+l^R-l^E-1} G^E(IP_t^E + q_t^{E,\pi} + q^{E,\pi}[t+1, u] + q^R[t+1-l^R+l^E, u-l^R+l^E] - D[t, u]) \right] \\
& - E \left[\sum_{u=t}^{t+l^R-l^E-1} G^E(IP_t^E + q^2(Q) + q^{E,\pi}[t+1, u] + q^R[t+1-l^R+l^E, u-l^R+l^E] - D[t, u]) \right].
\end{aligned}$$

Recall that $q_u^{E,\pi} + q_u^{R,\pi} = D_{u-1}$ for all $u \geq 2$, by definition. Therefore, $q^{E,\pi}[t+1, u] \leq D[t, u]$ with probability one. Thus, by the convexity of G^E , the definition of $q^2(Q)$ and the assumption that $q_t^{E,\pi} < q^2(Q)$, we get $DIFF \geq 0$. This proves the desired result. \square .

Theorem 4.3. *Consider a given period t and a given vector $(IP_t^E, q_{t-1}^R, \dots, q_{t-l^R+l^E+1}^R)$. Assume that the total order quantity in this period, $q_t^R + q_t^E$, is constrained to be Q . Then, for any finite horizon problem with a horizon length greater than $t + l^R - 1$ periods, there exists an optimal choice of q_t^E which is bounded below by $q^3(Q)$.*

Proof: We first define the function $B^E(y)$ as

$$B^E(y) = b \cdot E[\mathcal{D}^E - y]^+.$$

Notice that this is a decreasing function and that $G^E(y) = H^E(y) + B^E(y)$. Let policy π pick some $q_t^{E,\pi}$ from $[0, q^3(Q))$. Consider a policy $\tilde{\pi}$ that uses the same order quantities as π in all periods except t . In period t , let

$$q_t^{E,\tilde{\pi}} = q^3(Q) \text{ and } q_t^{R,\tilde{\pi}} = Q - q^3(Q).$$

Here, $DIFF$, the difference in the costs of the two policies, can be written as

$$\begin{aligned}
DIFF = & \\
& c^E \cdot (q_t^{E,\pi} - q^3(Q)) \\
& + G^E(IP_t^E + q_t^{E,\pi}) - G^E(IP_t^E + q^3(Q)) \\
& + E \left[\sum_{u=t+1}^{t+l^R-l^E-1} G^E(IP_t^E + q_t^{E,\pi} + q^{E,\pi}[t+1, u] + q^R[t+1-l^R+l^E, u-l^R+l^E] - D[t, u]) \right] \\
& - E \left[\sum_{u=t+1}^{t+l^R-l^E-1} G^E(IP_t^E + q^3(Q) + q^{E,\pi}[t+1, u] + q^R[t+1-l^R+l^E, u-l^R+l^E] - D[t, u]) \right].
\end{aligned}$$

Using the fact that B^E is an increasing function and that $q_t^{E,\pi} < q^3(Q)$, we get

$$\begin{aligned}
DIFF \geq & \\
& c^E \cdot (q_t^{E,\pi} - q^3(Q)) \\
& + G^E(IP_t^E + q_t^{E,\pi}) - G^E(IP_t^E + q^3(Q)) \\
& + E \left[\sum_{u=t+1}^{t+l^R-l^E-1} H^E(IP_t^E + q_t^{E,\pi} + q^{E,\pi}[t+1, u] + q^R[t+1-l^R+l^E, u-l^R+l^E] - D[t, u]) \right] \\
& - E \left[\sum_{u=t+1}^{t+l^R-l^E-1} H^E(IP_t^E + q^3(Q) + q^{E,\pi}[t+1, u] + q^R[t+1-l^R+l^E, u-l^R+l^E] - D[t, u]) \right].
\end{aligned}$$

The right hand side of the inequality above is non-negative by the convexity of H^E , the definition of $q^3(Q)$ and the assumption that $q_t^{E,\pi} < q^3(Q)$. Thus, $DIFF \geq 0$, which proves the desired result. \square

Based on Theorems 4.1-4.3, we propose two heuristic choices for q_t^E (recall that the total order quantity in period t is equal to D_{t-1} under an order-up-to policy for the system, assuming that we start with a total inventory position of S^R units in the system): they are $q^1(D_{t-1})$ and $\max(q^2(D_{t-1}), q^3(D_{t-1}))$. We, therefore, refer to these policies as *Demand Allocation (U)* and *Demand Allocation (L)* respectively.

5 Computational Results

In this section, we examine the performance of our heuristic policies numerically. We use the long run average cost performance measure. We do not compute the optimal cost or the optimal policy. Indeed, if we were to compute the cost of the optimal policy to benchmark our heuristics, we would have to restrict our computational study due to the computational complexity of the dynamic program (see Veeraraghavan and Scheller-Wolf (2007)). Instead, we compare the performance of the heuristics proposed here with the best available heuristic in the literature, the Best Dual Index policy. For each problem instance, we evaluate the percentage improvement over the Best Dual Index policy:

$$\frac{Best\ Dual\ Index - Heuristic}{Best\ Dual\ Index} * 100.$$

We discuss the following six heuristic policies: (1) Best Vector Base-stock policy, (2) Best Weighted Dual Index policy, (3) Standard Dual Index policy, (4) Standard Vector Base-stock policy, (5) Demand Allocation (U) policy and (6) Demand Allocation (L) policy.

We test our heuristics on the following cost parameters and demand distributions for lead time differences between the regular and emergency supplier of 2, 3 and 4 periods and $\frac{b}{b+h}$ ratios of 75%, 85% and 95%.

Table: Parameters of the numerical study.

c^R	c^E	h	b	D
0	20, 40, 60	5	15, 85/3, 95	Normal(3,1), Geometric(0.4), Geometric(0.5)

Tables 1-5 (see Appendix B) provide the costs of our heuristic policies and the percentage improvement over the Best Dual Index policy (a negative percentage implies that the policy is worse than the Best Dual Index policy for that particular problem instance).

We first summarize the performance of our heuristic policies over the collection of problem instances. Table 6 below provides the average cost reduction, the maximum cost reduction

and the minimum cost reduction relative to the Best Dual Index policy as well as the percentage of cases in which the policy performs better than the Best Dual Index policy. The last row in the table provides the performance of the better of the Demand Allocation (U) policy and the Demand Allocation (L) policy.

Table 6: Summary of Heuristic Performance

Heuristic	Average	Best Case	Worst Case	% Cases Better
Best Vector Base-stock policy	1.1	4.5	-0.7	92
Best Weighted Dual Index policy	0.8	6.1	0	100
Standard Dual Index policy	- 8.1	-1.8	- 22.9	0
Standard Vector Base-stock policy	- 1.3	3.9	- 12.0	46
Demand Allocation (U) policy	- 0.3	3.4	- 2.9	41
Demand Allocation (L) policy	- 0.4	3.5	- 5.4	37
Better Demand Allocation policy	0.1	3.5	- 2.4	51

The computational complexity of the Best Vector Base-Stock is the same as the Best Dual Index policy, but it provides an average cost reduction of 1.1 % over the Best Dual Index policy. Interestingly, it performs better than the Best Dual Index policy in 92 % of the problem instances that we test. All the problem instances where the Best Vector Base-stock policy performs worse than the Best Dual Index policy correspond to the case of $\delta = 2$. The policy performs better than the Best Dual Index policy for large values of δ .

We next discuss the performance of the Demand Allocation (both U and L) policy. Recall that both these policies are single parameter policies. Their computational complexity is significantly better than the Best Dual Index policy. Simulation of the overshoot distribution to compute the order quantity from the emergency supplier E is not required. The Demand Allocation (U) and Demand Allocation (L) policies perform, on average, marginally worse (0.3 % and 0.4 % respectively) than the Best Dual Index policy. Interestingly, the policies

complement each other - the Demand Allocation (L) outperforms the Demand Allocation (U) policy when the service level is “low” and the cost of emergency sourcing is “high”, while the reverse is true when the service level is “high” and the cost of emergency sourcing is “low”. Comparing the better of these two policies with the Best Dual Index policy, we obtain a marginal improvement of 0.1 %, on average (and the better of the two performs better than the Best Dual Index policy on 51 % of the problem instances).

Next, we discuss the performance of the Best Weighted Dual Index policy. Recall that this policy class contains the Best Dual Index policy, and thus the policy performs as good as or better than the Best Dual Index policy in all problem instances. While the average improvement of the Best Weighted Dual Index² policy over the Best Dual Index policy is 0.8 %, the best case improvement is as high as 6.1 %. The problem instances corresponding to a high coefficient of variation show significant improvement in performance. We will discuss this in more detail later in this section.

Finally, note that the Standard Vector Base-stock policy is the easiest to compute - it requires no search, and a simulation of the overshoot distribution only once. This policy has a cost performance which is 1.3 % worse than the Best Dual Index policy. In comparison, the Standard Dual Index policy, whose computational complexity is the “same” as the Standard Vector Base-stock policy, is 8.1 % worse than the Best Dual Index policy, on average.

In the remainder of this section, we discuss the performance of the policies in more detail. We study when the policies perform better / worse compared to the Best Dual Index policy. The effect of varying the lead time of the regular supplier, the service level or the cost of the emergency supplier and the coefficient of variation of the demand distribution on the

²The policy is implemented by searching for the best weight, β , over $n = 6$ values, for computational ease.

performance of the policies will become clear from our discussion.

The performance of the Best Vector Base-stock policy, the Standard Vector Base-stock policy and Best Weighted Dual Index policy improves relative to the Best Dual Index policy with δ . To see why this may be the case, suppose that the lead time of the emergency supplier l^E is fixed and the lead time difference between the suppliers δ is varied. Following our observation that the state space of the dynamic program for the dual sourcing problem has a dimension of $l^R - l^E$, we expect the highest improvement over the Best Dual Index policy when δ is high. This would, indeed, be the case if the policy kept better track of the pipeline from the regular supplier (beyond l^E) instead of just the sum $(q_{t-1}^R + \dots + q_{t-l^R+l^E+1}^R)$, as in the case of the Dual Index policy. Recall that the Vector Base-stock policies keep track of all the partial sums of $(q_{t-1}^R, \dots, q_{t-l^R+l^E+1}^R)$ while the Weighted Dual Index policy discounts order quantities. Let us now examine how the Standard Dual Index policy and the Demand Allocation policies perform relative to the Best Dual Index policy with respect to δ . From Tables 1-5, we are unable to observe any consistent pattern for these policies.

Based on these observations, we suggest using either the Standard Vector Base-stock policy (when a computationally simple policy is desired) or the Best Vector Base-stock policy / Best Weighted Dual Index policy (when using more computationally complex policies is acceptable) when the values of δ are high.

We also note that the the Best Dual Index policy, the Best Weighted Dual Index policy, the Demand Allocation (U) and Demand Allocation (L) achieve the optimal cost when $c^E = b \cdot \delta$. Indeed, sourcing solely from a single supplier is optimal in such cases. Thus, we suggest using the policies that are computationally simpler (such as the Demand Allocation (U) or Demand Allocation (L)) when either c^E is “very high” or b is “very low” and using the more computationally complex policies (such as Best Vector Base-stock or Best Weighted

Dual Index) otherwise.

It is also interesting to observe the performance of our new policies when the coefficient of variation is “high” in our experiments. In fact, the average and maximum improvement of the Best Vector Base Stock policy (Best Weighted Dual Index policy) in this case (Geometric(0.4)) is 1.4 % and 4.5 % (2.2 % and 6.1 %) respectively. This, indeed, demonstrates that the Dual Index policy performs poorly for high coefficient of variation. This is because the Vector Base-stock policy, the Weighted Dual Index policy and the Demand Allocation (U) policy have a less aggregated view of the pipeline of the regular supplier, rather than the sum of the components further than l^E away.

Finally, we summarize the above recommendations in Tables 7 below. We use A to denote the Demand Allocation (U) policy and the Demand Allocation (L) policy, B to denote the Standard Vector Base-stock policy, and C to denote the Best Vector Base-stock policy and the Best Weighted Dual Index policy. The trade-off between performance and computational complexity is implicit in these tables.

Table 7: Recommendations when b , c^E , δ or the Coefficient of Variation (COV) is varied

Parameter	Low	High
b	A	C
c^E	C	A
δ	A,B	C
COV	A,B	C

6 Conclusions

In this paper, we propose and evaluate the performance of three new policies for the dual sourcing problem, namely, the Vector Base-stock policy, the Weighted Dual Index policy and the Demand Allocation policy.

The Vector Base-stock policy and the Weighted Dual Index policy use an order-up-to rule for the emergency supplier and the state information of the vector of orders, unlike the Dual Index policies that use the sum of orders, for placing an order on the regular supplier. The policies that we propose for R are based on the connections that we establish between the dual sourcing problem and the lost sales problem. The Best Vector Base-stock policy and the Best Weighted Dual Index policy provide an average cost improvement of 1.1 % and 0.8 % relative to the Best Dual Index policy over the set of our experiments. Interestingly, both these policies perform well when the coefficient of variation of the demand distribution is high. The computational complexity of the Best Vector Base-stock policy (Best Weighted Dual Index policy) is the same as (no more than n times, where n is the number of values of the weight, β) that of the Best Dual Index policy.

The Demand Allocation policies (both (U) and (L)) use an order-up-to rule for the system. The demand in period t is allocated in period $t + 1$ between E and R based on the “myopic cost” (the procurement, holding and / or backordering cost) over the next δ periods. The Demand Allocation (L) policy and the Demand Allocation (U) policy complement each other - the Demand Allocation (L) policy performs better when c^E is “high” or service level is “low”. Comparing the better of these two policies with the Best Dual Index policy we obtain a marginal improvement of 0.1 %, on average (and the better of the two performs better than the Best Dual Index policy in 51 % of the problem instances).

References

- Beyer, D., and J. Ward. 2000. Network Server Supply Chain at HP: A Case Study. Hewlett Packard Software Technology Lab Report, HPL 2000-84, Palo Alto.
- Chiang, C., and G. Gutierrez. 1995. A Periodic Review Inventory System with Two Supply Modes. *European Journal of Operations Research* 94:527–547.

- Daniel, K. H. 1962. A Delivery-Lag Inventory Model with Emergency Order. *In Multistage Models and Techniques (Chapter 2)*, Scarf, Gilford, Shelly (eds.), Stanford University Press.
- Downs, B., R. Metters, and S. Semple. 2001. Managing Inventory with Multiple Products, Lags in Delivery, Resource Constraints and Lost Sales: A Mathematical Programming Approach. *Management Science* 47 (3): 464–479.
- Feng, Q., S. Sethi, H. Yan, and H. Zhang. 2006. Are Base-stock Policies Optimal in Inventory Problems with Multiple Delivery Modes. *Operations Research* 15 (4): 801–807.
- Fukuda, Y. 1964. Optimal Policies for the Inventory Problem with Negotiable Lead Times. *Management Science* 10 (4): 690–707.
- Groenevelt, H., and N. Rudi. 2003. A Base-stock Inventory Model with the Possibility of Rushing Part of Order. Working Paper.
- Jain, A., H. Groenevelt, and N. Rudi. 2007. A Continuous Review Inventory Model with Two Feight Modes. Working Paper.
- Janakiraman, G., and R. Roundy. 2004. Lost-Sales Problems with Stochastic Lead Times: Convexity Results for Base-Stock Policies. *Operations Research* 52:795–803.
- Karlin, S., and H. Scarf. 1958. Inventory Models of the Arrow-Harris-Marschak Type with Time Lag. In *Studies in the Mathematical Theory of Inventory and Production*, ed. K. J. Arrow, S. Karlin, and H. Scarf, Chapter 9, 155–178. Stanford University Press.
- Lawson, D., and E. Porteus. 2000. Multistage Inventory Management with Expediting. *Operations Research* 48 (6): 878–893.
- Morton, T. E. 1969. Bounds on the Solution of the Lagged Optimal Inventory Equation with no Demand Backlogging and Proportional Costs. *SIAM Review* 11 (4): 572–596.

- Morton, T. E. 1971. The Near-myopic Nature of the Lagged-Proportional-Cost Inventory Problem With Lost Sales. *Operations Research* 19 (7): 1708–1716.
- Muharremoglu, A., and J. Tsitsiklis. 2003. Dynamic Leadtime Management in Supply Chains. Working Paper.
- Nahmias, S. 1979. Simple Approximations for a Variety of Dynamic Leadtime Lost-Sales Inventory Models. *Operations Research* 27 (5): 904–924.
- Neuts, F. 1964. An Inventory Model with Optimal Time Lag. *SIAM Journal on Applied Mathematics* 12:179–185.
- Rao, U., A. Scheller-Wolf, and S. Tayur. 2000. Development of a Rapid-Response Supply Chain at Caterpillar. *Operations Research* 48:189–204.
- Scheller-Wolf, A., S. Veeraraghavan, and G. V. Houtum. 2005. Dual Sourcing: The Simple Yet Effective Single Index Policy. Working Paper.
- Tagaras, G., and D. Vlachos. 2001. A Periodic Review Inventory System with Emergency Replenishments. *Management Science* 47 (3): 415–429.
- Threatte, K., and S. Graves. 2002. Tactical Shipping and Scheduling at Polaroid with Dual Lead-Times. *Innovation in Manufacturing Systems and Technology*.
- Veeraraghavan, S., and A. Scheller-Wolf. 2007. Now or later: Dual Index Policies for Capacitated Dual Sourcing Systems. *Forthcoming, Operations Research*.
- Whittemore, A., and S. Saunders. 1977. Optimal Inventory under Stochastic Demand with Two Supply Options. *SIAM Journal on Applied Mathematics* 32 (2): 293–305.
- Zipkin, P. 2006. Old and New Methods for Lost-Sales Inventory Systems. Working Paper, Duke University.

A Capacitated Systems

We now extend the results of Section 3 to the case when E and R are capacitated, i.e., in any period, the maximum order quantities that can be procured from E and R are K^E and K^R respectively. Recall that the policy for E is order-up-to S^E . Similar to the observation in Section 3, an overshoot may arise in any period, i.e.,

$$OV_t = (IP_t^E - S^E)^+.$$

In addition, there might be some periods in which the expedited inventory position, IP_t^E , falls short of S^E (even after ordering) even if the state of the system in period 1 is carefully chosen. Therefore, there can be a *shortfall*, which we define as

$$SH_t = (IP_t^E + q_t^E - S^E)^-.$$

We first extend Lemma 3.1 to the capacitated case. (Veeraraghavan and Scheller-Wolf (2007) introduced the notion of shortfall in the context of Dual Index policies with capacities.)

Lemma A.1. *Consider the infinite horizon (discounted cost or average cost) problem for the dual sourcing system with $l^E \geq 0$. Assume that we follow an order-up-to S^E policy from E, i.e., $q_t^E = \min((S^E - IP_t)^+, K^E)$ for every t . Then, for a given S^E , it is sufficient to consider policies in which q_t^R depends on the vector $(w_t, q_{t-1}^E, \dots, q_{t-l^E}^E, q_{t-1}^R, \dots, q_{t-l^R}^R)$ only through the vector $(OV_t - SH_t, q_{t-1}^R, \dots, q_{t-l^R+l^E+1}^R)$.*

PROOF: The proof follows from the fact that

$$OV_t - SH_t = IP_t^E + q_t^E - S^E$$

and that the costs in periods $\{t, t+1, \dots, t+l^E-1\}$ are *sunk costs*, as seen from period t , and the state of the system in any period after period $t+l^E-1$ depends on the state in period t only through the “compressed” state vector, $(IP_t^E, q_{t-1}^R, q_{t-2}^R, \dots, q_{t-l^R+l^E+1}^R)$. \square .

Let the difference between the overshoot and the shortfall be defined as the *offset*, i.e.,

$$OF_t = OV_t - SH_t.$$

As a consequence of Lemma A.1, we will restrict attention to order-up-to policies for E and policies that use the information $(OF_t, q_{t-1}^R, \dots, q_{t-l^R+l^E+1}^R)$ for R. Let us use π_R to refer to any such ordering policy from R and $q_{\pi_R}(\cdot)$ to denote the order quantity as a function of this compressed information vector. We now extend the independence result of Lemma 3.2.

Lemma A.2. *Assume $OF_1 = 0$. Consider a given stationary ordering policy from R, π^R . For any sample path of demands (D_1, \dots, D_t) , the overshoot in period t , OF_t , does not depend on the choice of S^E . The same result also applies to q_t^R , the quantity ordered from R.*

Again, note that Veeraraghavan and Scheller-Wolf (2007) show the same result *within the class of Dual Index policies for capacitated systems*, which is a subset of our class of policies. We now derive the recursions of OF_t , q_t^R and q_t^E .

$$OF_{t+1} = OF_t - D_t + q_{t-l^R+l^E+1}^R + q_{t+1}^E. \quad (\text{A.8})$$

$$q_{t+1}^R = \min(K^R, q_{\pi_R}(OF_{t+1}, q_t^R, \dots, q_{t-l^R+l^E+2}^R)). \quad (\text{A.9})$$

$$q_{t+1}^E = \min(K^E, (D_t - (OF_t + q_{t+1-l^R+l^E}^R))^+). \quad (\text{A.10})$$

Let $Z_t = OF_t + q_{t+1-l^R+l^E}^R$. Then we can write

$$OF_{t+1} = Z_t - D_t + \min(K^E, (D_t - Z_t)^+)$$

and

$$Z_{t+1} = Z_t - D_t + \min(K^E, (D_t - Z_t)^+) + q_{t+2-l^R+l^E}^R. \quad (\text{A.11})$$

Consider a lost sales inventory system with i.i.d. demands with the same distribution as D with a lead time of $(l^R - l^E - 1)$ periods and capacity K^R . The evolution of the inventory on-hand at the beginning of a period is exactly the same as the processes $\{Z_t\}$ under the ordering policy π^R when $K^E = \infty$. To see this, note that when $K^E = \infty$,

$$Z_{t+1} = (Z_t - D_t)^+ + q_{t+2-l^R+l^E}^R. \quad (\text{A.12})$$

Let us now assume that the processes $\{OF_t\}$ and $\{Y_t\}$ converge to stationary distributions represented by the random variables OF_∞ and Z_∞ , respectively. Notice that both

these stationary random variables are also independent of the choice of S^E . Thus, we can derive a formula for the optimal S^E for a given π_R .

Lemma A.3. *For a given ordering policy from R , π^R , the optimal value of S^E for the infinite horizon average cost problem is the solution to*

$$P(S^E \geq D[1, l^E + 1] - OF_\infty) = b/(b + h) .$$

We can now derive an upper bound on the optimal cost of a dual sourcing system in terms of the optimal costs of a lost sales system and a backorder system using a result analogous to Lemma 3.4.

Let $C^*(\mathcal{DS}(h, b, c^E, D, l^E, l^R; K^E, K^R))$ be the infinite horizon average cost of a dual sourcing system, where K^E and K^R are the capacities of the emergency and regular supplier. Let $C^*(\mathcal{L}(h, p, D, \tau; K))$ be the infinite horizon average cost of a lost sales system with capacity K .

Theorem A.1. *The optimal infinite horizon average cost for $\mathcal{DS}(h, b, c^E, D, l^E, l^R; K^E = \infty, K^R)$ is bounded above by the sum of the optimal infinite horizon average costs of $\mathcal{L}(h, c^E, D, l^R - l^E - 1; K^R)$ and $\mathcal{B}(h, b, D, l^E)$, i.e.,*

$$C^*(\mathcal{DS}(h, b, c^E, D, l^E, l^R)) \leq C^*(\mathcal{L}(h, c^E, D, l^R - l^E - 1; K^R)) + C^*(\mathcal{B}(h, b, D, l^E)) .$$

B Numerical Results

Table 1: Performance of the heuristics for the Geometric (0.5) distribution.

I^E	I^R	c^E	b	Best Dual Index	Best Vector Base-stock		Best Weighted Dual Index		Standard Dual Index		Standard Vector Base-stock		Demand Allocation (U)		Demand Allocation (L)	
				Cost	Cost	%	Cost	%	Cost	%	Cost	%	Cost	%	Cost	%
0	2	20	15	16.55	16.54	0.02	16.55	0	19.78	-19.5	17.95	-8.5	16.39	0.93	16.32	1.37
0	3	20	15	17.6	17.56	0.26	17.6	0	19.49	-10.7	17.56	0.26	18.03	-2.4	17.73	-0.7
0	4	20	15	18.85	18.56	1.56	18.82	0.17	19.78	-4.94	18.56	1.56	19.35	-2.6	19.06	-1.1
0	2	20	85/3	20.28	20.36	-0.37	20.28	0	23.77	-17.2	21.39	-5.4	20.21	0.38	20.13	0.75
0	3	20	85/3	21.59	21.42	0.76	21.58	0.04	23.54	-9.03	21.42	0.76	21.77	-0.9	21.64	-0.3
0	4	20	85/3	22.66	22.21	2.01	22.63	0.14	23.61	-4.19	22.38	1.23	22.9	-1	22.72	-0.2
0	2	20	95	28.37	28.45	-0.27	28.37	0	31.79	-12	29.33	-3.4	28.28	0.34	28.38	-0
0	3	20	95	29.61	29.54	0.23	29.61	0	31.25	-5.54	29.74	-0.5	29.83	-0.8	29.97	-1.2
0	4	20	95	30.77	30.4	1.19	30.77	0	31.31	-1.77	30.56	0.66	30.67	0.31	31.62	-2.8
0	3	40	15	19.81	19.71	0.5	19.7	0.54	21.3	-7.56	20.64	-4.2	20.15	-1.8	20.1	-1.5
0	4	40	15	21.83	21.68	0.68	21.83	0	24.55	-12.4	21.78	0.21	22.45	-2.8	22.2	-1.7
0	2	40	85/3	22.42	22.34	0.33	22.41	0.02	24	-7.04	24.03	-7.2	22.11	1.36	22.08	1.5
0	3	40	85/3	24.22	24.03	0.77	24.21	0.02	25.29	-4.43	24.14	0.33	24.72	-2.1	24.45	-1
0	4	40	85/3	26	25.59	1.59	25.93	0.3	28.14	-8.2	25.59	1.59	26.43	-1.6	26.14	-0.5
0	2	40	95	30.01	29.88	0.42	30.01	0	31.98	-6.58	32.15	-7.2	30	0.02	30.13	-0.4
0	3	40	95	32.14	32.03	0.33	32.14	0	33.31	-3.66	32.03	0.33	32.15	-0	32.3	-0.5
0	4	40	95	33.96	33.45	1.5	33.96	0	36.25	-6.74	33.73	0.67	33.61	1.04	34.09	-0.4
0	3	60	85/3	25.52	25.32	0.76	25.44	0.33	28.99	-13.6	25.61	-0.3	25.62	-0.4	25.55	-0.1
0	4	60	85/3	27.81	27.45	1.3	27.81	0	30.6	-10	27.45	1.3	28.35	-1.9	27.94	-0.5
0	2	60	95	31.16	30.97	0.63	31.16	0	35.5	-13.9	32.79	-5.2	31.18	-0	31.05	0.37
0	3	60	95	33.52	33.28	0.71	33.52	0	37.02	-10.4	33.63	-0.3	33.76	-0.7	33.57	-0.2
0	4	60	95	35.64	35.03	1.69	35.64	0	38.72	-8.66	35.59	0.12	35.91	-0.8	35.86	-0.6

Table 2: Performance of the heuristics for the Geometric (0.4) distribution.

I^E	I^R	c^E	b	Best Dual Index	Best Vector Base-stock		Best Weighted Dual Index		Standard Dual Index		Standard Vector Base-stock		Demand Allocation (U)		Demand Allocation (L)	
				Cost	Cost	%	Cost	%	Cost	%	Cost	%	Cost	%	Cost	%
0	2	20	15	22.8	22.8	-0.01	22.8	0	24.92	-9.29	23.71	-4	22.63	0.76	22.63	0.76
0	3	20	15	24.54	24.53	0.02	24.54	0	26.11	-6.43	24.56	-0.1	24.9	-1.47	24.66	-0.5
0	4	20	15	26.18	25.68	1.9	25.65	2.03	28.86	-10.3	25.78	1.5	26.68	-1.9	26.3	-0.47
0	2	20	85/3	27.76	27.81	-0.19	27.76	0	29.89	-7.68	28.68	-3.3	27.62	0.49	27.66	0.36
0	3	20	85/3	29.54	29.53	0.06	29.45	0.32	31.09	-5.22	29.61	-0.2	29.81	-0.91	29.64	-0.34
0	4	20	85/3	31.36	30.79	1.81	30.54	2.59	33.8	-7.81	30.89	1.5	31.51	-0.5	31.47	-0.36
0	2	20	95	38.55	38.82	-0.71	38.26	0.73	40.57	-5.24	39.4	-2.2	38.48	0.18	38.65	-0.26
0	3	20	95	40.54	40.51	0.06	39.92	1.53	41.8	-3.12	40.7	-0.4	40.61	-0.18	41.03	-1.21
0	4	20	95	42.06	41.57	1.16	41.25	1.93	44.32	-5.38	41.79	0.64	41.91	0.36	43.31	-2.98
0	3	40	15	27.65	27.03	2.26	26.9	2.71	30.05	-8.66	28.06	-1.5	27.66	-0.03	27.5	0.56
0	4	40	15	30.47	29.89	1.89	29.33	3.75	32.81	-7.67	30.07	1.3	30.91	-1.43	30.44	0.1
0	2	40	85/3	30.45	30.43	0.07	30.43	0.05	35.04	-15.1	32.3	-6.1	30.22	0.77	30.15	0.98
0	3	40	85/3	33.2	33.14	0.17	32.75	1.36	35.06	-5.61	33.14	0.17	33.57	-1.13	33.32	-0.38
0	4	40	85/3	36.05	35.21	2.33	34.61	3.98	37.91	-5.16	35.21	2.33	36.24	-0.53	35.75	0.82
0	2	40	95	41.11	41.06	0.11	40.92	0.46	45.67	-11.1	43.31	-5.4	41.04	0.16	41.11	0
0	3	40	95	44.22	44.04	0.41	43.43	1.8	46.21	-4.5	44.26	-0.1	44.16	0.14	44.34	-0.28
0	4	40	95	46.68	46.01	1.43	45.25	3.06	48.61	-4.13	46.04	1.36	46.41	0.58	46.95	-0.59
0	3	60	85/3	35.64	34.9	2.09	34.44	3.38	37.9	-6.34	35.44	0.56	35.38	0.74	35.13	1.43
0	4	60	85/3	39.57	37.8	4.49	37.17	6.08	41.5	-4.86	38.02	3.92	38.81	1.92	38.39	3
0	2	60	95	42.66	42.41	0.61	42.45	0.51	47.41	-11.1	44.89	-5.2	42.78	-0.28	42.87	-0.48
0	3	60	95	46.8	46.07	1.55	45.5	2.77	48.69	-4.05	46.22	1.23	46.48	0.68	46.45	0.73
0	4	60	95	50.28	48.28	3.98	47.82	4.9	52.29	-3.99	48.54	3.47	49.09	2.37	49.33	1.89

Table 3: Performance of the heuristics for the Normal (3,1) distribution.

I^E	I^R	c^E	b	Best Dual Index	Best Vector Base-stock		Best Weighted Dual Index		Standard Dual Index		Standard Vector Base-stock		Demand Allocation (U)		Demand Allocation (L)	
				Cost	Cost	%	Cost	%	Cost	%	Cost	%	Cost	%	Cost	%
0	2	20	15	10.86	10.76	0.91	10.86	0	12.18	-12.2	11.39	-4.95	10.91	-0.49	10.86	-0.03
0	3	20	15	12.35	12.2	1.25	12.34	0.08	13.29	-7.63	12.35	0.03	12.59	-1.89	12.5	-1.16
0	4	20	15	13.51	13.22	2.15	13.49	0.18	14.35	-6.16	13.22	2.15	13.85	-2.48	13.87	-2.62
0	2	20	85/3	12.96	12.88	0.62	12.96	0	13.87	-7.01	13.16	-1.54	12.98	-0.15	13.02	-0.44
0	3	20	85/3	14.36	14.2	1.09	14.35	0.06	15.06	-4.94	14.23	0.906	14.54	-1.28	14.81	-3.17
0	4	20	85/3	15.53	15.18	2.27	15.5	0.24	16.1	-3.63	15.24	1.857	15.77	-1.53	16.38	-5.43
0	2	20	95	16.23	16.14	0.53	16.23	0	16.78	-3.4	16.21	0.089	16.27	-0.26	16.31	-0.51
0	3	20	95	17.6	17.43	0.95	17.59	0.03	18.09	-2.8	17.43	0.921	17.66	-0.35	17.87	-1.54
0	4	20	95	18.74	18.37	1.96	18.67	0.39	19.11	-1.99	18.6	0.75	18.96	-1.19	19.47	-3.9
0	3	40	15	12.68	12.56	0.96	12.68	0.01	14.33	-13.1	13.32	-5.05	12.9	-1.76	12.69	-0.07
0	4	40	15	14.28	14.08	1.39	14.28	0	15.92	-11.5	14.46	-1.26	14.68	-2.81	14.24	0.292
0	2	40	85/3	13.26	13.25	0.14	13.26	0	14.6	-10.1	14.02	-5.66	13.29	-0.2	13.26	0.003
0	3	40	85/3	15.27	15.16	0.68	15.25	0.08	16.27	-6.58	15.41	-0.96	15.46	-1.27	15.36	-0.63
0	4	40	85/3	17	16.79	1.29	16.98	0.13	17.9	-5.24	16.82	1.109	17.36	-2.07	17.25	-1.43
0	2	40	95	17.08	17.04	0.23	17.08	0	17.75	-3.93	17.31	-1.35	17.1	-0.11	17.14	-0.35
0	3	40	95	18.96	18.93	0.15	18.96	0	19.5	-2.87	18.95	0.065	19.06	-0.55	19.18	-1.15
0	4	40	95	20.61	20.4	1.02	20.53	0.38	21.17	-2.72	20.44	0.804	20.83	-1.09	21.27	-3.23
0	3	60	85/3	15.49	15.37	0.72	15.49	0	16.93	-9.36	15.97	-3.14	15.68	-1.27	15.45	0.23
0	4	60	85/3	17.48	17.26	1.27	17.48	0	18.89	-8.05	17.54	-0.35	17.97	-2.75	17.43	0.29
0	2	60	95	17.38	17.42	-0.25	17.38	0	18.33	-5.48	17.82	-2.54	17.33	0.274	17.39	-0.03
0	3	60	95	19.57	19.56	0.06	19.57	0	20.34	-3.93	19.68	-0.58	19.75	-0.91	19.84	-1.39
0	4	60	95	21.61	21.3	1.46	21.59	0.1	22.3	-3.17	21.39	1.045	21.84	-1.05	22.22	-2.82

Table 4: Performance of the heuristics for the Geometric (0.5) distribution.

I^E	I^R	c^E	b	Best Dual Index	Best Vector Base-stock		Best Weighted Dual Index		Standard Dual Index		Standard Vector Base-stock		Demand Allocation (U)		Demand Allocation (L)	
				Cost	Cost	%	Cost	%	Cost	%	Cost	%	Cost	%	Cost	%
1	3	20	15	19.52	19.45	0.33	19.52	0	23.99	-22.9	21.3	-9.2	19.47	0.24	19.43	0.46
1	4	20	15	20.95	20.79	0.78	20.95	0	23.28	-11.1	21	-0.3	21.01	-0.27	20.96	-0
1	5	20	15	22.07	21.73	1.53	22.03	0.16	23.4	-6.03	21.82	1.14	22.08	-0.03	22.05	0.09
1	3	20	85/3	24.48	24.41	0.32	24.48	0	28.76	-17.5	26.04	-6.4	24.46	0.1	24.48	0
1	4	20	85/3	25.89	25.56	1.27	25.89	0	28.05	-8.38	25.83	0.2	25.92	-0.14	26.03	-0.6
1	5	20	85/3	27.05	26.51	1.99	26.94	0.41	28.19	-4.2	26.65	1.48	26.87	0.65	27.27	-0.8
1	3	20	95	34.01	33.98	0.1	34.01	0	38.11	-12.1	35.39	-4.1	33.95	0.17	34.08	-0.2
1	4	20	95	35.26	34.86	1.15	35.26	0	37.39	-6.04	35.48	-0.6	35.68	-1.2	36.03	-2.2
1	5	20	95	36.57	35.81	2.09	36.55	0.07	37.59	-2.78	36.18	1.06	37	-1.16	37.72	-3.1
1	4	40	15	22.09	21.96	0.59	21.96	0.58	24.65	-11.6	23.37	-5.8	22.23	-0.63	22.15	-0.3
1	5	40	15	23.86	23.73	0.52	23.86	0	28.14	-17.9	23.95	-0.4	24.3	-1.83	24.07	-0.9
1	3	40	85/3	25.33	25.25	0.33	25.33	0	28.31	-11.8	28.34	-12	25.33	0.01	25.28	0.21
1	4	40	85/3	27.59	27.39	0.71	27.59	0	29.62	-7.36	28.1	-1.9	27.63	-0.14	27.6	-0.1
1	5	40	85/3	29.33	28.94	1.33	29.33	0	32.87	-12.1	28.94	1.33	29.27	0.2	29.14	0.62
1	3	40	95	35.18	35.05	0.37	35.18	0	37.91	-7.76	38.01	-8	35.12	0.17	35.25	-0.2
1	4	40	95	37.3	36.91	1.06	37.3	0	39.28	-5.3	37.47	-0.4	37.5	-0.53	37.67	-1
1	5	40	95	39.18	38.51	1.7	39.18	0	42.23	-7.8	38.51	1.7	39.35	-0.43	39.51	-0.8
1	4	60	85/3	28.49	28.3	0.67	28.34	0.53	33.32	-16.9	29	-1.8	28.56	-0.22	28.43	0.23
1	5	60	85/3	30.36	30.14	0.74	30.36	0	34.86	-14.8	30.8	-1.4	30.66	-1.01	30.5	-0.5
1	3	60	95	35.9	35.74	0.44	35.9	0	41.44	-15.4	38.41	-7	35.93	-0.06	35.9	0.01
1	4	60	95	38.4	38.06	0.88	38.4	0	43	-12	38.49	-0.2	38.44	-0.11	38.54	-0.4
1	5	60	95	40.39	39.83	1.39	40.28	0.29	44.41	-9.95	40.37	0.05	40.35	0.1	40.37	0.05

Table 5: Performance of the heuristics for the Geometric (0.4) distribution.

I^E	I^R	c^E	b	Best Dual Index	Best Vector Base-stock		Best Weighted Dual Index		Standard Dual Index		Standard Vector Base-stock		Demand Allocation (U)		Demand Allocation (L)	
				Cost	Cost	%	Cost	%	Cost	%	Cost	%	Cost	%	Cost	%
1	3	20	15	26.6	26.6	-0.02	26.56	0.16	30.01	-12.8	28.48	-7.1	26.45	0.57	26.45	0.57
1	4	20	15	28.8	28.58	0.76	28.33	1.62	31.26	-8.54	29	-0.7	28.79	0.01	28.78	0.05
1	5	20	15	30.47	29.99	1.58	29.54	3.04	34.14	-12	30.08	1.29	30.39	0.26	30.38	0.28
1	3	20	85/3	33.52	33.51	0.04	33.31	0.63	36.52	-8.96	34.76	-3.7	33.42	0.32	33.51	0.02
1	4	20	85/3	35.52	35.3	0.62	34.93	1.68	37.78	-6.34	35.77	-0.7	35.58	-0.16	35.68	-0.45
1	5	20	85/3	37.15	36.56	1.59	36.01	3.07	40.37	-8.65	36.64	1.39	37.16	-0.03	37.8	-1.73
1	3	20	95	46.7	46.62	0.17	46.7	0	48.66	-4.2	47.95	-2.7	46.83	-0.28	47.02	-0.69
1	4	20	95	48.5	48.19	0.64	48.32	0.38	50.83	-4.8	48.72	-0.5	48.95	-0.92	49.63	-2.32
1	5	20	95	50.2	49.65	1.1	49.55	1.31	52.28	-4.14	49.65	1.1	51.21	-2.01	52.1	-3.78
1	4	40	15	31.11	30.23	2.83	29.63	4.77	34.56	-11.1	31.66	-1.8	30.63	1.53	30.51	1.95
1	5	40	15	33.35	32.71	1.94	32.06	3.86	37.29	-11.8	33.48	-0.4	33.33	0.07	33.03	0.95
1	3	40	85/3	35.13	34.82	0.89	34.46	1.92	41.12	-17	38.24	-8.9	34.9	0.67	34.87	0.74
1	4	40	85/3	38.27	37.61	1.73	37.17	2.88	41.45	-8.31	38.47	-0.5	37.98	0.74	37.97	0.78
1	5	40	85/3	40.63	39.99	1.56	39.11	3.74	43.97	-8.23	40.11	1.28	40.35	0.68	40.35	0.68
1	3	40	95	48.49	48.35	0.29	48.35	0.29	52.54	-8.36	51.63	-6.5	48.42	0.14	48.6	-0.23
1	4	40	95	51.44	50.75	1.34	51.02	0.82	54.4	-5.75	51.34	0.2	51.41	0.06	51.88	-0.85
1	5	40	95	54.47	53.15	2.42	52.99	2.71	55.66	-2.18	53.2	2.32	54.17	0.55	54.55	-0.15
1	4	60	85/3	40.39	38.82	3.89	37.93	6.09	43.68	-8.13	40.14	0.63	39.02	3.39	38.96	3.54
1	5	60	85/3	42.9	41.43	3.44	40.75	5.01	47.31	-10.3	42.35	1.28	42.01	2.09	41.69	2.81
1	3	60	95	49.9	49.24	1.32	49.01	1.78	55.87	-12	52.98	-6.2	49.46	0.89	49.43	0.95
1	4	60	95	53.62	52.25	2.55	52.4	2.27	56.67	-5.7	53.22	0.74	52.98	1.2	53.16	0.86
1	5	60	95	57.54	55.01	4.4	55.01	4.4	60.38	-4.94	55.66	3.27	55.65	3.28	55.92	2.81