Optimal Joint Inventory and Transshipment Control under Uncertain Capacity

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Abstract

In this paper, we address the optimal joint control of inventory and transshipment for a firm that produces in two locations and faces capacity uncertainty. Capacity uncertainty (e.g., due to downtime, quality problems, yield, etc.) is a common feature of many production systems but its effects have not been explored in the context of a firm that has multiple production facilities. We first characterize the optimal production and transshipment policies and show that uncertain capacity leads the firm to ration the inventory that is available for transshipment to the other location and characterize the structure of this rationing policy. Then we characterize the optimal production policies at both locations which are defined by state-dependent produce-up-to thresholds. We also describe sensitivity of the optimal production and transshipment policies to problem parameters and, in particular, explain how uncertain capacity can lead to counterintuitive behavior, such as produce-up-to limits decreasing for locations that face stochastically higher demand. We finally explore, through a numerical study, when applying the optimal policy is most likely to yield significant benefits compared to simple policies. In particular, we consider two simple straw policies: 1) a policy that disallows transshipment and 2) a policy that disallows rationing and forces the two locations to transship inventory to satisfy the other location's shortage.

1 Introduction

Consider a firm that produces the same product in multiple locations, but faces demand and capacity uncertainty. The capacity uncertainty is caused by factors such as downtime, quality problems, yield, etc. The firm faces two related decisions: 1) how much should it produce at each location and 2) how much should it transship from one location to another? Even though the literature on transshipment is rich, it usually ignores the effects of capacity uncertainty and our aim is to gain insight into how capacity uncertainty affects both these decisions.

The problem we describe is very common in industry. For example, we recently worked with a diesel engine manufacturer that has multiple locations where castings are made. The capacity of the plants making the castings in any week was random due to quality problems and, therefore, the company was exploring transshipment from one location to another to satisfy engine plants' demands for castings. We observed similar issues in the case of a major paper manufacturer that produces paper cups in multiple locations in the U.S., as well as a major newspaper ink manufacturer with over 20 plants in the U.S. In all cases, products would be transshipped from one plant to another plant's markets, when capacity in a plant was low in a given period. However, we observed that the actual production policies of the plants did not take into account the fact that such transshipments may occur. In some situations, we also observed that plant management was reluctant to transship beyond a certain amount, due to fear that they may face a shortage next period if their inventory levels are down significantly. All of these observations motivated us to explore how optimal transshipment and production decisions should be made jointly and how the level of demand and capacity uncertainty affects the behavior of optimal policies.

We consider a centralized system with two facilities that operate in two markets, produce the same product, and sell it at constant prices. Both facilities, in addition to demand uncertainty, face uncertain production capacities. Inventory can be held from period to period, but unsatisfied demand is lost. All decisions are made by a central planner, who has full access to the stock status at the two facilities. Her objective is to maximize the expected discounted joint profits over a finite horizon. At the beginning of any period, she determines the production quantities for both facilities. After the production and demand uncertainties are revealed, she decides how much inventory should be transshipped from one location to another. Demands are satisfied after the transshipment.

We examine the structure of the optimal production and transhipment policies for both facilities and find it different from the previous research. Under various assumptions, Robinson [32], Tagaras [35], and others verify optimality of the "complete pooling" policy for transshipment: transshipment occurs when one location has excess stock and the other is short and the transshipped quantity is equal to minimum of the surplus and the shortage. Due to uncertain capacity in our setting, even when all the "complete pooling" assumptions (listed at the end of Section 4.1) are satisfied, the optimal transshipment follows a floor-rationing policy. In other words, the whole system may be better off with one facility keeping some safety stock and not satisfying the shortage of the other facility. Also, it may be beneficial to ship some inventory from the facility with higher holding cost to the other one, even when the latter does not need it, in order to decrease holding cost across multiple periods.

Unlike the base-stock policy established in the literature, our optimal production policies for two facilities are based on switching curves – each facility's production quantity is a non-increasing function of the other facility's starting inventory – and may also depend on its own starting inventory. In the special case when one of the facilities has infinite capacity, the optimal policy for that facility is an up-to level. The up-to level is decreasing in the inventory level of the facility facing uncertain capacity. For another special case, when both facilities have deterministic but limited capacities, we show that modified order-up-to policies are optimal. In addition to showing that the uncertainty in capacity changes the structure of the optimal production policy, we also show that optimal policies behave differently (sometimes counterintuitively) in the presence of uncertain capacity – e.g., stochastically larger demand, lower holding costs, or higher revenue may result in strictly lower production targets. In our numerical study, we examine both the direction and size of impact that different parameters have on the policy and on the total profit. Since the optimal policy is fairly complicated, we consider two simple straw policies (often used in practice) and compare them to the optimal policy, which allows us to describe when the optimal policy is most beneficial and when simple policies perform well.

We review the literature in Section 2 and state the assumptions and formulate the model in Section 3. In Section 4 we establish the structure of the optimal transshipment and production policies. Then, we derive analytical results for the sensitivity of the optimal policy in Section 5. In Section 6 we use a numerical study to describe additional properties of the optimal policy. Finally, in Section 7, we discuss possible extensions.

2 Literature Review

In multi-location stochastic inventory systems, lateral transshipment across locations allows one to better match supply and demand. Typically, transshipment helps a firm to deal with potential shortage of products and takes place after demand is realized but before it is satisfied. The commonly considered costs include linear production, holding, shortage, and transshipment costs. Krishnan and Rao [27] study a single-period two-location problem and include an extension to N-location scenario. The costs at all locations are equal. Robinson [32] extends their work to the multi-period multi-location case, with varying costs across the outlets. Tagaras [35] considers a similar model and focuses on the pooling effects created by allowing transshipment, on service levels in a two-location system. He also establishes a set of assumptions to guarantee the complete pooling (i.e., complete transshipment without rationing).

Gross [18] may be the first to consider a two-location problem in which transshipment occurs before the demand realization. The corresponding multi-period multi-location problem is studied by Karmarkar and Patel [26], Showers [34], and Karmarkar [25]. Das [7] allows one-time transshipment in the middle of the period when demand is partially disclosed. Lee [28] and Axsater [4] examine a continuous-review system with transshipment triggered by stockouts. Archibald, Sassen, and Thomas [1] combine Das' [7] work with Lee's [28] and Axsater's [4] to a multi-period two-location periodic-review model in which demand is disclosed continuously during the review period and the transshipment or emergency-order decision is made whenever stockout occurs.

A number of extensions have been studied in the literature. Fixed joint replenishment costs are explicitly considered by Herer and Rashit [21] in a two-location single-period problem. Tagaras and Cohen [36] study the effect of replenishment lead times. Axsater [5] considers a centralized system with more than two locations and develops an effective heuristic decision rule for lateral transshipment. The first paper, we are aware of, that considers decentralized decision makers is Rudi, Kapur, and Pyke [33]. They analyze a two-location single-period model where each location maximizes its own profits. The authors identify transshipment prices which induce both locations to choose inventory levels consistent with joint-profit maximization. Anupindi *et al* [2] addresses a similar problem. More general multiple location decentralized distribution systems are studied by

Anupindi *et al* [3], Granot and Sosic [17], and Zhao *et al* [39].

All of the above papers consider stochastic demand and assume that replenishment capacity is infinite and certain. Capacitated inventory systems and systems with uncertain capacity are considered in a separate group of papers, but these papers do not consider transshipment among locations. We first list papers that consider deterministic but limited capacity. Federgruen and Zipkin [11][12] study a system with the stationary stochastic demand and capacity restrictions and show that order-up-to policies are optimal for the infinite-horizon case. Glasserman and Tayur [14][15][16] assume order-up-to policies for a multi-echelon system and describe how to find optimal up-to levels. Parker and Kapuscinski [31] show that up-to policy is optimal for a 2-echelon capacitated system. Kapuscinski and Tayur [24] consider a capacitated production-inventory system with non-stationary demand. A multiproduct version is analyzed by DeCroix and Arreola-Risa [8]. None of these papers considers multiple locations.

Capacity/production uncertainty has been modeled in two different ways. One approach has been heavily influenced by yield issues in electronics manufacturing and uses the concept of stochastically proportional yield or random yield, as defined in Henig and Gerchak [20]. The other approach regards the capacity in a given time interval as a random variable. Random-yield models assume that a random fraction of a quantity ordered (or attempted to produce) is actually good (Henig and Gerchak). This is an appropriate model when the uncertainty is due to uncertain quality of individual items produced in a batch. Lee and Yano [30] extend the approach to multi-echelon systems with yield losses. Lee [29] and Gerchak, Wang, and Yano [13] consider the case when components are assembled and the suppliers have uncertain yields. Yano [37] allows for random lead times. Grosfeld-Nir, Gerchak, and He [19] take inspection of the possibly defective units into account. Yano and Lee [38] provide an excellent review of random yield literature in the context of lot sizing.

Ciarallo *et al* [6] and Duenyas *et al* [10] regard capacity in a given time interval as a random variable, i.e., maximum production in any given time interval is uncertain. These papers, however, do not consider lateral transshipment. In a chapter of a recent dissertation, Zhao [40] allows for transshipment between two M/M/1 queues and characterizes optimal policies.

In this paper, we focus on an inventory/production model with general demand and capacity distributions. We follow the approach in Ciarallo *et al*, and Duenyas *et al* and model the firm's capacity every period as a random variable. Unlike in those two papers however, we focus on a firm that produces in two locations instead of one. Our aim is to explore how the level of capacity uncertainty affects optimal transshipment decisions and optimal production when goods can be transshipped.

3 The Model

We consider two manufacturing facilities, each serving its individual market, through multiple time periods. The facilities face uncertain capacity – they do not know with certainty how much they will be able to produce in any given period (e.g., due to machine downtime or quality and yield issues). The uncertain capacities in each facility are characterized by capacity distributions which are independent in time and of each other. The facilities also face demand uncertainty. The stochastic demand distributions are independent in time but can be correlated for any given period across the two facilities. This assumption is relaxed in Section 7 by allowing dependence across the time periods (using a Markov-Modulated process).

In any period, production decisions are made first: the firm decides how much it will *attempt* to produce in each of the facilities that period. Then, the capacities and demands are realized for both facilities. The actual production is the minimum of planned production and the realized capacity. Finally, decisions are made regarding transshipment of inventory between facilities. We assume that demand that is unsatisfied after transshipment is lost. The firm earns linear revenues on satisfied demand and incurs linear production, holding, and transshipment costs. The objective is to maximize the joint discounted profit for both facilities. Let $i, j = 1, 2$ denote the facilities and

$c_i =$ variable production cost for facility i;

$h_i =$ variable holding cost for facility i;

$s_{ij} =$ variable transshipment cost from facility i to j, $i \neq j$;

$r_i =$ unit revenue for facility i.

We assume that marginal profit is always higher when using a unit to satisfy demand at the facility where it was produced rather than transhipping to satisfy demand at the other facility i.e.,

$$r_2 - s_{12} - r_1 \leq 0, \qquad r_1 - s_{21} - r_2 \leq 0.$$

(This assumption is a subset of a bigger set of “complete pooling” assumptions made in other work, e.g. Tagaras [35], and it only slightly influences our results. In Section 7, we present a more-detailed discussion of this assumption.) Obviously, in any specific period we do not transship in both directions and, thus, the cost of transshipment can be expressed as a function of the change in the inventory before and after transshipment.

We use boldface notation to represent two-dimensional vectors. Consider $N+1$ periods, where the ending period is $N+1$. For facility i and period k we denote:

$x_i^k =$ starting inventory level;

$y_i^k =$ planned production target (starting inventory + planned production);

$T_i^k =$ stochastic capacity with *pdf* f_i^k and *cdf* F_i^k;

$\bar{y}_i^k =$ achieved target, i.e., $\bar{y}_i^k$ is the realization of $\bar{Y}_i^k = y_i^k \wedge (x_i^k + T_i^k)$;

$D_i^k =$ stochastic demand, with *pdf* q_i^k and *cdf* Q_i^k;

$z_i^k =$ intermediate inventory position after demand is realized but before transhipment, i.e., z_i^k is the realization of $Z_i^k = \bar{Y}_i^k - D_i^k$;

$\hat{z}_i^k =$ inventory position after transshipment, i.e., $z_1^k + z_2^k = \hat{z}_1^k + \hat{z}_2^k$ and $(z_i^k - \hat{z}_i^k)^+$ is the quantity transshipped from facility i to the other one;

$\alpha_k=$ discount rate, $0 \leq \alpha_k \leq 1$.

Realizations of D_i^k and of T_i^k are denoted by d_i^k and t_i^k.

We analyze the problem using two-stage backward induction and denote $G_*^k(\mathbf{x}^k)$ as the optimal discounted profit-to-go from period k with starting inventory $\mathbf{x}^k$. We can formulate the model as follows:

$$\textbf{Stage One: } G_*^k(\mathbf{x}^k) = \max_{\mathbf{y}^k \geq \mathbf{x}^k} \mathrm{E}_{\mathbf{T}^k,\mathbf{D}^k}\{-\mathbf{c}(\mathbf{y}^k \wedge (\mathbf{x}^k+\mathbf{T}^k)-\mathbf{x}^k)+\mathbf{r}\mathbf{D}^k+G_v^k(\mathbf{y}^k \wedge (\mathbf{x}^k+\mathbf{T}^k)-\mathbf{D}^k)\} \tag{1}$$

$$\textbf{Stage Two: } G_v^k(\mathbf{z}^k) = \max_{\hat{z}_1^k+\hat{z}_2^k=z_1^k+z_2^k} \bar{G}^k(\mathbf{z}^k, \hat{\mathbf{z}}^k) \tag{2}$$

$$\text{where } \bar{G}^k(\mathbf{z}^k, \hat{\mathbf{z}}^k) = -\mathbf{r}(\hat{\mathbf{z}}^k)^- - \mathbf{h}(\hat{\mathbf{z}}^k)^+ - \mathbf{s}(\mathbf{z}^k - \hat{\mathbf{z}}^k)^+ + \alpha_k G_*^{k+1}((\hat{\mathbf{z}}^k)^+) \tag{3}$$

and $G_*^{N+1}(\mathbf{x}^{N+1}) \equiv 0$.

In stage one, production quantities are decided by choosing inventory target $\mathbf{y}^k \geq \mathbf{x}^k$. The first term on the right hand side of (1) is the production cost that the firm incurs given how much it produces once the two facilities realize their capacities. $\mathbf{r}\mathbf{D}^k$ represents the full revenue for any realized demands (the revenue for the unsatisfied demands is deducted later in stage two). Once the firm realizes its demands, it has the intermediate inventory position $\mathbf{z}^k$, which is the realization of $\mathbf{y}^k \wedge (\mathbf{x}^k + \mathbf{T}^k) - \mathbf{D}^k$, and needs to decide on transhipment quantities. In stage two, the firm decides on final inventory positions after transhipment (or equivalently on transhipment quantities). The firm deducts revenue for each unit of demand it is unable to satisfy, incurs holding costs for each unit of inventory it carries and transhipment costs for each unit transhipped. Note that the profit function for stage one is based on optimal decisions in stage two and is expressed using $G_v^k(\mathbf{z}^k)$.

4 Optimal Policy

We now characterize the structure of the optimal policy for the model given in (1)–(3). We will divide our analysis into two subsections. The first one (4.1) contains a full characterization of the structure of the transshipment policy (Theorem 1) and a derivation of some properties of the value function $G_v^k(\mathbf{z}^k)$ which is used in the second subsection (4.2). We characterize the structure of the optimal production policy in each facility in the second subsection (Theorems 2-4) and show how the structure changes depending on whether each of the facilities faces uncertain capacity. Theorems 1 to 4 along with our sensitivity results in Section 5 clarify the role that capacity uncertainty plays in production and transshipment planning, which has not been addressed in the previous literature.

We prove the structure of the optimal policy by induction. As a part of our inductional step, we assume that, from period $k+1$ on, the profit-to-go function $G_*^{k+1}(\mathbf{x}^{k+1})$ has the following three properties. (We denote the partial derivatives of function $K(\mathbf{y})$ as $K_i'(\mathbf{y}) = \dfrac{\partial}{\partial y_i}K(\mathbf{y})$ and $K_{ij}''(\mathbf{y}) = \frac{\partial^2}{\partial y_j \partial y_i}K(\mathbf{y})$.)

$\mathbb{A}_1^{k+1}$: $G_*^{k+1}(\mathbf{x}^{k+1})$ is jointly concave in $\mathbf{x}^{k+1}$, and

$$(G_*^{k+1})_{11}''(\mathbf{x}^{k+1}) \leq (G_*^{k+1})_{12}''(\mathbf{x}^{k+1}), (G_*^{k+1})_{22}''(\mathbf{x}^{k+1}) \leq (G_*^{k+1})_{21}''(\mathbf{x}^{k+1});$$

$\mathbb{A}_2^{k+1}$: $G_*^{k+1}(\mathbf{x}^{k+1})$ is submodular and $(G_*^{k+1})''_{12}(\mathbf{x}^{k+1}) = (G_*^{k+1})''_{21}(\mathbf{x}^{k+1})$;

$\mathbb{A}_3^{k+1}$: $(G_*^{k+1})'_i(\mathbf{x}^{k+1}) \leq r_i$, for $i = 1, 2$.

Notice that joint concavity of $G_*^{k+1}(\mathbf{x}^{k+1})$ implies the existence and continuity of one-sided first and second derivatives everywhere except on a set of measure 0. Whenever one-sided derivatives are not equal, all of the inequalities hold for the sets of supergradients. (In order to show concavity, in our inductional step we prove that the first derivative is non-increasing, even at the discontinuity points.)

Properties $\mathbb{A}_1^{k+1}$ to $\mathbb{A}_3^{k+1}$ guarantee that profit function behaves predictably, even though the function maximized in Stage 1 is not concave in $\mathbf{y}^k$. Using them, we first derive the optimal transshipment and production policies for period k, and then prove that the same properties also hold for the profit-to-go function $G_*^k(\mathbf{x}^k)$. Notice that for the ending period $N+1$, all three properties trivially hold.

4.1 Transshipment Policy

Let the current period be k and assume that $\mathbb{A}_1^i, \mathbb{A}_2^i$, and $\mathbb{A}_3^i$ hold, for $i = k+1, \dots, N+1$. The optimal transshipment policy for given $\mathbf{z}^k$ is identified by solving (2). Hence, it is critical to specify the first derivative of $\bar{G}^k(\mathbf{z}^k, \hat{\mathbf{z}}^k)$, defined in (3), with respect to $\hat{\mathbf{z}}^k$. Notice that $\hat{z}_1^k$ and $\hat{z}_2^k$ are related through $\hat{z}_1^k + \hat{z}_2^k = z_1^k + z_2^k$ and we always consider only one of them (depending on the scenario) as a decision variable. We have the following first derivatives of $\bar{G}^k(\mathbf{z}^k, \hat{\mathbf{z}}^k)$ $(i = 1, 2)$:

$$\begin{aligned} g_i(\mathbf{z}^k, \hat{\mathbf{z}}^k) :=& \frac{\partial}{\partial \hat{z}_i^k} \bar{G}^k(\mathbf{z}^k, \hat{\mathbf{z}}^k) \\ =& [-h_i + \alpha_k (G_*^{k+1})'_i((\hat{\mathbf{z}}^k)^+)] \mathbf{1}_{\{\hat{z}_i^k \geq 0\}} + [h_{3-i} - \alpha_k (G_*^{k+1})'_{3-i}((\hat{\mathbf{z}}^k)^+)] \mathbf{1}_{\{\hat{z}_{3-i}^k > 0\}} \\ &+ r_i \mathbf{1}_{\{\hat{z}_i^k < 0\}} - r_{3-i} \mathbf{1}_{\{\hat{z}_{3-i}^k \leq 0\}} + s_{i,3-i} \mathbf{1}_{\{z_i^k - \hat{z}_i^k \geq 0\}} - s_{3-i,i} \mathbf{1}_{\{z_{3-i}^k - \hat{z}_{3-i}^k > 0\}}. \end{aligned} \tag{4}$$

The policy defined below is the basic structure of the optimal transshipment policy for our model:

Definition 1 *Consider intermediate inventories (z_1, z_2), with $z = z_1 + z_2$. Define*

(a) **state-dependent rationing** *policy for facility i, SRi($\chi_i(z)$), as follows: Facility i transships $(z_i - \chi_i(z))^+$ to facility $3-i$.*

(b) **floor-rationing** *policy for facility i, FRi($\underline{\chi}_i, \bar{\chi}_i(z)$), as a state-dependent rationing policy SRi($\chi_i(z)$) where $\underline{\chi}_i$ is a constant, and $\chi_i(z) = \underline{\chi}_i$ for $z < \underline{\chi}_i$, and $\chi_i(z) = \bar{\chi}_i(z)$ for $z \geq \underline{\chi}_i$. We refer to $\underline{\chi}_i$ as facility i's floor.*

Before formally presenting Theorem 1, first observe that under our assumptions, given production decision $\mathbf{y}^k$, a facility will never tranship if 1) its intermediate inventory position is negative (by our assumption on revenues, it is better to satisfy first all of its own demand than to tranship) or 2) the intermediate inventory position of the other facility is positive and it costs the same or more to hold inventory in the other facility. Equivalently, once production decisions are made, facilities will consider transhipping only if they have surplus inventory and 1) the other facility needs the inventory to satisfy an immediate demand in the present period, i.e., negative intermediate inventory position or 2) the other facility does not need the inventory to satisfy demand this period but is a cheaper place to hold inventory.

For the remainder of Section 4, we assume $h_1 \geq h_2$. For $h_2 \geq h_1$, all the statements and the arguments hold symmetrically.

Theorem 1 *(Optimal transhipment for multi-period problem, $h_1 \geq h_2$)*
In period k, let the total intermediate inventories be $z^k = z_1^k + z_2^k$. The optimal transshipment policy is defined by floor rationing policies, FR1($\underline{\chi}_1^k, \bar{\chi}_1^k(z^k)$) for facility 1, and FR2($\underline{\chi}_2^k, z^k$) for facility 2, where

1. *For $z^k > \underline{\chi}_1^k$, $\underline{\chi}_1^k \leq \bar{\chi}_1^k(z^k) \leq z^k$; particularly, if $h_1 = h_2$, $\bar{\chi}_1^k(z^k) = z^k$.*
2. $0 \leq \dfrac{\partial \bar{\chi}_1^k}{\partial z^k} \leq 1$.
3. *$\underline{\chi}_i^k$ is non-increasing in the current-period h_i and r_{3-i}, non-decreasing in the current-period $s_{i,3-i}$, and independent of current-period h_{3-i}, r_i, and $s_{3-i,i}$.*
4. *$\bar{\chi}_1^k(z^k)$ is non-increasing in the current-period h_1 and non-decreasing in the current-period h_2 and s_{12}.*

Proof: First of all we observe that $\bar{G}^k(\hat{\mathbf{z}}^k, \mathbf{z}^k)$ is concave in $(\hat{\mathbf{z}}^k, \mathbf{z}^k)$. It suffices to justify that function $-r_1(\hat{z}_1^k)^- - r_2(\hat{z}_2^k)^- + \alpha_k G_*^{k+1}((\hat{z}_1^k)^+, (\hat{z}_2^k)^+)$ is jointly concave in $\hat{\mathbf{z}}^k$, which is guaranteed by inductional hypothesis $\mathbb{A}_1^{k+1}$ and $\mathbb{A}_3^{k+1}$.

Now we characterize the optimal transshipment from facility 1 to 2. As we have observed above, we only need to consider the case where $z_1^k > 0$. From (4), the first derivative of $\bar{G}^k(\mathbf{z}^k, \hat{\mathbf{z}}^k)$ w.r.t. $\hat{z}_1^k$ becomes

$$g_1(\mathbf{z}^k, \hat{\mathbf{z}}^k) = \begin{cases} -h_1 - r_2 + s_{12} + \alpha_k (G_*^{k+1})_1'(\hat{z}_1^k, 0) & \text{if } \hat{z}_2^k < 0 \text{ (i.e., } \hat{z}_1^k > (z^k)^+) \\ -h_1 + h_2 + s_{12} + \alpha_k (G_*^{k+1})_1'(\hat{z}_1^k, z^k - \hat{z}_1^k) - \alpha_k (G_*^{k+1})_2'(\hat{z}_1^k, z^k - \hat{z}_1^k) & \\ & \text{if } \hat{z}_2^k \geq 0 \text{ (i.e., } 0 \leq \hat{z}_1^k \leq z^k) \end{cases}$$

Clearly, since $\bar{G}^k(\hat{\mathbf{z}}^k, \mathbf{z}^k)$ is concave in $(\hat{\mathbf{z}}^k, \mathbf{z}^k)$, $g_1(\mathbf{z}^k, \hat{\mathbf{z}}^k)$ is non-increasing in $\hat{z}_1^k$ for given z^k. Denote $g_1^1(\hat{z}_1^k) = -h_1 - r_2 + s_{12} + \alpha_k (G_*^{k+1})_1'(\hat{z}_1^k, 0)$ and then let $\underline{\chi}_1^k = \infty$ if $g_1(\hat{z}_1^k) > 0$ for all $\hat{z}_1^k$, otherwise $\underline{\chi}_1^k = \min\{\hat{z}_1^k \geq 0 |\, g_1^1(\hat{z}_1^k) \leq 0\}$. Furthermore, denote $g_1^2(z^k, \hat{z}_1^k) = -h_1 + h_2 + s_{12} + \alpha_k (G_*^{k+1})_1'(\hat{z}_1^k, z^k - \hat{z}_1^k) - \alpha_k (G_*^{k+1})_2'(\hat{z}_1^k, z^k - \hat{z}_1^k)$ and then let $\bar{\chi}_1^k(z^k) = z^k$, if $g_1^2(z^k, \hat{z}_1^k) > 0$, for all $\hat{z}_1^k \in [0, z^k]$, otherwise $\bar{\chi}_1^k(z^k) = \min\{\hat{z}_1^k |\, g_1^2(z^k, \hat{z}_1^k) \leq 0,\ \hat{z}_1^k \in [0, z^k]\}$.

Given a z^k, since $g_1(\mathbf{z}^k, \hat{\mathbf{z}}^k)$ is independent of z_1^k, $\underline{\chi}_1^k$ is facility 1's optimal rationing level when $z^k < \underline{\chi}_1^k$ and $\bar{\chi}_1^k(z^k)$ is facility 1's optimal rationing level when $z^k \geq \underline{\chi}_1^k$. Thus, $FR1(\underline{\chi}_1^k, \bar{\chi}_1^k(z^k))$ describes the optimal policy.

The following is the justification for *1-4*:

1. $\bar{\chi}_1^k(z^k) \leq z^k$ holds trivially due to definition of $\bar{\chi}_1^k(z^k)$. From $\mathbb{A}_1^{k+1}$ and $\mathbb{A}_2^{k+1}$, $g_1^2(z^k, \hat{z}_1^k)$ is non-decreasing in z^k, hence, as $z^k > \hat{z}_1^k$, $g_1^1(\hat{z}_1^k) < g_1^2(\hat{z}_1^k, \hat{z}_1^k) \leq g_1^2(z^k, \hat{z}_1^k)$. Therefore, substituting

$\hat{z}_1^k = \underline{\chi}_1^k$, we have that $g_1^1(\underline{\chi}_1^k) < g_1^2(z^k, \underline{\chi}_1^k)$, which implies that for $z^k > \underline{\chi}_1^k$, $\bar{\chi}_1^k(z^k) \geq \underline{\chi}_1^k$. When $h_1 = h_2$, facility 1 transships nothing when $z_2^k \geq 0$. Hence, $\bar{\chi}_1^k(z^k) = z^k$.

2. According to the definition of $\bar{\chi}_1^k(z^k)$, since $g_1^2(z^k, \hat{z}_1^k)$ is non-decreasing in z^k, $\bar{\chi}_1^k(z^k)$ is also non-decreasing in z^k. Thus, $0 \leq \dfrac{\partial \bar{\chi}_1^k}{\partial z^k}$. To prove $\dfrac{\partial \bar{\chi}_1^k}{\partial z^k} \leq 1$, consider function $g_1^2(z^k, z^k - \hat{z}_2^k)$ for $\hat{z}_2^k > 0$. Define $\hat{z}_2^{*k}(z^k)$ as the optimal inventory position at facility 2 after transshipment. Clearly, $\hat{z}_2^{*k}(z^k) = z^k - \bar{\chi}_1^k(z^k)$, and $\hat{z}_2^{*k}(z^k)$ is either the point that satisfies $g_1^2(z^k, z^k - \hat{z}_2^{*k}(z^k)) = 0$ or is one the end points of $[0, z^k]$. From $\mathbb{A}_1^{k+1}$ and $\mathbb{A}_2^{k+1}$, $g_1^2(z^k, z^k - \hat{z}_2^k)$ is non-decreasing in $\hat{z}_2^k$ and non-increasing in z^k. Thus, $\hat{z}_2^{*k}(z^k)$ is non-decreasing in z^k, implying $1 - \dfrac{\partial \bar{\chi}_1^k}{\partial z^k} \geq 0$.

3. Immediately follows since $\underline{\chi}_1^k$ is determined by $g_1^1(\hat{z}_1^k)$, which is a non-increasing function of h_1, r_2, non-decreasing function of s_{12}, and independent of h_2, r_1, and s_{21}.

4. It follows since $g_1^2(z^k, \hat{z}_1^k)$ is non-increasing in h_1 and non-decreasing in h_2 and s_{12}.

Now, consider transshipment policy for facility 2. All the arguments are symmetric. Since $h_2 \leq h_1$, by our observations, transshipment is used only to eliminate potential shortages and we immediately have that $\bar{\chi}_2^k(z^k) = z^k$ for $z^k > \underline{\chi}_2^k$. ■

Theorem 1 implies that it may be profitable to ship inventory that is not immediately needed from higher holding cost facility 1 to facility 2. At the same time, both facilities may also have an incentive to ration their inventory. What makes this problem interesting is that depending on the intermediate inventory (after production and demands are realized), a range of behaviors may be optimal: rationing, transshipping full needed amounts, and transshipping inventory even though none is immediately needed by the other facility. Figure 1 illustrates the optimal transshipment

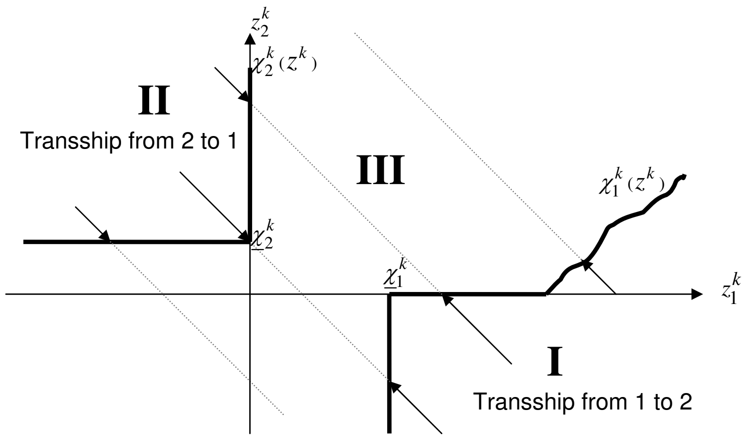

Figure 1: The optimal transshipment policy as a function of intermediate inventory positions.

policy. For each facility, the optimal rationing level $\chi_i^k(z^k)$ is composed of the floor $\underline{\chi}_i^k$ and a non-decreasing function $\bar{\chi}_i^k(z^k)$. (For lower-holding-cost facility, $\bar{\chi}_i^k(z^k) = z^k$.) If intermediate inventory level exceeds the rationing level, transshipment down-to that level will be made. The two optimal rationing levels divide the space of (z_1^k, z_2^k) into three areas – I: tranship from 1 to 2, corresponding to $z_1^k > \chi_1^k(z^k)$; II: tranship from 2 to 1, when $z_2^k > \chi_2^k(z^k)$; and III: do not tranship, $z_1^k \leq \chi_1^k(z^k)$

and $z_2^k \le \chi_2^k(z^k)$. By inserting the optimal transshipment policy for each area, we can rewrite the stage-two value function $G_v^k(\mathbf{z}^k)$ in (2) as follows:

$$\begin{aligned}
G_v^k(\mathbf{z}^k)=&\mathbf{1}_{\{z_1^k>\chi_1^k\}}\Big[-h_1\chi_1^k - h_2(z_1^k+z_2^k-\chi_1^k)^+ - r_2(z_1^k+z_2^k-\chi_1^k)^- - s_{12}(z_1^k-\chi_1^k)\\
&\qquad +\alpha_k G_*^{k+1}(\chi_1^k,(z_1^k+z_2^k-\chi_1^k)^+)\Big]\\
&+\mathbf{1}_{\{z_2^k>\chi_2^k\}}\Big[-h_2\chi_2^k - r_1(z_1^k+z_2^k-\chi_2^k)^- - s_{21}(z_2^k-\chi_2^k) + \alpha_k G_*^{k+1}((z_1^k+z_1^k-\chi_2^k)^+,\chi_2^k)\Big]\\
&+\mathbf{1}_{\{z_1^k\le\chi_1^k\}}\mathbf{1}_{\{z_2^k\le\chi_2^k\}}\Big[-\mathbf{h}(\mathbf{z}^k)^+ - \mathbf{r}(\mathbf{z}^k)^- + \alpha_k G_*^{k+1}((\mathbf{z}^k)^+)\Big].
\end{aligned}\tag{5}$$

The following Proposition 1 describes the properties of $G_v^k(\mathbf{z}^k)$, which we use in our proof of the structure of the optimal production policy in the next subsection.

Proposition 1 *$G_v^k(\mathbf{z}^k)$ has the following properties ($i=1,2$):*

1. *$G_v^k(\mathbf{z}^k)$ is concave in $\mathbf{z}^k$ and $\lim_{z_i^k\to\infty} G_v^k(\mathbf{z}^k) = -\infty$ ($i=1,2$).*
2. *Its first derivatives satisfy:*
   (a) $(G_v^k)'_i(\mathbf{z}^k) \le r_i$.

   For period k's holding cost h_i^k, revenue r_i^k, and transshipping cost $s_{i,3-i}^k$,

   (b) $\dfrac{\partial}{\partial h_i^k}(G_v^k)'_i(\mathbf{z}^k) \le \dfrac{\partial}{\partial h_i^k}(G_v^k)'_{3-i}(\mathbf{z}^k) \le 0.$

   (c) $\dfrac{\partial}{\partial r_i^k}(G_v^k)'_i(\mathbf{z}^k) \ge \dfrac{\partial}{\partial r_i^k}(G_v^k)'_{3-i}(\mathbf{z}^k) \ge 0.$

   (d) $\dfrac{\partial}{\partial s_{i,3-i}^k}(G_v^k)'_i(\mathbf{z}^k) \le 0,\ \dfrac{\partial}{\partial s_{i,3-i}^k}(G_v^k)'_{3-i}(\mathbf{z}^k) \ge 0.$
3. *Its second derivatives satisfy:*
   (a) $(G_v^k)''_{ii}(\mathbf{z}^k) \le (G_v^k)''_{12}(\mathbf{z}^k) = (G_v^k)''_{21}(\mathbf{z}^k) \le 0.$

   (b) $\dfrac{\partial}{\partial d_i^k}(G_v^k)'_i(\mathbf{z}^k) \ge \dfrac{\partial}{\partial d_{3-i}^k}(G_v^k)'_i(\mathbf{z}^k) = \dfrac{\partial}{\partial d_i^k}(G_v^k)'_{3-i}(\mathbf{z}^k) \ge 0.$

Proof: See Appendix. ■

Before we discuss the production policy, we note that our floor-rationing policy is different from results in Tagaras [35] and Robinson [32]. Under a set of "complete pooling" conditions, these papers show that the transshipment takes place only when one facility has excess inventory and the other is short, and that the transshipped quantity is the minimum of the surplus and the shortage quantity. Thus, in all of these papers, a facility that has excess inventory always satisfies as much of the shortage in the other facility as it can and no rationing takes place. The assumptions made in those papers correspond to the following for our model $h_i+r_{3-i}-s_{i,3-i}\ge 0$, $c_{3-i}-c_i-s_{i,3-i}\le 0$, $h_{3-i}-h_i-s_{i,3-i}\le 0$, and $r_{3-i}-r_i-s_{i,3-i}\le 0$, ($i=1,2$). As we emphasized before, we only use the last of these assumptions. However, a unique feature of our paper is our consideration of uncertain capacity. Thus, even if all of the above assumptions were satisfied, this still does not result in complete pooling when capacity is uncertain. For instance, consider a 5-period two-facility system, with $c_1 = c_2 = 1$, $h_1 = h_2 = 0.005$, $r_1 = r_2 = 1.11$, and $s_{12} = s_{21} = 0.1$. Both facilities face demand $(0, 1, 2)$ with probabilities $(0.2, 0.6, 0.2)$, and capacity $(1, 2, 3)$ with probabilities $(0.8, 0.1, 0.1)$. It is easy to see that all of the above "complete pooling" assumptions are satisfied. However, the transshipment policy for facility i is FR$i(1, z)$, where z is the sum of the intermediate inventories, i.e., either facility always uses rationing such that it will reserve 1 unit and never ship this unit to the other facility, even when the other facility is short.

4.2 Production Policy

In this section, we characterize the optimal production policy. The production decisions are based on the structure of function $G_v^k(\mathbf{z}^k)$, which is the profit function, with the revenue for actual demand excluded. The optimal production targets are determined by (1). By defining $G^k(\mathbf{y}^k) = \mathrm{E}_{\mathbf{D}^k}[G_v^k(\mathbf{y}^k - \mathbf{D}^k) + \mathbf{r}\mathbf{D}^k] - \mathbf{c}\mathbf{y}^k$, (1) can be expressed as:

$$G_*^k(\mathbf{x}^k) = \max_{\mathbf{y}^k \geq \mathbf{x}^k} E_{\mathbf{T}^k} G^k(\mathbf{y}^k \wedge (\mathbf{x}^k + \mathbf{T}^k)) + \mathbf{c}\mathbf{x}^k.$$

It is easy to check that all the properties listed in Proposition 1 also apply to $G^k(\mathbf{y}^k)$ w.r.t. $\mathbf{y}^k$, except 2(a) becomes $(G^k)_i'(\mathbf{y}^k) \leq r_i - c_i$. Due to capacity uncertainties, the objective function $E_{\mathbf{T}^k} G^k(\mathbf{y}^k \wedge (\mathbf{x}^k + \mathbf{T}^k)) + \mathbf{c}\mathbf{x}^k$ is not concave in $\mathbf{y}^k$. We prove, however, that it is unimodal in $\mathbf{y}^k$. Let us define function $G1^k(\mathbf{x}^k, \mathbf{y}^k) := \mathrm{E}_{T_1^k} G^k(y_1^k \wedge (x_1^k + T_1^k), y_2^k)$ and set $A1^k(\mathbf{x}^k) := \{(y_1^k, x_2^k \vee \hat{y}_2^k(y_1^k)) | G1^k(\mathbf{x}^k, y_1^k, \hat{y}_2^k(y_1^k)) = \max_{y_2^k} G1^k(\mathbf{x}^k, \mathbf{y}^k), y_1^k \geq x_1^k\}$, as well as symmetric $G2^k(\mathbf{x}^k, \mathbf{y}^k)$ and $A2^k(\mathbf{x}^k)$ in the same way. We have the following:

Proposition 2 *(a) The objective function of (1) is unimodal in* $\mathbf{y}^k$. *(b)* $A^k(\mathbf{x}^k) := A1^k(\mathbf{x}^k) \cap A2^k(\mathbf{x}^k)$ *is not empty and is a subset of (1)'s maximizers. (c)* $\mathbf{y}^{*k} \in A^k(\mathbf{x}^k) \Leftrightarrow \mathbf{y}^{*k}$ *satisfies (6) to (9):*

$$\lambda_1^k(\mathbf{x}^k, \mathbf{y}^{*k}) := \mathrm{E}_{T_2^k}(G^k)_1'(y_1^{*k}, y_2^{*k} \wedge (x_2^k + T_2^k)) \leq 0 \tag{6}$$

$$\lambda_2^k(\mathbf{x}^k, \mathbf{y}^{*k}) := \mathrm{E}_{T_1^k}(G^k)_2'(y_1^{*k} \wedge (x_1^k + T_1^k), y_2^{*k}) \leq 0 \tag{7}$$

$$y_1^{*k} \geq x_1^k \textit{ and } (y_1^{*k} - x_1^k)\lambda_1^k(\mathbf{x}^k, \mathbf{y}^{*k}) = 0 \tag{8}$$

$$y_2^{*k} \geq x_2^k \textit{ and } (y_2^{*k} - x_2^k)\lambda_2^k(\mathbf{x}^k, \mathbf{y}^{*k}) = 0. \tag{9}$$

Proof: See Appendix. ■

KKT conditions can be obtained from conditions (6) to (9) by multiplying (8) and (9) by $1 - F_i^k(y_i^k - x_i^x)$. Thus, as long as the optimal production quantity $\mathbf{y}^k - \mathbf{x}^k$ is reachable with positive probability, conditions (6) to (9) are not only sufficient, but also necessary. In the remainder of the paper, we use the fact that (6) to (9) define the set $A^k(\mathbf{x}^k)$. Since set $A^k(\mathbf{x}^k)$ may have multiple elements, for the purpose of consistency, let us consider a specific maximizer $\mathbf{y}^{*k}(\mathbf{x}^k) := \arg\max_{y_2^k}\{\arg\max_{y_1^k} A^k(\mathbf{x})\}$.

The following Proposition 3 derives properties of the optimal solution $\mathbf{y}^{*k}$ which we will use in proving the optimal production policy.

Proposition 3 *Consider period k. The optimal solution of (1), $\mathbf{y}^*(=\mathbf{y}^{*k})$, has the following properties $(i=1,2)$:*

1. $-1 \leq \dfrac{\partial y_i^*}{\partial x_i} - 1 \leq \dfrac{\partial y_i^*}{\partial x_{3-i}} \leq 0$ *and* $\dfrac{\partial y_1^*}{\partial x_i} + \dfrac{\partial y_2^*}{\partial x_i} \leq 0$.

2. *For a given $\mathbf{x}^0$ and the corresponding $\mathbf{y}^{0*}(\mathbf{x}^0)$:*

   (a) *If either*

   (i) $F_i(y_i^{0*} - x_i^0) = 0$ (facility i can reach y_i^{0*} with probability 1), or

   (ii) $y_{3-i}^{0*} = x_{3-i}^0$ (optimal policy at $3-i$ is to produce nothing),

   then for all $x_i \geq x_i^0$, we have $y_i^* = y_i^{0*} \vee x_i$ (produce up to y_i^{0*}).

   (b) *If $F_{3-i}(y_{3-i}^{0*} - x_{3-i}^0) = 1$ (facility $3-i$'s capacity does not exceed $y_{3-i}^{0*} - x_{3-i}^0$), then for all $x_i \leq x_i^0$, we have $y_i^* = y_i^{0*}$ (produce up to y_i^{0*}).*

3. *For a given $\mathbf{x}^0$ and the corresponding $\mathbf{y}^{0*}(\mathbf{x}^0)$:*

   If $F_i(y_i^{0*} - x_i^0) = 0$ (case a(i) above), then for all $x_i^0 \leq x_i \leq y_i^{0*}$, we have $y_{3-i}^* = y_{3-i}^{0*}$.

Proof: See Appendix. ∎

Essentially, Proposition 3 shows that if the desired inventory targets are reachable with probability 1, then the optimal production policies become up-to policies. Obviously, when capacity is uncertain, the facility may not reach its target, and we therefore need to characterize further the policy in that situation. Before doing so, to complete the inductional proof, we need establish the following Proposition 4 which verifies that all of the inductional assumptions $\mathbb{A}_1^k$, $\mathbb{A}_2^k$, and $\mathbb{A}_3^k$ hold.

Proposition 4 *$G_*^k(\mathbf{x}^k)$ has the following properties: $(i=1,2)$*

$\mathbb{A}_1^k$: *$G_*^k(\mathbf{x}^k)$ is jointly concave in (x_1^k, x_2^k) and $(G_*^k)''_{ii}(\mathbf{x}^k) \leq (G_*^k)''_{i,i-3}(\mathbf{x}^k)$;*

$\mathbb{A}_2^k$: *$G_*^k(\mathbf{x}^k)$ is submodular and $(G_*^k)''_{12}(\mathbf{x}^k) = (G_*^k)''_{21}(\mathbf{x}^k)$;*

$\mathbb{A}_3^k$: *$(G_*^k)'_i(\mathbf{x}^k) \leq r_i$, for $i=1,2$.*

Proof: See Appendix. ∎

We are now ready to characterize the structure of the optimal production policy completely. We are interested in how our results depend on the capacity distribution. We define **uncertain** capacity as the most general case where capacity is stochastic. We define **certain-limited** capacity as the case where capacity in every period is deterministic but finite. Finally, the third case is where we assume no capacity limitation, i.e., **infinite** capacity. We divide our characterization into several cases: (1) when both facilities have uncertain capacity and (2), when one facility has uncertain capacity and the other has infinite capacity. Finally, we note two interesting special cases: (3) when both facilities have certain-limited capacities and (4) both facilities have infinite capacities (previously addressed in the literature).

Two Facilities with Uncertain Capacities
Consider first the general case where both facilities may have uncertain capacities, i.e., $Pr(T_i^k = \infty) < 1$.

Theorem 2 *The optimal production policy at facility i is a function of the other facility's inventory x_{3-i}^k and is defined by two thresholds $\underline{x}_i^k \le \bar{x}_i^k$, (also functions of x_{3-i}^k)* [1]

$$\bar{x}_i^k = \bar{x}_i^k(x_{3-i}^k)=(\inf\{x_i^k : F_i^k(y_i^{*k} - x_i^k) = 0\}) \wedge (\inf\{x_i^k : y_{3-i}^{*k} = x_{3-i}^k\})$$
$$\underline{x}_i^k = \underline{x}_i^k(x_{3-i}^k)=\sup\{x_i^k : F_{3-i}^k(y_{3-i}^{*k} - x_{3-i}^k) = 1\} \wedge \bar{x}_i^k(x_{3-i}^k),$$

two order-up-to levels $\underline{y}_i^{*k} \le \bar{y}_i^{*k}$, and one function $y_i^{*k}(\mathbf{x}^k)$ such that

1. *if $x_i^k < \underline{x}_i^k$, then produce up-to $\underline{y}_i^{*k}$;*

2. *if $x_i^k > \bar{x}_i^k$, then produce up-to $\bar{y}_i^{*k}$;*

3. *if $\underline{x}_i^k \le x_i^k \le \bar{x}_i^k$, then produce to $y_i^{*k}(\mathbf{x}^k)$.*

Furthermore, all thresholds, up-to levels, and $y_i^{*k}(\mathbf{x}^k)$ are non-increasing in x_{3-i}^k, $y_i^{*k}(\mathbf{x}^k)$ is non-decreasing in x_i^k, but $y_i^{*k}(\mathbf{x}^k) - x_i^k$ is non-increasing in x_i^k.

Proof: We only consider $i = 2$. Clearly, from the definition of $\underline{x}_2^k$, $\underline{x}_2^k \le \bar{x}_2^k$. $\underline{y}_2^{*k} \le \bar{y}_2^{*k}$ follows immediately from Proposition 3 property *1* and $\bar{x}_2^k \le \underline{x}_2^k$.

1. $x_2^k < \underline{x}_2^k$. Note that $\underline{x}_2^k \le \sup\{x_2^k : F_1^k(y_1^{*k} - x_1^k) = 1\}$. Since y_1^{*k} is non-increasing in x_2^k, for any $x_2^k < \underline{x}_2^k$, the corresponding $F_1^k(y_1^{*k} - x_1^k) = 1$. Hence, from *2(b)* of Proposition 3, facility 2 produces up to $\underline{y}_2^{*k}$.

2. $x_2^k > \bar{x}_2^k$. Consider each of the two cases: 1) If $\bar{x}_2^k = \inf\{x_2^k : y_1^{*k} = x_1^k\}$, since y_1^{*k} is non-increasing in x_2^k (from Proposition 3 point *1*), then as x_2^k increases from $\bar{x}_2^k$ to ∞, $y_1^{*k} = x_1^k$. Thus, from Proposition 3 *2(a)ii*, facility 2 produces up to $\bar{y}_2^{*k}$. 2) If $\bar{x}_2^k = \inf\{x_2^k : F_2^k(y_2^{*k} - x_2^k) = 0\}$, since $y_2^{*k} - x_2^k$ is non-increasing in x_2^k (from Proposition 3 point *1*), for any $x_2^k > \bar{x}_2^k$, the corresponding $F_2^k(y_2^{*k} - x_2^k) = 0$. Hence, from Proposition 3 *2(a)i*, facility 2 produces up to $\bar{y}_2^{*k}$.

3. follows directly from Proposition 3 property *1.*

Note that the properties of $y_i^{*k}(\mathbf{x}^k)$ follow from Proposition 3 property *1*. Now we prove that all the thresholds and up-to levels are non-increasing in x_1^k. For the up-to levels, the conclusion immediately follows since y_2^{*k} is non-increasing in x_1^k (Proposition 3 property *1*). For the thresholds, suppose $\tilde{x}_1^k \ge x_1^k$. It suffices to show that $\{x_2^k : F_2^k(y_2^{*k}(x_1^k, x_2^k) - x_2^k) = 0\} \subseteq \{x_2^k : F_2^k(y_2^{*k}(\tilde{x}_1^k, x_2^k) - x_2^k) = 0\}$, $\{x_2^k : y_1^{*k}(x_1^k, x_2^k) = x_1^k\} \subseteq \{x_2^k : y_1^{*k}(\tilde{x}_1^k, x_2^k) = \tilde{x}_1^k\}$, and $\{x_2^k : F_1^k(y_1^{*k}(\tilde{x}_1^k, x_2^k) - \tilde{x}_1^k) = 1\} \subseteq \{x_2^k :$

[1] $\inf \emptyset = \sup \mathbb{R} = \infty$ and $\sup \emptyset = \inf \mathbb{R} = 0$.

$F_1^k(y_1^{*k}(x_1^k, x_2^k) - x_1^k) = 1\}$. We present the verification of $\{x_2^k : F_2^k(y_2^{*k}(x_1^k, x_2^k) - x_2^k) = 0\} \subseteq \{x_2^k : F_2^k(y_2^{*k}(\tilde{x}_1^k, x_2^k) - x_2^k) = 0\}$. Let x_2^k satisfy $F_2^k(y_2^{*k}(x_1^k, x_2^k) - x_2^k) = 0$. y_2^{*k} is non-increasing in x_1^k (from Proposition 3 point *1*), implies $0 = F_2^k(y_2^{*k}(x_1^k, x_2^k) - x_2^k) \geq F_2^k(y_2^{*k}(\tilde{x}_1^k, x_2^k) - x_2^k) \geq 0$ and $x_2^k \in \{x_2^k : F_2^k(y_2^{*k}(\tilde{x}_1^k, x_2^k) - x_2^k) = 0\}$. The other two subset inequalities follow from $\frac{\partial y_1^{*k}}{\partial x_1^k} \leq 1$. ■

Figure 2 illustrates the structure of the optimal production policy for one of the facilities when both facilities have uncertain capacities. The solid lines in Figure 2 correspond to the two thresholds and the dotted lines correspond to the up-to-levels. Consider a fixed initial inventory at location 1, $x_1 = a$. For the initial inventory of supplier 2, x_2, below $\underline{x}_2(a)$ and above $\bar{x}_2(a)$ the optimal production is up to $\underline{y}_2^{*k}(a)$ and $\bar{y}_2^{*k}(a)$, respectively.

The logic for both cases bears some similarities. In general, the target is a function of both locations' starting inventories $x_1 = a$ and x_2. However, when $x_2 \leq \underline{x}_2(a)$, $\underline{x}_2(a)$ is such that supplier 1 can never reach her desired target and will produce as much as capacity allows. Thus, given fixed initial inventory (and all possible realizations of capacity) of supplier 1, the objective function depends solely on the ending inventory of supplier 2 and the best such level is supplier 2's up-to level. (This is illustrated by starting points (a, b_1) and (a, b_2) in Figure 2 where in both cases, the up-to-level for supplier 2 is $\underline{y}_2^{*k}(a)$). On the other hand, when $x_2 \geq \bar{x}_2(a)$, either supplier 1 does not need to produce or supplier 2 can surely reach her desired target. When supplier 1 does not produce and supplier 2 is the sole source, the intuition is exactly the same as above. When supplier 2 can surely reach her desired level, the situation is equivalent to having infinite capacity in the current period. Consider starting inventories of (a, b_4) and the corresponding optimal targets, with the lowest total cost. Decreasing starting inventory at location 2, from b_4 to b_3 does not disable supplier 2 from reaching the same target point (which has the lowest cost). Finally, if the starting inventory is between the two solid lines (representing the thresholds), then the policy is to try to bring the inventory to $y_i^{*k}(\mathbf{x}^k)$ (not shown in Figure 2).

Note that if only one of the facilities faces an uncertain capacity and the other has infinite capacity, the optimal policy is simplified considerably. It corresponds to having a single up-to curve, as illustrated in Figure 3 and described below.

One Facility with Uncertain Capacity

Theorem 3 *Let facility 1 have uncertain capacity and facility 2 have infinite capacity.*

1. *Facility 1's optimal policy is exactly as defined in Theorem 2 with $\underline{x}_1^{*k} = 0$ and $\underline{y}_1^{*k} = 0$. Furthermore, when $x_2^k < y_2^{*k}(x_1^k)$, production target of facility 1 is independent of facility 2's starting inventory x_2^k.*

2. *Facility 2 produces up to $y_2^{*k}(x_1^k)$, and $y_2^{*k}(x_1^k)$ decreases in facility 1's inventory x_1^k.*

Proof: Facility 2 has infinite capacity. Since $Pr(T_2^k = \infty) = 1$, from definition of $\bar{x}_2^k$ and $\underline{x}_2^k$, we have $\bar{x}_2^k = \underline{x}_2^k = 0$. Hence, facility 2's optimal policy is an order-up-to policy, which is a function of facility 1's starting inventory x_1^k. Similarly, we know $\underline{x}_1^k = 0$ and $\underline{y}_1^{*k} = 0$. From Proposition 3 property *3*, clearly, when $x_2^k < y_2^{*k}(x_1^k)$, y_1^{*k} is independent of x_2^k.

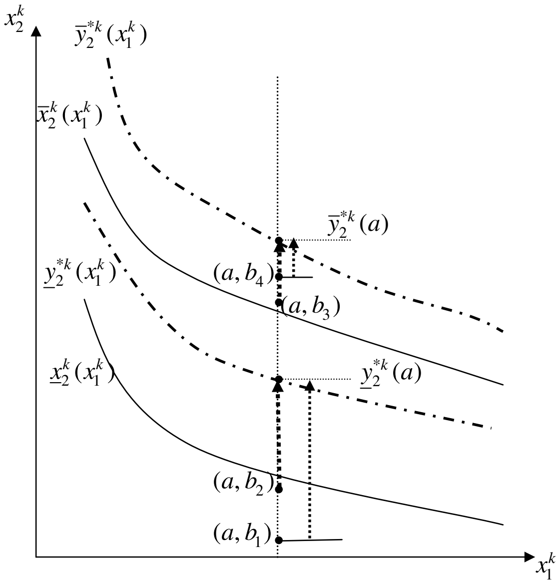

Figure 2: Facility 2's thresholds and up-to levels (with uncertain capacity).

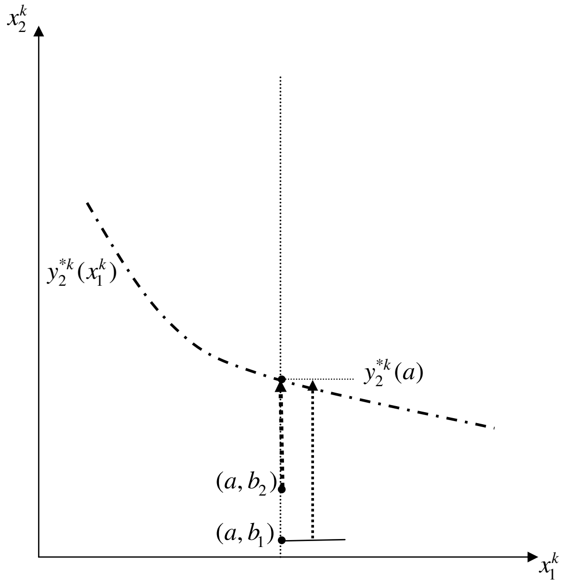

Figure 3: The up-to levels of facility 2 (with infinity capacity).

■

Figure 3 illustrates the production policy for facility 2 (with infinite capacity). Theorem 3 also shows that the production policy of the facility facing uncertain capacity is basically determined by whether or not the other facility needs to produce. If facility 2 (with infinite capacity) already has sufficient or excess inventory (i.e., $x_2^k \geq y_2^{*k}(x_1^k)$), then facility 1's policy is to take this excess into account and to try to produce up to $\bar{y}_1^{*k}(x_2)$. However, if the reverse is true, then facility 2 is below its target and since it can produce *any* quantity with probability 1, then facility 1 does not need to take into account the starting inventory in facility 2 when making its own production decision.

Two Facilities with Certain-Limited Capacities

Let now both facilities have deterministic capacities and facility i's capacity is limited to C_i^k in period k, that is, $Pr(T_i^k = C_i^k) = 1$ for $i = 1, 2$. This may be viewed as a special case of the scenario with both facilities having uncertain capacities. By Proposition 3 point *2*, modified order-up-to policies, similar to that in Kapuscinski and Tayur [24], is optimal:

Theorem 4 *Consider two facilities with certain-limited capacities* C_i^k $(i = 1, 2)$. *The optimal policy is defined by modified order-up-to levels which are functions of the other facility's starting inventory, i.e., facility* i *produces* $\min((y_k^{*i}(x_{3-i}^k) - x_i^k)^+, C_i^k)$.

Two Facilities with Infinite Capacities

The case when both facilities have infinite capacities has been well studied. Robinson [32] shows that each facility uses a base stock policy, and can ignore the other facility's inventory level in determining its production quantity, as long as both starting levels are below the base stocks.

5 Sensitivity of Optimal Policy

In this section we discuss how the facilities' capacity and demand levels, and cost and revenue coefficients affect the optimal policy. While we address the optimal policy for the multi-period case, we analyze changes in capacity and demand, cost and revenue coefficients only in a single period and, without loss of generality, only at facility 1. We are particularly interested in how changes in these variables affect the optimal inventory levels in each of these facilities. For example, does higher unit holding cost in a facility imply that target inventory levels at that facility should drop? Although this would be intuitive, and it is actually true if both facilities have infinite or certain-limited capacities, this turns out not necessarily to be the case when capacities are uncertain. This is why we formally analyze the effect of stochastically higher demand or capacity, higher holding costs, higher revenues, and higher production and transshipment costs on the parameters of the optimal policy.

All of our analysis relies on the following technical result. Consider a problem P_a:

$$(P_a) \qquad \begin{cases} K_1(a, \mathbf{y}) = 0 \\ K_2(a, \mathbf{y}) = 0 \end{cases}$$

and denote its solution set as $A(a) = \{\hat{\mathbf{y}}(a) = (\hat{y}_1(a), \hat{y}_2(a)) | (P_a)\}$. If solution set $A(a)$ is convex (which is always the case in the propositions below), then choose $\mathbf{y}^*(a) := \arg\max_{\hat{y}_2}\{\arg\max_{\hat{y}_1} A(a)\}$ to analyze how $\mathbf{y}^*(a)$ is changed by changing parameter a.

Proposition 5 *Suppose problem P_a satisfies*

$$v_{ii}(a, \mathbf{y}) \le v_{i,3-i}(a, \mathbf{y}) \le 0, \tag{10}$$

where $v_{ij}(a, \mathbf{y}) = \dfrac{\partial}{\partial y_j} K_i(a, \mathbf{y})$ $(i, j = 1, 2)$*, and its solution set is convex. Denote* $v_{ij} = v_{ij}(a, \mathbf{y}^*)$ *and* $u_i = u_i(a, \mathbf{y}^*) = \dfrac{\partial}{\partial a} K_i(a, \mathbf{y})|_{\mathbf{y}=\mathbf{y}^*}$*.*

1. *If $K_1(a, \mathbf{y})$ is non-increasing (non-decreasing) in a and $K_2(a, \mathbf{y})$ is non-decreasing (non-increasing) in a, then $y_1^*(a)$ is non-increasing (non-decreasing) in a and $y_2^*(a)$ is non-decreasing (non-increasing) in a.*

2. *If both $K_1(a, \mathbf{y})$ and $K_2(a, \mathbf{y})$ are non-increasing (non-decreasing) in a, then $y_1^*(a) + y_2^*(a)$ is non-increasing (non-decreasing) in a. Furthermore, if there exist $\beta_i > 0$, such that (P_a) satisfies*

$$\beta_i v_{ii} \le \beta_{3-i} v_{3-i,i} \tag{11}$$

   then

   (a) if $\beta_1|u_1| \ge \beta_2|u_2|$, $y_1^*(a)$ is non-increasing (non-decreasing) in a.

   (b) if $\beta_1|u_1| \le \beta_2|u_2|$, $y_2^*(a)$ is non-increasing (non-decreasing) in a.

Proof: See Appendix. ■

In the remainder of this section, $K_i(a, \mathbf{y}^k) = \lambda_k^i(\mathbf{x}^k, \mathbf{y}^k)$, where $\lambda_k^i(\mathbf{x}^k, \mathbf{y}^k)$ is defined in (6) and (7) and is a function of parameter a which will be the parameter that we are interested in changing in our sensitivity analysis. For example, a is capacity distribution T_1^k in Section 5.1, demand distribution D_1^k in Section 5.2, and any of cost or revenue coefficients, h_1, r_1 c_1 and s_{12}, in Section 5.3. Hence, $v_{ij}(a, \mathbf{y}^k) = \dfrac{\partial}{\partial y_j^k}\lambda_k^i(\mathbf{x}^k, \mathbf{y}^k)$. From Proposition 1 point *3(a)*, (10) clearly holds. Let $\beta_i = 1 - F_i^k(y_i^{*k} - x_i^k) > 0$, for $i = 1, 2$. We have

$$\beta_1 v_{11} = \beta_1\beta_2(G^k)''_{11}(\mathbf{y}^{*k}) + \beta_1\int_0^{y_2^{*k}-x_2^k}(G^k)''_{11}(y_1^{*k}, x_2^k + t_2^k) \leq \beta_1\beta_2(G^k)''_{21}(\mathbf{y}^{*k}) = \beta_2 v_{21}.$$

Thus, (11) holds for $i = 1$. Inequality for $i = 2$ holds by symmetry. Also note that for $K_i(a, \mathbf{y}^k) = \lambda_k^i(\mathbf{x}^k, \mathbf{y}^k)$ solution set to (P_a) is guaranteed to be convex due to Theorem 2(a). In the remainder of this section, we use "increase" and "decrease" in the non-strict sense.

5.1 Sensitivity to Capacity

Since transshipment decisions are made after both capacity and demand are realized, capacity uncertainty *in the current period* has no effect on the current-period transshipment policy. It, however, influences both facilities' current-period optimal production policy.

Proposition 6 *If in the current period k, facility 1's capacity stochastically increases, then facility* 1*'s optimal inventory target increases and facility* 2*'s target decreases.*

Proof: Consider the optimal production targets satisfying conditions (6) to (9). Notice that facility 1's capacity distribution appears only in (7). Assume first that "=" does not hold in (7), i.e., $\lambda_k^2(\mathbf{x}^k, y_1^{*k}, x_2^k) < 0$ and $y_2^{*k} = x_2^k$. Suppose (y_1^{**}, x_2^k) satisfies (6) and (8). Then any stochastic increase in facility 1's capacity guarantees $\lambda^2(\mathbf{x}^k, y_1^{**}, x_2^k) < 0$, since $G^k(\mathbf{y}^k)$ is submodular in $\mathbf{y}^k$, while expressions (6) and (8) are not influenced. Thus, (y_1^{**}, x_2^k) satisfies (6) to (9) and remains the optimal target.

Let "=" hold in (7). Denote $K_i(T_1^k, \mathbf{y}^{*k}) = \lambda_k^i(\mathbf{x}^k, \mathbf{y}^{*k})$. The optimal production targets for both facilities $\mathbf{y}^*$ must satisfy $K_1(T_1^k, \mathbf{y}^{*k}) = 0$ (or $y_1^k = x_1^k$) and $K_2(T_1^k, \mathbf{y}^{*k}) = \lambda_k^2(\mathbf{x}^k, \mathbf{y}^{*k}) = 0$, where facility 1's capacity distribution T_1^k is the parameter in K_1 and K_2. As T_1^k stochastically increases, the value of $K_2(T_1^k, \mathbf{y}^k)$ decreases, and the value of $K_1(T_1^k, \mathbf{y}^k)$ remains the same, since T_1^k does not appear in its formulation. Since (10) holds, from Proposition 5 point *1*, y_1^{*k} increases and y_2^{*k} decreases. ■

5.2 Sensitivity to Demand

Now we consider how the current-period demand influences the optimal policy. Once again, optimal transshipment policy is not influenced. Consider facility 1's demand distribution to be stochastically increased, while facility 2's demand distribution is unchanged. Intuitively, we expect facility 1 to raise its inventory target and the sum of the two facilities' targets to increase.

We find that while an increase in demand always increases the sum of the two facilities' inventory targets, the facility that experiences the increased demand may actually lower its inventory target. However, this cannot happen when the facility has infinite or certain-limited capacity.

Proposition 7 *If facility* 1*'s demand stochastically increases in the current period k, then the sum of the production targets at the two facilities is increased. Furthermore, if facility* 1 *has infinite or certain-limited capacity, its production target is increased.*

Proof: Note that for given y_2^k, we can define a function $\hat{y}_1^k(y_2^k)$ such that $(\hat{y}_1^k(y_2^k), y_2^k)$ satisfies (6) and (8). It is easy to prove that $-1 \le \dfrac{\partial \hat{y}_1^k(y_2^k)}{y_2^k} \le 0$. Symmetrically, from (7) and (9), we can define $\hat{y}_2^k(y_1^k)$ and $-1 \le \dfrac{\partial \hat{y}_2^k(y_1^k)}{y_1^k} \le 0$. Then, the intersection of $\hat{y}_1^k(y_2^k)$ and $\hat{y}_2^k(y_1^k)$ is the optimal target $\mathbf{y}^{*k}$ satisfying (6) to (9). Since $\hat{y}_i^k(y_{3-i}^k)$ $(i = 1, 2)$ and $\mathbf{y}^{*k}$ are also functions of demand distribution D_1^k, i.e., $\hat{y}_i^k(D_1^k, y_{3-i}^k)$ and $\mathbf{y}^{*k}(D_1^k)$, it suffices to discuss how these functions change when demand distribution stochastically decreases from D_1^k to $\bar{D}_1^k$, i.e., $D_1^k \ge_{s.t.} \bar{D}_1^k$

Let $K_i(D_1^k, \mathbf{y}^k) = \lambda_i^k(\mathbf{x}^k, \mathbf{y}^k)$, (10) holds and $K_i(D_1^k, \mathbf{y}^k) \ge K_i(\bar{D}_1^k, \mathbf{y}^k)$ (from Proposition 1 point *3(b)*). Hence, we know that $\hat{y}_i^k(D_1^k, y_{3-i}^k) \ge \hat{y}_i^k(\bar{D}_1^k, y_{3-i}^k)$ (from (10), K_i decreases in y_i^k). To prove the proposition, we consider two cases:

1) If either $K_1(D_1^k, \mathbf{y}^{*k}(D_1^k)) < 0$ or $K_2(D_1^k, \mathbf{y}^{*k}(D_1^k)) < 0$, then either $\hat{y}_1^k(D_1^k, y_2^k) = x_1^k$ or $\hat{y}_2^k(D_1^k, y_1^k) = x_2^k$. Without loss of generality, suppose $\hat{y}_1^k(D_1^k, y_2^k) = x_1^k$, then for $\bar{D}_1^k \le_{s.t.} D_1^k$, we have $\hat{y}_1^k(\bar{D}_1^k, y_2^k) = x_1^k$ and $\hat{y}_2^k(\bar{D}_1^k, y_2^k) \le \hat{y}_2^k(D_1^k, y_1^k)$, and clearly, $y_1^{*k}(\bar{D}_1^k) = y_1^{*k}(D_1^k) = x_1^k$ and $y_2^{*k}(\bar{D}_1^k) \le y_2^{*k}(D_1^k)$.

2) If $K_i(D_1^k, \mathbf{y}^{*k}(D_1^k)) = 0$ for $i = 1, 2$, then

1. The monotonicity of the sum of the targets follows from Proposition 5 point *2*.

2. Suppose that facility 1 has infinite capacity, we have either $F_2^k(y_2^{*k} - x_2^k) < 1$ or $F_2^k(y_2^{*k} - x_2^k) = 1$. Consider first $F_2^k(y_2^{*k} - x_2^k) < 1$. Letting $\beta_1 = 1 - F_1^k(y_1^{*k} - x_1^k) = 1$ and $\beta_2 = 1 - F_2^k(y_2^{*k} - x_2^k)$, (11) follows. From Proposition 1 *3(b)*, we have

$$
\begin{aligned}
&\mathrm{E}_{\mathbf{D}^k} \frac{\partial}{\partial y_1^k} G_v^k(\mathbf{y}^k - \mathbf{D}^k) - \mathrm{E}_{\bar{D}_1^k, D_2^k} \frac{\partial}{\partial y_1^k} G_v^k(y_1^k - \bar{D}_1^k, y_2^k - D_2^k) \\
\ge &\mathrm{E}_{\mathbf{D}^k} \frac{\partial}{\partial y_2^k} G_v^k(\mathbf{y}^k - \mathbf{D}^k) - \mathrm{E}_{\bar{D}_1^k, D_2^k} \frac{\partial}{\partial y_2^k} G_v^k(y_1^k - \bar{D}_1^k, y_2^k - D_2^k) \ge 0,
\end{aligned}
$$

which implies $\beta_1[K_1(D_1^k, \mathbf{y}^{*k}) - K_1(\bar{D}_1^k, \mathbf{y}^{*k})] \ge \beta_2[K_2(D_1^k, \mathbf{y}^{*k}) - K_2(\bar{D}_1^k, \mathbf{y}^{*k})]$. Thus, from Proposition 5 point *2(a)*, $y_1^{*k}(D_1^k) \ge y_1^{*k}(\bar{D}_1^k)$. Consider now $F_2^k(y_2^{*k} - x_2^k) = 1$, i.e., facility 2 reaches its capacity limit. We reverse the comparison and consider stochastically increasing D_1^k. From *1*, $y_1^{*k} + y_2^{*k}$ increases. In this case, y_2^{*k} remains the same for D_1^k and $\bar{D}_1^k$, and therefore y_1^{*k} must increase.

Finally suppose that facility 1 has a certain capacity limit C_1^k, then (6) can be rewritten as: $\lambda_1^k(\mathbf{x}^k, \mathbf{y}^{*k}, \mu_1^k) := \mathrm{E}_{T_2^k}(G^k)'_1(y_1^{*k}, y_2^{*k} \wedge (x_2^k + T_2^k)) - \mu_1^k \le 0$, and $\mu_1^k \ge 0$, $\mu_1^k \times (y_1^{*k} - x_1^k - C_1^k) = 0$. It suffices to consider the scenario that $y_1^{*k}(\bar{D}_1^k) - x_1^k = C_1^k$. Using $\mathbb{A}_1^{k+1}$, it is straightforward to verify that $(C_1^k + x_1^k, \hat{y}_2^k(D_1^k, C_1^k + x_1^k))$ also satisfies (6) to (9), hence y_1^{*k} does not decrease when demand stochastically increases from $\bar{D}_1^k$ to D_1^k.

■

To show that Proposition 7 fails when facility 1 has uncertain capacity, consider the situation where facility 2 has stochastically larger capacity than facility 1. Facing stochastically larger demand, it may be optimal for facility 1 to rely on facility 2's capacity to help deal with production uncertainty. This may result in pushing down facility 1's target and pushing up facility 2's target. Below we show how to generate a class of examples that result in this behavior.

Example 1 Consider a single-period problem where facility 2 has infinite capacity. Let holding costs be $h = h_1 = h_2 > 0$, production costs be $c_2 \geq c_1 = 0$, shipping costs be $s_{12} \geq s_{21} = 0$ (with $s_{12} \geq c_2 - c_1 = c_2$), and revenues be $r = r_1 = r_2 > c_2$. Denote $\delta = Q_1^{-1}(\frac{h}{h+r})$. Facility 2 has fixed demand $D_2 = \gamma \geq 0$. Two demand distributions D_1 (*cdf* Q_1) and $\bar{D}_1$ (*cdf* $\bar{Q}_1$) at facility 1 satisfy:

- $\forall\, d_1 \geq \delta$, $1 - \bar{Q}_1(d_1) = 1 - Q_1(d_1)$,
- $\forall\, 0 \leq d_1 < \delta$, $1 - \bar{Q}_1(d_1) \geq 1 - Q_1(d_1)$.

Clearly, $\bar{D}_1 \geq_{st} D_1$. Under this setting, transshipment occurs only from facility 2 to 1, $y_2^* \geq \gamma$, and the rationing levels are 0, since only a single period is considered. The total profit function is as follows ($c_1 = s_{21} = 0$):

$$\begin{aligned}\max_{\mathbf{y} \geq \mathbf{x}} -c_2(y_2 - x_2) + \mathrm{E}_{T_1, D_1}[r(D_1 + \gamma) - h(y_1 \wedge (x_1 + T_1) + y_2 - D_1 - \gamma)^+ \\ -r(y_1 \wedge (x_1 + T_1) + y_2 - D_1 - \gamma)^-].\end{aligned}$$

This scenario is a special case of problem considered in Hu at el [23], where facility 2 is an outsourcer. Based on Proposition 8 in [23], $y^* := y_1^* + y_2^* = \delta + \gamma$ is the same for both D_1 and $\bar{D}_1$. Taking derivatives w.r.t. y_2, we obtain:

$$\int_0^{y^* - y_2^*} [r - (r+h) Q_1(y_2^* - \gamma + t_1)] f_1(t_1) \mathrm{d}t_1 = c_2.$$

The LHS is a non-increasing function of y_2 (for given y). For $\bar{D}_1$ instead of D_1, we have $y_2^*(\bar{D}_1) \geq y_2^*(D_1)$ and for appropriate chosen c_2 the inequality is strict. Consequently, $y_1^*(\bar{D}_1) < y_1^*(D_1)$. ■

5.3 Sensitivity to Cost and Revenue Coefficients

Even though the transhipment decision is independent of current-period demand and capacity distributions, changes in the current-period cost and revenue coefficients influence both the transshipment policy and production policy. The effect of individual changes in the revenue and cost coefficients on the transshipment policy is transparent (see Theorem 1). The following subsections evaluate the sensitivity of optimal production policy to each of the cost and revenue coefficients.

5.3.1 Sensitivity to Holding Cost

Assume facility 2's holding cost does not exceed that of facility 1's, $h_2 \leq h_1$. If facility 1's holding cost increases, we expect its inventory target to decrease. With uncertain capacity, the opposite may be the case. The following proposition is a counterpart of Proposition 7 showing the result is "intuitive" when the facility has infinite or finite-limited capacity while Example 2 demonstrates that an increase in holding cost may actually increase the inventory target at the facility with uncertain capacity.

Proposition 8 *As facility 1's current-period holding cost increases,*

1. *the sum of the production targets at both facilities is decreased;*

2. *if facility 1 has infinite or certain-limited capacity, then facility 1's production target decreases.*

Proof: Let $K_i(h_1^k, \mathbf{y}^k) = \lambda_k^i(\mathbf{x}^k, \mathbf{y}^k)$, $i = 1, 2$. Using Proposition 1 point *2(b)*, the proof is similar to the proof of Proposition 7 and is omitted for space. ■

The following example shows that if facility 1 has uncertain capacity, increasing its holding cost may lead to a higher production target at facility 1:

Example 2 Consider a single-period problem where facility 2 has infinite capacity, and stochastic demand, while facility 1 has uncertain capacity and faces no demand. Let facility 1's capacity be $(0, 1, 2)$ with probabilities $(0.2, 0.6, 0.2)$ and facility 2's demand be $(0, 1, 2)$ with probabilities $(0.28, 0.22, 0.5)$. Furthermore, let $c_1 = 0$, $c_2 = 0.9$, $r_1 = r_2 = 1.5$, $s_{12} = s_{21} = 1$ $(c_2 - s_{12} - c_1 < 0)$, $h_2 = 0.1$. Table 1 illustrates the profit for the system as a function of inventory targets, and (in bold face) the optimal profits for $h_1 = 0.1$ and $h_1 = 0.4$. When $h_1 = 0.1$, the optimal production quantities are $(y_1^*, y_2^*) = (1, 1)$ with profit 0.312. For $h_1 = 0.4$ (note that $h_1 - s_{12} - h_2 < 0$ still holds), we have $(y_1^*, y_2^*) = (2, 0)$ with profit 0.2084.

y_1	y_2					
	$h_1 = 0.1$			$h_1 = 0.4$		
	0	1	2	0	1	2
0	0	0.152	-0.048	0	0.152	-0.048
1	0.2656	**0.312**	-0.128	0.1984	0.192	-0.368
2	0.3056	0.292	-0.148	**0.2084**	0.112	-0.448

Table 1: Production quantities and profits under different holding costs.

To explain this behavior, note that due to uncertain capacity, increased target production level at facility 1 may result in lower total expected remaining inventory in the system. Let us compare the two scenarios $(y_1, y_2) = (2, 0)$ and $(1, 1)$, with the sum of production levels equal to 2. For the case of $(2, 0)$, facility 1's expected remaining inventory is 0.324 and facility 2's expected remaining inventory is 0. For the case of $(1, 1)$, facility 1's expected remaining inventory is 0.4 and facility

2's expected remaining inventory is 0.28. Thus, with higher target at facility 1, in scenario $(2, 0)$, the remaining inventory is lower (due to uncertain capacity). Hence, the expected total holding costs increase when shifting from the target of $(2, 0)$ to $(1, 1)$, but when $h_1 = 0.4$ the increase is larger than that when $h_1 = 0.1$. On the other hand, given the policy $(2, 0)$ (or $(1, 1)$), the expected revenue, incurred transshipment costs and production costs when $h_1 = 0.4$ is the same as for $h_1 = 0.1$. Therefore, the change in revenues and production and transshipment costs, by shifting from the policy $(2, 0)$ to the policy $(1, 1)$, is the same for holding cost $h_1 = 0.1$ and for $h_1 = 0.4$. As a result, when $h_1 = 0.4$, $(2, 0)$ is a more profitable policy than $(1, 1)$.

5.3.2 Sensitivity to Revenue

Intuitively, a facility's inventory target should be increasing in its revenue, but uncertainty in capacity complicates this effect. Similar as before, the inventory target of a facility with infinite or certain-limited capacity increases in its revenue. The sum of the inventory targets is always increased with increasing revenue, as shown below in Proposition 9. A facility with uncertain capacity, however, may decrease its inventory target (see Example 3).

Proposition 9 *As facility 1's current-period revenue increases,*

1. *the sum of the inventory targets at both facilities is increased;*
2. *if facility* 1 *has infinite or certain-limited capacity, then its inventory target is increased.*

Proof: Let $K_i(r_1, \mathbf{y}^k) = \lambda_i^k(\mathbf{x}^k, \mathbf{y}^k)$. Proposition 9 follows from Proposition 1 *2(c)* and the logic similar to that in the proof of Proposition 7. ■

Example 3 Consider a single-period problem where facility 1 has uncertain capacity and facility 2 has infinite capacity. Assume also, equal holding costs $h_1 = h_2 > 0$, production costs $c_2 \geq c_1 = 0$, shipping costs $s_{12} \geq s_{21} = 0$ $(c_2 < s_{12})$, equal revenues $r > c_2$, stochastic demand D_1 at facility 1, and fixed zero demand $D_2 = 0$ at facility 2. Let $r' > r$, and denote $\delta = \dfrac{r}{r+h}$ and $\delta' = \dfrac{r'}{r'+h}$. Suppose the *cdf* Q_1 "jumps over???? δ and δ', i.e., $\{d_1 | Q_1(d_1) \leq \delta\} = \{d_1 | Q_1(d_1) \leq \delta'\}$. Then $y^* := y_1^* + y_2^* = (Q_1)^{-1}(\delta) = (Q_1)^{-1}(\delta')$. As in Example 1, $y_2^*(r)$ must satisfy:

$$\int_0^{y^*-y_2^*} [r - (r+h)Q_1(y_2^* + t_1)] f_1(t_1) \mathrm{d}t_1 = c_2.$$

The LHS is a non-increasing function of y_2 and non-decreasing function of r, implying $y_2^*(r) \leq y_2^*(r')$ with strict inequality for some r'. Hence, $y_1^*(r) > y_1^*(r')$.

5.3.3 Sensitivity to Production and Transshipment Costs

In the previous subsections, we showed how inventory target could move in one direction with changes in a parameter when the facility has infinite or certain-limited capacity but in the other direction when capacity is uncertain. However, when production and transshipment costs are changed, we do not see nonintuitive behavior.

Proposition 10 *Facility* 1*'s current-period production cost and transshipment cost have the following effects:*

1. *If facility* 1*'s production cost is increased, then (1) both facilities' transshipment policies remain the same, (2) facility* 1*'s inventory target is decreased, and (3) facility* 2*'s inventory target is increased.*

2. *If facility* 1*'s transshipment cost is increased, then (1) facility* 1*'s inventory target is decreased and (2) facility* 2*'s inventory target is increased.*

Proof: 1. Consider $K_i(c_1, \mathbf{y}^k) = \lambda_k^i(\mathbf{x}^k, \mathbf{y}^k)$. From $G^k(\mathbf{y}^k) = \mathrm{E}_{\mathbf{D}^k}[G_v^k(\mathbf{y}^k - \mathbf{D}^k) + \mathbf{r}\mathbf{D}^k] - \mathbf{c}\mathbf{y}^k$ and (5), $K_1(c_1, \mathbf{y}^k)$ is decreasing in c_1 and $K_2(c_1, \mathbf{y}^k)$ is independent of c_1. Hence, *1.* follows from Proposition 5 point *1.*

2. Consider $K_i(s_{12}, \mathbf{y}^k) = \lambda_k^i(\mathbf{x}^k, \mathbf{y}^k)$. From Proposition 1 *2(d)* and Proposition 5 point *1*, *2* follows. ∎

6 The Benefits of the Optimal Policy

In the previous sections we have characterized how a firm can coordinate two facilities with uncertain capacity through the use of optimal production planning and transshipment. However, the policies described in Section 4 are fairly complex as they require judicious use of transshipment and rationing. That is, the firm has to decide when units can be transshipped and when it has to keep units at its own location to ensure that the facility can meet future demand.

In this section, we explore how significant (and when) this coordination relying on transshipment and production is. We do this through comparing the optimal policy against two simple straw policies. To understand the value of transshipment, we first explore a simple "no transshipment" policy – the facilities are separate from each other and cannot aid each other. When no transshipment is possible, it is straightforward to show that both facilities attempt to bring their inventory to a base stock level each period. The second straw policy is a "no rationing" policy, where facilities are not allowed to ration and transshipment takes place whenever there is any surplus at one facility and shortage at the other. Unlike the optimal policy established in Section 4, the transshipment is enforced, and the facility that has the surplus cannot reserve any inventory for future period as long as the other facility needs it. When making production decisions for both facilities, the central planner should take into account the fact that rationing is not allowed and adjust the inventory level for each facility appropriately.

We first focus on symmetric systems, where the two facilities face the same capacity and demand distributions, as well as identical cost/revenue coefficients, and explore the benefits of transshipment. We then investigate the benefits of rationing in asymmetric systems. The benefits of transshipment (or rationing) are evaluated as a relative profit decrease from the profit of the optimal policy to that of no transshipment (or no rationing) policy, as follows:

$$\frac{\text{profit(Optimal)} - \text{profit(Straw Policy)}}{\text{profit(Optimal)}}. \tag{12}$$

All three policies (optimal, no transshipment, and no rationing) and their profits are determined by solving 10-period dynamic programs using backward induction, with starting inventory of 0 at both facilities. In each period, a 2-step backward induction takes place. First, the optimal transshipment policy is calculated for any realization of demand and capacity, and then the inventory targets at both facilities are computed. Based on these two steps, the value function is computed.

Table 2 shows the parameters of demand and capacity distributions we use. Both capacity

Demand			Capacity		
mean	stand. devi.	variability	mean	variability	utilization
18	1.8	0.1	30	0.1	0.6
	5.4	0.3	22.5	0.3	0.8
	10.8	0.6	20	0.6	0.9
	16.2	0.9	18	0.9	1.0
	21.6	1.2	15	1.2	1.2

Table 2: Demand and capacity distributions.

and demand are triangular non-negative integer random variables (we use inverted triangle to achieve high standard deviations). The mean demand is fixed at 18 and we generate the triangular distributions with values between 0 and 68 and with the desired coefficient of variability. For each of the demand distributions, we adjust the capacity distribution to achieve the desired system utilizations and capacity variabilities. The capacity distribution is also triangular with values between 0 and 114. We fix the production cost at 1. We define nominal service level (NSL) as follows:

$$NSL = \frac{\text{revenue} - \text{production cost}}{\text{revenue} - \text{production cost} + \text{holding cost}} \tag{13}$$

Note that this is the service level that a facility will aim at if no transshipment was allowed and capacity was infinite. Thus, changes in NSL for fixed holding and transshipment costs are used to evaluate the effects of changes in revenue on transshipment and rationing. Table 3 lists the values of NSL, holding cost, and transshipment cost. In total, we ran $24,000$ experiments comparing the

Nominal Service Level (NSL)	0.4, 0.5, 0.8, 0.9, 0.95, 0.99
Holding Cost	0.005, 0.02, 0.08, 0.32
Transshipment Cost	0.1, 0.3, 0.5, 0.7

Table 3: Nominal service level (NSL), holding cost, and transshipment cost.

no transshipment and no rationing policies with the optimal policy.

6.1 The Benefits of Transshipment

By comparing the optimal policy and the no transshipment policy, we identify when transshipment is most beneficial. The benefits vary across the range of parameters studied with the averages between 0% and 14%. We observe that, as expected, increased capacity variability and demand variability lead to increased benefits. The effect of NSL is, however, slightly more complicated.

Figure 4 illustrates the effects of capacity variability and of demand variability. Each point represents the average relative benefit across all utilizations and values of NSL in Table 2 and Table 3. As capacity variability or demand variability increases, the benefits of transshipment increase. For low capacity variability, e.g., around 0.1, the benefits span the range of 0-11%. As capacity variability increases, the benefits are higher; however, their range shrinks.

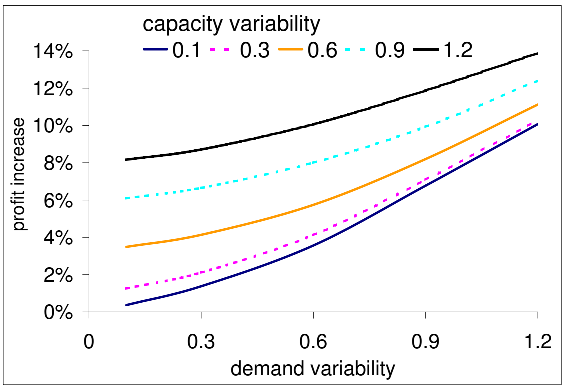

Figure 4: Effects of capacity variability and demand variability.

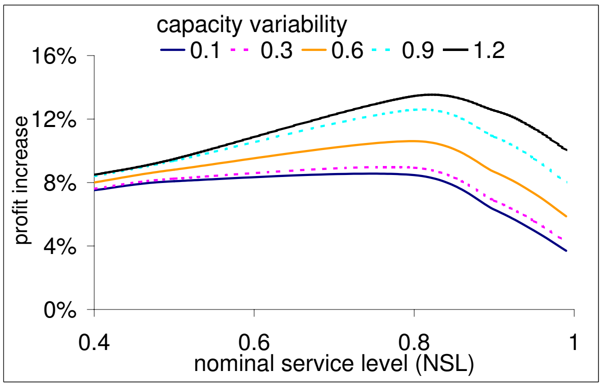

Figure 5: Effects of nominal service level.

The reasons for this behavior are the same when considering capacity or demand variability. The uncertainties in capacity and demand increase the chances of lost sales and necessitate inventory to be held. The transshipment between two facilities helps them pool their capacities and demands together and therefore reduce both uncertainties. The larger the uncertainties, the larger are the benefits of transshipment.

Figure 5 shows the effects of the nominal service level on the benefits of transshipment. These benefits increase first and then decrease. Recall that a low value of NSL implies a low value of revenue compared to costs and in this situation, transshipping to save one unit of demand from being lost is not that valuable. As NSL increases, the unit revenue is higher relative to unit cost and transshipment becomes more appealing. However, at very high service levels, losing a customer implies such a big revenue loss (relative to cost) that each facility carries a lot of inventory and does not have to rely as much on transshipment in the first place, thereby decreasing the relative value of transshipment.

6.2 The Benefits of Rationing

Rationing is caused by a facility's need to protect itself from capacity uncertainty. By rationing, the firm gives up the current-period revenue that would be generated by satisfying the current-period demand shortage. But at the same time, the facility that rations hedges against its future capacity shortage, caused either by unexpected capacity failure or demand surge. Clearly, the benefits of rationing increase in demand variability (see Figure 6), capacity variability (see Figure 8), and

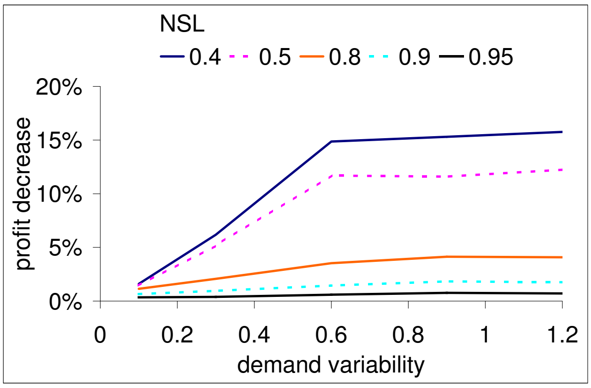

Figure 6: Effects of demand variability.

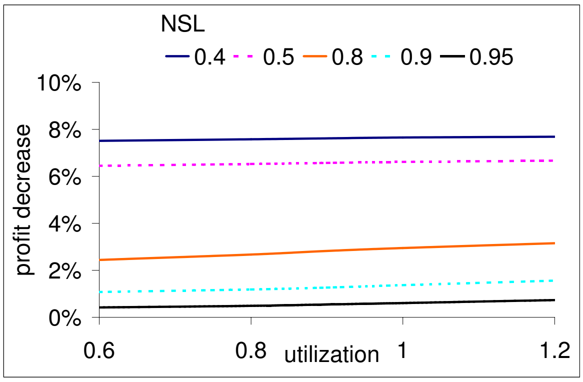

Figure 7: Effects of utilization.

utilization (see Figure 7). The effect of utilization is, however, fairly insignificant compared to the effects of demand variability or capacity variability.

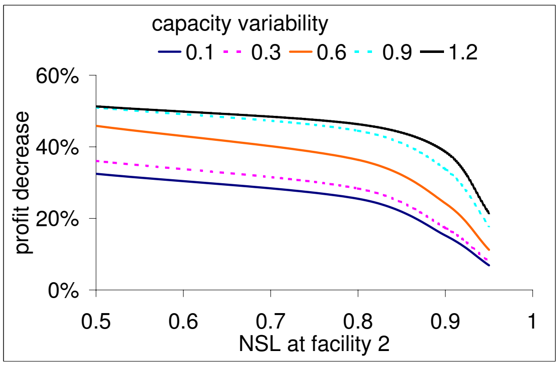

Figure 8: Effects of nominal service level and capacity variability.

Note that a high NSL translates into a high unit revenue, and therefore a stronger incentive for the facility not to ration. Hence, the benefits of rationing decrease in NSL. This behavior can be observed in Figure 8 where facility 1 has fixed NSL of 90%, and both facilities have $h_1 = h_2 = 0.02$, $s_{12} = s_{21} = 0.5$, as well as identical demand and capacity variability and utilization. As facility 2's NSL increases from 50% to 95% (and the corresponding revenue increases from 1.02 to 1.38), the benefits of rationing keeps decreasing, but even at the lowest capacity variability of 0.1, we can still observe a significant 7% benefit.

Figures 4 through 8 show how transshipment and rationing used optimally in a supply chain with capacity uncertainty can lead to significantly increased profits. Therefore, the policies described in Section 4 can have a significant impact, especially as capacity and demand variability increase.

7 Discussion and Extensions

Our model considers optimal production and transshipment control in a centralized system with stationary linear cost and revenue coefficients and lost sales. As compared to other papers, our model is much more general. We impose only one assumption on the facilities' revenues, $r_2 - s_{12} - r_1 \leq 0$ and $r_1 - s_{21} - r_2 \leq 0$, compared to several assumptions in Tagaras [35], Robinson [32], and Rudi *et al* [33], etc., and even this assumption does not influence the structure of our optimal policy. In fact, the same structure also holds with a model that allows backlogging, nonstationary and some nonlinear cost-revenue coefficients. The model can also be generalized to allow Markov-Modulated capacity and demand processes. Therefore, we now discuss how the model in Section 4 can be generalized in multiple dimensions while still resulting in the same optimal structure.

1. Revenue assumption.

As mentioned in Section 3, the assumption $r_2 - s_{12} - r_1 \leq 0$ and $r_1 - s_{21} - r_2 \leq 0$ can be relaxed. Without loss of generality, we consider that $r_2 - s_{12} - r_1 > 0$, which, under our lost-sale setting, implies that it is always beneficial to satisfy facility 2's demand prior to facility 1's demand. As $r_1 - s_{21} - r_2 \leq 0$ still holds, transshipment from facility 2 to 1 occurs only when facility 2's own demand is fully satisfied by its own inventory. Consequently, $\mathbb{A}_3^{k+1}$ (for $i = 1$) is replaced by $(G_*^{k+1})'_1(\mathbf{x}^{k+1}) \leq r_2 - s_{12}$, where $\mathbb{A}_3^{k+1}$ for $i = 2$ remains $(G_*^{k+1})'_2(\mathbf{x}^{k+1}) \leq r_2$. This condition guarantees concavity of the problem ($\mathbb{A}_1^{k+1}$ and $\mathbb{A}_2^{k+1}$ remain the same). It is easy to argue by induction that facility 1 rations nothing and satisfies facility 2s shortage before its own demand, that the transshipment policy for facility 2 is FR2$(\underline{\chi}_2^k, \bar{\chi}_2^k(z^k))$, and that the optimal production policy for both facilities has the same structure.

2. Non-stationary or non-linear cost-revenue coefficients.

Stationarity of the cost coefficients, h_i, c_i, and $s_{i,3-i}$, is not essential to establish the structure of the optimal transshipment and production policy, since none of the cost coefficients plays any role in determining the concavity and submodularity of the model. However, under the lost-sale setting, it is critical to have $(G_*^k)'_i(\mathbf{x}^k) \leq r_i^k$ to ensure the concavity of the problem. Hence, we require either a stationary revenue coefficient r_i or non-increasing ordered r_i^k's: $r_1^1 \geq \ldots \geq r_1^{N+1}$. For the structure of the production policy, linearity of the production cost c_i is fundamental, since it allows the inventory target y_i to be independent of initial inventory x_i. Linearity of revenue r_i and transshipment cost $s_{i,3-i}$ is important to establish the structure of the transshipment policy. We can relax the linear holding cost h_i to any convex and non-decreasing holding cost function without changing the structure of the transshipment policy.

3. Backlogging.

Allowing for backlogging makes the problem much easier and the needed concavity holds without $\mathbb{A}_3^k$. Based on $\mathbb{A}_1^{k+1}$ and $\mathbb{A}_2^{k+1}$, we can verify that the transshipment policy for facility i is SR$i(\chi^k(z^k))$, and the structure of the optimal production policy for both facilities is the same as under lost-sale setting. The only assumption we require is the linearity of transshipment cost $s_{i,3-i}^k$ and production cost c_i^k, as well as convexity of holding cost function and concavity of revenue function.

4. Markov-Modulated Process.

The assumption of demand and capacity independence across the periods can be relaxed if we allow a Markov chain driving demand and capacity distributions. Suppose there are L states of the world with Markov transition matrix $[p(l, l')]$ and corresponding demands $\mathbf{D}^l$, and capacities $\mathbf{T}^l$, $l = 1, 2, \cdots, L$. Based on (1)–(3), the model can be reformulated as follows:

$$\textbf{Stage One: } G_*^k(l, \mathbf{x}^k) = \max_{\mathbf{y}^k \geq \mathbf{x}^k} \mathrm{E}_{\mathbf{T}^{lk}, \mathbf{D}^{lk}} \{-\mathbf{c}(\mathbf{y}^k \wedge (\mathbf{x}^k + \mathbf{T}^{lk}) - \mathbf{x}^k) + \mathbf{r}\mathbf{D}^{lk} + G_v^k(l, \mathbf{y}^k \wedge (\mathbf{x}^k + \mathbf{T}^{lk}) - \mathbf{D}^{lk})\}$$

$$\textbf{Stage Two: } G_v^k(l, \mathbf{z}^k) = \max_{\hat{z}_1^k + \hat{z}_2^k = z_1^k + z_2^k} \bar{G}^k(l, \mathbf{z}^k, \hat{\mathbf{z}}^k)$$

where $\bar{G}^k(l, \mathbf{z}^k, \hat{\mathbf{z}}^k) = -\mathbf{r}(\hat{\mathbf{z}}^k)^- - \mathbf{h}(\hat{\mathbf{z}}^k)^+ - \mathbf{s}(\mathbf{z}^k - \hat{\mathbf{z}}^k)^+ + \sum_{l'=1}^{L} \alpha_k p(l, l') G_*^{k+1}(l', (\hat{\mathbf{z}}^k)^+)$

and $G_*^{N+1}(l, \mathbf{x}^{N+1}) \equiv 0$, $l = 1, 2, \cdots, L$.

By the same arguments used for the basic model, we can justify that in period k, given state of the world l, the structure of optimal production and transshipment policies remains the same as in Section 4. Obviously, parameters of the optimal polices would now depend on the state of the world l.

Appendix: Proofs of Propositions

Proposition 1

$G_v^k(\mathbf{z}^k)$ has the following properties:

1. *$G_v^k(\mathbf{z}^k)$ is concave in $\mathbf{z}^k$ and $\lim_{z_i^k \to \infty} G_v^k(\mathbf{z}^k) = -\infty$ $(i = 1, 2)$.*
2. *Its first derivatives $(i = 1, 2)$ satisfy:*
   (a) $(G_v^k)_i'(\mathbf{z}^k) \leq r_i$.

   For period k's holding cost h_i^k, revenue r_i^k, and transshipping cost $s_{i,3-i}^k$,

   (b) $\frac{\partial}{\partial h_i^k}(G_v^k)_i'(\mathbf{z}^k) \leq \frac{\partial}{\partial h_i^k}(G_v^k)_{3-i}'(\mathbf{z}^k) \leq 0$.

   (c) $\frac{\partial}{\partial r_i^k}(G_v^k)_i'(\mathbf{z}^k) \geq \frac{\partial}{\partial r_i^k}(G_v^k)_{3-i}'(\mathbf{z}^k) \geq 0$.

   (d) $\frac{\partial}{\partial s_{i,3-i}^k}(G_v^k)_i'(\mathbf{z}^k) \leq 0$, $\frac{\partial}{\partial s_{i,3-i}^k}(G_v^k)_{3-i}'(\mathbf{z}^k) \geq 0$.
3. *Its second derivatives $(i = 1, 2)$ satisfy:*
   (a) $(G_v^k)_{ii}''(\mathbf{z}^k) \leq (G_v^k)_{12}''(\mathbf{z}^k) = (G_v^k)_{21}''(\mathbf{z}^k) \leq 0$.

   (b) $\frac{\partial}{\partial d_i^k}(G_v^k)_i'(\mathbf{z}^k) \geq \frac{\partial}{\partial d_{3-i}^k}(G_v^k)_i'(\mathbf{z}^k) = \frac{\partial}{\partial d_i^k}(G_v^k)_{3-i}'(\mathbf{z}^k) \geq 0$.

Proof: The properties hold for both $i = 1, 2$. We present only the case $i = 1$, since the other case can be justified using symmetric logic.

1. Concavity.

Joint concavity of $\bar{G}^k(\hat{\mathbf{z}}^k, \mathbf{z}^k)$ in $\hat{\mathbf{z}}^k$, $\mathbf{z}^k$, is a sufficient condition for concavity of $G_v^k(\mathbf{z}^k)$, see Heyman and Sobel [22], Property B-4, page 525. To justify concavity of $\bar{G}^k(\hat{\mathbf{z}}^k, \mathbf{z}^k)$, it suffices to show that function $-r_1(\hat{z}_1^k)^- - r_2(\hat{z}_2^k)^- + \alpha_k G_*^{k+1}((\hat{z}_1^k)^+, (\hat{z}_2^k)^+)$ is jointly concave in $\hat{\mathbf{z}}^k$, which is guaranteed by inductional hypothesis $\mathbb{A}_1^{k+1}$ and $\mathbb{A}_3^{k+1}$.

Given z_2^k, as $z_1^k \to \infty$, from Theorem 1, we have $z_1^k \geq \chi_1^k$ and either $\chi_1 \to \infty$ or $z_1^k + z_2^k - \chi_1^k \to \infty$. Hence, according to (5), $\lim_{z_1^k \to \infty} G_v^k(\mathbf{z}^k) = -\infty$ follows.

2. Properties of $G_v^k(\mathbf{z}^k)$'s first derivatives.

(a) is straightforward since the profit from any additional inventory never surpasses its revenue.

The first derivatives of $G_v^k(\mathbf{z}^k)$ (see (5)) w.r.t. z_1^k and z_2^k are evaluated based on definition of χ_1^k and χ_2^k and, after some algebra, result in intuitive expressions, as shown in Figure 9. Note that different forms of $\chi_i^k(z_1^k + z_2^k)$ translate into different formulae for the derivatives.

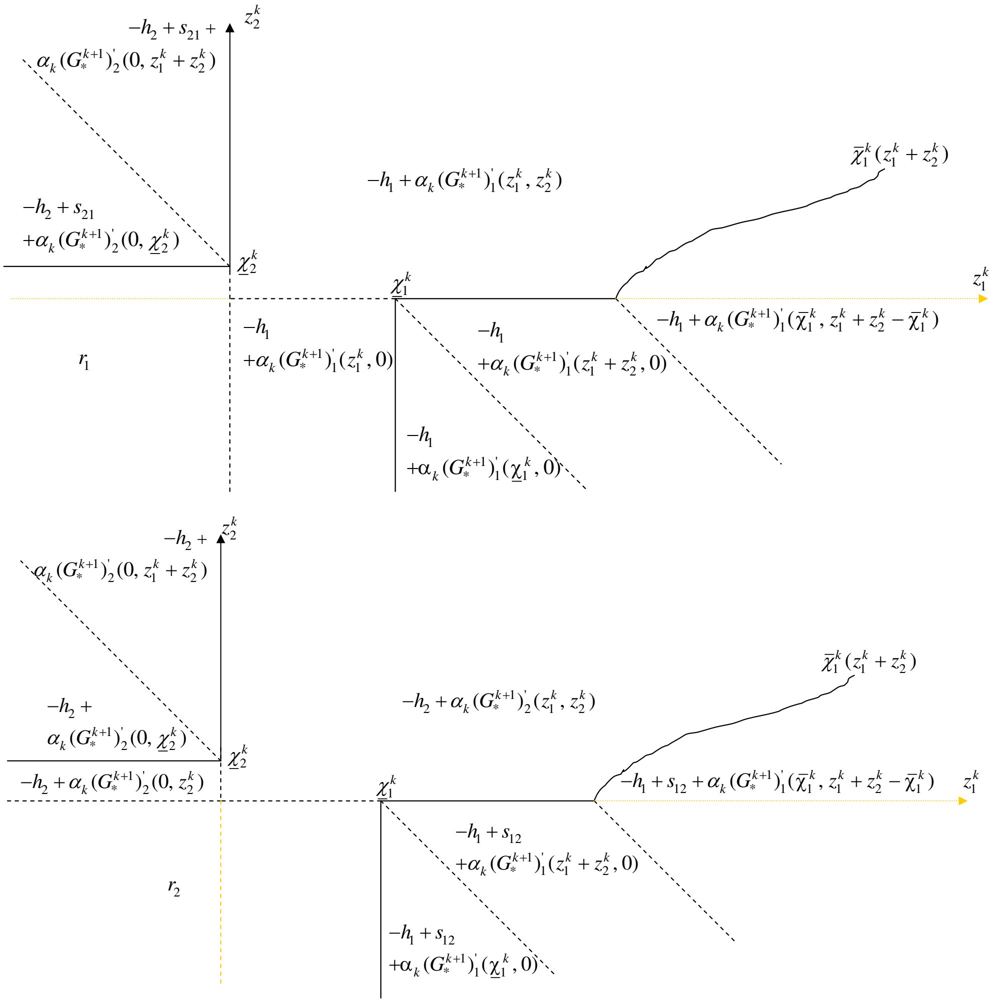

Figure 9: First derivatives of $G_v^k(\mathbf{z}^k)$ w.r.t. z_1^k (above) and w.r.t. z_2^k (below).

(b) Consider $(G_v^k)_i'(\mathbf{z}^k)$ as a function of the current-period holding costs $\mathbf{h}$ and denote it as $(Gh_v^k)_i(\mathbf{h}, \mathbf{z}^k)$. It suffices to prove that $(Gh_v^k)_1(\bar{h}_1, h_2, \mathbf{z}^k) - (Gh_v^k)_1(h_1, h_2, \mathbf{z}^k) \leq (Gh_v^k)_2(\bar{h}_1, h_2, \mathbf{z}^k) -$

$(Gh_v^k)_2(h_1, h_2, \mathbf{z}^k) \le 0$, for any $h_1 < \bar{h}_1$. From Theorem 1, $\underline{\chi}_1^k(h_1) \ge \underline{\chi}_1^k(\bar{h}_1)$, $\bar{\chi}_1^k(h_1) \ge \bar{\chi}_1^k(\bar{h}_1)$, and $\chi_2^k(h_1) = \chi_2^k(\bar{h}_1)$. When $z_1^k \le 0$, from Figure 9, $(Gh_v^k)_i(\bar{h}_1, h_2, \mathbf{z}^k) - (Gh_v^k)_i(h_1, h_2, \mathbf{z}^k) = 0$. Hence, we concentrate on $z_1^k > 0$.

We first prove $(Gh_v^k)_i(\bar{h}_1, h_2, \mathbf{z}^k) - (Gh_v^k)_i(h_1, h_2, \mathbf{z}^k) \le 0$. Figure 10 illustrates how the holding cost affects $\chi_1^k(z^k)$. In the shaded areas the expression for $(Gh_v^k)'_1(h_1, h_2, \mathbf{z}^k)$ changes when h_1 is

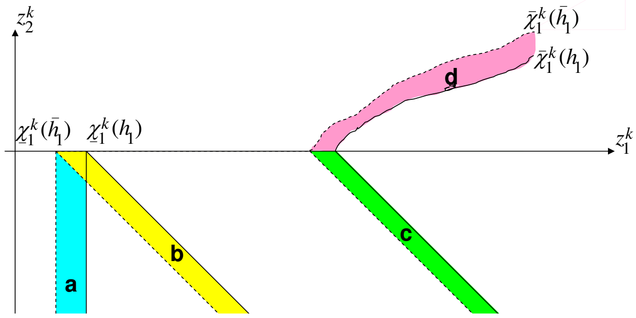

Figure 10: Effects of h_1 on $\underline{\chi}_1^k$ and $\bar{\chi}_1^k$.

replaced with $\bar{h}_1$. Hence, we prove the inequality for two cases: i) if the formulae do not change when moving from h_1 to $\bar{h}_1$ ii) if the formulae change (the shaded areas).

i) From Figure 9 (above), the two regions with
$(Gh_v^k)'_1(h_1, h_2, \mathbf{z}^k) = -h_1 + \alpha_k (G_*^{k+1})'_1(\underline{\chi}_1^k(h_1), 0)$ and
$(Gh_v^k)'_1(h_1, h_2, \mathbf{z}^k) = -h_1 + \alpha_k (G_*^{k+1})'_1(\bar{\chi}_1^k(h_1), z_1^k + z_2^k - \bar{\chi}_1^k(h_1))$
are the only ones where monotonicity w.r.t. h_1 is not trivial. For the first of them note that $-h_1 + \alpha_k (G_*^{k+1})'_1(\underline{\chi}_1^k(h_1), 0) = r_2 - s_{12}$ (= const), due to definition of $\underline{\chi}_1^k$. For the second one, from definition of $\bar{\chi}_1^k$, $-h_1 + \alpha_k (G_*^{k+1})'_1(\bar{\chi}_1^k, z_1^k + z_2^k - \bar{\chi}_1^k) = -h_2 - s_{12} + \alpha_k (G_*^{k+1})'_2(\bar{\chi}_1^k, z_1^k + z_2^k - \bar{\chi}_1^k)$. Hence, it suffices to show that $(G_*^{k+1})'_2(\bar{\chi}_1^k(h_1), z_1^k + z_2^k - \bar{\chi}_1^k(h_1))$ is non-increasing, which follows since $\bar{\chi}_1^k(h_1)$ is non-increasing and $\mathbb{A}_1^{k+1}$ implies that $(G_*^{k+1})'_2(\hat{z}_1^k, z^k - \hat{z}_1^k)$ is non-decreasing in $\hat{z}_1^k$.

ii) Now, for each of the shaded regions a, b, c, and d we justify $(Gh_v^k)_1(\bar{h}_1, h_2, \mathbf{z}^k) - (Gh_v^k)_1(h_1, h_2, \mathbf{z}^k) \le 0$. In region a, $\underline{\chi}_1^k(\bar{h}_1) \le z_1^k \le \underline{\chi}_1^k(h_1)$. Definition of $\underline{\chi}_1^k$ implies that, for $\underline{\chi}_1^k(\bar{h}_1) \le z_1^k$, $(Gh_v^k)_1(\bar{h}_1, h_2, \mathbf{z}^k) = -\bar{h}_1 + \alpha_k (G_*^{k+1})'_1(\underline{\chi}_1^k(\bar{h}_1), 0) = r_2 - s_{12}$, and for $z_1^k \le \underline{\chi}_1^k(h_1)$, $(Gh_v^k)_1(h_1, h_2, \mathbf{z}^k) = -h_1 + \alpha_k (G_*^{k+1})'_1(z_1^k, 0) \ge r_2 - s_{12}$. For the same reasons, in region b, $\underline{\chi}_1^k(\bar{h}_1) \le z_1^k + z_2^k \le \underline{\chi}_1^k(h_1)$ implies $-\bar{h}_1 + \alpha_k (G_*^{k+1})'_1(z_1^k + z_2^k, 0) \le r_2 - s_{12} = -h_1 + \alpha_k (G_*^{k+1})'_1(\underline{\chi}_1^k(h_1), 0)$. In region c,

$$\begin{aligned} -\bar{h}_1 + \alpha_k (G_*^{k+1})'_1(\bar{\chi}_1^k(\bar{h}_1), z_1^k + z_2^k - \bar{\chi}_1^k(\bar{h}_1)) &= \\ -h_2 - s_{12} + \alpha_k (G_*^{k+1})'_2(\bar{\chi}_1^k(\bar{h}_1), z_1^k + z_2^k - \bar{\chi}_1^k(\bar{h}_1)) &\le -h_2 - s_{12} + \alpha_k (G_*^{k+1})'_2(z_1^k + z_2^k, 0) \\ &\le -h_1 + \alpha_k (G_*^{k+1})'_1(z_1^k + z_2^k, 0), \end{aligned}$$

where the "=" is based on definition of $\bar{\chi}_1^k$, the following "$\le$" is due to $\bar{\chi}_1^k(\bar{h}_1) \le z_1^k + z_2^k$ and function $(G_*^{k+1})'_2(\hat{z}_1^k, z^k - \hat{z}_1^k)$ non-decreasing in $\hat{z}_1^k$, the last "$\le$" is implied by $\bar{\chi}_1^k(h_1) = z_1^k + z_2^k$. Similarly, in region d,

$$\begin{aligned} -\bar{h}_1 + \alpha_k (G_*^{k+1})'_1(\bar{\chi}_1^k(\bar{h}_1), z_1^k + z_2^k - \bar{\chi}_1^k(\bar{h}_1)) &= \\ -h_2 - s_{12} + \alpha_k (G_*^{k+1})'_2(\bar{\chi}_1^k(\bar{h}_1), z_1^k + z_2^k - \bar{\chi}_1^k(\bar{h}_1)) &\le -h_2 - s_{12} + \alpha_k (G_*^{k+1})'_2(z_1^k, z_2^k) \\ &\le -h_1 + \alpha_k (G_*^{k+1})'_1(z_1^k, z_2^k). \end{aligned}$$

The comparison between $(Gh_v^k)_2(h_1, h_2, \mathbf{z}^k)$ and $(Gh_v^k)_2(\bar{h}_1, h_2, \mathbf{z}^k)$ are very similar and omitted.

To prove $(Gh_v^k)_1(\bar{h}_1, h_2, \mathbf{z}^k) - (Gh_v^k)_1(h_1, h_2, \mathbf{z}^k) \le (Gh_v^k)_2(\bar{h}_1, h_2, \mathbf{z}^k) - (Gh_v^k)_2(h_1, h_2, \mathbf{z}^k)$, consider first the area $\{\mathbf{z}^k | \, 0 \le z_1^k \le \chi_1^k(\bar{h}_1)\}$. From Figure 9 (above), $(Gh_v^k)_1(\bar{h}_1, h_2, \mathbf{z}^k) - (Gh_v^k)_1(h_1, h_2, \mathbf{z}^k) \le 0$ and from Figure 9 (below), $(Gh_v^k)_2(\bar{h}_1, h_2, \mathbf{z}^k) - (Gh_v^k)_2(h_1, h_2, \mathbf{z}^k) = 0$. In all the remaining areas, the two differences are equal.

(c) and (d) The proofs proceed similarly to those for *(b)* and are omitted.

3. Properties of $G_v^k(\mathbf{z}^k)$'s second derivatives.

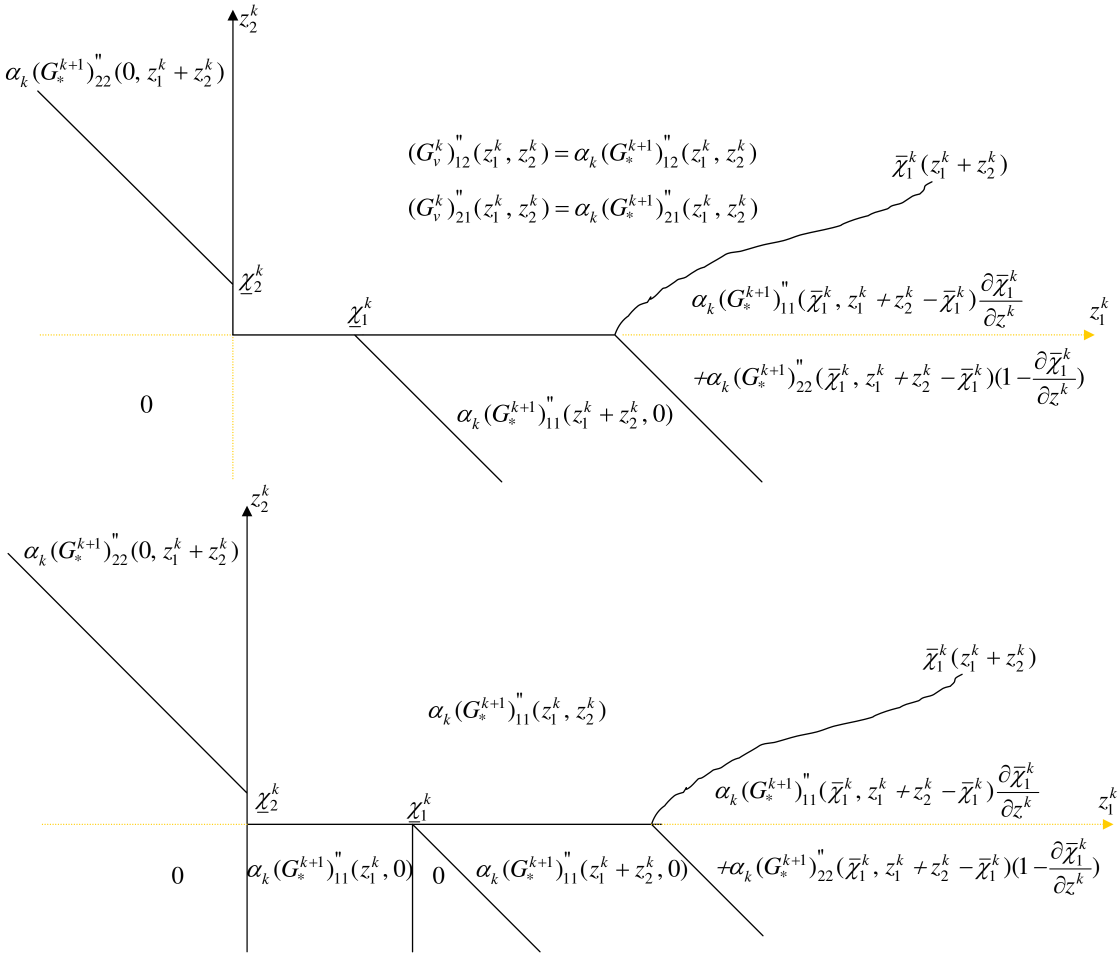

Figure 11: Second cross derivatives $(G_v^k)''_{i,3-i}(\mathbf{z}^k)$ (above) and $(G_v^k)''_{11}(\mathbf{z}^k)$ (below).

It suffices to prove (a). (b) immediately follows since $z_i^k = \bar{y}_i^k - d_i^k$.

Based on the first derivatives in Figure 9, Figure 11 (above) shows the second cross derivatives of $G_v^k(\mathbf{z}^k)$. From $\mathbb{A}_2^{k+1}$, $(G_v^k)''_{12}(\mathbf{z}^k) = (G_v^k)''_{21}(\mathbf{z}^k)$. By $\mathbb{A}_1^{k+1}$ and $\mathbb{A}_2^{k+1}$, and using $\dfrac{\partial \bar{\chi}_i^k}{\partial z^k} \ge 0$ and $1 - \dfrac{\partial \bar{\chi}_i^k}{\partial z^k} \ge 0$, $(G_v^k)''_{21}(\mathbf{z}^k) \le 0$ in part (a) immediately follows.

To prove $(G_v^k)''_{11}(\mathbf{z}^k) \le (G_v^k)''_{i,3-i}(\mathbf{z}^k)$, we need to consider $z_1^k = 0$ separately, since $(G_v^k)'_1(\mathbf{z}^k)$ is not continuous there. For $z_1^k \ne 0$, comparing $(G_v^k)''_{i,3-i}(\mathbf{z}^k)$, Figure 11 (above), with $(G_v^k)''_{11}(\mathbf{z}^k)$ Figure 11 (below), $\mathbb{A}_1^{k+1}$ implies that $(G_v^k)''_{11}(\mathbf{z}^k) \leq (G_v^k)''_{i,3-i}(\mathbf{z}^k)$ in all regions. For $z_1^k = 0$, it is sufficient to show that, for $\epsilon^- < 0 < \epsilon^+$ sufficiently close to 0, $(G_v^k)'_1(\epsilon^+, z_2^k) - (G_v^k)'_1(\epsilon^-, z_2^k) \leq (G_v^k)'_2(\epsilon^+, z_2^k) - (G_v^k)'_2(\epsilon^-, z_2^k)$. Since $(G_v^k)'_2(\mathbf{z}^k)$ is continuous in z_1^k, it suffices to show that $\lim_{\epsilon \to 0^+} (G_v^k)'_1(\epsilon, z_2^k) \leq \lim_{\epsilon \to 0^-} (G_v^k)'_1(\epsilon, z_2^k)$. For $z_2^k \leq \underline{\chi}_2^k$, from Figure 9 (above), $\lim_{\epsilon \to 0^+} (G_v^k)'_1(\epsilon, z_2^k) = -h_1 + \alpha_k (G_*^{k+1})'_1(0, (z_2^k)^+)$ and $\lim_{\epsilon \to 0^-} (G_v^k)'_1(\epsilon, z_2^k) = r_1$. $\mathbb{A}_3^{k+1}$ implies that the desired inequality holds. When $z_2^k > \underline{\chi}_2^k$, $\lim_{\epsilon \to 0^+} (G_v^k)'_1(\epsilon, z_2^k) = -h_1 + \alpha_k (G_*^{k+1})'_1(0, z_2^k)$ and $\lim_{\epsilon \to 0^-} (G_v^k)'_1(\epsilon, z_2^k) = -h_2 + s_{21} + \alpha_k (G_*^{k+1})'_2(0, z_2^k)$. Optimality of $\bar{\chi}_2^k$ implies $-h_2 + h_1 + s_{21} + \alpha_k (G_*^{k+1})'_2(z_1^k + z_2^k - \bar{\chi}_2^k, \bar{\chi}_2^k) - \alpha_k (G_*^{k+1})'_1(z_1^k + z_2^k - \bar{\chi}_2^k, \bar{\chi}_2^k) \geq 0$. Since $\bar{\chi}_2^k = z_1^k + z_2^k$, the desired inequality holds. ∎

Proposition 2 *(a) The objective function of (1) is unimodal in* $\mathbf{y}^k$. *(b)* $A^k(\mathbf{x}^k) := A1^k(\mathbf{x}^k) \cap A2^k(\mathbf{x}^k)$ *is not empty and is a subset of (1)'s maximizers. (c)* $\mathbf{y}^{*k} \in A^k(\mathbf{x}^k) \Leftrightarrow \mathbf{y}^{*k}$ *satisfies (6) to (9):*

$$\lambda_1^k(\mathbf{x}^k, \mathbf{y}^{*k}) := \mathrm{E}_{T_2^k} (G^k)'_1(y_1^{*k}, y_2^{*k} \wedge (x_2^k + T_2^k)) \leq 0 \tag{6}$$
$$\lambda_2^k(\mathbf{x}^k, \mathbf{y}^{*k}) := \mathrm{E}_{T_1^k} (G^k)'_2(y_1^{*k} \wedge (x_1^k + T_1^k), y_2^{*k}) \leq 0 \tag{7}$$
$$y_1^{*k} \geq x_1^k \text{ and } (y_1^{*k} - x_1^k)\lambda_1^k(\mathbf{x}^k, \mathbf{y}^{*k}) = 0 \tag{8}$$
$$y_2^{*k} \geq x_2^k \text{ and } (y_2^{*k} - x_2^k)\lambda_2^k(\mathbf{x}^k, \mathbf{y}^{*k}) = 0. \tag{9}$$

Proof: For the simplicity of notation, in this proof, we skip the period superscript k. Since our conclusions hold for any given $\mathbf{x}$, we also abbreviate $Gi(\mathbf{x}, \mathbf{y})$ to $Gi(\mathbf{y})$, $Ai(\mathbf{x})$ to Ai $(i = 1, 2)$, $A(\mathbf{x})$ to A, and the objective function of (1) is denoted as $\hat{G}(\mathbf{y}) = \mathrm{E}_{\mathbf{T}} G(\mathbf{y} \wedge (\mathbf{T} + \mathbf{x})) + \mathbf{cx}$.

(a) Recall that a function defined on a convex set is unimodal if its set of local maxima is convex. To prove unimodality of $\hat{G}(\mathbf{y})$, we first justify its unimodality in y_2 for given y_1. For that purpose we use $G1(\mathbf{y})$, which allows us to define the maximizer $\hat{y}_2(y_1)$. Then, to show joint unimodality, we prove that $\hat{G}(y_1, \hat{y}_2(y_1))$ is unimodal in y_1.

Clearly, $G1(\mathbf{y})$ is concave in y_2. Suppose $\hat{y}_2(y_1)$ is a maximizer of $G1(\mathbf{y})$. (Whenever the context is clear, we abbreviate $\hat{y}_2(y_1)$ to $\hat{y}_2$.) Since T_1 and T_2 are independent, for any realization t_2 of T_2, $\hat{y}_2(y_1)$ is also one of the maximizers of $G1(y_1, y_2 \wedge (t_2 + x_2))$. For $y_2 \leq \hat{y}_2$, $G1(\mathbf{y})$ is concave and non-decreasing in y_2, which implies that $G1(y_1, y_2 \wedge (t_2 + x_2))$ is also concave and non-decreasing. For $y_2 > \hat{y}_2$, $G1(\mathbf{y})$ is non-increasing implying that $G1(y_1, y_2 \wedge (t_2 + x_2))$ is also non-increasing. Thus, $\hat{G}(\mathbf{y})$ is unimodal in y_2 with the maximizer being $\hat{y}_2(y_1)$.

Suppose $\mathbf{T}$ has *pdf* (f_1, f_2) and *cdf* (F_1, F_2). From definition of $\hat{y}_2(y_1)$, $\mathrm{E}_{T_1} G'_2(y_1 \wedge (T_1 + x_1), \hat{y}_2) = 0$. Thus, $\hat{G}'_2(y_1, \hat{y}_2) = (1 - F_2(\hat{y}_2 - x_2))\mathrm{E}_{T_1} G'_2(y_1 \wedge (T_1 + x_1), \hat{y}_2) = 0$. Hence, treating $\hat{y}_2$ as an implicit function and using envelope theorem, we have $\frac{\mathrm{d}}{\mathrm{d}y_1}\hat{G}(y_1, \hat{y}_2) = \hat{G}'_1(y_1, \hat{y}_2) = (1 - F_1(y_1 - x_1))\mathrm{E}_{T_2} G'_1(y_1, \hat{y}_2 \wedge (T_2 + x_2))$. In order to show unimodality in y_1, it suffices to justify that $\mathrm{E}_{T_2} G'_1(y_1, \hat{y}_2 \wedge (T_2 + x_2))$ is non-increasing in y_1. First note that

$$\frac{\mathrm{d}}{\mathrm{d}y_1}\mathrm{E}_{T_2} G'_1(y_1, \hat{y}_2 \wedge (T_2 + x_2)) = (1 - F_2(\hat{y}_2 - x_2))(G''_{11}(y_1, \hat{y}_2) + G''_{12}(y_1, \hat{y}_2)\frac{\mathrm{d}\hat{y}_2}{\mathrm{d}y_1})$$
$$+ \int_0^{\hat{y}_2 - x_2} G''_{11}(y_1, t_2 + x_2) f_2(t_2) \mathrm{d}t_2.$$

By taking derivatives of $\hat{G}_2'(\mathbf{y})$ with respect to y_1 and y_2, we obtain:

$$\frac{\mathrm{d}\hat{y}_2}{\mathrm{d}y_1} = -\frac{\hat{G}_{21}''(y_1,y_2)}{\hat{G}_{22}''(y_1,y_2)}\Big|_{y_2=\hat{y}_2} = -\frac{(1-F_1(y_1-x_1))G_{21}''(y_1,\hat{y}_2)}{(1-F_1(y_1-x_1))G_{22}''(y_1,\hat{y}_2)+\int_0^{y_1-x_1}G_{22}''(t_1+x_1,\hat{y}_2)f_1(t_1)\mathrm{d}t_1}$$
$$\geq -\frac{G_{21}''(y_1,\hat{y}_2)}{G_{22}''(y_1,\hat{y}_2)}.$$

Since G is defined in terms of G_v, using the above inequalities, concavity of G_v, and *3(a)* of Proposition 1, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}y_1}\mathrm{E}_{T_2}G_1'(y_1,\hat{y}_2\wedge(T_2+x_2)) &\leq (1-F_2(\hat{y}_2-x_2))(G_{11}''(y_1,\hat{y}_2)+G_{12}''(y_1,\hat{y}_2)\frac{\mathrm{d}\hat{y}_2}{\mathrm{d}y_1})\\ &\leq (1-F_2(\hat{y}_2-x_2))\Big(G_{11}''(y_1,\hat{y}_2)-G_{12}''(y_1,\hat{y}_2)\frac{G_{21}''(y_1,\hat{y}_2)}{G_{22}''(y_1,\hat{y}_2)}\Big)\\ &\leq 0,\end{aligned}$$

which ends our proof of *(a)*.

(b) Suppose there exists $\mathbf{y}^* \in A1 \cap A2$, then y_{3-i}^* maximizes $Gi(\mathbf{y})$ on the set of $\mathbf{y} \geq \mathbf{x}$, given fixed value of y_i^*. Continuing with the same y_i^*, point y_{3-i}^* also maximizes $\hat{G}(\mathbf{y})$ on the set of $\mathbf{y} \geq \mathbf{x}$. Since $\hat{G}(\mathbf{y})$ is defined on the convex set and unimodal in $\mathbf{y}$, $\mathbf{y}^*$ is one of the maximizers of $\hat{G}(\mathbf{y})$ on the set of $\mathbf{y} \geq \mathbf{x}$. Hence, A is a subset of (1)'s maximizers.

Now we prove that $A \neq \emptyset$. Since $G1(\mathbf{y})$ is concave in y_2 and $\lim_{y_2\to\infty} G1(\mathbf{y}) = -\infty$ (due to point *1.* of Proposition 1), we have $A1 \neq \emptyset$. If there exists any pair $(y_1, x_2^k \vee \hat{y}_2) \in A1$ such that $\mathrm{E}_{T_2}G_1'(y_1,(x_2^k\vee\hat{y}_2)\wedge(T_2+x_2)) = 0$ (including the supergradients), then due to concavity of $G2(\mathbf{y})$ in y_1, $G2(\mathbf{y})$ is maximized at this y_1 for this given $x_2 \vee \hat{y}_2$ and $(y_1, x_2 \vee \hat{y}_2) \in A2$, which implies $A \neq \emptyset$. If, on the other hand, no $(y_1, x_2 \vee \hat{y}_2(y_1))$ makes $\mathrm{E}_{T_2}G_1'(y_1,(x_2\vee\hat{y}_2)\wedge(T_2+x_2)) = 0$, then since $\mathrm{E}_{T_2}G_1'(y_1,\hat{y}_2\wedge(T_2+x_2))$ is non-increasing in y_1 (proved in *(a)*) and $\lim_{y_1\to\infty} G1(\mathbf{y}) = -\infty$, we have that for all $y_1 \geq x_1$, $\mathrm{E}_{T_2}G_1'(y_1,(x_2\vee\hat{y}_2)\wedge(T_2+x_2)) < 0$. Correspondingly, given $y_2 = x_2 \vee \hat{y}_2(x_1)$, $G2(\mathbf{y})$ is maximized at $y_1 = x_1$. Hence, $(x_1, x_2 \vee \hat{y}_2(x_1)) \in A2$ and $A \neq \emptyset$ follows.

(c) Since $G1(\mathbf{y})$ is concave in y_2, $\hat{y}_2(y_1)$ is in the set of its maximizers if and only if $(G1)_2'(\mathbf{x}, y_1, \hat{y}_2(y_1)) = 0$, i.e., $\lambda_2(\mathbf{x}, y_1, \hat{y}_2(y_1)) = \mathrm{E}_{T_1}G_2'(y_1\wedge(x_1+T_1), \hat{y}_2(y_1)) = 0$. Hence, $A1 = \{(y_1, x_2\vee\hat{y}_2(y_1))|\lambda_2(\mathbf{x}, y_1, \hat{y}_2(y_1)) = 0, y_1 \geq x_1\}$. Let $y_2^*(y_1) = x_2\vee\hat{y}_2(y_1)$. (We omit the obvious dependence of y_2^* on $\mathbf{x}$.) Using this redefinition, we have $A1 = \{(y_1, y_2^*(y_1))|\lambda_2(\mathbf{x}, y_1, y_2^*(y_1)) \leq 0, y_2^*(y_1) \geq x_2, (y_2^*(y_1)-x_2)\lambda_2(\mathbf{x}, y_1, \hat{y}_2(y_1)) = 0, y_1 \geq x_1\}$, which is equivalent to (7) and (9), with added constraint $y_1 \geq x_1$.

Symmetrically, we define $y_1^*(y_2) = x_1 \vee \hat{y}_1(y_2)$. This implies $A2 = \{(y_1^*(y_2), y_2)|$ $\lambda_1(\mathbf{x}, y_1^*(y_2), y_2) \leq 0, y_1^*(y_2) \geq x_1, (y_1^*(y_2) - x_1)\lambda_1(\mathbf{x}, y_1^*(y_2), y_2) = 0, y_2 \geq x_2\}$, which is equivalent to (6)and (8), with extra condition $y_2 \geq x_2$. Combining these two results, $\mathbf{y}^* \in A(\mathbf{x}) \Leftrightarrow \mathbf{y}^*$ satisfies (6) to (9). ■

Proposition 3 *Consider period k. The optimal solution of (1), $\mathbf{y}^*(= \mathbf{y}^{*k})$, has the following properties $(i = 1, 2)$:*

1. $-1 \leq \dfrac{\partial y_i^*}{\partial x_i} - 1 \leq \dfrac{\partial y_i^*}{\partial x_{3-i}} \leq 0$ *and* $\dfrac{\partial y_1^*}{\partial x_i} + \dfrac{\partial y_2^*}{\partial x_i} \leq 0.$

2. *For a given* $\mathbf{x}^0$ *and the corresponding* $\mathbf{y}^{0*}(\mathbf{x}^0)$*:*

   (a) *If either*

   (i) $F_i(y_i^{0*} - x_i^0) = 0$ *(facility* i *can reach* y_i^{0*} *with probability* 1*), or*

   (ii) $y_{3-i}^{0*} = x_{3-i}^0$ *(optimal policy at* $3-i$ *is to produce nothing),*

   then for all $x_i \geq x_i^0$*, we have* $y_i^* = y_i^{0*} \vee x_i$ *(produce up to* y_i^{0*}*).*

   (b) *If* $F_{3-i}(y_{3-i}^{0*} - x_{3-i}^0) = 1$ *(facility* $3-i$*'s capacity does not exceed* $y_{3-i}^{0*} - x_{3-i}^0$*), then for all* $x_i \leq x_i^0$*, we have* $y_i^* = y_i^{0*}$ *(produce up to* y_i^{0*}*).*

3. *For a given* $\mathbf{x}^0$ *and the corresponding* $\mathbf{y}^{0*}(\mathbf{x}^0)$*:*

   If $F_i(y_i^{0*} - x_i^0) = 0$ *(case a(i) above), then for all* $x_i^0 \leq x_i \leq y_i^{0*}$*, we have* $y_{3-i}^* = y_{3-i}^{0*}$*.*

Proof: 1. We consider four cases of the value of the optimal solution $\mathbf{y}^*$, depending on whether (6) and (7) are binding and $\mathbf{y}^* = \mathbf{x}$. We assume that all the inequalities in Proposition 1 point *3* are strict, e.g., $(G_v)''_{i,3-i}(\mathbf{z}) < 0$ everywhere. For each of the statements, however, if it holds for functions satisfying strict inequality, then it also holds for the limit of such functions (which satisfies weak inequality).

A. Assume '=' holds in (6) and (7).

Note that we refer to the region in which the above equalities hold as "region A". Taking derivatives of (6) and (7) w.r.t. x_1, we get:

$$0 = \frac{\partial}{\partial x_1}\lambda_1(\mathbf{x},\mathbf{y}^*) = \mathrm{E}_{T_2} G''_{11}(y_1^*, y_2^* \wedge (T_2 + x_2))\frac{\partial y_1^*}{\partial x_1} + [1-F_2(y_2^*-x_2)]G''_{12}(\mathbf{y}^*)\frac{\partial y_2^*}{\partial x_1}$$
$$= a_{11}\frac{\partial y_1^*}{\partial x_1} + a_{12}\frac{\partial y_2^*}{\partial x_1} \tag{14}$$
$$0 = \frac{\partial}{\partial x_1}\lambda_2(\mathbf{x},\mathbf{y}^*) = [1-F_1(y_1^*-x_1)]G''_{21}(\mathbf{y}^*)\frac{\partial y_1^*}{\partial x_1} + \mathrm{E}_{T_1} G''_{22}(y_1^* \wedge (T_1 + x_1), y_2^*)\frac{\partial y_2^*}{\partial x_1}$$
$$+\int_0^{y_1^*-x_1} G''_{21}(x_1+t_1, y_2^*) f_1(t_1)\mathrm{d}t_1$$
$$= a_{21}\frac{\partial y_1^*}{\partial x_1} + a_{22}\frac{\partial y_2^*}{\partial x_1} + b_2, \tag{15}$$

where $a_{11} = \mathrm{E}_{T_2} G''_{11}(y_1^*, y_2^* \wedge (T_2 + x_2))$, $a_{22} = \mathrm{E}_{T_1} G''_{22}(y_1^* \wedge (T_1 + x_1), y_2^*)$, $a_{i,3-i} = [1-F_{3-i}(y_{3-i}^* - x_{3-i})]G''_{i,3-i}(\mathbf{y}^*)$, and $b_2 = \int_0^{y_1^*-x_1} G''_{21}(x_1+t_1, y_2^*) f_1(t_1)\mathrm{d}t_1$. Note that $G(\mathbf{y}) = \mathrm{E}_{\mathbf{D}}[G_v(\mathbf{y}-\mathbf{D}) + \mathbf{r}\mathbf{D}] - \mathbf{c}\mathbf{y}$, and from *3(a)* of Proposition 1, taking expectations with respect to T_{3-i} and $\mathbf{D}$, we have $a_{ii} < a_{i,3-i} \leq 0$ and $b_2 \leq 0$. Hence, $\frac{\partial y_1^*}{\partial x_1} = \frac{a_{12}b_2}{a_{11}a_{22} - a_{12}a_{21}} \geq 0$, $\frac{\partial y_2^*}{\partial x_1} = \frac{-a_{11}b_2}{a_{11}a_{22} - a_{12}a_{21}} \leq 0$, and $\frac{\partial y_1^*}{\partial x_1} + \frac{\partial y_2^*}{\partial x_1} \leq 0$.

By symmetry, we have $\frac{\partial y_2^*}{\partial x_2} = \frac{a_{21}b_1}{a_{11}a_{22} - a_{12}a_{21}}$ and $\frac{\partial y_1^*}{\partial x_2} = \frac{-a_{22}b_1}{a_{11}a_{22} - a_{12}a_{21}}$, where $b_1 = \int_0^{y_2^*-x_2} G''_{12}(y_1^*, x_2 + t_2) f_2(t_2)\mathrm{d}t_2$. From Proposition 1-*3(a)*, straightforward calculation shows $a_{ii} \leq a_{i,3-i} + b_i \leq 0$. Hence, $\frac{\partial y_i^*}{\partial x_{3-i}} - (\frac{\partial y_i^*}{\partial x_i} - 1) = \frac{a_{3-i,3-i}(a_{ii} - b_i) - a_{i,3-i}(a_{3-i,i} + b_{3-i})}{a_{11}a_{22} - a_{12}a_{21}} \geq 0$.

Thus, *1* holds within region A. Note that for any given x_i, as x_{3-i} decreases, $(\mathbf{x}, \mathbf{y}^*)$ remains in region A.

B. Assume '=' holds for (6) and $y_2^* = x_2$.

Within this region (region B), $\mathbf{y}^*$ is the solution of $\lambda_1(\mathbf{x}, \mathbf{y}) = 0$, $y_2^* - x_2 = 0$, and $\lambda_2(\mathbf{x}, \mathbf{y}^*) \leq 0$. Since $\lambda_1(\mathbf{x}, \mathbf{y})$ and $y_2 - x_2$ are independent of x_1, $\frac{\partial y_1^*}{\partial x_1} = \frac{\partial y_2^*}{\partial x_1} = 0$. Since $\frac{\partial y_2^*}{\partial x_2} = 1$, from the derivative of $\lambda_1(\mathbf{x}, \mathbf{y}^*)$ w.r.t. x_2: $G''_{11}(y_1^*, x_2)\frac{\partial y_1^*}{\partial x_2} + G''_{12}(y_1^*, x_2) \times 1 = 0$, and from Proposition 1 property *3(a)*, we have $\frac{\partial y_1^*}{\partial x_2} = \frac{-G''_{12}(y_1^*, x_2)}{G''_{11}(y_1^*, x_2)} \in (-1, 0)$ (recall that Proposition 1 property *3(a)* also holds for function $G(\mathbf{y})$). Hence, *1* holds within the region.

Now consider how the region shifts as x_i decreases for any given x_{3-i}. For given x_1, $-1 < \frac{\partial y_1^*}{\partial x_2} < 0$ and Proposition 1 property *3(a)* implies,

$$\frac{\partial}{\partial x_2}\lambda_2(\mathbf{x}, y_1^*, x_2) = [1 - F_1(y_1^* - x_1)]G''_{21}(y_1^*, x_2)\frac{\partial y_1^*}{\partial x_2} + E_{T_1}G''_{22}(y_1^* \wedge (x_1 + T_1), x_2) \leq 0.$$

Similarly, for given x_2, $\frac{\partial}{\partial x_1}\lambda_2(\mathbf{x}, y_1^*, x_2) \leq 0$. Hence, for given x_i, as x_{3-i} decreases, we remain in region B or move to A. Since the intersection of region A and B is not empty, consisting of the points that make both $\lambda_2(\mathbf{x}, \mathbf{y}^*) = 0$ and $y_2^* = x_2$, and the problem we optimize is unimodal, hence all of the desired inequalities also hold when crossing region A and B.

C. Assume '=' holds for (7) and $y_1^* = x_1$.

This case is symmetric to case 2: $\frac{\partial y_1^*}{\partial x_2} = \frac{\partial y_2^*}{\partial x_2} = 0$, $\frac{\partial y_1^*}{\partial x_1} = 1$, and $-1 < \frac{\partial y_2^*}{\partial x_1} < 0$.

D. Assume $y_1^* = x_1$ and $y_2^* = x_2$.

While $\mathbf{y}^*$ remains in region D, *1* straightforwardly follows. If $\mathbf{y}^*$ shift out of D, all of the inequalities hold since the intersection between D and A (or B or C) is not empty and the optimized problem itself is unimodal.

Before proving *2* and *3*, we first describe how $(\mathbf{x}, \mathbf{y}^*)$ shifts between the above four cases. For given x_2, 1) if $(\mathbf{x}, \mathbf{y}^*)$ is in the region A, then as x_1 decreases, it remains in A; 2) when x_1 is increasing, then $(\mathbf{x}, \mathbf{y}^*)$ must exit A; 3) increasing x_1 either keeps $(\mathbf{x}, \mathbf{y}^*)$ in region B or C or moves it into region D; 4) if $(\mathbf{x}, \mathbf{y}^*)$ is in region D, then as x_1 increases, it remains in D. Note that when $(\mathbf{x}, \mathbf{y}^*)$ is outside of A, both y_1^* and y_2^* are order-up-to levels. Also, due to non-empty intersections among the "neighboring" regions, $\mathbf{y}^*$ is a continuous function of $\mathbf{x}$ across the regions.

2. Due to continuity at the intersections of regions and the fact that an order-up-to policy is optimal in other regions, it is sufficient to justify the property for region A alone. Consider $(\mathbf{x}^0, \mathbf{y}^{0*})$ on region A.

Let $i = 1$. *(a)-(i)* Since $0 \leq \frac{\partial y_1^*}{\partial x_1} \leq 1$, $F_1(y_1^{0*} - x_1^0) = 0$ implies that for $x_1 \geq x_1^0$, $F_1(y_1^* - x_1) = 0$. Consider $x_1 \geq x_1^0$ such that $(\mathbf{y}^*, \mathbf{x})$ is also in region A. $F_1(y_1^* - x_1) = 0$ implies $b_2 = 0$. Since in A, $\frac{\partial y_1^*}{\partial x_1} = \frac{a_{12}b_2}{a_{11}a_{22} - a_{12}a_{21}}$, we have $\frac{\partial y_1^*}{\partial x_1} = 0$ (corresponding to an order-up-to policy.)

(a)-(ii) If $y_2^{0*} = x_2^0$, then from *1*, for $x_1 \geq x_1^0$, $y_2^* = x_2^0$. Hence, $(\mathbf{x}, \mathbf{y}^*)$ is in region B or D (where facility 1's policy is an order-up-to).

(b) If $F_2(y_2^{0*}-x_2^0)=1$, from *1*, for $x_1\le x_1^0$, we have $F_2(y_2^*-x_2^0)=1$. Clearly, $(\mathbf{x},\mathbf{y}^*)$ must be in A or C. Since $F_2(y_2^*-x_2^0)=1$ implies $a_{12}=0$, we have (in A) $\dfrac{\partial y_1^*}{\partial x_1}=0$.

3. Note that $F_1(y_1^{0*}-x_1^0)=0$ implies that for $x_1^0\le x_1\le y_1^{0*}$, $F_1(y_1^*-x_1)=0$. Consequently, $b_2=0$ and (in A) $\dfrac{\partial y_2^*}{\partial x_1}=\dfrac{-a_{11}b_2}{a_{11}a_{22}-a_{12}a_{21}}=0$. ∎

Proposition 4 *$G_*^k(\mathbf{x}^k)$ has the following properties: $(i=1,2)$*

$\mathbb{A}_1^k$: *$G_*^k(\mathbf{x}^k)$ is jointly concave in (x_1^k,x_2^k) and $(G_*^k)''_{ii}(\mathbf{x}^k)\le(G_*^k)''_{i,i-3}(\mathbf{x}^k)$;*

$\mathbb{A}_2^k$: *$G_*^k(\mathbf{x}^k)$ is submodular and $(G_*^k)''_{12}(\mathbf{x}^k)=(G_*^k)''_{21}(\mathbf{x}^k)$;*

$\mathbb{A}_3^k$: *$(G_*^k)'_i(\mathbf{x}^k)\le r_i$, for $i=1,2$.*

Proof: We again omit the period label k. Recall that $G(\mathbf{y})=\mathrm{E}_{\mathbf{D}}[G_v(\mathbf{y}-\mathbf{D})+\mathbf{rD}]-\mathbf{cy}$, and $G_*(\mathbf{x})=\max_{\mathbf{y}\ge\mathbf{x}}E_{\mathbf{T}}G(\mathbf{y}\wedge(\mathbf{x}+\mathbf{T}))+\mathbf{cx}$. By symmetry between x_1 and x_2, we only discuss the properties with respect to x_1. To prove $\mathbb{A}_3^k$, notice that 2(a) of Proposition 1 implies $G'_1(\mathbf{y})\le r_1-c_1$. The first order derivative of $G_*(\mathbf{x})$ w.r.t. x_1 is:

$$
\begin{aligned}
(G_*)'_1(x_1,x_2)=&\mathbf{1}_{\{y_1^*>x_1\}}\int_0^{y_1^*-x_1}\mathrm{E}_{T_2}G'_1(x_1+t_1,y_2^*\wedge(x_2+T_2))f_1(t_1)\mathrm{d}t_1+\mathbf{1}_{\{y_1^*=x_1\}}\mathrm{E}_{T_2}G'_1(x_1,y_2^*\wedge(x_2+T_2))+c_1\\
\le&\mathbf{1}_{\{y_1^*>x_1\}}F_1(y_1^*-x_1^k)(r_1-c_1)+\mathbf{1}_{\{y_1^*=x_1\}}(r_1-c_1)+c_1\le r_1.
\end{aligned}
$$

and $\mathbb{A}_3^k$ is proved.

When $y_1^*>x_1$, we have:

$$
\begin{aligned}
(G_*)''_{11}(x_1,x_2)=&\int_0^{y_1^*-x_1}\Big\{[1-F_2(y_2^*-x_2)][G''_{11}(x_1+t_1,y_2^*)+G''_{12}(x_1+t_1,y_2^*)\frac{\partial y_2^*}{\partial x_1}]\\
&+\int_0^{y_2^*-x_2}G''_{11}(x_1+t_1,x_2+t_2)f_2(t_2)\mathrm{d}t_2\Big\}f_1(t_1)\mathrm{d}t_1\le 0,\\
(G_*)''_{12}(x_1,x_2)=&\int_0^{y_1^*-x_1}\Big\{[1-F_2(y_2^*-x_2)]G''_{12}(x_1+t_1,y_2^*)\frac{\partial y_2^*}{\partial x_2}\\
&+\int_0^{y_2^*-x_2}G''_{12}(x_1+t_1,x_2+t_2)f_2(t_2)\mathrm{d}t_2\Big\}f_1(t_1)\mathrm{d}t_1\le 0.
\end{aligned}
$$

When $y_1^*=x_1$, we have:

$$
\begin{aligned}
(G_*)''_{11}=&[1-F_2(y_2^*-x_2)][G''_{11}(x_1,y_2^*)+G''_{12}(x_1,y_2^*)\frac{\partial y_2^*}{\partial x_1}]+\int_0^{y_2^*-x_2}G''_{11}(x_1,x_2+t_2)f_2(t_2)\mathrm{d}t_2\le 0,\\
(G_*)''_{12}=&[1-F_2(y_2^*-x_2)]G''_{12}(x_1,y_2^*)\frac{\partial y_2^*}{\partial x_2}+\int_0^{y_2^*-x_2}G''_{12}(x_1,x_2+t_2)f_2(t_2)\mathrm{d}t_2\le 0.
\end{aligned}
$$

As $G(\mathbf{y})=\mathrm{E}_{\mathbf{D}}[G_v(\mathbf{y}-\mathbf{D})+\mathbf{rD}]-\mathbf{cy}$, using Proposition 1 property *3(a)*, we have $(G_*)''_{11}(\mathbf{x})\le(G_*)''_{12}(\mathbf{x})\le 0$, which used jointly with the symmetric condition, $(G_*)''_{22}(\mathbf{x})\le(G_*)''_{21}(\mathbf{x})\le 0$, validates $\mathbb{A}_1^k$ and the first part of $\mathbb{A}_2^k$.

Furthermore, the second part of point $\mathbb{A}_2^k$, $(G_*)''_{12}(\mathbf{x})=(G_*)''_{21}(\mathbf{x})$ is trivial for $y_1^*=x_1$ and $y_2^*=x_2$.

When $y_1^* > x_1$ and $y_2^* = x_2$,

$$G''_{21}(\mathbf{x}) = [1-F_1(y_1^*-x_1)]G''_{21}(y_1^*, x_2)\frac{\partial y_1^*}{\partial x_1} + \int_0^{y_1^*-x_1} G''_{21}(x_1+t_1, x_2)f_1(t_1)\mathrm{d}t_1.$$

Since $\frac{\partial y_1^*}{\partial x_1} = 0$ and $\frac{\partial y_2^*}{\partial x_2} = 1$, we have $G''_{21}(\mathbf{x}) = G''_{12}(\mathbf{x})$.

When $y_1^* > x_1$ and $y_2^* > x_2$, it suffices to prove that

$$\int_0^{y_1^*-x_1}[1-F_2(y_2^*-x_2)]G''_{12}(x_1+t_1, y_2^*)\frac{\partial y_2^*}{\partial x_2}f_1(t_1)\mathrm{d}t_1 = \int_0^{y_2^*-x_2}[1-F_1(y_1^*-x_1)]G''_{21}(y_1^*, x_2+t_2)\frac{\partial y_1^*}{\partial x_1}f_2(t_2)\mathrm{d}t_2.$$

From (14) and (15), we know that $\frac{\partial y_i^*}{\partial x_i} = \frac{a_{i,3-i}b_{3-i}}{a_{ii}a_{3-i,3-i} - a_{i,3-i}a_{3-i,i}}$ with $a_{i,3-i} = [1-F_{3-i}(y_{3-i}^* - x_{3-i})]G''_{i,3-i}(\mathbf{y}^*)$, $b_1 = \int_0^{y_2^*-x_2} G''_{12}(y_1^*, x_2+t_2)f_2(t_2)\mathrm{d}t_2$, and $b_2 = \int_0^{y_1^*-x_1} G''_{21}(x_1+t_1, y_2^*)f_1(t_1)\mathrm{d}t_1$. Straightforward substitution shows that $(G_*)''_{12}(\mathbf{x}) = (G_*)''_{21}(\mathbf{x})$. ■

Proposition 5 *Suppose problem P_a satisfies*

$$v_{ii}(a, \mathbf{y}) \le v_{i,3-i}(a, \mathbf{y}) \le 0, \tag{10}$$

where $v_{ij}(a, \mathbf{y}) = \frac{\partial}{\partial y_j}K_i(a, \mathbf{y})$ $(i, j = 1, 2)$, and its solution set is convex. Let $v_{ij} = v_{ij}(a, \mathbf{y}^*)$ and $u_i = u_i(a, \mathbf{y}^*) = \frac{\partial}{\partial a}K_i(a, \mathbf{y})|_{\mathbf{y}=\mathbf{y}^*}$.

1. *If $K_1(a, \mathbf{y})$ is non-increasing (non-decreasing) in a and $K_2(a, \mathbf{y})$ is non-decreasing (non-increasing) in a, then $y_1^*(a)$ is non-increasing (non-decreasing) in a and $y_2^*(a)$ is non-decreasing (non-increasing) in a.*

2. *If both $K_1(a, \mathbf{y})$ and $K_2(a, \mathbf{y})$ are non-increasing (non-decreasing) in a, then $y_1^*(a) + y_2^*(a)$ is non-increasing (non-decreasing) in a. Furthermore, if there exist $\beta_i > 0$, such that (P_a) satisfies*

$$\beta_i v_{ii} \le \beta_{3-i}v_{3-i,i} \tag{11}$$

then

   (a) if $\beta_1|u_1| \ge \beta_2|u_2|$, $y_1^*(a)$ is non-increasing (non-decreasing) in a.

   (b) if $\beta_1|u_1| \le \beta_2|u_2|$, $y_2^*(a)$ is non-increasing (non-decreasing) in a.

Proof: Due to convexity of the solution set of (P_a) it is sufficient to show the property for a single point within the set (say, a boundary point). Taking derivative of (P_a) with respect to a and evaluating at $\mathbf{y}^*(a)$, we have $u_i + v_{ii}\frac{\partial y_i^*}{\partial a} + v_{i,3-i}\frac{\partial y_{3-i}^*}{\partial a} = 0$. Straightforward calculation shows that $\frac{\partial y_i^*}{\partial a} = \frac{v_{i,3-i}u_{3-i} - v_{3-i,3-i}u_i}{v_{11}v_{22} - v_{12}v_{21}}$ and $\frac{\partial y_1^*}{\partial a} + \frac{\partial y_2^*}{\partial a} = \frac{(v_{12} - v_{11})u_2 + (v_{21} - v_{22})u_1}{v_{11}v_{22} - v_{12}v_{21}}$.

1. $u_1 \leq 0$, $u_2 \geq 0$, then from (10) and from the formula for $\frac{\partial y_i^*}{\partial a}$, we have $\frac{\partial y_1^*}{\partial a} \leq 0$ and $\frac{\partial y_2^*}{\partial a} \geq 0$.

2. Consider the case where both $K_1(a, \mathbf{y})$ and $K_2(a, \mathbf{y})$ are non-increasing, i.e., $u_1, u_2 \leq 0$. From (10), $\frac{\partial y_1^*}{\partial a} + \frac{\partial y_2^*}{\partial a} \leq 0$, which proves the first part of point *2.*

(a) According to the formula for $\frac{\partial y_1^*}{\partial a}$, from (10), it suffices to prove that $v_{12}u_2 - v_{22}u_1 \leq 0$. Since $u_1, u_2 \leq 0$, $\beta_1|u_1| \geq \beta_2|u_2|$ implies $\frac{u_1}{u_2} \geq \frac{\beta_2}{\beta_1}$. From (11) ($i = 2$), $\frac{v_{22}}{v_{12}} \geq \frac{\beta_1}{\beta_2}$. Thus, $\frac{v_{22}u_1}{v_{12}u_2} \geq 1$ and (a) follows. Proof of (b) is similar. ■

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