A Decomposition Approach for a Class of Capacitated Serial Systems[1]

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Abstract

We study a class of two echelon, serial systems with identical ordering capacities or limits for both echelons. For the case where the lead time to the upstream echelon is one period, the optimality of state-dependent, modified echelon base-stock policies is proved using a decomposition approach. For the case where the upstream lead time is two periods, we introduce a new class of policies called "two-tier, base-stock policies," and prove their optimality. Some insight about the inventory control problem in N echelon, serial systems with identical capacities at all stages and arbitrary lead times everywhere is also provided. We argue that a generalization of two-tier, base-stock policies, which we call "multi-tier, base-stock policies," are optimal for these systems; we also provide a bound on the number of parameters required to specify the optimal policy.

1 Introduction

We consider a periodic review inventory control problem for a three stage supply chain consisting of one supplier, one distribution center and one retailer. The supplier is considered as being external; that is, we are interested only in optimally managing inventory at the distribution center and the retailer. Consequently, we call this a two-echelon, serial system. The supplier and the distribution center can ship up to C units in any period. The retailer is only an inventory storage stage with unlimited storage capacity. We label the supplier, distribution center and retailer as L_3, L_2, and, L_1, respectively. Inventory at L_1 is used to meet customer demand. Excess demand at L_1 is assumed to be backordered. The costs considered are linear holding costs and linear backorder costs. Customer demands are Markov modulated and lead times are deterministic.

We prove the following results: (a) the optimal inventory control problem for this system can be decomposed into C problems, each one of which represents a subsystem that consists of a two echelon serial system with unit capacity at each stage, (b) under the additional assumption that the lead time between L_3 and L_2 is one period, the optimal policy is a modified echelon base-stock policy at L_1 and L_2 and (c) when the lead time between L_3 and L_2 is two periods, the optimal policy is a "two-tier, base-stock policy" (we will define this term later) at L_1 and L_2. The decomposition technique used for these two-echelon systems also holds for N echelon, serial systems with arbitrary lead times and identical capacities everywhere. Moreover, we provide a bound on the number of parameters required to describe the optimal policies in such N echelon systems.

The approach we use is an extension of the "single-unit, single-customer" approach introduced by Axsater (1990) and subsequently used by Katircioglu and Atkins (1998) and Muharremoglu and Tsitsiklis (2003). Axsater (1990) develops a cost evaluation technique which is based on examining the costs associated with an individual unit and uses this to optimize base-stock levels for two-echelon inventory systems with one-for-one replenishment rules. He extends this technique to systems with batch-ordering in Axsater (1993). While Axsater uses this approach to evaluate costs and to find optimal parameters within the class of one-for-one replenishment policies or re-order point, re-order quantity policies, Katircioglu and Atkins (1998), Muharremoglu and Tsitsiklis (2003) and this paper are concerned with the derivation of the structure of optimal policies using the single-unit, single-customer approach. Katircioglu and Atkins (1998) study a continuous review, single-stage system with arbitrary inter-arrival distributions with increasing failure rates. Muharremoglu and Tsitsiklis (2003) study uncapacitated, serial systems under periodic-review.

Next, we briefly review the related literature. We refer the reader to Muharremoglu and Tsitsiklis (2003) and Kapuscinski and Tayur (1999) for more extensive reviews.

In their seminal paper, Clark and Scarf (1960) showed that echelon base-stock policies are optimal for uncapacitated serial systems with deterministic lead times under the assumption that demands are independent and identically distributed from period to period and procurement costs are linear. The infinite horizon extensions were achieved by Federgruen and Zipkin (1984). A key extension of this result is Chen and Song (2001), where the optimality of state dependent echelon base-stock policies is proved when the demands are driven by a Markov Chain (also known as Markov modulated demand). This result has recently been extended by Muharremoglu and Tsitsiklis (2003) to systems where lead times are stochastic and non-crossing. They allow both lead times and demands to be Markov modulated.

The optimality of modified base-stock policies for a single stage, capacitated system with deterministic lead times and stationary demand was proved by Federgruen and Zipkin (1986a) and Federgruen and Zipkin (1986b). This work was extended to the case of periodic demand processes and Markov modulated demand processes by Aviv and Federgruen (1997) and Kapuscinski and Tayur (1998), respectively. Tayur (1992) uses the "shortfall distribution", applying the theory of stochastic storage processes (see Prabhu (1998)), and provides a method to compute the optimal base-stock level for the stationary case.

The only result that has been proved about the structure of the optimal policy for a serial system with capacities is due to Parker and Kapuscinski (2004). They consider a two echelon serial system of the type we described earlier assuming the lead time between L_3 and L_2 is one period and assuming the lead time between L_2 and L_1 is an arbitrary, deterministic integer. There is a capacity of C units per period at L_3 and L_2. (Note: Their model allows for a higher capacity at L_3 than L_2; but, the optimal policy is the same as the optimal policy when the capacity at L_3 is replaced by the capacity at L_2.) They show that a modified echelon base-stock policy, specified by two parameters S_1 and S_2, is optimal for this system for both the finite and infinite horizon cases with Markov modulated demands. This policy suggests that L_1 should order up to the level S_1, if possible. L_2 should order as much as possible to raise the echelon inventory position to S_2 or enough to raise the inventory on hand at L_2 to C, whichever is smaller. We present an alternate proof of this result. A key difference between their paper and ours is that they use the dynamic programming approach to obtain their results, while we use a decomposition approach to establish ours.

Glasserman and Tayur have made significant contributions to the analysis of multi-echelon inventory systems that have capacities and that follow echelon base-stock policies. In Glasserman and Tayur (1994), they study stability conditions and long-run convergence properties. In Glasserman and Tayur (1995), they show how IPA (Infinitesimal Perturbation Analysis) can be used to find near-optimal base-stock levels. They develop simple approximations in Glasserman and Tayur (1996) to find base-stock levels.

The literature on the control of tandem queues is also related to the problem studied here. Please see Parker and Kapuscinski (2004) for a brief discussion on this connection.

The remainder of the paper is organized as follows. Section 2 describes the inventory systems to be studied in greater detail and the notation used throughout the paper. In Sections 3 and 4, we study the capacitated, two echelon, serial system when the lead time between L_3 and L_2 is one period and two periods, respectively. Specifically, we prove the optimality of modified, echelon base-stock policies (MEBS policies, in short) and two-tier, base-stock policies for these systems, respectively. Our proof methodology is based on a decomposition of a capacitated two-echelon serial system into a collection of two-echelon serial systems with unit capacities at both echelons. In Section 5, we discuss how the analysis can be extended to longer supply chains, that is, serial systems with more stages and/or longer lead times; in particular, we explain the optimality of multi-tier policies, a generalization of two-tier policies, and provide a bound on the number of tiers or parameters required to describe these policies.

2 Notation and Preliminaries

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The most general system we consider in this paper is a serial system with $N+1$ stages, L_1, L_2, …, L_{N+1}, in series where L_1 is the closest to the customers and L_{N+1} is the farthest from the customer. L_{N+1} is an external supplier with infinite supply. We are interested in determining or characterizing the structure of an optimal inventory policy for stages L_1, L_2, …, L_N. Every stage L_n, $n \in \{2, 3, \ldots, N+1\}$, has a shipping capacity of C units per period. L_1 is simply an inventory storage stage that serves the customers and has infinite storage capacity. The amount ordered by L_n, $n \in \{1, 2, \ldots, N\}$, in period t is shipped by L_{n+1} in the same period and this inventory reaches L_n after l_n periods, the lead time for stage L_n. L_n orders q_{nt} units from L_{n+1} in period t only if there are at least q_{nt} units available to be shipped by L_{n+1} in that period and q_{nt} is no larger than the capacity C. We refer to L_2, L_3,

[3] At the end of the paper, individual glossaries of notation are provided in tabular format for the main sections of the paper.

..., L_{N+1} as "physical stages".

We initially assume the planning horizon consists of T periods, numbered $t = 1, 2, \ldots, T$ in that order. In Section 3.4, we examine the infinite horizon case.

We assume that there is an exogenous, finite-state, ergodic Markov Chain $\{s_t\}$ that governs the demand process. s_t is observed at the beginning of each period t. Ω is the sample space of s_t. The transition probabilities for the Markov Chain $\{s_t\}$ are assumed to be known. Furthermore, given s_t, the probability distribution of d_t, the demand in period t, is known. Demand is assumed to be an integer.

2.1 Customers and Distances

Our analysis is motivated by the "single-unit, single-customer" approach. In this and in the following sub-section, we introduce the concepts of customers and units, and also the associated concepts of distances and locations that are the basis for our analysis. This construction is identical to the one presented in Muharremoglu and Tsitsiklis (2003).

We consider each unit of demand as an individual customer. Suppose at the beginning of period 1 there are v_0 customers waiting to have their demand satisfied. We index these customers 1, 2, ..., v_0 in any order. All subsequent customers are indexed $v_0 + 1$, $v_0 + 2$, ... in the order of the period of their arrivals, arbitrarily breaking ties among customers that arrive in the same period.

Next, we define the concept of *the distance of a customer* at the beginning of any period. (See Figure 1.) Every customer who has been served is at distance 0; every customer who has arrived, placed an actual order, but who has not yet received inventory, is at distance 1; all customers arriving in subsequent periods are said to be at distances 2, 3, ..., corresponding to the sequence in which they will arrive. Distances are assigned to customers that arrive in the same period in the same order as their indices. This ensures that customers with higher indices are always at "higher" distances.

We assume that there is a backorder cost of b associated with every unit backordered at the end of a period, that is, every unit at distance 1.

2.2 Units and Locations

Next, we discuss the concepts of "units" and "locations". Inventory is considered to be discrete throughout this paper and every unit of inventory is referred to as "unit". If a unit has been used to satisfy a customer's order, the unit is in location 0. If it is part of the inventory on hand at L_1, it is said to be in location 1. If it has been shipped by L_{n+1} (in other words, ordered by L_n) t periods ago ($1 \leq t \leq l_n$), it is said to be in location $1 + l_1 + \ldots + (l_n - t)$. If the unit is waiting at L_{n+1}, it is said to be in location $1 + l_1 + l_2 + \ldots + l_n$. *For compactness, let us denote* $1 + l_1 + l_2 + \ldots + l_{n-1}$ *by* M_n*; that is,* M_n *is the location of stage* L_n. Thus, there are $2 + \sum_{n=1}^{N} l_n$ possible "locations" at which a unit can exist in a N echelon serial system. (See Figure 1 for an example of a two echelon serial system.) Observe that there are $l_{n-1} - 1$ locations in the pipeline between L_n and L_{n-1}; therefore, if $l_2 = 1$, L_3 and L_2 will be adjacent to each other.

At the beginning of period 1, we assign an index to all units in a serial manner, starting with units at location 1, then location 2, ..., location M_{N+1}, and arbitrarily assign an order to units present at the same location. We assume a countably infinite number of units is available at the supplier, that is, location M_{N+1}, at the beginning of period 1.

There is an echelon holding cost h_n associated with each unit of inventory downstream of L_{n+1} at the end of a period.

2.3 Sequence of Events

We now define the sequence of events in a period. We will use j and k to denote the indices of both units and customers. We define z_{jt} to be the location of unit j and y_{jt} to be the distance of customer j at the beginning of period t.

Let $\mathcal{S}$ refer to the entire system with all the units and all the customers and the capacity constraint of C units per period at stages 2, 3, $\ldots$, $N+1$. The state of the system at the beginning of period t is given by the vector $\mathbf{x}_t = (s_t, (z_{1t}, y_{1t}), (z_{2t}, y_{2t}), \ldots)$. Let Z_{nt} be the amount of inventory on hand at L_n at the beginning of period t. That is, $Z_{nt} = |\{j : z_{jt} = M_n\}|$. The number of backorders at the start of period t is $|\{k : y_{kt} = 1\}|$.

Next, we explain the sequence of events in period t. (Though redundant at this point, we repeat the phrase "in $\mathcal{S}$" for the sake of conciseness later in the paper.)
(1) $\mathbf{x}_t$ is observed. **(2)** Next, L_1 places an order for q_{1t} units from L_2, where $0 \leq q_{1t} \leq \min(Z_{2t}, C)$, and integer. All units in any of the locations M_1+1, M_1+2, $\ldots$, M_2-1 move to the next location. The q_{1t} units move from location M_2 to location M_2-1. Then, L_2 places an order for q_{2t} units from L_3, where $0 \leq q_{2t} \leq \min(Z_{3t}, C)$, and integer. All units in any of the locations M_2+1, M_2+2, $\ldots$, M_3-1 move to the next location. The q_{2t} units move from location M_3 to location M_3-1. This process continues sequentially until L_N places an order on the external supplier L_{N+1} for q_{Nt} units and the corresponding movement of units occurs, where q_{Nt} is constrained only by C, since L_{N+1} is assumed to carry infinite inventory. The ordering decisions can formally be represented as follows: for each $n \geq 2$, $u_{jt} \in \{0,1\}$ is decided for all $j \in \mathcal{S}$ [4] such that $z_{jt} = M_n$. Unit j is ordered (we will use "released" from stage n and "ordered" by stage $n-1$ interchangeably since they mean the same action) *if and only if* $u_{jt} = 1$. We will use *hold* to refer to the action of not releasing

[4] By $j \in \mathcal{S}$, we refer to any unit that belongs to $\mathcal{S}$. This becomes more relevant when we define a subsystem, $\mathcal{S}_w$ in the following section and say $j \in \mathcal{S}_w$.

a unit. The number of units released from L_{n+1} $(n \geq 1)$ is

$$q_{nt} \;=\; \sum_{j \in \mathcal{S}: z_{jt} = M_{n+1}} u_{jt} \;.$$

For *capacity feasibility*, q_{nt} is C or less. **(3)** Demand d_t is realized. That is, customers in $\mathcal{S}$ at distances 2, 3, ..., $2 + d_t - 1$ all arrive and are by definition at distance 1. Customers in $\mathcal{S}$ currently at distances $2 + d_t$, $3 + d_t$, ... move d_t steps towards distance 1. **(4)** Units on-hand, at stage L_1, in $\mathcal{S}$ are matched with waiting customers in $\mathcal{S}$ to the extent possible. That is, as many waiting customers are satisfied as possible and as many units on hand are consumed as possible. *Without loss of generality, we assume that units and customers in $\mathcal{S}$ at location 1 and distance 1, respectively, are matched in a first-come, first-serve order based on their indices, starting from the lowest index.* Let E'_{nt} be the echelon-n inventory position at this point in time. That is,

$$E'_{nt} \;=\; |\{j : 1 \leq z'_{jt} \leq M_{n+1} - 1\}| \;-\; |\{k : y'_{kt} = 1\}| \;,$$

where z'_{jt} and y'_{jt} denote the location of unit j and the distance of customer j at the end of period t, respectively. **(5)** h_n dollars are charged per unit of inventory downstream of stage L_{n+1} in $\mathcal{S}$ and b dollars are charged per waiting customer (at distance 1) in $\mathcal{S}$. The cost incurred in period t can be written as

$$\sum_{n=1}^{N} h_n \cdot E'_{nt} \;+\; (b \;+\; \sum_{n=1}^{N} h_n) \cdot |\{k : y'_{kt} = 1\}| \;.$$

Note: Though we have not mentioned purchase costs or transportation costs in the model, linear purchase or transportation costs payable at the time of receipt of inventory can easily be accommodated. (See Janakiraman and Muckstadt (2004) for a general discussion.)

The performance measure under consideration is the expected sum (discounted or undiscounted) of costs over the T period planning horizon. A set of mappings, one for every t, from $\mathbf{x}_t$ to (u_{jt}) is called a *policy*. A *feasible policy* is one that satisfies the constraints $q_{nt} \;\leq\; \min(Z_{(n+1)t}, C)$ for all $n \leq N$, t and $\mathbf{x}_t$. A *monotone policy* is one that satisfies the constraint $u_{jt} \geq u_{(j+1)t}$ for all j, t and $\mathbf{x}_t$ such that $z_{jt} = z_{(j+1)t}$. That is, a monotone policy always releases a lower indexed unit no later than a higher indexed unit. Similarly, we define a *monotone state* to be one where lower indexed units are in the same or lower indexed locations. That is, $z_{kt} \leq z_{jt}$ if $k \leq j$. Next we state a lemma with some facts about monotone policies. The proofs are trivial and hence omitted.

Lemma 1 *(i) For every feasible policy, we can construct a monotone, feasible policy that incurs the same cost in every period along every sample path. Consequently, the class of monotone policies contains an optimal policy. (ii) When a monotone policy is used in every period, no unit other than j can satisfy customer j's demand since customer demands are satisfied based on the indices. Thus, unit j and customer j are matched when monotone policies are used. (iii) When a monotone policy is used in every period, $\mathbf{x}_t$ is a monotone state for all t.*

From now on, our attention is restricted to monotone states when analyzing the system $\mathcal{S}$ without any loss of generality.

We are now ready to derive the optimal policy in two-echelon serial systems. The following preliminary lemma bounds the amount of inventory between consecutive stages when an optimal policy is followed and is important for our analysis.

Proposition 1 *(Parker and Kapuscinski (2004): Corollary 1(b) and Remark 2)*
Let E_{nt} be the echelon n inventory position at the start of period t. Assume that $E_{n1} - E_{(n-1)1} \leq l_n \cdot C$. Then, any policy that leads to a state where $E_{nt} - E_{(n-1)t} > l_n \cdot C$ in some period, t, is not optimal.

The proof, which is omitted, is a consequence of the fact that at most $l_n \cdot C$ units can be processed by L_n within the next lead time number of periods; if $E_{nt} - E_{(n-1)t}$ exceeds this quantity, the holding cost can be reduced without increasing the backorder cost by reducing $E_{nt} - E_{(n-1)t}$.

Assumption 1 *We assume throughout that* $E_{n1} - E_{(n-1)1} \leq l_n \cdot C$.

Consequently, when an optimal policy is followed in periods $1, 2, \ldots, t-1$, we know that the condition $E_{nt} - E_{(n-1)t} \leq l_n \cdot C$ will be satisfied.

Before proceeding further, we introduce a useful definition.

Definition 1 *For all* n, T_n *is the smallest positive integer such that when there are at least* T_n *periods left in the horizon, it is optimal to have a non-negative echelon* n *inventory position after ordering, if possible.*

In other words, it is optimal to release enough inventory into the pipeline below stage $n+1$ to meet all existing backorders at stage 1, if possible, if there are T_n or more periods remaining in the planning horizon. If the number of periods remaining in the horizon is less than T_n, it is optimal NOT to release any more units from stage $n+1$. Mathematically, T_n is the smallest positive integer such that the discounted cost of backordering a customer and holding the unit at L_{n+1} for T_n periods exceeds the discounted holding costs accumulated by a unit from the period it is released from L_{n+1} until the period it is received by L_1 and the backorder costs incurred by the customer during that time, assuming that the unit is released from every intermediate stage as soon as it is received. For example, T_1 is the smallest positive integer T such that $(b + h_2)(1 + \alpha + \alpha^2 + \ldots + \alpha^T) > (b + h_1 + h_2)(1 + \alpha + \alpha^2 + \ldots + \alpha^{l_1 - 2})$, where α is the discount factor. T_n can be computed in a similar way using b, h_1, h_2, $\ldots$, h_n.

Next we state a simple property of the sequence $\{T_n\}$; the proof is straight forward and hence omitted.

Proposition 2 *There exists* $\alpha_0 < 1$ *such that the sequence* $\{T_n\}$ *increases in* n *for all* $\alpha \geq \alpha_0$.

Assumption 2 *We assume throughout that* $\alpha \geq \alpha_0$. *In other words, we assume the sequence* $\{T_n\}$ *increases in* n.

This assumption is necessary only for the finite horizon results; even there, the purpose of the assumption is to ensure that a separate and elaborate analysis is not required for periods close to the end of the horizon.

3 Two Echelon Serial Systems with a One Period Upstream Leadtime

In this section, we examine in detail a two echelon serial system (see Figure 2) where l_2, the lead time between L_3 and L_2, is exactly one period. We study the optimal policy structure for such systems using a decomposition approach.

Parker and Kapuscinski (2004) prove the optimality of "modified echelon base-stock policies" for this system. A modified echelon base-stock policy has the following structure. In period t, echelon 1 raises its inventory position to a target level, $S_1(t, s_t)$, if sufficient capacity and inventory are available. If not, the inventory position is raised to the maximum possible level. Furthermore, L_2 should order enough to raise its (i.e., echelon 2) inventory position to $S_2(t, s_t)$, or enough to raise the inventory on hand at L_2 to C, whichever is smaller. In this section, we will provide an alternate proof of this result.

Our proof of this result has the following key steps. We first show that the system can be decomposed into C two-echelon subsystems, each having unit capacity. Subsequently, we prove that each subsystem can be managed optimally by using a "critical distance" policy at each echelon. We also prove that when the same "critical distance" policy is used to manage each subsystem, the original system follows a modified echelon base-stock policy.

Note: Throughout this section, we will assume that the number of units at stage 2, that is L_2, is less than or equal to C at the start of period t. Proposition 1 justifies this assumption. This is identical to the assumption of "being within the band" in Parker and Kapuscinski

(2004).

We proceed to discuss how the system under consideration can be decomposed into C subsystems of unit capacity.

3.1 Decomposition into Unit Capacity Subsystems

We start by defining a subsystem.

Definition 2 *Subsystem w, represented by $\mathcal{S}_w$, $1 \leq w \leq C$, refers to the subset of unit-customer pairs with indices w, $w+C$, $w+2C$, $\ldots$. Each subsystem has a unit capacity at stages L_3 and L_2.*

The intuitive reason for defining a subsystem in this way is the fact that when a monotone policy is used in $\mathcal{S}$, unit j can be affected by the *capacity constraint* at stage $L_3(L_2)$ in a period if unit $j-C$ has still not been released from stage $L_3(L_2)$. This provides a natural connection between unit j and unit $j-C$ for any j.

The sequence of events in $\mathcal{S}$ are steps **(1)**-**(5)** of Section 2.3 as applied to a two echelon system with $l_2 = 1$. The sequence of events in $\mathcal{S}_w$ are the same with the additional modifications: $\mathcal{S}$ is replaced by $\mathcal{S}_w$ and C is replaced by 1. Note that we still assume that $\mathbf{x}_t$, the information about the entire system $\mathcal{S}$, is available when managing $\mathcal{S}_w$.

For subsystem $\mathcal{S}_w$, a policy is *monotone* if unit j is released no later than unit $j+C$ from stages L_3 and L_2 for any unit j in $\mathcal{S}_w$. Note that the class of monotone policies is optimal to each subsystem $\mathcal{S}_w$ and these policies ensure that unit j is matched with customer j.

We now claim that the subsystems can be optimally managed separately even though the demand processes of different subsystems are *not stochastically independent* and that these policies, when combined, form an optimal policy for $\mathcal{S}$. Let us first define $\mathbf{x}_t^w =_{def}$

$(s_t, (z_{wt}, y_{wt}), (z_{(w+C)t}, y_{(w+C)t}), \ldots)$, that is, the information in $\mathbf{x}_t$ that pertains to $\mathcal{S}_w$.

Theorem 2 *For any monotone state $\mathbf{x}_t$, the optimal expected discounted (undiscounted) cost in periods t, $t+1$, $\ldots$ T for system $\mathcal{S}$ equals the optimal expected discounted (undiscounted) cost in periods t, $t+1$, $\ldots$ T for the group of subsystems $\{\mathcal{S}_w\}$. $\mathcal{S}_w$ can be optimally managed using $\mathbf{x}_t^w$ instead of $\mathbf{x}_t$. Furthermore, when each $\mathcal{S}_w$ is managed optimally using $\mathbf{x}_t^w$ in periods t, $t+1$, $\ldots$ T, the resulting policy is optimal for the entire system, $\mathcal{S}$.*

Proof : A *feasible* policy for subsystem $\mathcal{S}_w$ can be constructed from any *feasible, monotone* policy in $\mathcal{S}$ by implementing the (u_{jt}) actions suggested by the latter policy on the elements of $\mathcal{S}_w$. Similarly, a *feasible* policy for $\mathcal{S}$ can be constructed from any set of *feasible* policies for $\{\mathcal{S}_w\}$ by combining these policies as follows: for every unit $j \in \mathcal{S}$ implement the u_{jt} action suggested by the policy for the subsystem to which j belongs. Furthermore, note that the cost incurred by $\mathcal{S}$ in any period is the sum of the costs incurred by the units and customers belonging to the C subsystems. Combining these three observations with the optimality of the class of *monotone* policies in $\mathcal{S}$ proves the first statement.

Next, notice that the cost incurred in $\mathcal{S}_w$ in period t depends only on $\mathbf{x}_t^w$, and the probabilities necessary to describe the transition from a state $\mathbf{x}_t^w$ to $\mathbf{x}_{t+1}^w$ depend only on the actions in $\mathcal{S}_w$ and the information in $\mathbf{x}_t^w$. This proves the second statement.

The last statement in the theorem is a direct consequence of the first two statements. $\square$

Note: Theorem 2 and the proof hold for serial systems with deterministic lead times and an arbitrary number of stages as long as the capacities are identical.

3.2 Analysis of Subsystem $\mathcal{S}_w$

We will now show the existence of an optimal policy with a special structure for every subsystem.

Before examining an individual subsystem, we first observe that all subsystems are identical in the sense that (i) they have identical cost structures and (ii) given a state $\mathbf{x}_t^w$ and a fixed operating policy for a subsystem, the stochastic evolution of the subsystem is independent of the index w. Consequently, the optimal policy(ies) is(are) identical across all subsystems.

Next, we develop some necessary preliminaries about optimal policies for the subsystems by examining a subsystem $\mathcal{S}_w$. We consider only the class of monotone policies for the subsystem, which contains at least one optimal policy. Let us assume that we have used such an optimal policy in periods 1, 2, ..., $t-1$. Therefore, in any period t, the state $\mathbf{x}_t^w$ is monotone. That is, $z_{wt} \leq z_{(w+C)t} \leq \ldots$. Therefore, the units in location $2 + l_1$, that is, L_3, are indexed in a serial manner with consecutive indices differing by C. Let j_{wt} be the lowest such index, that is, unit j_{wt} is the candidate for being released from L_3 in period t in subsystem w. There are two possibilities regarding L_2: either unit $j_{wt} - C$ is present at L_2 or L_2 is empty. L_2 cannot contain more than one unit because both stages have a unit capacity and consequently, it is never optimal to have more than one unit at stage 2 (see Proposition 1).

Recall that in every period, we make the stage 2 decision before the stage 3 decision. If $\mathbf{x}_t^w$ is such that unit $j_{wt} - C$ is present at stage 2, that is, at location $1 + l_1$, then a *Release/Hold* decision has to be made for that unit at stage 2. We define $U_{2t}^*(\mathbf{x}_t^w) \subseteq \{1, 0\}$ to be the set of optimal stage 2 decisions at time t, where 1 refers to ordering/releasing the unit and 0 refers to holding the unit. If state $\mathbf{x}_t^w$ is such that there is no unit at stage 2, there is no decision to take at stage 2 and consequently, $U_{2t}^*(\mathbf{x}_t^w) = \emptyset$. Similarly, $U_{3t}^*(\tilde{\mathbf{x}}_t^w) \subseteq \{1, 0\}$ is the set of optimal stage 3 decisions for subsystem w in period t, where $\tilde{\mathbf{x}}_t^w$ is the state of subsystem w after stage 2 has taken its Release/Hold decision. That is, if the stage 2 decision were to release a unit, then we are examining the subsystem after the unit has been released from stage 2. For example, if $U_{3t}^*(\tilde{\mathbf{x}}_t^w) = \{1\}$ and subsystem w is in state $\tilde{\mathbf{x}}_t^w$ at time t after the stage 2 decision, then it is optimal to release unit j_{wt} from location $2 + l_1$, that is, L_3, and suboptimal to hold it there.

3.2.1 Sufficient Information Vectors

Let us now examine the information that is actually required to manage subsystem w using an optimal, monotone policy. Consider a given t, s_t and j_{wt}. Through a sequence of incremental observations, we will show that the information required to optimally manage $\mathcal{S}_w$ is much smaller than $\mathbf{x}_t^w$.

Observation 1 $(s_t, j_{wt}, z_{j_{wt}-C,t}, y_{j_{wt}-C,t}, y_{j_{wt},t}, y_{j_{wt}+C,t}, \ldots)$ *is a sufficient information vector for optimally managing $\mathcal{S}_w$ from period t.*

Proof: Since there is at most one unit at stage 2, monotonicity implies that all units indexed below $j_{wt} - C$ in subsystem w have already been released from location $1 + l_1$ (stage 2). Consequently, the expected costs associated with all these units and the corresponding customers are sunk; that is, these costs are the same for all policies from period t onward. Therefore, having information about the locations(distances) of units(customers) in subsystem w with indices below $j_{wt} - C$ is unnecessary. Furthermore, we know that the location of all units with indices higher than j_{wt} is $2 + l_1$. □

That is, the knowledge of the location of unit $j_{wt} - C$ and the distances of all customers in w with indices $j_{wt} - C$ and higher is sufficient for this subsystem. Even this information turns out to be more than needed, as we will see next.

Since unit j_{wt} is still at location $2 + l_1$, $y_{j_{wt},t}$ cannot be 0. Assume $y_{j_{wt},t} > 1$, that is, customer j_{wt} has not yet arrived.

Observation 2 *If* $y_{j_{wt},t} > 1$, $(s_t, j_{wt}, y_{j_{wt},t}, z_{j_{wt}-C,t})$ *is sufficient to manage* $\mathcal{S}_w$ *optimally from period* t.

Proof: $y_{j_{wt},t} > 1$ implies that all customers with indices higher than j_{wt} have also not arrived and that the subsequent customer in w is at distance $y_{j_{wt},t} + C$, the next one at $y_{j_{wt},t} + 2C$ and so on. Also, this means that customer $j_{wt} - C$ is at distance $\max(y_{j_{wt},t} - C, 1)$ or 0. If unit $j_{wt} - C$ is in location $1 + l_1$ (stage 2), then the distance of this customer cannot be zero and is therefore $\max(y_{j_{wt},t} - C, 1)$. If unit $j_{wt} - C$ is downstream of stage 2, the cost associated with the unit-customer pair $j_{wt} - C$ is sunk and the distance of customer $j_{wt} - C$ is not required for the decision in this period. □

Let us now assume that $y_{j_{wt},t} = 1$.

Observation 3 *If* $y_{j_{wt},t} = 1$, $(s_t, j_{wt}, y_{j_{wt},t}, z_{j_{wt}-C,t})$ *is sufficient to optimally manage* $\mathcal{S}_w$ *from period* t.

Proof: In this case, customers j_{wt} and $j_{wt} - C$ have arrived and it is not known whether some subsequent customers have also arrived. However, since customer j_{wt} has arrived, it is optimal to release unit j_{wt} from stage 3 if and only if $T - t \geq T_2$ (see Definition 1 for the meaning of T_2) and release unit $j_{wt} - C$ from stage 2, if it is located there, if and only if $T - t \geq T_1$. Consequently, any information about other customer distances is unnecessary. □

The two observations above show that $y_{j_{wt},t}$ alone provides us with sufficient information about all customer distances from the point of view of finding the optimal decisions.

Let us now define $i_{wt} \in \{0, 1\}$ to be an indicator of whether unit $j_{wt} - C$ is located at stage 2 or not. In other words, i_{wt} is the indicator of whether stage 2 is empty or not.

We are now able to compress the information requirement even further.

Lemma 3 *$(s_t, y_{j_{wt},t}, i_{wt})$ is a sufficient information vector for managing $\mathcal{S}_w$ optimally from period t onwards.*

Proof: If $i_{wt} = 0$, it means unit $j_{wt} - C$ has already departed stage 2 and the cost associated with that unit-customer pair is sunk; so, in this case $z_{j_{wt}-C,t}$ does not provide any additional information for managing $\mathcal{S}_w$. If $i_{wt} = 1$, then $z_{j_{wt}-C,t}$ is immediately known to be $1 + l_1$ (stage 2). So, all useful information about $z_{j_{wt}-C,t}$ is obtained from i_{wt} itself.

It is now clear that $(s_t, j_{wt}, y_{wt}, i_{wt})$ is a minimally sufficient information vector to optimally manage subsystem w from period t using a monotone policy. Furthermore, since all subsytems and units are identical, w (a subsystem index) and j_{wt} (a unit index) do not provide useful information for decision making purposes; so, we can use a more compact information vector (s_t, y, i) where $y = y_{j_{wt},t}$ and $i = i_{wt}$. $\square$

3.2.2 Optimal Policy for $\mathcal{S}_w$

We define $R^*_{2t}(s_t, y, i) \subseteq \{1, 0\}$ as the set of optimal stage 2 decisions at time t if the state of the exogenous Markov Chain is s_t and if $y_{j_{wt},t}$ is y and i_{wt} is i. $R^*_{2t}(s_t, y, 0) = \emptyset$ since there is no decision to take at stage 2 if i is zero. Similarly, let $\tilde{i}_{wt} \in \{0, 1\}$ be an indicator of whether unit $j_{wt} - C$ is located at stage 2 or not after the stage 2 decision. $R^*_{3t}(s_t, y, \tilde{i}) \subseteq \{1, 0\}$ is the set of optimal stage 3 decisions at time t if the state of the exogenous Markov Chain is s_t and if $y_{j_{wt},t}$ is y and $\tilde{i}_{wt}$ is $\tilde{i}$. Proposition 1 implies that $R^*_{3t}(s_t, y, 1)$ is $\{0\}$. That is, if a unit is present at L_2 in a subsystem, it will not be optimal for L_3 to release a unit.

Next, we show that there is a "critical distance" policy that is optimal for a subsystem. We need the following Lemma to prove this fact. The lemma states that if it is (uniquely) optimal for subsystem w to release unit $j_{wt} - C$ from L_2 in period t when the system is in the Markovian-state s_t and customer j_{wt} is at a distance $y + 1$, then it would be (uniquely) optimal to release it if the customer were any closer. An equivalent claim can be made about releasing unit j_{wt} from L_3.

Lemma 4 *$R^*_{3t}(s_t, y+1, 0) = (\supseteq)\ \{1\}$, for some $y > 1$, implies that $R^*_{3t}(s_t, y, 0) = (\supseteq)\ \{1\}$. Also, $R^*_{2t}(s_t, y+1, 1) = (\supseteq)\ \{1\}$ for some $y > C+1$, implies that $R^*_{2t}(s_t, y, 1) = (\supseteq)\ \{1\}$.*

Proof: Consider the statement $R^*_{3t}(s_t, y{+}1, 0) = \{1\}$, for some $y > 1$, implies that $R^*_{3t}(s_t, y, 0) = \{1\}$. We will prove this by contradiction.

Assume $\exists\ y > 1$ such that $R^*_{3t}(s_t, y+1, 0) = \{1\}$, and $\{0\} \in R^*_{3t}(s_t, y, 0)$. Consider two unit capacity subsystems, $\mathcal{S}_1$ and $\mathcal{S}_2$ at the time of making the stage 3 release decision. Assume that $\tilde{i}_{1t} = \tilde{i}_{2t} = 0$, $y_{j_{1t},t} = y$, $y_{j_{2t},t} = y+1$, $j_{1t} = j$ and $j_{2t} = j+1$. Let $\mathcal{S}_1$ and $\mathcal{S}_2$ follow some optimal policies, say Π_1 and Π_2, respectively. In particular, in period t, Π_1 holds unit j_{1t} at L_3 in $\mathcal{S}_1$ and Π_2 releases unit j_{2t} from L_3 in $\mathcal{S}_2$. By our assumption on R^*_{3t}, we know that the decision in $\mathcal{S}_1$ is *optimal* and the decision in $\mathcal{S}_2$ is *strictly optimal*.

Now consider the *combined system* $\mathcal{S}_1 \bigcup \mathcal{S}_2$, that contains the units and customers belonging to both $\mathcal{S}_1$ and $\mathcal{S}_2$. (Please see Figure 2.) This new system has a capacity of two units in each period.

Let us now construct a policy for the *combined system* $\mathcal{S}_1 \bigcup \mathcal{S}_2$. In any period from t onwards, $\mathcal{S}_1 \bigcup \mathcal{S}_2$ releases the same number of units from $L_3(L_2)$ as the number of units released by $\mathcal{S}_1$ plus the number of units released in $\mathcal{S}_2$ from $L_3(L_2)$. Furthermore, units in $\mathcal{S}_1 \bigcup \mathcal{S}_2$ are always released in a monotone fashion. Let us refer to this policy as $\tilde{\Pi}$ and its applications to $\mathcal{S}_1$ and $\mathcal{S}_2$ as $\tilde{\Pi}_1$ and $\tilde{\Pi}_2$, respectively.

The construction of $\tilde{\Pi}$ and the fact that customers arrive in the order of their indices imply that the cost incurred by $\mathcal{S}_1 \bigcup \mathcal{S}_2$ in any period under $\tilde{\Pi}$ is no larger than the sum of the costs incurred by $\mathcal{S}_1$ under Π_1 and $\mathcal{S}_2$ under Π_2 in that period, with probability 1. Also, notice that the cost of $\mathcal{S}_1 \bigcup \mathcal{S}_2$ under $\tilde{\Pi}$ equals the sum of the costs for $\mathcal{S}_1$ under the policy $\tilde{\Pi}_1$ and $\mathcal{S}_2$ under the policy $\tilde{\Pi}_2$.

However, notice that $\tilde{\Pi}$ releases unit j, which belongs to $\mathcal{S}_1$, and holds unit $j+1$, which belongs to $\mathcal{S}_2$, in period t. From the underlined statement earlier, it is clear that the action in $\mathcal{S}_2$ is *strictly suboptimal*. This implies that when $\mathcal{S}_1$ and $\mathcal{S}_2$ use Π_1 and Π_2, respectively, the expected sum of the costs incurred by $\mathcal{S}_1$ and $\mathcal{S}_2$ is strictly smaller than the corresponding expected sum under $\tilde{\Pi}_1$ and $\tilde{\Pi}_2$.

The conclusions of the preceding two paragraphs contradict each other. This proves the first statement. The proofs of the remaining statements are identical. $\Box$

Next, we use this lemma to develop the notion of "critical distance" policies. Let us define

$$Y^*_{2t}(s) \stackrel{\text{def}}{=} \max\{y \geq C+1 : \; R^*_{2t}(s,y,1) \supseteq \{1\}\ \} \text{ if } T-t \geq T_1 \text{ and } -\infty \text{ otherwise;}$$
$$Y^*_{3t}(s) \stackrel{\text{def}}{=} \max\{y \geq 1 : \; R^*_{3t}(s,y,0) \supseteq \{1\}\ \} \text{ if } T-t \geq T_2 \text{ and } -\infty \text{ otherwise.}$$

$Y^*_{2t}(s)$ is defined in such a way that it is optimal to release unit $j_{wt}-C$ from L_2 if and only if customer j_{wt} is at a distance of $Y^*_{2t}(s)$ or closer. This distance $Y^*_{2t}(s)$ is a "critical distance" for L_2 at time t and Markovian state s_t for every subsystem. Similarly, $Y^*_{3t}(s)$ is a critical distance for stage 3.

Now, consider the policy

$$\begin{aligned} R_{2t}(s,y,1) &= \{1\} \text{ if } y \leq Y^*_{2t}(s) \text{ and } \{0\} \text{ o.w.} \\ R_{3t}(s,y,0) &= \{1\} \text{ if } y \leq Y^*_{3t}(s) \text{ and } \{0\} \text{ o.w., and,} \\ R_{3t}(s,y,1) &= \{0\}\ \forall\, y\,. \end{aligned}$$

It is clear that we have defined the functions R_{2t} and R_{3t} such that they constitute an optimal policy for a subsystem. We now prove this result.

Theorem 5 *Consider subsystem $\mathcal{S}_w$. Assume that there are no units or one unit located at stage 2, that is L_2, at the start of period t. Then, an optimal policy for $\mathcal{S}_w$ is to use R_{2t} at stage 2 and R_{3t} at stage 3 in period t.*

Proof: When i_{wt} is zero, there is no decision to take at L_2. When i_{wt} is one, R_{2t} prescribes an optimal stage 2 decision and this can be seen directly from the definitions of $R_{2t}(y,s,1)$ and $Y^*_{2t}(s)$. Furthermore, if it is optimal to hold unit $j_{wt} - C$ at L_2, Proposition 1 implies that the optimal decision at stage 3 is to hold unit j_{wt} at L_3. For the case where $\tilde{i}_{wt}$ is zero, R_{3t} prescribes an optimal stage 3 decision and this can be seen from the definitions of $R_{3t}(y,s,0)$ and $Y^*_{3t}(s)$. $\square$

3.3 Optimality of Modified Echelon Base-Stock Policies in System $\mathcal{S}$

We are ready to prove that when each subsystem follows the policies prescribed by R_{2t} and R_{3t} in every period t, the resulting policy for the original system is of the echelon base-stock type, with the exception that the number of units shipped from either of the two stages, L_3 and L_2, and the inventory at L_2 are never allowed to exceed C. Furthermore, this policy is optimal for the entire system $\mathcal{S}$ according to Theorem 2. Parker and Kapuscinski (2004) introduced the term "Modified Echelon Base-stock Policies" to refer to such policies.

Theorem 6 *Assume that there are $\gamma_2 \leq C$ units at stage 2, that is L_2, at the start of period t. Consider the beginning of period t and some state $s \in \Omega$. An optimal policy for the system $\mathcal{S}$ from this state is ordering q_1 units at stage 1 and ordering q_2 units at stage 2, that is, shipping q_1 and q_2 units from L_2 and L_3, respectively, where:*

$$\begin{aligned} q_1 &= \min(\gamma_2, (Y^*_{2t}(s) - (C+1) - E_{1t})^+) \text{ and} \\ q_2 &= \min((Y^*_{3t}(s) - 1 - E_{2t})^+, C - \gamma_2 + q_1)\ , \end{aligned}$$

and E_{1t} and E_{2t} are the echelon-1 and echelon-2 inventory positions at the start of period t, respectively. That is, a state-dependent, modified echelon base-stock policy with base-stock levels $Y^*_{2t}(s) - (C+1)$ and $Y^*_{3t}(s) - 1$ at echelons 1 and 2, respectively, is optimal for $\mathcal{S}$.

Proof: It is sufficient to prove that the policy stated in the theorem will be followed by $\mathcal{S}$ when each subsystem $\mathcal{S}_w$ follows the policy prescribed by R_{2t} and R_{3t} at stages, 2 and 3, respectively.

If there are no backorders at L_1, we can renumber the existing units and customers $1, 2, \ldots$ and we will have the relationship $y_{jt} = j + 1$ for all j. If backorders exist, we can start numbering the units and customers beyond the backordered customers and the corresponding units as $1, 2, \ldots$; again, we will have $y_{jt} = j + 1$ for all $j \geq 1$. (Customers with indices zero or below are backordered, i.e. at distance 1.)

We consider three cases: (i) $E_{1t} + \gamma_2 \leq C$ and $E_{1t} \geq 0$, (ii) $E_{1t} \leq 0$, and (iii) $E_{1t} + \gamma_2 \geq C$ and $E_{1t} \geq 0$.

Case (i): $E_{1t} + \gamma_2 \leq C$ and $E_{1t} \geq 0$.

Now, $\mathcal{S}_{E_{1t}+1}, \ldots, \mathcal{S}_{E_{1t}+\gamma_2}$ are the only subsystems that can take release decisions at L_2 in t. Furthermore, using the definition of j_{wt} (recall that j_{wt} is the unit waiting at L_3 whereas the release decision at stage 2 is taken on unit $j_{wt} - C$), we get

$$y_{j_{wt},t} = w + 1 + C \ \forall \ w \ \in \ \{E_{1t} + 1, \ldots, E_{1t} + \gamma_2\} \ .$$

The unit at L_2 in $\mathcal{S}_w$ in this set will be released if and only if $w + 1 + C \leq Y^*_{2t}(s)$. So, we get

$$q_1 \ = \ \min\{\gamma_2, (Y^*_{2t}(s) - (E_{1t} + C + 1))^+\} \ .$$

Case (ii): $E_{1t} \leq 0$. Therefore, $E_{1t} + \gamma_2 \leq C$. We now have two subcases.

Subcase (iia): $E_{1t} + \gamma_2 \ \leq \ 0$: In this case, since the total inventory in the system is negative after accounting for backorders, the customers corresponding to the γ_2 units at L_2 have already arrived and are therefore at a distance of 1. So, all γ_2 units are released if $T - t \geq T_1$ and are held otherwise. So,

$$q_1 = 0 \text{ if } Y^*_{2t}(s) = -\infty \text{ and } = \gamma_2 \ \text{ otherwise.}$$

Using the fact that $Y^*_{2t}(s) \geq C + 1$ when $T - t \geq T_1$, it is easy to check that this also satisfies the required formula under the assumptions of the subcase.

Subcase (iib): $E_{1t}+\gamma_2 > 0$: Here again, $q_1 = 0$ if $Y^*_{2t}(s) = -\infty$. Otherwise, the first $-E_{1t}$ units from L_2 will be released because the corresponding customers have already arrived. In addition, unit $j_{wt} - C$ will be released from $\mathcal{S}_w$, $w \in \{-E_{1t}+1, -E_{1t}+2, \ldots, \gamma_2\}$, if and only if $y_{j_{wt},t} \leq Y^*_{2t}(s)$. For these w, $y_{j_{wt},t}$ can be verified to be $w + C + E_{1t} + 1$. So, the unit in $\mathcal{S}_w$, $w \in \{-E_{1t} + 1, -E_{1t} + 2, \ldots, \gamma_2\}$, is released if and only if $w \leq Y^*_{2t}(s) - E_{1t} - (C + 1)$. In total, the number of units released from L_2 over the γ_2 subsystems is

$$q_1 \;=\; \min\{\gamma_2, Y^*_{2t}(s) - E_{1t} - (C+1)\}\ .$$

Case (iii): $E_{1t} + \gamma_2 \geq C$. Therefore, $E_{1t} \geq 0$. Here again, $q_1 = 0$ if $Y^*_{2t}(s) = -\infty$. Otherwise, of units in $\{E_{1t}+1, \ldots, E_{1t}+\gamma_2\}$, we release unit $E_{1t}+j$ if and only if $y_{E_{1t}+j+C,t} \leq Y^*_{2t}(s)$. This is equivalent to releasing units $E_{1t}+j$ from $j \in \{1, \ldots, \gamma_2\}$ such that $j \leq Y^*_{2t}(s) - E_{1t} - (C+1)$. Therefore,

$$q_1 \;=\; \min\{\gamma_2, Y^*_{2t}(s) - E_{1t} - (C+1)\}\ .$$

The expression for q_1 agrees with the formula in the statement in all three cases. The derivation for q_2 is similar. □

3.4 Infinite Horizon

Let us briefly discuss two results on the infinite horizon, discounted problem for the two-echelon system studied in this section.

Theorem 7 *Assume that $\{s_t\}$ is a time-homogeneous Markov Chain and that there are C units or less at stage 2, that is L_2, at the start of period t. The class of state-dependent, modified echelon base-stock policies, as described in Theorem 6, is optimal for the system $\mathcal{S}$ when the planning horizon is infinite and the performance measure is the total expected discounted cost. Furthermore, since $\{s_t\}$ is time-homogeneous, the policy does not depend on the period index, t. That is, for every state $s \in \Omega$, there exist parameters $y^*_2(s)$ and $y^*_3(s)$ such that*

$$\begin{aligned} q_1 &= \min(\gamma_2, (y^*_2(s) - (C+1) - E_{1t})^+) \textit{ and} \\ q_2 &= \min((y^*_3(s) - 1 - E_{2t})^+, C - \gamma_2 + q_1)\ , \end{aligned}$$

where γ_2, q_1 and q_2 represent the same quantities as in Theorem 6.

Proof: The existence of optimal stationary policies for the system and the subsystems is a consequence of Theorem 4.1.4 of Sennott (1999). The rest of the proof is the same as the finite horizon proof. $\Box$.

Next, we show an additional result for systems where $|\Omega|$ is one, that is, $\{d_t\}$ is a sequence of independent and identically distributed random variables. We show the existence of an optimal policy where the base-stock levels for the two echelons do not differ by more than C. This result is similar to Proposition 1 in Glasserman and Tayur (1994).

Lemma 8 *In addition to the assumptions of Theorem 7, assume $|\Omega|$ is one, that is, the sequence of random variables $\{d_t\}$ is independent and identically distributed. In this case, an optimal policy can be defined using two stationary parameters, y_2^* and y_3^*. Also, assume that $E_{1t} \leq y_2^* - (C+1)$ and $\gamma_2 \leq C$, at the start of period 1, that is, echelon 1's inventory position is lower than its base-stock level and the on-hand inventory at L_2 is not more than C. Then, there exists an optimal policy such that the base-stock level for echelon 2 is at most C in excess of the base-stock level for echelon 1. In particular, using the same notation as Theorem 7, the optimal policy in period t is given by*

$$
\begin{aligned}
q_1 &= \min(\gamma_2, (y_2^* - (C+1) - E_{1t})^+) \text{ and} \\
q_2 &= \min[(\min\{y_3^*, y_2^*\} - 1 - E_{2t})^+, C] \,.
\end{aligned}
$$

That is, an echelon base-stock policy is optimal.

Proof: The optimal policy is prescribed in Theorem 7, where $y_2^*(s)$ and $y_3^*(s)$ are replaced by y_2^* and y_3^*, respectively. Since E_{1t} is smaller than the base-stock level at the start of period 1, it will always be smaller than the base-stock level when this policy is followed. That is, $E_{1t} \leq y_2^* - (C+1)$ at the start of any period t. Since the policy limits the amount of inventory that can be stocked at L_2 to be less than or equal to C, it is clear that the maximum value that echelon 2's inventory position can reach at any time is $y_2^* - 1$. Therefore, we can replace echelon 2's target base-stock level of $y_3^* - 1$ with $\min\{y_3^* - 1, y_2^* - 1\}$. It can be verified that the expression for q_2 stated in this lemma is exactly the same as the corresponding expression in Theorem 7, though the expression is more compact here. □

Let us summarize the main structural results of this section. We showed that a state-dependent, modified echelon base-stock policy is optimal for the system $\mathcal{S}$ for the finite horizon problem and the infinite horizon discounted cost problem. When demands are stationary through time, we showed that the optimal policy is an echelon base-stock policy for the infinite horizon, discounted cost problem. The optimality of echelon base-stock policies, in the stationary, infinite horizon discounted model, is a refinement of the optimality of modified echelon base-stock policies shown in Corollary 3 of Parker and Kapuscinski (2004) for the same model.

This concludes our discussion of the two echelon system with a one period lead time between L_3 and L_2. Next, we present some results for the case where this lead time is two periods.

4 Two Echelon Serial Systems with a Two Period Upstream Leadtime

In this section, we consider two echelon serial systems with identical capacities at stages 2 and 3 and a two period lead time between these stages. Clearly, the important question is whether the class of MEBS policies is optimal for these problems. If not, what can we say about the optimal policy? How complicated can the structure of the optimal policy be? We answer these questions here.

Let us first extend the definition of MEBS policies to these systems. When the lead time between L_3 and L_2 was one period, we knew that it was never optimal to stock L_2 in excess of C. Now, since the lead time is two periods, it is never optimal for the stock on hand at L_2 plus the stock in the pipeline from L_3 to L_2 to exceed $2C$. This should be the $modification$ to echelon base-stock policies. However, this does not appear to be the form of the optimal policy according to an example presented in Speck and van der Wal (1991). They show that in the optimal policy, the number of units released from stage 2 depends non-trivially on the number of units in stock at L_2 and in the pipeline between L_3 and L_2. In particular, as this total amount of inventory increases, the amount released from stage 2 may increase and this is observed to happen even when the quantity released from stage 2 is initially strictly less than the stock there. This clearly violates the conditions of an MEBS policy. As Parker and Kapuscinski (2004) comment, it is still possible that the structure of the policy is a modification of echelon base-stock policies in some other way. From a more abstract perspective, it might be possible to find an optimal policy that depends only on two parameters, one for each echelon for a given set of problem data, in every period and state $s \in \Omega$. We show that an optimal policy will depend on a maximum of four parameters, rather than two. The rest of this section is devoted to the development of this result. We omit those proofs that are identical to corresponding proofs for the one period lead time case.

First, we know from Lemma 1 that monotone policies are optimal for this system. We also know from Theorem 2 that this system can be decomposed into C subsystems, each with unit capacities at L_2 and L_3. Optimally managing each of these subsystems is an optimal policy for the entire system.

Note: Throughout this section, we will assume that the number of units at L_2 and in the pipeline between L_3 and L_2 is less than or equal to $2 \cdot C$ at the start of period t. This assumption is justified by Proposition 1.

4.1 Analysis of a Subsystem

Let us now examine subsystem w's decision problem in period t. We start by finding a sufficient information vector for the subsystem.

As before, let j_{wt} be the lowest index of the units in subsystem w located at stage 3, that is, location $l_1 + 3$. In other words, unit j_{wt} is the only candidate for being released from L_3 in period t. Note that location $l_1 + 2$ represents the *pipeline* between L_3 and L_2. Location $l_1 + 1$ is L_2. There are five possibilities regarding units being present at locations $l_1 + 2$ and $l_1 + 1$. Let $(i_1, i_2) \in \mathcal{I}_2$, where $\mathcal{I}_2$ is $\{(0, 2), (1, 1), (1, 0), (0, 1), (0, 0)\}$, represent the number of units at these two locations. That is, at the time of stage 2's release decision, there are i_1 units in the pipeline between stages 2 and 3, and, i_2 units at stage 2. Note that $\mathcal{I}_2$ represents all the possible realizations of (i_1, i_2) if an optimal policy has been followed in periods 1, 2, $\ldots$ $t - 1$ and there were 2 units or less, in total, in locations $2 + l_1$ and $1 + l_1$ at the start of period 1. This can be seen from the following facts: (i) i_1 is the number of units shipped by L_3 in the previous period and is constrained to be 0 or 1 because of the unit capacity restriction and (ii) $i_1 + i_2 \leq 2$ when an optimal policy is followed, as proved in Proposition 1.

Let y be the distance of customer j_{wt}. Using exactly the same arguments as in Section 3, we can see that (t, s, i_1, i_2, y) is a sufficient information vector required by subsystem w at the start of period t. Similarly, let i_2' be the number of units at L_2 after the stage 2 shipments are sent out and the i_1 units are moved from the pipeline to stage 2 inventory, but before the stage 3 decision. Clearly $i_2' \leq 2$ and (t, s, i_2', y) is a sufficient information vector required at the time of stage 3's decision.

Let us now examine stage 2's decision problem closely. Clearly, the only states where a decision needs to be taken at stage 2 are states such that $i_2 \geq 1$, that is, $(i_1, i_2) \in \{(0, 1), (1, 1), (0, 2)\}$. Observe that the optimal release decision for stage 2 is the same when (i_1, i_2) is either $(1, 1)$ or $(0, 2)$ because any pair of stage 2 and stage 3 decisions in $\{1, 0\}^2$

leads to exactly the same state of the system at the beginning of the next period, in either case.

Let $R^*_{2t}(s, y, i_1, i_2)$ and $R^*_{3t}(s, y, i_2^{'}) \subseteq \{1, 0\}$ be the set of optimal stage 2 and stage 3 decisions given the information vectors. Using arguments similar to those in the proof of Lemma 4, we can prove the following lemma.

Lemma 9

$$
\begin{aligned}
&\textit{(0)} \quad R^*_{2t}(s_t, y+1, 1, 1) \;=\; R^*_{2t}(s_t, y+1, 0, 2) \\
&\textit{(i)} \quad R^*_{3t}(s_t, y+1, 0) \;= (\supseteq)\; \{1\}\;,\ \textit{for some}\, y \;>\; 1 \;\Rightarrow\; R^*_{3t}(s_t, y, 0) \;= (\supseteq)\; \{1\}\;, \\
&\textit{(ii)} \quad R^*_{3t}(s_t, y+1, 1) \;= (\supseteq)\; \{1\}\;,\ \textit{for some}\, y \;>\; 1 \;\Rightarrow\; R^*_{3t}(s_t, y, 1) \;= (\supseteq)\; \{1\}\;, \\
&\textit{(iii)} \quad R^*_{2t}(s_t, y+1, 0, 1) \;= (\supseteq)\; \{1\}\;,\ \textit{for some}\, y \;>\; C+1 \\
&\qquad \Rightarrow\; R^*_{2t}(s_t, y, 0, 1) \;= (\supseteq)\; \{1\}\ \textit{and} \\
&\textit{(iv)} \quad R^*_{2t}(s_t, y+1, 1, 1) \;=\; R^*_{2t}(s_t, y+1, 0, 2) \;= (\supseteq)\; \{1\}\;,\ \textit{for some}\, y \;>\; 2C+1 \\
&\qquad \Rightarrow\; R^*_{2t}(s_t, y, 1, 1) \;= R^*_{2t}(s_t, y, 0, 2) \;= (\supseteq)\; \{1\}\;,
\end{aligned}
$$

It is also clear that $R^*_{3t}(s_t, y, 2) = \{0\}$ due to Proposition 1. Furthermore, we know that $R^*_{3t}(s_t, 1, 0) = R^*_{3t}(s_t, 1, 1)$ is $\{1\}$ if $T - t \;\geq\; T_2$ because the customer corresponding to the unit under consideration has arrived already. Similarly, when $T - t \;\geq\; T_1$, $R^*_{2t}(s_t, y, 0, 1)$ is $\{1\}$ if $y \leq C + 1$, and, $R^*_{2t}(s_t, y, 1, 1)$ and $R^*_{2t}(s_t, y, 0, 2)$ are both $\{1\}$ if $y \leq 2C + 1$. This is because if $i_1 + i_2$ is two, then the unit waiting to be released at stage 2 is $j_{wt} - 2C$. So, if customer j_{wt} is within a distance of $2C + 1$, customer $j_{wt} - 2C$ has arrived.

Let us now proceed in exactly the same fashion as the one period lead time case and develop the notion of "critical distance" policies. Let us define

$$
\begin{aligned}
Y^*_{2t}(s, 2) &\stackrel{\text{def}}{=} \max\{y \geq 2 \cdot C + 1 : R^*_{2t}(s, y, 1, 1) = R^*_{2t}(s, y, 0, 2) \supseteq \{1\}\} \text{ if } T - t \geq T_1 \text{ and} \\
&\stackrel{\text{def}}{=} -\infty \text{ o.w.};
\end{aligned}
$$

$$
\begin{aligned}
Y^*_{2t}(s,1) &\stackrel{\text{def}}{=} \max\{y \geq C+1 : R^*_{2t}(s,y,0,1) \supseteq \{1\}\} \text{ if } T-t \geq T_1 \text{ and } -\infty \text{ o.w.;}\\
Y^*_{3t}(s,0) &\stackrel{\text{def}}{=} \max\{y \geq 1 : R^*_{3t}(s,y,0) \supseteq \{1\}\} \text{ if } T-t \geq T_2 \text{ and } -\infty \text{ o.w.;}\\
Y^*_{3t}(s,1) &\stackrel{\text{def}}{=} \max\{y \geq 1 : R^*_{3t}(s,y,1) \supseteq \{1\}\} \text{ if } T-t \geq T_2 \text{ and } -\infty \text{ o.w..}
\end{aligned}
$$

Now, consider the policy

$$
\begin{aligned}
R_{2t}(s,y,1,1) &= R_{2t}(s,y,0,2) = \{1\} \text{ if } y \leq Y^*_{2t}(s,2) \text{ and } \{0\} \text{ o.w.}\\
R_{2t}(s,y,0,1) &= \{1\} \text{ if } y \leq Y^*_{2t}(s,1) \text{ and } \{0\} \text{ o.w.}\\
R_{3t}(s,y,0) &= \{1\} \text{ if } y \leq Y^*_{3t}(s,0) \text{ and } \{0\} \text{ o.w.}\\
R_{3t}(s,y,1) &= \{1\} \text{ if } y \leq Y^*_{3t}(s,1) \text{ and } \{0\} \text{ o.w., and,}\\
R_{3t}(s,y,2) &= \{0\}\ \forall\, y\,.
\end{aligned}
$$

This is clearly an optimal policy for the subsystem. We state this result formally.

Lemma 10 *Assume that the total amount of inventory at stage* 2 *and in the pipeline to stage* 2 *is not more than two units at the start of period* t*. Then, an optimal policy for any subsystem is to use* R_{2t} *at stage* 2 *and* R_{3t} *at stage* 3 *in period* t*.*

We are now ready to state a lemma that relates the critical distances for each echelon through inequalities.

Lemma 11 *(i)* $Y^*_{2t}(s,2) \geq Y^*_{2t}(s,1) + C$ *and (ii)* $Y^*_{3t}(s,0) \geq Y^*_{3t}(s,1)$.

Before proving this lemma, let us discuss the intuition. First, consider the inequality relating the stage 2 critical distances. Let us consider two scenarios, (A) and (B), at the start of period t in state $s \in \Omega$. In (A), subsystem w has units j and $j+C$ located at stage 2 and unit $j+2C$ waiting at stage 3. In (B), unit j is located at stage 2 while unit $j+C$ is waiting at stage 3. Let us assume the vector of customer distances are the same in both the scenarios. In (A), observe that unit j is constraining unit $j+C$, in the sense that unit $j+C$ cannot be released prior to unit j and is forced to wait at stage 2 until unit j is released. Consequently, if we *Hold* unit j in this period, we are constraining unit $j+C$

for at least one more period. In (B), if we *Hold* unit j, we will not constrain unit $j+C$ in the next period because it cannot reach stage 2 in the next period even if it is released from stage 3 in the current period. So, intuitively, a unit is more likely to be released from stage 2 if one more unit belonging to the same subsystem is located at stage 2 or is in transit to stage 2 than otherwise because of the constraint the unit imposes on the next unit in the same subsystem. It is easy to see that this notion can be formalized by the inequality $Y_{2t}^*(s,2) \geq Y_{2t}^*(s,1) + C$ (Statement (i) of the lemma). Using identical reasoning, we can argue intuitively that a unit is less likely to be released from stage 3 if a unit belonging to the same subsystem is located at stage 2 at the time of the stage 3 decision than otherwise. This notion could be expressed by the inequality $Y_{3t}^*(s,0) \geq Y_{3t}^*(s,1)$ (Statement (ii) of the lemma).

Proof: We prove statement (i) by using an approach similar to the proof of Lemma 4. Assume $Y_{2t}^*(s,2) < Y_{2t}^*(s,1) + C$. We will contradict this statement by constructing two *new* systems $\mathcal{A}$ and $\mathcal{B}$ both with unit capacity. We index the units and customers in $\mathcal{A}(\mathcal{B})$ as $1A, 2A \ldots (1B, 2B, \ldots)$, respectively.

We assume that the customer distances in each of these two systems evolves as follows. Customers $1A$ and $1B$ are mapped to customer 1 in $\mathcal{S}$, customers $2A$ and $2B$ are mapped to customer $1+C$ in $\mathcal{S}$, customers $3A$ and $3B$ are mapped to customer $1+2\cdot C$ in $\mathcal{S}$, ... In other words, the customer arrival processes in $\mathcal{A}$ and $\mathcal{B}$ are identical to that in $\mathcal{S}_1$. Note that in $\mathcal{A}$ and $\mathcal{B}$, customers with consecutive indices are separated by a distance of C. In that sense, the construction of $\mathcal{A}$ and $\mathcal{B}$ is different from both $\mathcal{S}$ and $\mathcal{S}_1$.

We further assume that (a) $z_{1A,t} = 1 + l_1$ (stage L_2), $z_{2A,t} = 2 + l_1$ and $z_{3A,t} = 3 + l_1$ (stage L_3), (b) $y_{3A,t} = Y_{2t}^*(s,1) + C$ (this implies that $y_{3A,t} > Y_{2t}^*(s,2)$), (c) $z_{1B,t} = 1 + l_1$ (stage L_2), $z_{2B,t} = 3 + l_1$ (stage L_3), (d) $y_{2B,t} = Y_{2t}^*(s,1)$. (Please see Figure 3.)

Now, (d) implies it is optimal to release unit $1B$ from L_2 and (b) implies it is *strictly* optimal to hold unit $1A$ at L_2. Let $\mathcal{A}$ and $\mathcal{B}$ make these decisions in period t and also use an optimal policy in all subsequent periods.

We can now consider two new systems $\mathcal{C}$ and $\mathcal{D}$ that are identical initially to $\mathcal{A}$ and $\mathcal{B}$, respectively. We consider a policy on $\mathcal{C}$ and $\mathcal{D}$ such that the total number of units released from L_2 and from L_3 is the same as the corresponding quantities in $\mathcal{A}$ and $\mathcal{B}$; in addition, units are released from the union of $\mathcal{C}$ and $\mathcal{D}$ in the precedence order $1A < 1B < 2A < 2B < 3A < 3B \ldots$. Note that the states of $\mathcal{A} \bigcup \mathcal{B}$ and $\mathcal{C} \bigcup \mathcal{D}$, at the beginning of period t, are monotone with respect to this precedence order.

It is easy to see that the cost incurred by $\mathcal{C} \bigcup \mathcal{D}$ in any period is no larger than the cost incurred by $\mathcal{A} \bigcup \mathcal{B}$ in that period, with probability 1. However, the release decisions in $\mathcal{C} \bigcup \mathcal{D}$ are the reverse of the release decisions in $\mathcal{A} \bigcup \mathcal{B}$ in period t. However, the strict optimality of the decision to hold unit $1A$ at L_2 implies that the expected cost incurred by $\mathcal{A} \bigcup \mathcal{B}$ in periods $[t, T]$ is strictly smaller than the expected cost incurred by $\mathcal{C} \bigcup \mathcal{D}$. This is a contradiction.

The proof of statement (ii) is identical. $\square$

4.2 Optimality of Two-tier Base-stock Policies in $\mathcal{S}$

We now use the structure of the optimal policy for a subsystem to infer the structure of the optimal policy for the entire system, $\mathcal{S}$.

Notice that there are two stage 2 critical distances for a subsystem; the release decision is based on the critical distance corresponding to whether the inventory vector at locations $(2 + l_1, 1 + l_1)$ is $(0, 1)$ or $(1, 1)$. Similar distinctions exist for stage 3 also. So, in order to know how many units are released from stage 2 in the entire system, we need to know how many subsystems have the inventory vector compositions $(0, 1)$, $(1, 1)$ etc. This necessitates the classification of the subsystems into different sets. This is the next step in our study.

Let A_1 and A_2 be the number of units in the system in transit to stage 2 and at stage 2, respectively, at the beginning of period t. Without loss of generality, we can assume that the A_2 units at stage 2 are numbered 1, 2, ..., A_2 and the A_1 units in transit to 2 are numbered $A_2 + 1$, $A_2 + 2$, ..., $A_1 + A_2$. Notice that this numbering determines the number of units at stage 2, i_{2w}, and number of units in transit to stage 2, i_{1w}, in each subsystem w.

Let us now classify the C subsystems into 5 mutually exclusive categories. Let N_{ab} $= \{w : i_{1w} = a, i_{2w} = b\}$ for $(a, b) \in \{(0, 0), (0, 1), (1, 0), (1, 1), (0, 2)\}$. Let $n_{ab} = |N_{ab}|$. Clearly, $n_{02} = (A_2 - C)^+$, because subsystems 1, 2, ..., $(A_2 - C)^+$ have two units present at stage 2 and none in transit. That is, $N_{02} = \{1, 2, \ldots, (A_2 - C)^+\}$. Similarly, $N_{11} = \{(A_2 - C)^+ + 1, (A_2 - C)^+ + 2, \ldots, (A_1 + A_2 - C)^+\}$ and consequently, $n_{11} = (A_1 + A_2 - C)^+$ - $(A_2 - C)^+$. (Recall that $A_1 + A_2 \leq 2C$ by assumption and $A_1 \leq C$ because C is the maximum number of units that could have been released by stage 3 in the previous period. These inequalities are useful in verifying the composition of these 5 sets.) Furthermore, we have $n_{11} + n_{01} + 2n_{02} = A_2$ and consequently, $n_{01} = \min(A_2, C) - (A_1 + A_2 - C)^+$. Also, $N_{01} = \{(A_1 + A_2 - C)^+ + 1, (A_1 + A_2 - C)^+ + 2, \ldots, \min(A_2, C)\}$.

Let us first discuss the optimal ordering policy for echelon 1, that is, the optimal release policy for stage 2. The following theorem characterizes the structure of an optimal policy, which is derived by using policy R_{2t} at stage 2 in every subsystem in period t. The proof is similar to the proof of Theorem 6 and is omitted.

Theorem 12 *Assume that there are $2C$ units or less at stage 2 and in transit to stage 2 at the start of period t. Consider the beginning of period t and some state $s \in \Omega$. An optimal ordering policy for echelon 1, or equivalently, an optimal release policy for stage 2, dictates the release of q_1 units from stage 2, where $q_1 = q_1^{(1)} + q_1^{(2)}$, where, $q_1^{(1)}$ and $q_1^{(2)}$ are computed* *as follows.*

$$q_1^{(1)} = \min\left((Y_{2t}^*(s,2) - (2C+1) - E_{1t})^+, n_{02} + n_{11}\right).$$

If $q_1^{(1)} < n_{02} + n_{11}$*, then* $q_1^{(2)} = 0$*. Otherwise,*

$$q_1^{(2)} = \min\left((Y_{2t}^*(s,1) - (C+1) - (E_{1t} + q_1^{(1)}))^+, n_{01}\right).$$

In words, the optimal policy says the following: first, echelon 1 should order-up-to $Y_{2t}^*(s,2) - (2C+1)$*, if possible, using only the units that are elements of* $N_{02} \cup N_{11}$*. If all the elements of* $N_{02} \cup N_{11}$ *have now been released, then echelon 1 should order up to* $Y_{2t}^*(s,1) - (C+1)$*, if possible, using the elements of* N_{01}*. (We refer to this policy as a "two-tier, echelon base-stock policy".)*

Let us now proceed to analyze the structure of an optimal policy for echelon 2. That is, let us examine the stage 3 release decision after the stage 2 release decision has been taken, and the units released from stage 2 have moved to the subsequent location and the units in transit to stage 2 have moved to stage 2. So there are no units in transit to stage 2 at this point in time.

Let A be the number of units at stage 2. Without loss of generality, we can assume that the A units at stage 2 are numbered 1, 2, ..., A. Notice that this numbering determines $\tilde{i}_w$, the number of units at stage 2 in each subsystem w. Let us now classify the C subsystems into 3 mutually exclusive categories. Let $N_a = \{w : \tilde{i}_w = a\}$, $a \in \{0,1,2\}$ and let n_a denote $|N_a|$. It is easy to verify that $n_0 = (C-A)^+$ and $n_1 = \min(A,C) - (A-C)^+$. Since there are C subsystems in total, n_2 is $C - (n_0 + n_1)$, that is, $(A-C)^+$.

The following theorem characterizes the structure of an optimal policy, which is derived by using policy R_{3t} at stage 3 in every subsystem in period t. The proof is similar to the proof of Theorem 6 and is omitted.

Theorem 13 *Assume there are* $2C$ *units or less at stage* 2 *plus in transit to stage 2 at the start of period* t*. Consider the point in time when stage 3's release decision has to be made* *in period t and some state $s \in \Omega$. An optimal ordering policy for echelon 2, or equivalently, an optimal release policy for stage 3, dictates the release of q_2 units from stage 3, where $q_2 = q_2^{(1)} + q_2^{(2)}$, where, $q_2^{(1)}$ and $q_2^{(2)}$ are computed as follows.*

$$q_2^{(1)} \;=\; \min\Big((Y_{3t}^*(s,0) - 1 - E_{2t})^+, n_0\Big)\,.$$

If $q_2^{(1)} < n_0$, then $q_2^{(2)} = 0$. Otherwise,

$$q_2^{(2)} \;=\; \min\Big((Y_{3t}^*(s,1) - 1 - (E_{2t} + q_2^{(1)}))^+, n_1\Big)\;.$$

In words, the optimal policy says the following: first, echelon 2 should order-up-to $Y_{3t}^*(s,0)-1$, if possible, using only the units that are elements of N_0. If all the elements of N_0 have now been released, then echelon 2 should order up to $(Y_{3t}^*(s,1)-1)$, if possible, using the elements of N_1.

Theorem 13 indicates that this optimal policy for echelon 2 is also a "two-tier" echelon base-stock policy.

Note that Theorems 12 and 13 can also be shown to hold in the infinite horizon, discounted cost version of the problem. See the discussion in section 3.4.

One final comment about "two-tier" echelon base-stock policies: note that the policy for echelon 1 requires the knowledge of A_1 and A_2, the inventory in transit to stage 2 and on hand at stage 2 explicitly. An echelon base-stock policy would have required the knowledge of A_2 only. Similarly, the "two-tier" policy at echelon 2 requires the knowledge of A, the number of units at stage 2, which an echelon base-stock policy would not.

5 Longer Serial Systems: Optimality of Multi-Tier Base-stock Policies

Next, we comment briefly about the optimal policy when the lead time between stages 3 and 2 is an arbitrary integer and/or when there are more than three stages in the serial system.

We know Proposition 1 holds for these systems. Also, Theorem 2 can be used to decompose N echelon, serial systems with identical capacities at all physical stages and arbitrary lead times into serial systems with unit capacities at all physical stages. Now, we can use the proof technique we used for the two echelon system with a two-period upstream lead time. For every echelon n, we can classify the subsystems into several categories based on the positioning of inventories within each subsystem. For each of these categories, the optimality of monotone policies leads to the existence of a critical distance, which in turn, leads to a base-stock level. This is a multi-tier, base-stock policy, in the sense that there is a base-stock level corresponding to every category of subsystems. Thus, for each echelon, a "multi-tier base-stock policy" is optimal. The number of "tiers" grows exponentially in the total leadtime of the system between stages 2 and $N+1$. In fact, it is easy to show that the number of tiers at each echelon is less than $2^{M_{N+1}-M_2}$.

Lemma 14 *Consider any subsystem $\mathcal{S}_w$. Assume that for every n, the number of units at stage n plus the number of units in transit to stage n, at the start of period t, does not exceed l_n. Then, in period t, the optimal release decision at any stage is determined by at most $2^{M_N-M_2}$ parameters.*

Proof: Consider a particular subsystem, $\mathcal{S}_w$, at the time the stage n release decision is made. Let $\mathbf{a} = (a_{M_N}, a_{M_N-1}\ldots, a_{M_2+1}, a_{M_2})$ be the vector denoting the number of units located at each of the locations $(M_N, M_N-1\ldots, M_2+1, M_2)$. Each component in this vector corresponding to a transit location belongs to $\{0,1\}$. The component corresponding to any stage L_m belongs to $\{0, 1, \ldots, l_m\}$.

As in earlier sections, $\mathbf{a}$ and $y_{j_{wt},t}$ (the distance of the customer corresponding to the waiting unit at stage L_{N+1}) are sufficient to determine the optimal decision. Furthermore, as discussed for the simpler models earlier, the optimal decision for any vector that has multiple units at some stages is identical to the optimal decision for a vector whose components are all in $\{0, 1\}$. For example, consider a two echelon subsystem with a leadtime of 3 periods between L_3 and L_2. Let the vector $\mathbf{a} = (1, 0, 2)$ or $(0, 0, 3)$ or $(0, 1, 2)$. The optimal decision at any stage given any of these vectors is identical to the optimal decision given the vector $(1, 1, 1)$.

So, the optimal decisions for a subsystem in a period are determined by mapping $\mathbf{a}$ to the appropriate binary vector. Therefore, it suffices to determine the optimal decisions for the $2^{M_N - M_2}$ such vectors. $\square$

Let us now study the optimal policy for the entire system. From the lemma above, we know that for every stage there are at most $2^{M_N - M_2}$ critical distances determining the release decision. So, the aggregate number of units released optimally in a system is computed as follows. Determine the subsystems corresponding to each of the $2^{M_N - M_2}$ inventory configurations. For each such configuration, the critical distances determine a base-stock policy at each echelon. The aggregate number of units released from each stage as a result of these base-stock policies prescribes the optimal policy for the entire system. Since there are several base-stock levels possible for each stage, we call these policies *multi-tier* base-stock policies.

Corollary 1 *Assume there are $l_n \cdot C$ or fewer units at stage n plus in transit to stage n at the start of period t, for every n. The optimal policy in this period can be characterized by at most $2^{M_N - M_2} \cdot N$ parameters; in other words, for every stage, there are at most $2^{M_N - M_2}$ base-stock levels.*

6 Endnotes

1. Timewise Convex Penalty Costs: We briefly discuss the case of holding and backorder costs that are increasing and convex with respect to time. For example, consider a supplier who has an advertised delivery-time promise which is backed-up by discounts to the customer when the promise cannot be met. This could lead to convex backordering costs. Please see Bhargava et al. (2005) for several examples from the online retail industry that are similar in spirit.

Let $h_n(t)$ denote the holding cost associated with a unit if it stays at stage n for exactly t time periods. Similarly, $b(t)$ denotes the backorder cost associated with a unit of demand that is backordered for exactly t time periods. $h_n(0)$ is zero for all n and $b(0)$ is zero.

It can easily be verified that the results of this paper hold under the assumptions that (a) $h_n(t)$ is convex and non-decreasing for all n, and, (b) $b(t)$ is convex and non-decreasing. This is related to Derman and Klein (1958); they present conditions under which FIFO (first in, first out) or LIFO (last in, first out) release policies are optimal in an environment where units of different ages are available to meet future demands.

2. Stochastic Lead Times: Consider the following stochastic lead time model first introduced by Kaplan (1970) and subsequently redefined and simplified by Nahmias (1979), both in the context of single stage systems. The key feature of this model is that units do not overtake one another despite the randomness in the lead times.

Model $\mathcal{L}$: There is a random variable ρ_{nt}, whose distribution is determined completely by s_t, that specifies the least "age" of units that will be delivered in period t at L_n. This means all units shipped by L_{n+1}, or ordered by L_n, in period $t - \rho_{nt}$ or earlier are delivered at L_n no later than period t. The maximum value the random variable ρ_{nt} can take is l_n and the probability mass function of ρ_{nt} is known for every possible value of s_t. It can be verified that the maximum possible lead time to stage L_n under this model is also l_n.

All the results and the proofs in this paper hold under this model with the understanding that l_n now represents the maximum possible leadtime to L_n. In particular, the following result can be shown for two echelon systems. If the leadtime between L_3 and L_2 is random but never exceeds two periods, two-tier base-stock policies are optimal. For serial systems with arbitrary number of stages and stochastic leadtimes, multi-tier base-stock policies are optimal and the bound on the number of tiers is still the same.

7 Conclusions

We have extended the "single-unit single-customer" approach to a class of capacitated serial systems. In particular, two echelon serial systems with identical capacities at both echelons are studied in detail. When the lead time at the upstream echelon is one period, we show that modified echelon base-stock policies are optimal using a decomposition approach. For the stationary infinite horizon discounted cost model, we further refine this result by demonstrating that echelon base-stock policies are optimal. When the lead time to the upstream echelon is two periods, the class of "two-tier base-stock policies" are shown to be optimal. We also argue that a generalization of these policies, ones we call "multi-tier base-stock policies", are optimal for multi-echelon serial systems with identical capacities at all physical stages and stochastic, non-crossing lead times. We also provide a bound on the number of tiers or parameters required to specify these policies.

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Table 1: Glossary of General Notation

Notation	Meaning
n	index for a stage.
$N, N+1$	number of echelons and stages
C	shipping capacity in units per period
L_{N+1}	external supplier: infinite inventory and a capacity of C
$L_2, \ldots, L_N$	other stages with shipping capacity of C
L_1	inventory storage facility serving customers
l_n	lead time between L_{n+1} and L_n
t, T	period index and number of time periods
s_t	state of the exogenous Markov chain in t
d_t	demand in t
M_n	location of L_n, that is, $1 + l_1 + \ldots + l_{n-1}$
$y_{jt}(z_{jt})$	distance(location) of customer(unit) j in t
$\mathcal{S}$	system consisting of all units and customers
$\mathbf{x}_t$	$(s_t, (z_{1t}, y_{1t}), (z_{2t}, y_{2t}), \ldots)$
Z_{nt}	inventory on hand at L_n at the beginning of t
q_{nt}	quantity ordered by L_n and shipped by L_{n+1} in t
u_{jt}	1, if unit j is released from some stage in t and 0, otherwise
E_{nt}	echelon n inventory position at the beginning of t
E'_{nt}	echelon n inventory position at the end of t
$\{h_n\}, b$	echelon holding costs and backorder cost
T_n	min$\{T$: when there are at least T periods left in the horizon, it is optimal to have a non-negative echelon n inventory position, if possible.$\}$
α	discount factor

Table 2: Glossary of Notation for Section 3

Notation	Meaning
$\mathbf{x}_t^w$	$(s_t, (z_{wt}, y_{wt}), (z_{(w+C)t}, y_{(w+C)t}), \ldots)$
j_{wt}	$\min\{j \in \{w, w+C, w+2C, \ldots\} :$ unit j is at $L_3\}$
$U_{2t}^*(\mathbf{x}_t^w)$	$\subseteq \{0,1\}$; set of optimal stage 2 decisions at time t, if $\mathbf{x}_t^w$ is such that unit $j_{wt} - C$ is present at L_2
$\tilde{\mathbf{x}}_t^w$	state of $\mathcal{S}_w$ after the stage 2 decision
$U_{3t}^*(\tilde{\mathbf{x}}_t^w)$	$\subseteq \{0,1\}$; set of optimal stage 3 decisions for $\mathcal{S}_w$
i_{wt}	$\in \{0,1\}$; indicator of whether unit $j_{wt} - C$ is located at stage 2 or not.
$R_{2t}^*(s_t, y, i)$	$\subseteq \{1,0\}$; set of optimal stage 2 decisions if $y_{j_{wt},t}$ is y and i_{wt} is i.
$\tilde{i}_{wt}$	$\in \{0,1\}$; indicator of whether unit $j_{wt} - C$ is located at stage 2 or not at the time of the stage 3 decision.
$R_{3t}^*(s_t, y, \tilde{i})$	$\subseteq \{1,0\}$; set of optimal stage 3 decisions if $y_{j_{wt},t}$ is y and $\tilde{i}_{wt}$ is $\tilde{i}$
$Y_{2t}^*(s)$	$\max\{y \geq C+1 :\ R_{2t}^*(s,y,1) \supseteq \{1\}\}$ if $T - t \geq T_1$ and $-\infty$ otherwise
$Y_{3t}^*(s)$	$\max\{y \geq 1 :\ R_{3t}^*(s,y,0) \supseteq \{1\}\}$ if $T - t \geq T_2$ and $-\infty$ otherwise
$R_{2t}(s, y, 1)$	$\{1\}$ if $y \leq Y_{2t}^*(s)$ and $\{0\}$ o.w.
$R_{3t}(s, y, 0)$ $R_{3t}(s, y, 1)$	$\{1\}$ if $y \leq Y_{3t}^*(s)$ and $\{0\}$ o.w. $\{0\}\ \forall\, y$.
γ_2	number of units at L_2 in $\mathcal{S}$

Table 3: Glossary of Notation for Section 4

Notation	Meaning
$\mathbf{x}_t^w$	$(s_t, (z_{wt}, y_{wt}), (z_{(w+C)t}, y_{(w+C)t}), \ldots)$
j_{wt}	$\min\{j \in \{w, w+C, w+2C, \ldots\} :$ unit j is at $L_3\}$
(i_1, i_2)	$\in \mathcal{I}_2$, where $\mathcal{I}_2 = \{(0,2), (1,1), (1,0), (0,1), (0,0)\}$; represents the number of units at locations $(2+l_1, 1+l_1)$
$i_2^{'}$	number of units at L_2 after the stage 2 shipments are sent out and the i_1 units are moved from the pipeline to stage 2
$R_{2t}^*(s, y, i_1, i_2)$	$\subseteq \{1, 0\}$; set of optimal stage 2 decisions
$R_{3t}^*(s, y, i_2^{'})$	$\subseteq \{1, 0\}$; set of optimal stage 3 decisions
$Y_{2t}^*(s, 2)$	$\max\{y \geq 2 \cdot C + 1 : R_{2t}^*(s, y, 1, 1) = R_{2t}^*(s, y, 0, 2) \supseteq \{1\}\}$ if $T - t \geq T_1$ and $-\infty$ o.w.
$Y_{2t}^*(s, 1)$	$\max\{y \geq C + 1 : R_{2t}^*(s, y, 0, 1) \supseteq \{1\}\}$ if $T - t \geq T_1$ and $-\infty$ o.w.
$Y_{3t}^*(s, 0)$	$\max\{y \geq 1 : R_{3t}^*(s, y, 0) \supseteq \{1\}\}$ if $T - t \geq T_2$ and $-\infty$ o.w.
$Y_{3t}^*(s, 1)$	$\max\{y \geq 1 : R_{3t}^*(s, y, 1) \supseteq \{1\}\}$ if $T - t \geq T_2$ and $-\infty$ o.w.
$R_{2t}(s, y, 1, 1)$ $= R_{2t}(s, y, 0, 2)$	$\{1\}$ if $y \leq Y_{2t}^*(s, 2)$ and $\{0\}$ o.w.
$R_{2t}(s, y, 0, 1)$	$\{1\}$ if $y \leq Y_{2t}^*(s, 1)$ and $\{0\}$ o.w.
$R_{3t}(s, y, 0)$	$\{1\}$ if $y \leq Y_{3t}^*(s, 0)$ and $\{0\}$ o.w.
$R_{3t}(s, y, 1)$	$\{1\}$ if $y \leq Y_{3t}^*(s, 1)$ and $\{0\}$ o.w.
$R_{3t}(s, y, 2)$	$\{0\}\ \forall\ y$
A_1	number of units in $\mathcal{S}$ in transit to stage 2
A_2	number of units in $\mathcal{S}$ at stage 2
N_{ab}	$\{w : i_{1w} = a, i_{2w} = b\}$ for $(a, b) \in \{(0,0), (0,1), (1,0), (1,1), (0,2)\}$
n_{ab}	$\|N_{ab}\|$
A	number of units at L_2 in $\mathcal{S}$ at the time of the stage 3 decision
N_a	$\{w : \tilde{i}_w = a\}$, $a \in \{0, 1, 2\}$
n_a	$\|N_a\|$

Figure 1: Locations of Units and Distances of Customers

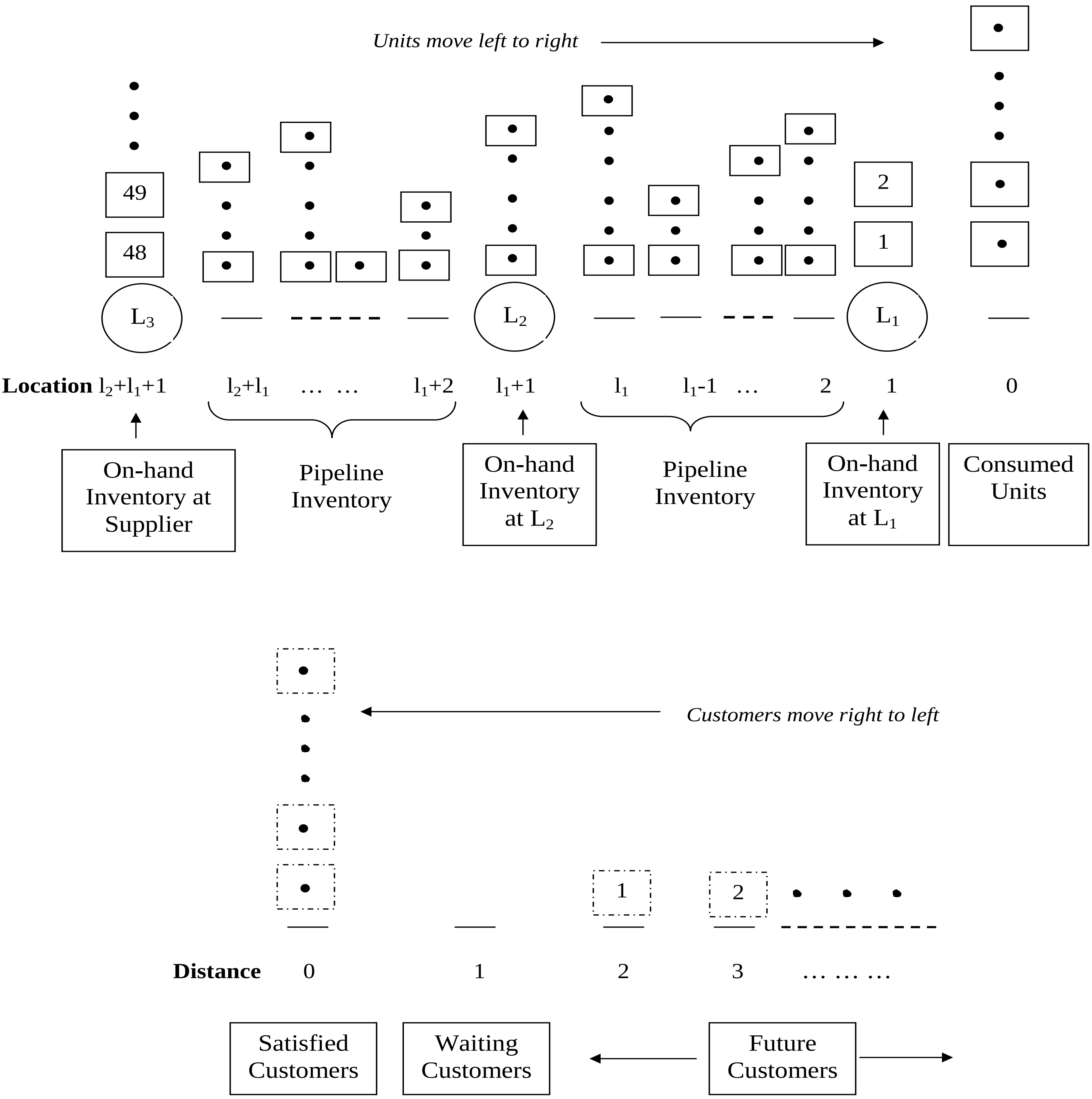


Figure 2: Proof of Lemma 4 – $\mathcal{S}_1 \bigcup \mathcal{S}_2$

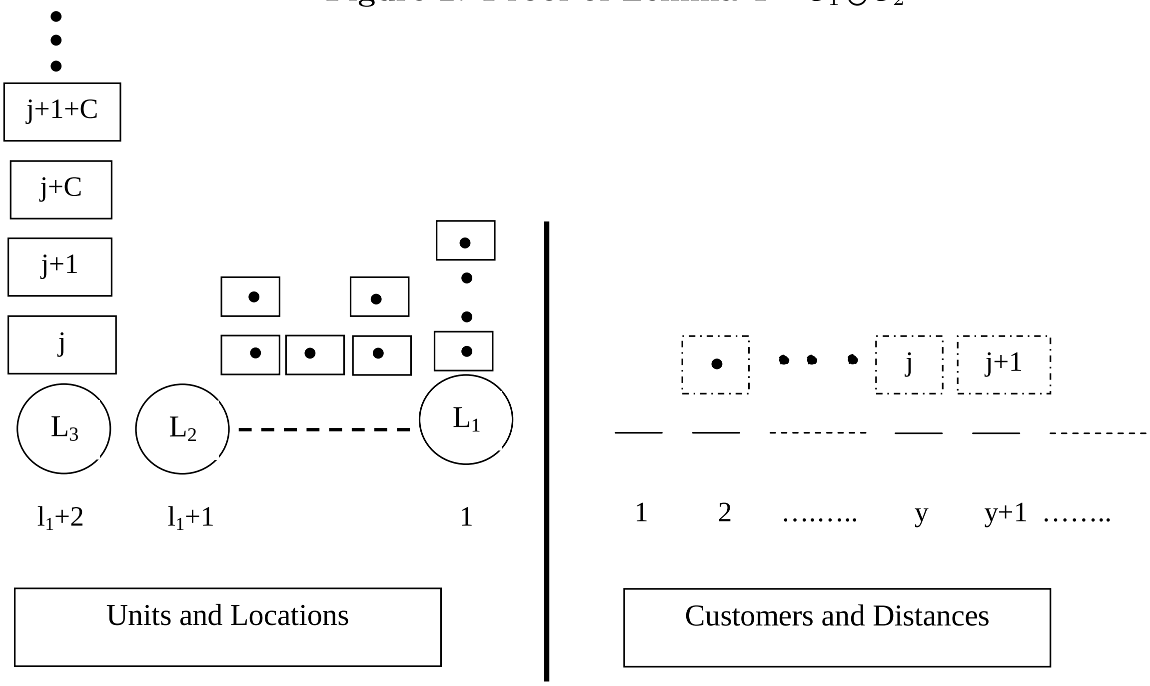
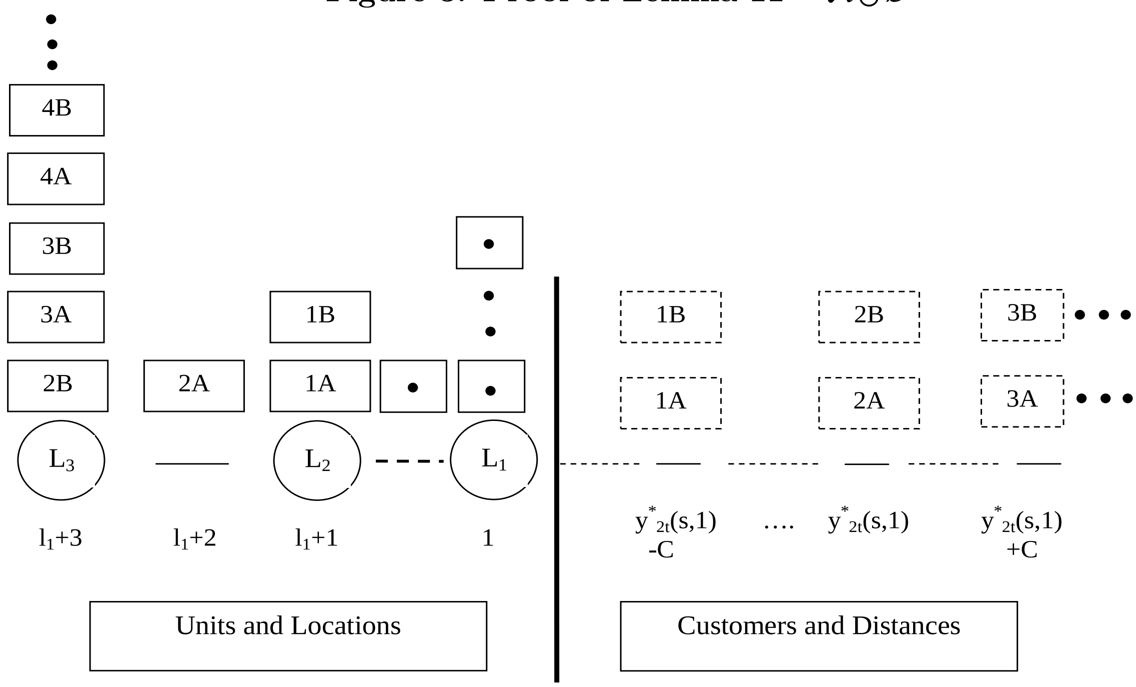


Figure 3: Proof of Lemma 11 – $\mathcal{A} \bigcup \mathcal{B}$



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