

# Inventory Management with Auctions and Other Sales Channels: Optimality of $(s, S)$ Policies

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## Abstract

We study periodic-review inventory replenishment problems with fixed setup costs, and show the optimality of  $(s, S)$  inventory replenishment policies. We consider several sales mechanisms, e.g., auction mechanisms, name-your-own-price mechanisms, and multiple heterogeneous sales channels. We prove this result by showing that these models satisfy the Unifying Assumption of Huh and Janakiraman (2004), which is a sufficient condition for the optimality of  $(s, S)$  policies. Thus, this paper shows the significance of  $(s, S)$  policies by demonstrating its optimality in new settings motivated by the increasing popularity of e-commerce.

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# 1 Introduction

In this paper, we study periodic-review inventory replenishment problems with fixed setup costs. In the classical literature on inventory theory, products are assumed to be sold at an exogenously fixed price. The demand distribution in each period is exogenously determined; the manager is assumed to have no control over demand. With the rise of e-commerce, managers have an increased capability to control demand by (i) dynamically changing prices (from now, we refer to this as the “posted price channel”), (ii) using electronic auctions, or (iii) a combination of such choices. When there is a fixed cost of ordering, we show that the optimal replenishment policy is of the  $(s, S)$  type, thereby demonstrating the robustness of this simple and popular policy in a variety of emerging settings.

We briefly discuss the possible rationales for dynamic demand control (e.g., dynamic pricing). For non-replenishable and perishable products like airline tickets, revenues can be enhanced significantly by changing prices dynamically based on the availability of inventory and the time until departure. The underlying economic reason is that the marginal value of holding a unit of inventory is continuously changing through time. For replenishable and non-perishable products, there has been a recent interest in studying inventory models with dynamic demand control. It is easy to see that the optimal demand control is static over time under the conditions of (i) a stationary demand environment, (ii) a sufficiently long planning horizon, and (iii) inventory replenishment every period. In such models, the value of holding inventory does not change over time. However, when there is a positive setup cost, the time between two successive orders is not a constant and can be large; therefore, the optimal demand control parameter will be dynamically changing. In this paper, we study the optimal inventory replenishment problem with a fixed setup cost and dynamic demand control.

As a motivating example, consider a seller who has a fixed cost for replenishing inventory and sells through an online auction channel like eBay. Each week, she auctions her inventory using a suitably determined reserve price. Any unsold inventory is carried over to the next week. The seller’s optimal control problem is to maximize her expected profit over a planning horizon. In each week, she has to decide whether to place a replenishment order or not, and, if so, how much to order; also, she has to determine the reserve price for that week’s auction depending on the available inventory. Also, consider an industrial parts distributor with a large number of

customers. Buyers arrive sequentially, and the sales manager needs to decide the minimum price at which an order will be accepted. This minimum price dynamically changes over time depending on the inventory availability. This paper shows that under reasonable assumptions, the optimal inventory replenishment policy in models such as these examples is of the  $(s, S)$  type.

In the classical approach, proving the optimality of  $(s, S)$  policies is based on a dynamic programming formulation. Its value function exhibits a certain structure, which is shown to be preserved by an inductive argument. When there is no sales lever, this structure is  $K$ -convexity, a concept due to Scarf (1960). Nearly half a century later, Chen and Simchi-Levi (2003a,b) have extended it to “symmetric  $K$ -convexity” to establish the optimality of  $(s, S)$  policies when the demand distribution in each period is a function of the price set by the manufacturer in that period. A handful of papers have emerged to show the same result for related models with dynamically posted prices, e.g., Chen et al. (2003); Feng and Chen (2003).

A recent paper by Huh and Janakiraman (2004) provides an alternate approach for proving optimality results in stochastic inventory models with multiple sales levers. (We call any decision variable that affects the demand distribution as a *sales lever*, for example, price.) This approach is constructive and works as follows. For any policy that is not of the  $(s, S)$  type, they construct an alternate policy such that it experiences strictly higher expected profits than the first in the initial periods, after which it couples with the original policy. They have identified a set of sufficient conditions, called the *Unifying Assumption*, for the optimality of  $(s, S)$ -type replenishment policies. They show that this Unifying Assumption is satisfied by existing models in the literature that combine pricing and inventory decisions.

While Huh and Janakiraman (2004) have only considered the single sales channel with dynamically posted prices, we show in this paper that the Unifying Assumption is applicable in more general settings. For example, we use it to demonstrate the optimality of  $(s, S)$  policies when demand is controlled using auctions with dynamically changing reserve prices. Furthermore, it is satisfied in an environment with multiple sales channels: for example, one channel which dynamically posts the unit price, and another channel which runs an auction in every period. Therefore, this paper, together with Huh and Janakiraman (2004), demonstrates that there is sufficient struc-

ture in inventory models for the optimality of  $(s, S)$  policies, even when sales are conducted through a variety of channels and influenced by a variety of decision variables.

The remainder of the paper is organized as follows. In section 2, we present a review of the literature in this area. Section 3 contains the basic model and the main result of Huh and Janakiraman (2004) since their result is essential to this paper. In section 4, we develop an inventory replenishment model in which sales are conducted through auctions. Section 5 contains a model where customers arrive and make price offers which can be accepted or rejected by the seller. Section 6 describes a model with multiple sales channels. In section 6.1, we study a model with sequential sales channels, where the first channel is a posted price channel, and the second channel is an auction channel. In sections 6.2 and 6.3, we analyze models in which sales are conducted through multiple channels simultaneously. Section 7 contains a modification of the dynamic pricing model in which the seller has the capability of dynamically changing the price between two successive ordering epochs. We conclude in section 8.

## 2 Literature Survey

**Combined Pricing-Inventory Control.** Recently, there has been a growing interest in periodic-review inventory models in which demand is stochastic and influenced by the pricing decision, i.e., demand  $D(p) = D(p, \epsilon)$  is a function of price  $p$ , where  $\epsilon$  is a stochastic component. When the fixed setup cost of inventory replenishment is zero and excess demand is backordered, Federgruen and Heching (1999) showed the optimality of the base stock inventory policy. Establishing the structure of optimal inventory policies is more challenging when the fixed setup cost is strictly positive. However, under the assumption of complete backlogging, Chen and Simchi-Levi (2003a,b) showed the optimality of  $(s, S)$  inventory policies for finite and infinite horizon models under different assumptions. They used the term  $(s, S, p)$  policy, which refers to an  $(s, S)$  inventory policy where the price  $p$  depends on  $y$ , the inventory level after ordering. Under the assumption of lost sales, Chen et al. (2003) showed a similar result using a finite horizon model. For stationary systems, Huh and Janakiraman (2004) used an alternate proof technique to obtain a few extensions: (i) the joint concavity of the expected single-period profit in Chen and Simchi-Levi (2003b) can be replaced by the weaker assumption of joint quasi-concavity, and (ii) the lost sales result of Chen et al. (2003)

can be extended to the infinite horizon discounted profit criterion. For more references, please see Huh and Janakiraman (2004).

**Auctions and Other Sales Channels.** Single-period auctions have been well-studied in the economics literature. We refer readers to a textbook by Krishna (2002) for details. Inspired by industry practices such as the buy-it-now feature of eBay and other online auctions, a number of recent papers have studied combining an auction and a buy-it-now price, e.g., Budish and Takeyama (2001); Matthews (2003); Hidvegi et al. (2002). They showed that the seller sets the buy-it-now price such that certain bidders may prefer buying at that posted price as opposed to participating in the auction provided that either the buyers are risk-averse or the seller is risk-averse. These three papers assume that only one unit is being auctioned.

There are several papers that have addressed maximizing the seller's revenue when a finite quantity is sold over a time horizon. Some of them use auctions exclusively (Segev et al. (2001); Lavi and Nisan (2004); Vulcano et al. (2002); Gallien (2002)) while others use a combination of auctions and posted prices (Etzion et al. (2004); Caldentey and Vulcano (2004)). The latter set of papers assumes that customers arrive in a single stream, observe both channels, and decide which channel to purchase from. In contrast to this assumption, in our multiple channel model of Section 6, customers arrive in each sales channel in a separate stream. Depending on the context of the application, one model is certainly more applicable than the other. For an excellent review of management science research on online auctions, see Pinker et al. (2003).

The only paper to our knowledge that considers inventory replenishment when the sales channel is not a posted price channel is Van Ryzin and Vulcano (2004). They proved the optimality of a base-stock policy with periodic auctions when there is *no fixed cost* for inventory replenishment. When a positive fixed cost is present and an auction is run every period, there is no known result on the structure of the optimal inventory policies. Furthermore, the optimal inventory replenishment problem has never been studied in the presence of multiple sales channels. In this paper, we address the inventory replenishment problem with a fixed setup cost and more general types of sales channels.

### 3 The Basic Model and the Unifying Assumption

The main result of this paper is the optimality of  $(s, S)$ -type inventory policies in systems with a variety of sales channels. In this section, we review the Unifying Assumption of Huh and Janakiraman (2004) and its implication on the optimality of  $(s, S)$  policies. In subsequent sections, we demonstrate that the systems we study satisfy the Unifying Assumption.

The basic model is the following. Let  $c$  and  $K$  be the variable cost and fixed setup cost of inventory replenishment. Without loss of generality, we assume  $c = 0$  by appropriately transforming the holding and back-order cost function and assuming that all inventory at the end of the horizon can be salvaged at  $c$  dollars per unit. We assume instantaneous inventory replenishment. Let  $x$  and  $y$  be the inventory levels before and after possible replenishment in a period, respectively. Let  $\mathbf{d}$  be the vector of sales levers (e.g. price), and  $D(\mathbf{d}, \epsilon)$  be the stochastic demand, where  $\epsilon$  is a random variable. (We denote vectors by a bold-face.) In this paper, the inventory levels ( $x$  and  $y$ ) and demand  $D(\mathbf{d}, \epsilon)$  are assumed to be *integers*. When demand exceeds inventory, excess demand is backlogged. The single-period expected profit, excluding inventory replenishment cost, is denoted by  $\pi(y, \mathbf{d})$ , which depends on both  $y$  and  $\mathbf{d}$ .

The following sequence of events takes place in each period:

- (1) The available inventory level  $x$  at the beginning of a period is observed.
- (2) A replenishment decision is made to raise the inventory to  $y \geq x$ . A fixed setup cost of  $K$  is incurred if  $y > x$ .
- (3) A sales lever  $\mathbf{d}$  is chosen. Demand is a random variable  $D(\mathbf{d}) = D(\mathbf{d}, \epsilon)$  whose distribution is determined by the choice of  $\mathbf{d}$ . In each of the models we study, the sales lever and the dependence of demand on it are further described in detail.
- (4) Demand is realized and generates revenue. A holding and back-order cost  $h(y - D(\mathbf{d}))$  is incurred. The expected single-period profit  $\pi(y, \mathbf{d})$  is the difference between the expected revenue and  $E[h(y - D(\mathbf{d}))]$ .

An inventory replenishment policy is said to be an  $(s, S)$  policy if there exists two numbers  $s$  and  $S$  such that the inventory level is raised to  $S$  whenever  $x \leq s$ . If these two numbers depend on the period index  $t$ , then we call this an  $(s_t, S_t)$  policy.

We state the Unifying Assumption, which ensures that the problem has a sufficient structure in terms of the expected single-period profit  $\pi$ .

**Assumption 1** (Unifying Assumption). *Let  $Q(y) := \max_{\mathbf{d}} \pi(y, \mathbf{d})$ , and let  $y^*$  be a maximizer of  $Q(\cdot)$ .*

(a)  *$Q(y)$  is quasi-concave, i.e.,  $Q(y)$  is increasing<sup>1</sup> for  $y \leq y^*$ , and decreasing for  $y \geq y^*$ , and*

(b) *for any  $y^1$  and  $y^2$  satisfying  $y^* \leq y^1 \leq y^2$  and any  $\mathbf{d}^2$ , there exists*

$$\mathbf{d}^1 \in \{ \mathbf{d} \mid \pi(y^1, \mathbf{d}) \geq \pi(y^2, \mathbf{d}^2) \}$$

*such that for any  $\epsilon$ ,*

$$y^1 - D(\mathbf{d}^1, \epsilon) \leq \max\{ y^2 - D(\mathbf{d}^2, \epsilon), y^* \}. \quad (1)$$

Part (a) ensures that the closer the after-replenishment inventory level is to  $y^*$ , the more the single-period profit  $\pi$  we can generate. Part (b) shows that if  $y^* \leq y^1 < y^2$ , then the after-replenishment inventory level  $y^1$  is “better than” (more precisely, not worse by more than the setup cost)  $y^2$ . In the current period, the  $y^1$ -system can generate more profit than the  $y^2$ -system. In the next period, the starting inventory of the  $y^1$ -system is closer to  $y^*$  than that of the  $y^2$ -system, or it is possible for the  $y^1$ -system to order up to the inventory level of the  $y^2$ -system. Observe that Assumption 1 is stated in terms of a single-period profit function.

We also state a stronger version of Assumption 1.

**Assumption 2** (Strong Unifying Assumption). *Assumption 1 is satisfied where (1) is replaced by*

$$y^1 - D(\mathbf{d}^1, \epsilon) \leq y^2 - D(\mathbf{d}^2, \epsilon).$$

It is easy to observe that Assumption 2 implies Assumption 1. We now state one of the main results of Huh and Janakiraman (2004):

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<sup>1</sup>In this paper, we use *increasing* (decreasing) to mean *non-decreasing* (non-increasing).

**Theorem 3** (Huh and Janakiraman (2004)). *Suppose Assumption 1 holds. If  $K > 0$ ,  $(s, S)$  replenishment policies are optimal with the infinite horizon discounted-profit model. If  $K = 0$ , the optimal replenishment policy is myopic and of the base-stock type with the finite horizon or the infinite horizon discounted-profit model. In addition, if Assumption 2 also holds, then an  $(s_t, S_t)$  policy is optimal for a finite-horizon model with  $K > 0$ .*

Theorem 3 simplifies the task of proving the optimality of base-stock and  $(s, S)$  policies. It is sufficient to focus on a single period and verify Assumption 1. Thus, the analysis of a multi-period dynamic program is interestingly reduced to examining a simpler single-period problem.

## 4 Periodic Auctions with Inventory Replenishment and Fixed Costs

In this section, we study systems in which inventory is sold through an auction in every period. The reserve prices used in each period are dynamically chosen based on the inventory levels. In section 4.1, we discuss the problem of determining the optimal auction to use in each period. In section 4.2, we restrict our attention to the so-called modified second-price auction, and show that the Unifying Assumption is satisfied. In section 4.3, we extend this result to the case where backorders are not allowed.

### 4.1 Optimal Auctions and Non-Concave Salvage Value Functions

Let us now consider a single-period multi-unit auction in which each customer bids for at most a single unit. Suppose there are  $y$  units available to be auctioned. If  $k$  is the number of units sold in the auction, then the ending inventory is  $y - k$  units. Let  $S(\cdot)$  be the salvage value obtained as a function of the ending inventory  $y - k$ . The seller's problem is to design an auction maximizing the expected revenue and the salvage value. When  $S(\cdot)$  is concave, Vulcano et al. (2002) have shown that an optimal auction is the modified second price auction with an appropriately chosen reserve price vector. However, with a positive setup cost  $K$ , the profit-to-go function from the dynamic program is not typically concave. Consequently, we are interested in the design of an optimal auction in the case where  $S(\cdot)$  is not necessarily concave.



Suppose customer  $n$  has the realized value of  $v_n$ , and submits a bid of  $b_n$ . We assume that customers' values are nonnegative, and have an identical and independent distribution with CDF  $F$  and PDF  $f$ . We let  $\mathbf{v} = (v_1, v_2, \dots)$  and  $\mathbf{b} = (b_1, b_2, \dots)$  be the *value vector* and the *bid vector*, respectively. Let  $b(i)$  be the  $i$ 'th highest bid, and  $v(i)$  be the  $i$ 'th highest value. Ties are broken arbitrarily. When there is no ambiguity, we also use  $\mathbf{v}$  and  $\mathbf{b}$  to denote  $(v(1), v(2), \dots)$  and  $(b(1), b(2), \dots)$ , respectively. Let  $i_n$  be the rank of  $b_n$  in  $\mathbf{b}$  when sorted from the highest, i.e.,  $b_n = b(i_n)$ .

Maskin and Riley (1989) have shown that the expected revenue of the seller is determined by the allocation, a result known as the *revenue equivalence theorem*. Let  $q_n(\mathbf{v})$  be the binary variable indicating whether a unit is allocated to customer  $n$  given a value vector  $\mathbf{v}$ . Let  $J(v) = v - 1/\rho(v)$  be the *virtual value* function where  $\rho(v) = f(v)/(1 - F(v))$  is the associated failure rate function. Then, the expected seller's revenue is given by

$$E_{\mathbf{v}} \left[ \sum_n J(v_n) \cdot q_n(\mathbf{v}) \right],$$

provided that (i) the bidder with  $v_n = 0$  has an expected equilibrium profit of 0, and (ii)  $q_n(\mathbf{v}) \leq q_n(\mathbf{v}')$  whenever  $\mathbf{v}$  and  $\mathbf{v}'$  are identical except  $v_n < v'_n$ .

Let  $\Delta S(x) = S(x+1) - S(x)$  be the marginal salvage value. Given that  $y$  units are available for auction, the seller's expected profit is

$$E_{\mathbf{v}} \left[ \sum_{i=1}^y \{J(v(i)) - \Delta S(y-i)\} \cdot q_i(\mathbf{v}) \right], \quad (2)$$

plus an additive constant. We assume, as in Maskin and Riley (1989) and Vulcano et al. (2002), that  $J$  is an increasing function. Therefore, if we sell  $k$  units to the bidders with the  $k$  highest values, the expected marginal profit from the  $i$ 'th unit is  $J(v(i)) - \Delta S(y-i)$ . If  $S$  is zero as in Maskin and Riley (1989), or concave as in Vulcano et al. (2002), the expected marginal profit is decreasing in  $i$ . Therefore, the modified second price auction with the reserve price vector  $(J^{-1}(\Delta S(y-1)), J^{-1}(\Delta S(y-2)), \dots)$  is optimal.<sup>2</sup>

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<sup>2</sup>In the modified second-price auction, the seller selects the reserve price vector  $\mathbf{r} = (r(1), r(2), \dots)$  where  $r(1) \leq r(2) \leq \dots$ , and the  $i$ 'th unit will be sold if and only if  $b(i) \geq r(i)$ . Given a particular bid vector  $\mathbf{b}$ , the number of sold units is given by  $\kappa(\mathbf{r}, \mathbf{b}) = \max\{i : b(i) \geq r(i)\}$ . The highest  $\kappa(\mathbf{r}, \mathbf{b})$  bidders receive a unit, and pay a uniform price of  $\max\{r(\kappa(\mathbf{r}, \mathbf{b})), b(\kappa(\mathbf{r}, \mathbf{b}) + 1)\}$ . It is well known that a dominant strategy for buyers is to bid their own values in this auction.

However, if  $S(\cdot)$  is not concave, then  $J^{-1}(\Delta S(y-i))$  may not be increasing in  $i$ , and there may be no reserve price vector that achieves an optimal auction. This is because it is possible that the  $i$ 'th unit will be sold even if the marginal profit  $J(v_i) - \Delta S(y-i)$  is negative since the marginal profit from subsequent units will be substantially high. In other words, a marginal loss from one unit may be offset by the marginal profits from selling the subsequent units. To our knowledge, the optimal auction with a non-concave salvage value function (or equivalently, a non-convex production cost function) is not known. This is an interesting research question in its own right. Also, it appears that even if such an optimal auction can be found, it would be a complex one. In this paper, we restrict our attention to auctions the modified second-price auction of Vulcano et al. (2002).

Thus, the seller's optimization problem is to set the reserve price vector to maximize the expected profit, i.e.,

$$\begin{aligned} \max_{\mathbf{r}} \quad & E_{\mathbf{v}} \left[ \sum_{i=1}^y \{J(v(i)) - \Delta S(y-i)\} \cdot I[v(i) \geq r(i)] \right] \\ \text{s. t.} \quad & r(1) \leq r(2) \leq \dots \leq r(y) \end{aligned} \tag{3}$$

where  $I$  is a binary indicator function.

We remark that our analysis also holds for the modified *first-price* auction (pay-your-bid auction) as in Van Ryzin and Vulcano (2004).

## 4.2 Periodic Auction Sales Channel: Backorders Allowed

In this section, we consider a model in which inventory is sold using an auction in each period. Inventory is reviewed every period and replenished, if necessary. We consider the class of modified second-price auctions with reserve price vectors.

The number of customers and the value vectors are independently and identically distributed across time periods, and each buyer submits his bid only once in the period he arrives. These assumptions are also used by Van Ryzin and Vulcano (2004).

In the *periodic auction* model, the following sequence of events takes place in each period.

- (1) The available inventory level  $x$  at the beginning of a period is observed.

- (2) An ordering decision is made to raise the inventory to  $y \geq x$ . A fixed setup cost of  $K$  is incurred if  $y > x$ .
- (3) A reserve price vector  $\mathbf{r} = (r(1), r(2), \dots)$  is chosen and announced to the bidders, where  $r(1) \leq r(2) \leq \dots$ .
- (4) Customers arrive and submit their bids. The bid vector  $\mathbf{b} = (b(1), b(2), \dots)$  is sorted in a descending order.
- (5) The allocation and payment decisions are made according to the modified second-price auction mechanism. A holding and back-order cost of  $h(y - k)$  is incurred.

By the incentive compatibility property, customers' bids are the same as their values, i.e.,  $b(i) = v(i)$ . Thus, recall that the number of units sold equals

$$\kappa(\mathbf{r}, \mathbf{v}) = \sum_{i=1}^{\infty} I[v(i) \geq r(i)] , \quad (4)$$

where  $I$  is the indicator function. We suppose  $h(\cdot)$  is nonnegative and convex in  $\mathcal{Z}$ , and satisfies  $h(0) = 0$ . Furthermore, we assume that the holding cost is linear, i.e., there exists  $h \geq 0$  such that  $h(z) = h \cdot z$  for  $z \geq 0$ . The back-order cost does not have to be linear. Let  $\Delta h(z) = h(z) - h(z-1)$ .

We note that our model allows the seller to sell more units than the inventory level. This would occur when buyers' bids are sufficiently high.

Let  $R(y, \mathbf{r})$  denote the seller's expected single-period profit when the inventory level after replenishment is  $y$  and the reserve price vector is  $\mathbf{r}$ . The value of  $R(y, \mathbf{r})$  includes the holding and back-order cost  $h(\cdot)$ , but excludes the fixed setup cost  $K$ . By the revenue equivalence theorem, we know that  $R(y, \mathbf{r}) = E_{\mathbf{v}} [R(y, \mathbf{r}, \mathbf{v})]$ , where

$$R(y, \mathbf{r}, \mathbf{v}) = -h(y) + \sum_{i=1}^{\infty} [J(v(i)) + \Delta h(y - i + 1)] \cdot I[v(i) \geq r(i)]. \quad (5)$$

Let

$$L(y) = \max_{\mathbf{r}} R(y, \mathbf{r}) .$$

Then,  $L(y)$  is the maximum possible expected profit in a single period. Let  $\Delta L(y) = L(y+1) - L(y)$ . Let  $y^*$  be the largest maximizer of  $L(y)$ .

We recall our assumption that the virtual value function  $J(\cdot)$  is a strictly increasing function. It follows that  $J(v(i))$  is decreasing in  $i$ . Let

$$v^* = J^{-1}(0) , \text{ and } v_y^*(i) = J^{-1}(-\Delta h(y - i + 1)). \quad (6)$$

We remark that if the starting inventory is  $y$ , then  $v_y^*(i)$  is the least value above which it is profitable, in this single period, to sell the  $i$ 'th unit to a customer with that value. Let

$$\mathbf{r}_y = (v_y^*(1), v_y^*(2), \dots). \quad (7)$$

Then, the components of  $\mathbf{r}_y$  are increasing since  $J(\cdot)$  is an increasing function and  $h(\cdot)$  is convex.

**Proposition 4.** *In the periodic auction model, the following statements are true:*

- (a)  $L(y) = R(y, \mathbf{r}_y)$ .
- (b)  $L(y)$  is concave.
- (c)  $\Delta L(y) \leq 0$  for  $y \geq y^*$ , and  $\Delta L(y) \geq 0$  for  $y < y^*$ .
- (d)  $y^* \geq 0$ .

*Proof.* To prove (a), observe from equation (5) that  $\mathbf{r}_y$  maximizes  $R(y, \mathbf{r}, \mathbf{v})$  for each  $\mathbf{v}$ . Therefore,  $\mathbf{r}_y$  clearly maximizes  $R(y, \mathbf{r})$ . The proof of (b) is given in appendix A.1. Statement (c) immediately follows from (b). To prove (d), consider the definition (5) of  $R(y, \mathbf{r}, \mathbf{v})$ :

$$\begin{aligned} R(0, \mathbf{r}, \mathbf{v}) &= -h(0) + \sum_{i=1}^{\infty} [J(v(i)) + \Delta h(1 - i)] \cdot I[v(i) \geq r(i)], \quad \text{and} \\ R(-1, \mathbf{r}, \mathbf{v}) &= -h(-1) + \sum_{i=1}^{\infty} [J(v(i)) + \Delta h(-i)] \cdot I[v(i) \geq r(i)]. \end{aligned}$$

For any integer  $z$ , we have  $h(z) \geq 0 = h(0)$ . By the convexity of  $h(\cdot)$ , we have  $\Delta h(-i+1) \geq \Delta h(-i)$ . it follows that  $R(0, \mathbf{r}, \mathbf{v}) \geq R(-1, \mathbf{r}, \mathbf{v})$  for any  $\mathbf{r}$  and  $\mathbf{v}$ . Thus,

$$R(-1, \mathbf{r}_{-1}) = E_{\mathbf{v}}[R(-1, \mathbf{r}_{-1}, \mathbf{v})] \leq E_{\mathbf{v}}[R(0, \mathbf{r}_{-1}, \mathbf{v})] = R(0, \mathbf{r}_{-1}) \leq R(0, \mathbf{r}_0).$$

Therefore,  $\Delta L(-1) = L(0) - L(-1) \geq 0$  from (a). Since  $y^*$  is the largest maximizer of  $L(\cdot)$ , we get  $y^* \geq 0$ . □

From (6) and (7), we recall  $\mathbf{r}_y = (v_y^*(1), v_y^*(2), \dots)$  where  $v_y^*(i) = J^{-1}(-\Delta h(y - i + 1))$ .

**Lemma 5.** For  $y^2 > y^*$ , consider a system  $\mathcal{A}$  with  $y^2$  units of inventory after replenishment and the reserve price vector  $\mathbf{r} = (r(1), r(2), \dots)$ . For  $y^1 \in [y^*, y^2)$ , let  $\tilde{\mathcal{A}}$  be another system with  $y^1$  units of inventory after replenishment and the reserve price vector  $\tilde{\mathbf{r}} = (\tilde{r}(1), \tilde{r}(2), \dots)$ , where

$$\tilde{r}(i) = \begin{cases} \min\{v_{y^1}^*(i), r(i)\}, & \text{if } 1 \leq i \leq y^1 \\ v_{y^1}^*(i), & \text{if } i \geq y^1 + 1. \end{cases}$$

For a given realization of  $\mathbf{v}$ , let  $z^1(\mathbf{v})$  and  $z^2(\mathbf{v})$  denote the ending inventories in systems  $\tilde{\mathcal{A}}$  and  $\mathcal{A}$ , respectively. Then, the single-period expected profits satisfy  $R(y^1, \tilde{\mathbf{r}}) \geq R(y^2, \mathbf{r})$ , and

$$z^1(\mathbf{v}) \leq \max\{0, z^2(\mathbf{v})\} \leq \max\{y^*, z^2(\mathbf{v})\}. \quad (8)$$

*Proof.* Since  $\mathbf{r}$  is a valid reserve price vector,  $r(i)$  is increasing in  $i$ . Since  $v_y^*(i)$  is increasing in  $i$  for any fixed  $y$ , it follows that the components of  $\tilde{\mathbf{r}}$  are increasing.

For any given  $\mathbf{v}$ , we recall that  $\kappa(\tilde{\mathbf{r}}, \mathbf{v})$  and  $\kappa(\mathbf{r}, \mathbf{v})$  are the number of units sold in  $\tilde{\mathcal{A}}$  and  $\mathcal{A}$ , respectively. We show (8) by considering two disjoint cases. In the first case, suppose  $\kappa(\mathbf{r}, \mathbf{v}) \leq y^1$ , i.e.,  $z^1(\mathbf{v}) \geq 0$ . By the construction of  $\tilde{r}$ , we have  $\tilde{r}(i) \leq r(i)$  for each  $i = 1, 2, \dots, y^1$ , implying  $\kappa(\tilde{\mathbf{r}}, \mathbf{v}) \geq \kappa(\mathbf{r}, \mathbf{v})$ . Thus,

$$z^1(\mathbf{v}) = y^1 - \kappa(\tilde{\mathbf{r}}, \mathbf{v}) \leq y^2 - \kappa(\mathbf{r}, \mathbf{v}) = z^2(\mathbf{v})$$

Thus, we obtain (8) in this case. In the second case, we have  $\kappa(\mathbf{r}, \mathbf{v}) > y^1$ . It follows  $v(y^1) \geq r(y^1) \geq \tilde{r}(y^1)$ . Therefore, we have  $\kappa(\tilde{\mathbf{r}}, \mathbf{v}) \geq y^1$ , and  $z^1(\mathbf{v}) \leq 0$ . Now (8) follows from  $y^* \geq 0$  in Proposition 4.

Next, we proceed to show  $R(y^1, \tilde{\mathbf{r}}) \geq R(y^2, \mathbf{r})$ . We recall that  $\kappa(\mathbf{r}_{y^1}, \mathbf{v})$  and  $\kappa(\mathbf{r}_{y^2}, \mathbf{v})$  are the number of units sold when the reserve price vectors are  $\mathbf{r}_{y^1}$  and  $\mathbf{r}_{y^2}$ , respectively. We claim that for any  $\mathbf{v}$ ,

$$R(y^2, \mathbf{r}, \mathbf{v}) - R(y^1, \tilde{\mathbf{r}}, \mathbf{v}) \leq R(y^2, \mathbf{r}_{y^2}, \mathbf{v}) - R(y^1, \mathbf{r}_{y^1}, \mathbf{v}). \quad (9)$$

The proof of this claim is given in appendix A.2. Then, it follows that

$$\begin{aligned} R(y^2, \mathbf{r}) - R(y^1, \tilde{\mathbf{r}}) &= E_{\mathbf{v}} [R(y^2, \mathbf{r}, \mathbf{v}) - R(y^1, \tilde{\mathbf{r}}, \mathbf{v})] \\ &\leq E_{\mathbf{v}} [R(y^2, \mathbf{r}_{y^2}, \mathbf{v}) - R(y^1, \mathbf{r}_{y^1}, \mathbf{v})] \\ &= L(y^2) - L(y^1), \end{aligned}$$

where the last equality comes from Proposition 4. Since  $y_2 > y_1 \geq y^*$ , we have  $L(y^1) \geq L(y^2)$ , from which we obtain  $R(y^1, \tilde{\mathbf{r}}) \geq R(y^2, \mathbf{r})$ .  $\square$

Lemma 5 establishes that the Unifying Assumption holds for the periodic auction sales channel model presented in this section. Thus we obtain the conclusion of Theorem 3, in particular the optimality of  $(s, S)$  policies.

**Theorem 6.** *The optimal inventory replenishment policy is of the  $(s, S)$  type under the following assumptions: (i) the class of auctions under consideration is the class of modified second-price auctions with increasing reserve price vectors, and (ii) the number of customers and the value vectors are independently and identically distributed across time periods.*

### 4.3 Periodic Auction Sales Channel: Backorders Not Allowed

In section 4.2, the seller was allowed to sell more units than the inventory level and backlog excess demand. In this section, we assume that the seller cannot sell any more units than the inventory level after replenishment in each period.

When back-orders are not allowed, the analysis of section 4.2 carries over by setting an arbitrarily high cost for back-orders, i.e.,  $h(z) = \infty$  for  $z < 0$ . (It is useful to define  $h(z) = -\infty$  for  $z \leq 0$ .) Thus, the seller does not want to backlog any demand, i.e.,  $r(i) = \infty$  for any  $i > y$ , where  $y$  is the inventory level after replenishment. In this case, a slightly stronger version of Lemma 5 holds. This result will be useful later in the analysis of dual channel models.

**Lemma 7.** *Under the assumptions and definitions in Lemma 5, if no back-orders are allowed, the conclusions of Lemma 5 and Theorem 6 hold. Moreover, we have*

$$z^1(\mathbf{v}) \leq z^2(\mathbf{v}) ,$$

*i.e., the Strong Unifying Assumption (Assumption 2) is satisfied.*

*Proof.* Since back-orders are not allowed, the reserve price vector  $\tilde{\mathbf{r}} = (\tilde{r}(1), \tilde{r}(2), \dots)$  in Lemma 5 is given by

$$\tilde{r}(i) = \begin{cases} \min\{v_{y^1}^*(i), r(i)\}, & \text{if } 1 \leq i \leq y^1 \\ \infty, & \text{if } i \geq y^1 + 1. \end{cases}$$

We consider two separate cases. If  $\kappa(\mathbf{r}, \mathbf{v}) \leq y^1$ , then the definition of  $\tilde{\mathbf{r}}$  implies that  $\kappa(\tilde{\mathbf{r}}, \mathbf{v}) \geq \kappa(\mathbf{r}, \mathbf{v})$ . Thus, we have  $y^1 - \kappa(\tilde{\mathbf{r}}, \mathbf{v}) \leq y^2 - \kappa(\mathbf{r}, \mathbf{v})$ , and the required inequality holds. Otherwise, if  $\kappa(\mathbf{r}, \mathbf{v}) > y^1$ , it follows  $\kappa(\tilde{\mathbf{r}}, \mathbf{v}) = y^1$ . Since back-orders are not allowed, we must have  $z^1(\mathbf{v}) = 0 \leq z^2(\mathbf{v})$ .  $\square$

Whereas the majority of this paper uses a increasing reserve price vector, we make a brief comment on the modified second-price auction with a uniform reserve price. Let us suppose that the seller is restricted to set the same reserve price for each of the  $y$  units, and does not sell more than  $y$  units. Consider Lemma 5. Suppose that for some  $r$ , the reserve price vector  $\mathbf{r}$  satisfies  $r(i) = r$  for  $i \leq y^2$ , and  $r(i) = \infty$  for  $i > y^2$ . Let  $\tilde{r} = \min\{J^{-1}(-h), r\}$ . Then, the construction of  $\tilde{\mathbf{r}}$  satisfies  $\tilde{r}(i) = \tilde{r}$  for  $i \leq y^1$ , and  $\tilde{r}(i) = \infty$  for  $i > y^1$ . In other words, if  $\mathbf{r}$  represents a uniform reserve price, then  $\tilde{\mathbf{r}}$  also represents a uniform reserve price. Thus, when the reserve price vector is restricted to a uniform price, the Strong Unifying Assumption result still holds, as in Lemma 7.

## 5 Bid-Price Control

Consider a *bid-price control* model. Suppose that in each period, the seller observes the available inventory  $x$ , and has an option of raising it to  $y > x$  incurring a fixed setup cost of  $K$ . In each period, there is at most one customer arrival, and the arrival probability is  $q$ . (This could be treated as a discrete-time approximation of a Poisson arrival model.) The customer submits a bid of  $v$  for a single unit, and we assume that the customer bids are independently and identically distributed as  $F(v)$ . The seller decides whether or not to accept this bid. Since the bids are independent, it is optimal to accept the bid if it exceeds a threshold that depends on the inventory level  $y$ . An accepted order is back-ordered if inventory is not available. If  $z$  is the inventory level at the end of the period, a holding and back-order cost of  $h(z)$  is incurred, where  $h(z) = h \cdot [z]^+ + b \cdot [z]^-$  for some nonnegative constants  $h$  and  $b$ . This model is motivated by the name-your-own-price practice of priceline.com.

We will now demonstrate that this model satisfies the Strong Unifying Assumption.

**Lemma 8.** *In the bid-price control model described in this section, the Strong Unifying Assumption (Assumption 2) is satisfied, and the conclusion of Theorem 3 holds.*

*Proof.* When the inventory level is  $y$ , the maximum expected single-period profit, excluding the setup cost, is given by

$$Q(y) = -h(y) + \max_r (E[v|v \geq r] + \Delta h(y)) \cdot (1 - F(r)).$$

We claim that  $Q(\cdot)$  is quasi-concave. To see this, let

$$\begin{aligned} \phi_1 &= \max_r (E[v|v \geq r] - b) \cdot (1 - F(r)) \quad \text{and} \\ \phi_2 &= \max_r (E[v|v \geq r] + h) \cdot (1 - F(r)). \end{aligned}$$

Note that both  $\phi_1$  and  $\phi_2$  are independent of  $y$ . It follows that

$$Q(y) = \begin{cases} b \cdot y + \phi_1 & \text{if } y \leq 0, \\ -h \cdot y + \phi_2 & \text{if } y \geq 1. \end{cases}$$

Thus,  $Q$  is increasing for  $y \leq 0$ , and decreasing for  $y \geq 1$ . Thus,  $Q$  is quasi-concave on the set of integers, proving the claim.

Furthermore, suppose  $y^* \leq y^1 < y^2$ . Consider two systems  $\tilde{\mathcal{A}}$  and  $\mathcal{A}$ , in which the inventory levels after replenishment are  $y^1$  and  $y^2$ , respectively. For the  $\tilde{\mathcal{A}}$  system, let  $\tilde{r}$  be the maximizer of  $\phi_1$  if  $y^1 \leq 0$ , and the maximizer of  $\phi_2$  if  $y^1 \geq 1$ . Thus,  $\tilde{\mathcal{A}}$  system achieves the maximum expected single-period cost. Furthermore, the ending inventory of the  $\tilde{\mathcal{A}}$  system is at most the ending inventory of the  $\mathcal{A}$ -system since the demand is at most one.  $\square$

The above lemma can be extended to allow for Markov-modulated arrival probabilities.

## 6 Multiple Sales Channels

In this section, we study the optimal inventory policy when there are multiple sales channels. We present a formal definition of a sales channel. A *sales channel*  $\mathcal{M} = (D, \Lambda)$  consists of demand function  $D$  and revenue function  $\Lambda$ :

- $D(\mathbf{d})$  is stochastic demand as a function of the sales lever  $\mathbf{d}$ . Let  $D(\mathbf{d}) = D(\mathbf{d}, \epsilon)$ , where  $\epsilon$  is a random variable.



- $\Lambda(\mathbf{d})$  is the expected single-period revenue when the sales lever is  $\mathbf{d}$ .

We provide a few examples of sales channels. The first example is the dynamic pricing channel with *discrete additive demand* (studied in Huh and Janakiraman (2004)). Here, the sales lever is the expected demand  $d$ , which belongs to a set of consecutive integers between two numbers,  $d_L$  and  $d_U$ . Then,

$$\begin{aligned} D(d) &= d + \epsilon, \\ \Lambda(d) &= p(d) \cdot d, \end{aligned}$$

where  $\epsilon$  is a random variable with  $E[\epsilon] = 0$ , and  $p(d)$  is the price corresponding to  $d$ . The second example is the auction model of section 4, where the sales lever is the reserve price vector. Let  $\mathbf{V}(\epsilon) = (V(1, \epsilon), V(2, \epsilon), \dots)$  be the sorted value vector parameterized by a random vector  $\epsilon$ , and let  $\mathbf{r}$  be a reserve price vector. We set

$$\begin{aligned} D(\mathbf{r}) &= \kappa(\mathbf{r}, \mathbf{V}(\epsilon)), \\ \Lambda(\mathbf{r}) &= E \left[ \sum_{i=1}^{D(\mathbf{r})} J(V(i, \epsilon)) \right], \end{aligned}$$

where  $\kappa(\cdot, \cdot)$  is the number of units sold as defined in (4).

In the next three subsections, we will present three different models of systems with multiple sales channels. These models differ with respect to whether inventory is shared or pre-allocated among channels, and whether the channels are operated simultaneously or sequentially.

## 6.1 Posted Pricing and Auctioning: A Sequential Model

In this section, we consider a dual sales channel system in which the first channel is a posted price channel and the second is an auction channel. In each period, inventory is first allocated to the posted price channel, and then the remaining inventory is sent to the auction channel. One may interpret the posted price channel as a primary channel and the auction channel as a secondary channel. This section is motivated by recent papers on online auctions such as Budish and Takeyama (2001), Etzion et al. (2004), Van Ryzin and Vulcano (2004), and Caldentey and Vulcano (2004).

In the *posted pricing and auction dual channel* model, the following sequence of events takes place in each period:

- (1) The available inventory level  $x$  at the beginning of a period is observed.
- (2) A replenishment decision is made to raise the inventory to  $y \geq x$ . A fixed setup cost of  $K$  is incurred if  $y > x$ .
- (3) The sales lever  $d$  (“demand level”) for the posted price channel is chosen, where  $d \in \mathcal{D}$  where  $\mathcal{D} = \{d_L, d_{L+1}, \dots, d_U\}$  for some integers  $d_L$  and  $d_U$ . Then, demand  $D(d) = D(d, \epsilon_P) = d + \epsilon_P$  is realized, where  $\epsilon_P$  is an integer random variable with  $E[\epsilon_P] = 0$ . We require  $\epsilon_P \geq -d_L$  so that  $D(d, \epsilon_P)$  is always nonnegative. Let  $d_P = D(d, \epsilon_P)$  be the realized posted price channel demand. (This is the discrete additive demand model referred to earlier.)
- (4) The reserve price vector  $\mathbf{s} = (s(y - d_P - 1), s(y - d_P - 2), \dots, s(0), s(-1), \dots)$  for the auction channel is chosen, where the components of  $\mathbf{s}$  are increasing. Then, the customer value vector  $\mathbf{v} = (v(1), v(2), v(3), \dots)$  is realized. The number of units sold through the auction equals

$$k(\mathbf{s}, \mathbf{v}, d_P) = \sum_{i=d_P+1}^{\infty} I[v(i - d_P) \geq s(y - i)].$$

- (5) Holding and back-order costs are charged based on the ending inventory level  $y - D(d, \epsilon_P) - k(\mathbf{s}, \mathbf{v}, d_P)$ . We assume a linear holding cost and a convex back-order cost function, as in section 4.

From (3) and (4), the first  $d_P$  units are sold through the posted price channel. For  $i > d_P$ ,  $s(y - i)$  acts as the reserve price for the  $(i - d_P)$ ’th unit in the auction channel, i.e., the  $(i - d_P)$ ’th is sold if and only if  $v(i - d_P) \geq s(y - i)$ . We assume a separate stream of customers for each sales channel. Within the posted price channel, the demand distribution in each period depends only on the sales lever  $d$  of that period. Similarly, in the auction channel, the number of bidders and their valuations are independent over time.

The expected revenue from the posted price channel is  $p(d) \cdot d$ , which we assume to be concave in  $d$ . Let  $t = y - d_P$  be the inventory level between steps (3) and (4), i.e., immediately after

demand from the posted price channel is satisfied. Let  $R_A(t, \mathbf{s})$  be the expected single-period profit from the auction channel, similar to  $R$  in section 4. The value of  $R_A(t, \mathbf{s})$  includes the holding and back-order cost. Let  $L_A(t)$  be the maximum expected single-period profit from the auction channel, i.e.,  $L_A(t) = \max_{\mathbf{s}} R_A(t, \mathbf{s})$ . This  $L_A(\cdot)$  corresponds to  $L(\cdot)$  in section 4. The maximum value  $L_A(\cdot)$  is achieved when the reserve price vector  $\mathbf{s}$  satisfies  $s(y - i) = v_y^*(i)$ , i.e.,

$$(s(y - 1), s(y - 2), \dots) = (v_y^*(1), v_y^*(2), \dots) = \mathbf{r}_y,$$

where we recall  $v_y^*(i) = J^{-1}(-\Delta h(y - i + 1))$ . It follows that the maximum total expected profit from both channels in a single period is

$$Q(y) = \max_{d \in \mathcal{D}} p(d) \cdot d + E[L_A(y - d - \epsilon_P)].$$

**Proposition 9.**  *$Q(y)$  is a concave function.*

*Proof.* The proof is based on constructing a continuous-space interpolation of the single period profit function with respect to  $y$  and  $d$ . For the details, see Appendix A.3.  $\square$

**Theorem 10.** *The sequential list-price and auction channel model described in this section satisfies the Unifying Assumption, and the conclusion of Theorem 3 holds.*

*Proof.* We have already shown the concavity of  $Q$ . Let  $t^*$  be the maximizer of  $E[L_A(t - \epsilon_P)]$  over  $t$ , and let  $d^*$  be the maximizer of  $p(d) \cdot d$  over  $d$ . It is straightforward to observe that  $Q(\cdot)$  is maximized at  $y^* = d^* + t^*$ .

Suppose  $y^1$  and  $y^2$  satisfy  $y^* \leq y^1 < y^2$ . Consider two systems  $\tilde{\mathcal{A}}$  and  $\mathcal{A}$ , in which the inventory levels after replenishment are  $y^1$  and  $y^2$ , respectively. Let  $(d, \mathbf{s})$  be the pair of posted price sales lever and reserve price vector used in  $\mathcal{A}$ . We show how to choose  $(\tilde{d}, \tilde{\mathbf{s}})$  for the  $\tilde{\mathcal{A}}$  system to satisfy the second part of the Unifying Assumption.

Let  $t^2 = y^2 - d$ . Set  $t^1 = \min\{y^1 - d^*, t^2\}$  and  $\tilde{d} = y^1 - t^1$ . We consider two disjoint cases. In the first case, we assume  $t^1 = t^2$ . Then, we must have  $d^* \leq \tilde{d} \leq d$ , which implies  $p(\tilde{d}) \cdot \tilde{d} \geq p(d) \cdot d$  by the concavity of revenue from the posted price channel. We set  $\tilde{\mathbf{s}} = \mathbf{s}$ , and it follows that

$$E[R_A(t^1 - \epsilon_P, \tilde{\mathbf{s}})] = E[R_A(t^2 - \epsilon_P, \mathbf{s})].$$

Furthermore, the ending inventories of  $\tilde{\mathcal{A}}$  and  $\mathcal{A}$  systems are the same.

In the second case, we have  $t^1 = y^1 - d^* < t^2$ , and  $\tilde{d} = d^*$ . Thus,  $p(\tilde{d}) \cdot \tilde{d} = p(d^*) \cdot d^* \geq p(d) \cdot d$ . Furthermore, we have  $t^* \leq t^1 < t^2$ . We set  $\tilde{s} = (\tilde{s}(y-1), \tilde{s}(y-2), \dots)$  such that

$$\tilde{s}(y-i) = \begin{cases} \min \{ J^{-1}(-\Delta h(y-i)), s(y-i + (t^2 - t^1)) \}, & \text{if } i \leq y; \\ J^{-1}(-\Delta h(y-i)), & \text{if } i > y. \end{cases}$$

Following an argument similar to the proof of Lemma 5, it can be shown that for every realization of  $\epsilon_P$ , (i)  $\tilde{\mathcal{A}}$  has a single-period profit no less than  $\mathcal{A}$ , and (ii) the ending inventory of  $\tilde{\mathcal{A}}$  is either negative or at most the ending inventory of  $\mathcal{A}$ .  $\square$

## 6.2 Multiple Sales Channels with Inventory Allocation

In this section, we consider a system in which inventory is pre-allocated among multiple sales channels at the beginning of each period.

We suppose that there are  $M$  sales channels indexed by  $m = 1, 2, \dots, M$ . In each period, the inventory level after possible replenishment is rationed among the  $M$  sales channels, and the seller sets the sales lever for each channel. The demand distribution faced by each mechanism is determined by its sales lever, and independent of one another. The following sequence of events takes place in each period:

- (1) The available inventory level  $x$  at the beginning of a period is observed.
- (2) A replenishment decision is made to raise the inventory to  $y \geq x$ . A fixed setup cost of  $K$  is incurred if  $y > x$ .
- (3) The inventory is allocated among sales channels. Let  $y_m$  be the amount of inventory allocated to mechanism  $m$ . Thus, we must have  $\sum_m y_m = y$ . For each mechanism  $m$ , a sales lever  $\mathbf{d}_m$  is chosen. The expected single-period revenue from mechanism  $m$  is  $\Lambda_m(\mathbf{d}_m)$ .
- (4) Demand  $D_m(\mathbf{d}_m)$  is realized from for each mechanism, and the appropriate single-period revenue is gained. Let  $z_m = y_m - D_m(\mathbf{d}_m)$  denote the ending inventory level of sales channel  $m$ .

(5) Holding cost of  $H(z^h)$  and back-order cost of  $B(z^b)$  are incurred, where

$$z^h = \sum_m [z_m]^+ \quad \text{and} \quad z^b = \sum_m [z_m]^-.$$

We assume that both the holding cost and the back-order cost are linear, i.e.,  $H(z^h) = h \cdot z^h$  and  $B(z^b) = b \cdot z^b$  for some nonnegative constants  $h$  and  $b$ . In the next period, inventory, if any, should first be allocated towards back-orders. The net inventory of the entire system at the beginning of the next period is  $z = z^h - z^b$ . *Note that inventory can be shared among channels before demand is realized, but not after it is realized.*

Let  $h(z) = h \cdot [z]^+ + b \cdot [z]^-$ . Then,  $H(z^h) + B(z^b) = \sum_m h(z_m)$ . If we charge the holding and back-order cost  $h(z_m)$  to sales channel  $m$ , the expected profit of sales channel  $m$  is given by

$$\pi_m(y_m, \mathbf{d}_m) = \Lambda_m(\mathbf{d}_m) - E[h(y_m - D_m(\mathbf{d}_m))]. \quad (10)$$

The seller's expected profit is the sum of expected profits in all sales channels. Let  $\mathbf{d} = (\mathbf{d}_1, \mathbf{d}_2, \dots, \mathbf{d}_M)$ . For fixed  $y$  and  $\mathbf{d}$ , the seller's maximum system-wide expected profit is

$$\pi(y, \mathbf{d}) = \max \left\{ \sum_m \pi_m(y_m, \mathbf{d}_m) \mid \sum_m y_m = y \right\}.$$

We proceed by assuming that the distribution  $D_m(\mathbf{d}_m)$  is discrete, and its support is a set of consecutive integers for each  $m$ . We need the following lemma about the optimal allocation problem which is useful in studying the multiple channel problem.

**Lemma 11.** *For each  $i = 1, 2, \dots, I$ , let  $f_i(\cdot)$  be a quasi-concave function defined on a set of consecutive integers. Let  $s_i^*$  be a maximizer of  $f_i(\cdot)$ . Then,  $f(s) = \max\{\sum_i f_i(s_i) \mid \sum_i s_i = s\}$  is quasi-concave, and achieves its maximum at  $\sum_i s_i^*$ .*

*Proof.* See appendix A.4. □

**Theorem 12.** *Consider the multiple sales channel model with inventory allocation. Suppose that the support of demand in each channel is a set of consecutive integers. If each channel satisfies the Strong Unifying Assumption (Assumption 2), then the multiple sales channel model also satisfies the Strong Unifying Assumption. Therefore, the conclusion of Theorem 3 holds.*

*Proof.* Let  $Q_m(y_m) = \max_{\mathbf{d}_m} \pi_m(y_m, \mathbf{d}_m)$ , and  $Q(y) = \max_{\mathbf{d}} \pi(y, \mathbf{d})$ . Since each sales channel  $m$  satisfies Assumption 2,  $Q_m(\cdot)$  is quasi-concave. Let  $y_m^*$  be a maximizer of  $Q_m$ . Then, by Lemma 11,  $Q(\cdot)$  is also quasi-concave, and achieves its maximum at  $y^* = \sum_m y_m^*$ .

Consider two systems  $\tilde{\mathcal{A}}$  and  $\mathcal{A}$ , in which the inventory levels after replenishment are  $y^1$  and  $y^2$ , respectively. Assume  $y^* \leq y^1 < y^2$ . Suppose that in the  $\mathcal{A}$  system, the seller chooses an allocation of  $\mathbf{y} = (y_1, \dots, y_M)$  where  $y^2 = \sum_m y_m$ , and the sales lever vector of  $\mathbf{d} = (\mathbf{d}_1, \dots, \mathbf{d}_M)$ . For the  $\tilde{\mathcal{A}}$  system, we specify the allocation vector  $\tilde{\mathbf{y}} = (\tilde{y}_1, \dots, \tilde{y}_M)$  and the sales lever vector  $\tilde{\mathbf{d}} = (\tilde{\mathbf{d}}_1, \dots, \tilde{\mathbf{d}}_M)$  such that they verify the second part of Assumption 2.

From  $\sum_m y_m^* = y^* \leq y^1 < y^2 = \sum_m y_m$ , there exists an allocation vector  $(\tilde{y}_1, \dots, \tilde{y}_M)$  satisfying  $y^1 = \sum_m \tilde{y}_m$ , and

$$\begin{aligned} \tilde{y}_m &= y_m, & \text{if } y_m < y_m^* \\ \tilde{y}_m &\in \{y_m^*, y_m^* + 1, \dots, y_m\}, & \text{if } y_m \geq y_m^*. \end{aligned}$$

We now construct the sales levers for the  $\tilde{\mathcal{A}}$  system. If  $\tilde{y}_m = y_m$ , set  $\tilde{\mathbf{d}}_m = \mathbf{d}_m$ , and we get, for every  $\epsilon_m$ ,

$$\begin{aligned} \tilde{y}_m - D_m(\tilde{\mathbf{d}}_m, \epsilon_m) &= y_m - D_m(\mathbf{d}_m, \epsilon_m), \quad \text{and} \\ \pi_m(\tilde{y}_m, \tilde{\mathbf{d}}_m) &= \pi_m(y_m, \mathbf{d}_m). \end{aligned}$$

Otherwise, we have  $y_m^* \leq \tilde{y}_m < y_m$ . By Assumption 2, there exists  $\tilde{\mathbf{d}}_m$  such that, for any  $\epsilon_m$ , we have

$$\begin{aligned} \tilde{y}_m - D_m(\tilde{\mathbf{d}}_m, \epsilon_m) &\leq y_m - D_m(\mathbf{d}_m, \epsilon_m), \quad \text{and} \\ \pi_m(\tilde{y}_m, \tilde{\mathbf{d}}_m) &\geq \pi_m(y_m, \mathbf{d}_m) \end{aligned}$$

Therefore, for every  $\epsilon = (\epsilon_1, \dots, \epsilon_M)$ , it follows

$$\begin{aligned} y^1 - \sum_m D_m(\tilde{\mathbf{d}}_m, \epsilon_m) &\leq y^2 - \sum_m D_m(\mathbf{d}_m, \epsilon_m), \quad \text{and} \\ \pi(y^1, \tilde{\mathbf{d}}) &= \sum_m \pi_m(\tilde{y}_m, \tilde{\mathbf{d}}_m) \geq \sum_m \pi_m(y_m, \mathbf{d}_m) = \pi(y^2, \mathbf{d}), \end{aligned}$$

satisfying the second part of Assumption 2 for the multiple sales channel model.  $\square$

We remark that these sales channels need not be identical. For example, one sales channel may be a channel with dynamically posted prices such as the discrete additive demand model. Another sales channel may be the periodic auction channel described in section 4.3.

### 6.3 Multiple List Price Channels with Inventory Sharing

The model presented here differs from section 6.1 since inventory is not allocated between sales channels before demand is realized.

In the multiple channel model in section 6.2, the inventory level in each period is allocated among sales channels before demand is realized, and the underage cost is incurred if demand in a channel exceeds inventory allocated to that channel. In this section, we consider a model in which inventory is shared among channels, and the underage cost is incurred only if the total demand exceeds the inventory level. We focus on posted price channels.

Suppose there are  $M$  sales channels, each of which is a list-pricing model with additive demand, i.e., demand in channel  $m$  is given by  $D_m(d_m) = D_m(d_m, \epsilon_m) = d_m + \epsilon_m$ . The expected single-period revenue from channel  $m$  is  $\Lambda_m(d_m)$  as a function of the sales lever  $d_m$ , which is assumed to be a concave function of  $d_m$ . The sequence of events is similar to section 6.2 with a few differences. First, in step (3), inventory level  $y$  is not allocated among sales channels. Second, the holding and back-order cost is given by  $h(z) = h \cdot [z]^+ + b \cdot [z]^-$  where  $z = y - \sum_m D_m(d_m)$ .

Let  $\mathbf{d} = (d_1, d_2, \dots, d_M)$ . The expected profit in a single period, as a function of the after-replenishment inventory level  $y$ , is given by

$$\begin{aligned} \pi(y, \mathbf{d}) &= \sum_m \Lambda_m(d_m) - E[h(y - \sum_m D_m(d_m, \epsilon_m))] \\ &= \sum_m \Lambda_m(d_m) - h \cdot E[y - \sum_m d_m - \sum_m \epsilon_m]^+ - b \cdot E[y - \sum_m d_m - \sum_m \epsilon_m]^- , \end{aligned}$$

where the expectation is taken over  $\epsilon = (\epsilon_1, \epsilon_2, \dots, \epsilon_M)$ . Thus, it is straightforward to show that  $\pi$  is jointly concave in  $(y, \mathbf{d})$ . It follows  $\max_{\mathbf{d}} \pi(y, \mathbf{d})$  is concave in  $y$ . Let  $y^*$  be its maximizer.

**Theorem 13.** *The multiple posted price channel model with inventory sharing satisfies the Unifying Assumption, and the conclusion of Theorem 3 holds.*

*Proof.* The first part of the Strong Unifying Assumption follows from the concavity of  $\max_{\mathbf{d}} \pi(y, \mathbf{d})$  in  $y$ . We will now prove the second part. Consider any  $y^1$  and  $y^2$  satisfying  $y^* \leq y^1 < y^2$ , and any  $\mathbf{d}^2 = (d_1^2, \dots, d_M^2)$ . Let  $\mathbf{d}^* = \arg \max_{\mathbf{d}} \pi(y^*, \mathbf{d})$  and  $\mathbf{d}^\circ = \arg \max_{\mathbf{d}} \{\pi(y^2, \mathbf{d}) \mid \sum_m d_m = \sum_m d_m^2\}$ . By an application of the proof of Lemma 11, we can assume, without loss of generality, that we have either (i)  $d_m^\circ \leq d_m^*$  for all  $m$ , or (ii)  $d_m^\circ \geq d_m^*$  for all  $m$ . If (i) occurs, set  $d_m^1 = d_m^*$  for each  $m$ . Otherwise, in case of (ii), define  $\lambda = (y^1 - y^*)/(y^2 - y^*)$ , and set  $d_m^1 = (1 - \lambda)d_m^* + \lambda d_m^\circ$ . In both cases, it is straightforward to show

$$\begin{aligned} \pi(y^1, \mathbf{d}^1) &\geq \pi(y^2, \mathbf{d}^\circ) \geq \pi(y^2, \mathbf{d}^2), \quad \text{and} \\ y^1 - \sum_m D_m(d_m^1, \epsilon_m) &\leq y^2 - \sum_m D_m(d_m^\circ, \epsilon_m) = y^2 - \sum_m D_m(d_m^2, \epsilon_m), \end{aligned}$$

where  $\mathbf{d}^1 = (d_1^1, \dots, d_M^1)$ . Therefore, the Strong Unifying Assumption is satisfied.  $\square$

## 7 Dynamic Pricing with Limited Ordering Opportunities

Suppose now that inventory can be replenished in every  $T > 2$  periods. For example, inventory replenishment is allowed at the beginning of every week, whereas prices can be changed every day. Alternatively, we assume that inventory is replenished in every period, and a single period is divided into  $T$  sub-periods, in which prices are set and sales take place. We study posted pricing with the additive demand model here.

The following sequence of events takes place in each period:

- (1) The available inventory level  $x$  at the beginning of a period is observed.
- (2) A replenishment decision is made to raise the inventory to  $y \geq x$ . A fixed setup cost of  $K$  is incurred if  $y > x$ .
- (3) For each sub-period  $t = 1, 2, \dots, T$ , let  $y_t$  be the inventory level at the beginning of sub-period  $t$ . A sales lever  $d_t$  is chosen, and demand is a random variable given by  $D_t(d_t) = D_t(d, \epsilon_t) = d_t + \epsilon_t$ . An appropriate revenue is realized and the holding and back-order cost is incurred. We assume full backlogging, and  $\alpha \in [0, 1]$  is the discount factor per sub-period.



We suppose that the expected single sub-period profit  $\pi_t(y_t, d_t)$ , the difference between the revenue and the holding and back-order cost, is jointly concave.

**Lemma 14.** *In the posted price model with limited ordering opportunities described in this section, the Unifying Assumption (Assumption 1) is satisfied, and the conclusion of Theorem 3 holds.*

*Proof.* For each  $t$ , we define  $L_t(\cdot)$  recursively as following:

$$L_t(y_t) = \begin{cases} \max_{d_t} \pi_t(y_t, d_t) + \alpha \cdot E[L_{t+1}(y_t - d_t - \epsilon_t)], & \text{if } t < T. \\ \max_{d_t} \pi_t(y_T, d_T), & \text{if } t = T. \end{cases}$$

By the convexity of  $\pi$ ,  $L_t$  is convex for each  $t$ . Let  $y^*$  be the maximizer of  $L_1$ .

Suppose  $y^* \leq y^1 \leq y^2$ . For a fixed sequence of  $\epsilon_1, \epsilon_2, \dots, \epsilon_T$ , let  $\mathcal{A}^*$  be the optimal system starting with the inventory level  $y^*$ . Let  $d_1^*, d_2^*, \dots, d_T^*$  be the optimal sequence of decisions in  $\mathcal{A}^*$ . (Clearly,  $d_t^*$  depends on  $\epsilon_1, \dots, \epsilon_{t-1}$ , but we suppress that dependence to simplify notation.) Let  $y_t^*$  be the beginning of sub-period inventory level in  $\mathcal{A}^*$ .

Consider two systems  $\tilde{\mathcal{A}}$  and  $\mathcal{A}$ , and suppose that the inventory levels at the beginning of sub-period  $t = 1$  are  $y^1$  and  $y^2$ , respectively. Suppose that for fixed  $\epsilon_1, \epsilon_2, \dots, \epsilon_T$ , the decisions of  $\mathcal{A}$  are given by  $d_1^2, d_2^2, \dots, d_T^2$ . Let  $\lambda \in [0, 1]$  such that  $y^1 = \lambda y^* + (1 - \lambda)y^2$ . For each  $t$ , choose the decision  $d_t^1$  of the  $\tilde{\mathcal{A}}$  system such that  $d_t^1 = \lambda d_t^* + (1 - \lambda)d_t^2$ . Let  $z_t^*$ ,  $z_t^1$  and  $z_t^2$  be the ending inventories in sub-period  $t$  in systems  $\mathcal{A}^*$ ,  $\tilde{\mathcal{A}}$  and  $\mathcal{A}$ , respectively. Thus, if  $y_t^1 = \lambda y_t^* + (1 - \lambda)y_t^2$ , then

$$\begin{aligned} z_t^1 &= y_t^1 - (d_t^1 + \epsilon_t) \\ &= [\lambda y_t^* + (1 - \lambda)y_t^2] - [\lambda d_t^* + (1 - \lambda)d_t^2 + \epsilon_t] \\ &= \lambda[y_t^* - d_t^* - \epsilon_t] + (1 - \lambda)[y_t^2 - d_t^2 - \epsilon_t] \\ &= \lambda z_t^* + (1 - \lambda)z_t^2. \end{aligned}$$

By induction, we show the above result for all  $t$ . Since  $z_T^* \leq y^*$ , it follows  $z_T^1 \leq \max\{z_T^2, y^*\}$ .

Furthermore, the expected single sub-period profit in  $t$  satisfies

$$\pi_t(y_t^1, d_t^1) \geq \lambda \cdot \pi_t(y_t^*, d_t^*) + (1 - \lambda) \cdot \pi_t(y_t^2, d_t^2),$$

where expectation is taken over  $\epsilon_t$ . Thus, the total expected profit in all  $T$  sub-periods satisfies

$$\sum_{t=1}^T \alpha^{t-1} \pi_t(y_t^1, d_t^1) \geq \lambda \cdot \sum_{t=1}^T \alpha^{t-1} \pi_t(y_t^*, d_t^*) + (1 - \lambda) \cdot \sum_{t=1}^T \alpha^{t-1} \pi_t(y_t^2, d_t^2).$$

Therefore, the total expected profit in the  $\tilde{\mathcal{A}}$  system (left-hand side) is at least the total expected profit in the  $\mathcal{A}$  system (last term on the right-hand side).  $\square$

## 8 Conclusion

In this paper, we establish the optimality of  $(s, S)$  inventory replenishment policies in the presence of a fixed setup cost. We allow for a variety of sales mechanisms (e.g., name-your-own-price mechanisms). In particular, this is the first paper that considers a fixed setup cost when the sales mechanism is a periodic auction. We further extend the optimality of  $(s, S)$  policies to systems in which multiple heterogeneous sales channels are present.

Our results are proved by demonstrating that these models satisfy the Unifying Assumption of Huh and Janakiraman (2004), a sufficient condition for the optimality of  $(s, S)$  policies. While this assumption was originally developed for the posted price demand models, the present paper shows that the applicability of this assumption extends far beyond traditional models.

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## A Appendix

### A.1 Concavity of $L(\cdot)$ in Proposition 4

In this section, we prove that the maximum single-period profit  $L(y)$  is concave in  $y$ . This result is one of the components of Proposition 4.

We note  $y \in \mathcal{Z}$ . By part (a) of Proposition 4 and the definition of  $R(y, \mathbf{r})$ , it follows

$$L(y) = R(y, \mathbf{r}_y) = E_{\mathbf{v}}[R(y, \mathbf{r}_y, \mathbf{v})]. \quad (11)$$

Now, we fix the vector  $\mathbf{v}$ . From (5),

$$R(y, \mathbf{r}_y, \mathbf{v}) = \max_{k \in \mathcal{Z}^+} \{J(v(1)) + J(v(2)) + \dots + J(v(k)) - h(y - k)\}.$$

For each  $k \in \mathcal{R}^+$ , we define  $\bar{J}(k, \mathbf{v})$  as follows:

$$\bar{J}(k, \mathbf{v}) = \begin{cases} 0, & \text{if } k = 0; \\ J(v(1)) + J(v(2)) + \dots + J(v(k)), & \text{if } k \in \mathcal{N}; \\ (\lceil k \rceil - k) \cdot \bar{J}(\lfloor k \rfloor) + (k - \lfloor k \rfloor) \cdot \bar{J}(\lceil k \rceil), & \text{otherwise.} \end{cases}$$

That is, for any fixed  $\mathbf{v}$ ,  $\bar{J}(k, \mathbf{v})$  is a piecewise linear function of  $k$  with slope changes only at integers. Let  $\tilde{h}(\cdot)$  be a piecewise-linear extension of  $h(\cdot)$  to  $\mathcal{R}$ . We also define, for  $y \in \mathcal{R}$  and  $k \in \mathcal{R}^+$ ,

$$\varphi(y, k, \mathbf{v}) = \bar{J}(k, \mathbf{v}) - \tilde{h}(y - k) \quad (12)$$

Thus, for each integer  $y$  and value vector  $\mathbf{v}$ , we have

$$R(y, \mathbf{r}_y, \mathbf{v}) = \max_{k \in \mathcal{Z}^+} \varphi(y, k, \mathbf{v}). \quad (13)$$

We claim that for any fixed  $\mathbf{v}$ ,  $\varphi(y, k, \mathbf{v})$  is jointly concave in  $(y, k)$  in  $\mathfrak{R} \times \mathfrak{R}^+$ . It is easy to see that  $\bar{J}(\cdot)$  is concave, and  $\tilde{h}(\cdot)$  is convex. Since  $y - k$  is a linear function of  $(y, k)$ . Equation (12) implies  $\varphi(y, k, \mathbf{v})$  is jointly concave in  $(y, k)$ .

Thus, for any fixed  $\mathbf{v}$  and integer  $y$ ,  $\varphi(y, k, \mathbf{v})$  is concave in  $k \in \mathfrak{R}^+$ . Since  $\varphi(y, k, \mathbf{v})$  is piecewise linear in  $k$  with slope changes only at integer points, there exists an integer value of  $k$  that maximizes  $\varphi(y, k, \mathbf{v})$ . Thus,  $\max_{k \in \mathcal{Z}^+} \varphi(y, k, \mathbf{v}) = \max_{k \in \mathfrak{R}^+} \varphi(y, k, \mathbf{v})$ .

Since  $\varphi(y, k, \mathbf{v})$  is a jointly concave function in  $(y, k) \in \mathfrak{R} \times \mathfrak{R}^+$ ,  $\max_{k \in \mathfrak{R}^+} \varphi(y, k, \mathbf{v})$  is concave with respect to  $y$  in  $\mathfrak{R}$ . Thus,  $\max_{k \in \mathfrak{R}^+} \varphi(y, k, \mathbf{v})$  is also concave, in the discrete sense, with respect to  $y$  in  $\mathcal{Z}$ . It follows from (13) that  $R(y, \mathbf{r}_y, \mathbf{v})$  is concave with respect to  $y$  in  $\mathcal{Z}$ . From (11), we conclude  $L(y)$  is concave in  $y$ .

## A.2 Proof of (9) in Lemma 5

In this section, we provide the proof of claim (9), i.e., for any  $\mathbf{v}$ ,

$$R(y^2, \mathbf{r}, \mathbf{v}) - R(y^1, \tilde{\mathbf{r}}, \mathbf{v}) \leq R(y^2, \mathbf{r}_{y^2}, \mathbf{v}) - R(y^1, \mathbf{r}_{y^1}, \mathbf{v}).$$

We consider two disjoint cases. In the first case, assume  $\kappa(\mathbf{r}_{y^1}, \mathbf{v}) = \kappa(\tilde{\mathbf{r}}, \mathbf{v})$ . It follows  $I[v(i) \geq v_{y^1}^*(i)] = I[v(i) \geq \tilde{r}(i)]$  for each  $i$ . Then,

$$\begin{aligned} & [J(v(i)) + \Delta h(y^1 - i + 1)] \cdot I[v(i) \geq v_{y^1}^*(i)] \\ &= [J(v(i)) + \Delta h(y^1 - i + 1)] \cdot I[v(i) \geq \tilde{r}(i)], \end{aligned}$$

implying  $R(y^1, \mathbf{r}_{y^1}, \mathbf{v}) = R(y^1, \tilde{\mathbf{r}}, \mathbf{v})$  by (5). Similarly,

$$\begin{aligned} & [J(v(i)) + \Delta h(y^2 - i + 1)] \cdot I[v(i) \geq v_{y^2}^*(i)] \\ &= [J(v(i)) + \Delta h(y^2 - i + 1)]^+ \\ &\geq [J(v(i)) + \Delta h(y^2 - i + 1)] \cdot I[v(i) \geq r(i)], \end{aligned}$$

implying  $R(y^2, \mathbf{r}_{y^2}, \mathbf{v}) \geq R(y^2, \mathbf{r}, \mathbf{v})$ . Thus, we obtain claim (9).

In the second case, we have  $\kappa(\tilde{\mathbf{r}}, \mathbf{v}) \neq \kappa(\mathbf{r}_{y^1}, \mathbf{v})$ . By the definition of  $\tilde{\mathbf{r}}$ , we get  $\tilde{r}(i) \leq v_{y^1}^*(i)$  for  $i \leq y^1$ , and  $\tilde{r}(i) = v_{y^1}^*(i)$  for  $i > y^1$ . Thus, from  $\kappa(\tilde{\mathbf{r}}, \mathbf{v}) \neq \kappa(\mathbf{r}_{y^1}, \mathbf{v})$ , we get

$$\kappa(\mathbf{r}_{y^1}, \mathbf{v}) < \kappa(\tilde{\mathbf{r}}, \mathbf{v}) \leq y^1.$$

Recall  $v_{y^1}^*(i) = J^{-1}(-\Delta h(y^1 - i + 1))$ . Thus, if  $i \leq \kappa(\mathbf{r}_{y^1}, \mathbf{v})$ , then we have  $i \leq y^1$ , and  $v(i) \geq v_{y^1}^*(i) = J^{-1}(-h)$ . Otherwise, we have  $i > \kappa(\mathbf{r}_{y^1}, \mathbf{v})$ , and thus

$$v(i) \leq v(\kappa(\mathbf{r}_{y^1}, \mathbf{v}) + 1) \leq v_{y^1}^*(\kappa(\mathbf{r}_{y^1}, \mathbf{v}) + 1) = J^{-1}(-h),$$

since  $i \geq \kappa(\mathbf{r}_{y^1}, \mathbf{v}) + 1$  and  $\kappa(\mathbf{r}_{y^1}, \mathbf{v}) + 1 \leq y^1$ .

For any  $i \in \{1, 2, \dots, y^1\}$ ,  $\tilde{r}(i)$  is either  $r(i)$  or  $v_{y^1}^*(i)$ . If  $\tilde{r}(i) = v_{y^1}^*(i)$ , then,

$$\begin{aligned} & [J(v(i)) + \Delta h(y^1 - i + 1)] \cdot I[v(i) \geq \tilde{r}(i)] \\ &= [J(v(i)) + h] \cdot I[v(i) \geq v_{y^1}^*(i)] \\ &\geq [J(v(i)) + h] \cdot I[v(i) \geq r(i)] \\ &= [J(v(i)) + \Delta h(y^2 - i + 1)] \cdot I[v(i) \geq r(i)]. \end{aligned}$$

Otherwise, if  $\tilde{r}(i) = r(i)$ , then

$$[J(v(i)) + \Delta h(y^1 - i + 1)] \cdot I[v(i) \geq \tilde{r}(i)] = [J(v(i)) + \Delta h(y^2 - i + 1)] \cdot I[v(i) \geq r(i)].$$

Now consider  $i \geq y^1$ . In that case, the event  $v(i) \geq \tilde{r}(i)$  is equivalent to  $i \leq \kappa(\tilde{\mathbf{r}}, \mathbf{v})$ , which never occurs. Furthermore, since  $i \geq y^1 > \kappa(\mathbf{r}_{y^1}, \mathbf{v})$ , we get

$$\begin{aligned} J(v(i)) + \Delta h(y^2 - i + 1) &\leq J(v(i)) + h \\ &\leq J(v(\kappa(\mathbf{r}_{y^1}, \mathbf{v}) + 1)) + h \\ &\leq 0. \end{aligned}$$

Thus, from  $I[v(i) \geq \tilde{r}(i)] = 0$ ,

$$\begin{aligned} [J(v(i)) + \Delta h(y^2 - i + 1)] \cdot I[v(i) \geq r(i)] &\leq 0 \\ &= [J(v(i)) + \Delta h(y^1 - i + 1)] \cdot I[v(i) \geq \tilde{r}(i)]. \end{aligned}$$

Finally, for any  $i$ ,

$$\begin{aligned} & [J(v(i)) + \Delta h(y^2 - i + 1)] \cdot I[v(i) \geq v_{y^2}^*(i)] \\ &= [J(v(i)) + \Delta h(y^2 - i + 1)]^+ \\ &\geq [J(v(i)) + \Delta h(y^1 - i + 1)]^+ \\ &= [J(v(i)) + \Delta h(y^1 - i + 1)] \cdot I[v(i) \geq v_{y^1}^*(i)]. \end{aligned}$$

Therefore from (5), we obtain (9).

### A.3 Proof of Proposition 9

Let  $\tilde{\mathcal{D}}$  be the compact interval  $[d_L, d_U]$ . Thus,  $\mathcal{D}$  is the set of integers in  $\tilde{\mathcal{D}}$ . For any fixed integer  $y$  and  $d \in \mathcal{D}$ , define  $\pi(y, d) = \pi^1(d) + \pi^2(y - d)$ , where

$$\begin{aligned}\pi^1(d) &= p(d) \cdot d, & \text{and} \\ \pi^2(r) &= E[L_A(r - \epsilon_P)] .\end{aligned}$$

Now, for any integer  $y$  and real  $d \in \tilde{\mathcal{D}}$ , define

$$\tilde{\pi}^1(d) = (1 - \lambda) \cdot \tilde{\pi}^1(\lfloor d \rfloor) + \lambda \cdot \tilde{\pi}^1(\lceil d \rceil)$$

where  $\lambda = d - \lfloor d \rfloor$ . Similarly, for any real  $r$ , define

$$\tilde{\pi}^2(r) = (1 - \lambda) \cdot \tilde{\pi}^2(\lfloor r \rfloor) + \lambda \cdot \tilde{\pi}^2(\lceil r \rceil)$$

where  $\lambda = r - \lfloor r \rfloor$ . For any real  $y$  and  $d \in \tilde{\mathcal{D}}$ , let

$$\tilde{\pi}(y, d) = \tilde{\pi}^1(d) + \tilde{\pi}^2(y - d) .$$

Since the expected revenue from the posted price channel,  $\pi^1(d) = p(d) \cdot d$ , is assumed to be concave with respect to  $d \in \mathcal{D}$ , its linear interpolation  $\tilde{\pi}^1(d)$  is also concave with respect to  $d \in \tilde{\mathcal{D}}$ . The concavity of  $L_A(t)$ , the maximum expected single-period profit from the auction channel, is proved in Proposition 4 of section 4. Thus,  $\tilde{\pi}(y, d)$  is jointly concave with respect to  $y$  and  $d$ . Thus,  $\max_{d \in \tilde{\mathcal{D}}} \tilde{\pi}(y, d)$  is concave with respect to  $y$ .

Moreover, from the construction of  $\tilde{\pi}$ , if  $y$  is an *integer*, then  $\tilde{\pi}(y, d)$  is a piece-wise linear interpolation of  $\pi(y, d)$  with respect to  $d$ . Thus, for fixed integer  $y$ , the single-dimensional function  $\tilde{\pi}(y, \cdot)$  has at least one integer maximizer, i.e.,

$$Q(y) = \max_{d \in \mathcal{D}} \pi(y, d) = \max_{d \in \tilde{\mathcal{D}}} \tilde{\pi}(y, d) .$$

The conclusions of the last two paragraphs together imply that  $Q(y)$  is concave with respect to  $y$ .

#### A.4 Proof of Lemma 11

First, we provide the proof assuming that the domain of  $f_i$  is the set of all integers. Since  $s_i^*$  is the maximizer of  $f_i(\cdot)$ , we have  $f_i(s_i) \leq f_i(s_i + 1)$  for  $s_i < s_i^*$ , and  $f_i(s_i) \geq f_i(s_i + 1)$  for  $s_i \geq s_i^*$ . Let  $s^* = \sum_i s_i^*$ .

Suppose  $s < s^*$ . Then, we claim that there exist  $s_1, s_2, \dots, s_I$  such that  $s = \sum_i s_i$  and  $f(s) = \sum_i f_i(s_i)$  satisfying  $s_i \leq s_i^*$  for each  $i$ . To see this claim, suppose that there exists  $j$  such that  $s_j > s_j^*$  and  $s_k < s_k^*$ . Then, by decreasing  $s_j$  by 1 and increasing  $s_k$  by 1, we weakly increase the objective function. By repeating this process, we prove the claim.

Furthermore, there exists  $i'$  such that  $s_{i'} < s_{i'}^*$ . Then,

$$f(s) = \sum_i f_i(s_i) = f_{i'}(s_{i'}) + \sum_{i \neq i'} f_i(s_i) \leq f_{i'}(s_{i'} + 1) + \sum_{i \neq i'} f_i(s_i) \leq f(s + 1).$$

Similarly, it can be argued that  $s > s^*$  implies  $f(s) \leq f(s - 1)$ .

If the domain of  $f_i$  is a subset of all integers, extend  $f_i$  by defining  $f_i(s_i) = -\infty$  for each  $s_i$  outside the domain.