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A column generation algorithm for choice-based network revenue management

Online Appendix

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A1 Complexity of the Column Generation Subproblem

Theorem 1 The 0–1 fractional programming problem

$$\max_{y\in\{0,1\}^n} \sum_{j=1}^{n} w_j y_j \left(\sum_{l=1}^{L} \frac{\lambda_l v_{lj}}{\sum_{i\in C_l} v_{li} y_i + v_{l0}} \right), \tag{A1}$$

where $w_j, v_{lj} > 0, l = 1, \ldots, L,\ j = 1, \ldots, n$, and $C_l \subset N$, is NP-Hard.

Proof. Given a connected graph $G = (V, E)$, with nodes $V = \{1, \ldots, v\}, v \geq 2$, and arcs $E \subset \{(i, j) \in V \times V, i < j\}$, a *minimum vertex cover* of G is a subset $V' \subset V$ such that every arc in E is incident to at least one node in V'.

Let I be an instance of *minimum vertex cover*. We will construct an instance J of problem (A1) corresponding to I. There are $v + 1$ binary decision variables $y_1, \ldots, y_v, y_{v+1}$, and $|E| + v$ summands in the objective function, where the first $|E|$ summands are:

$$\frac{y_i + y_j}{y_i + y_j + 1/v^2}, \quad \text{for every } (i, j) \in E,$$

and the last v summands are of the form

$$\frac{2y_{v+1} + y_1}{y_{v+1} + y_1 + 1/v^2} + \frac{2y_{v+1} + y_2}{y_{v+1} + y_2 + 1/v^2} + \cdots + \frac{2y_{v+1} + y_v}{y_{v+1} + y_v + 1/v^2}.$$

Note that we can recover formulation (A1) by taking $n = v + 1$, $w_{v+1} = 2$, $w_j = 1, 1 \leq j \leq v$, and by defining $L = v + |E|$; $C_k = \{i, j\},\ k = 1, \ldots, |E|$, where the arcs are labeled following the lexicographic order and $C_{|E|+i} = \{v + 1, i\},\ i = 1, \ldots, v$. For $l = 1, \ldots, L$, we also need $\lambda_l = 1$, $v_{l0} = 1/v^2$, and $v_{lj} = \mathbb{1}\{j \in V \bigcap C_l\}$.

The following facts hold:

1. An optimal solution of J must verify $y_{v+1} = 1$, since for all $i \in V$,

$$\frac{2 + y_i}{1 + y_i + 1/v^2} > \frac{y_i}{y_i + 1/v^2}, \quad \forall v \geq 2$$

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2. In the optimum of J, we cannot have $y_{i_0} = y_{j_0} = 0$ for an arc $(i_0, j_0) \in E$. To see this, we argue by contradiction: Fix i_0, let S be an optimal solution of J with $y_{i_0} = y_{j_0} = 0$, and let S' be the same as S but with $y_{i_0} = 1$. The terms affected by the increment in y_{i_0} would be:

$$\frac{2y_{v+1} + y_{i_0}}{y_{v+1} + y_{i_0} + 1/v^2}, \quad \text{and} \quad \frac{y_{i_0} + y_j}{y_{i_0} + y_j + 1/v^2}, \text{ for all } j \text{ such that } (i_0, j) \in E.$$

For the first one, we decrease the value of the solution S' with respect to the value of the solution S by less than 1/2 for $v \geq 2$ (recall that from fact 1 above, $y_{v+1} = 1$):

$$\frac{3v^2}{2v^2+1} - \frac{2v^2}{v^2+1} = -\frac{v^4 - v^2}{2v^4 + 3v^2 + 1} > -\frac{1}{2} \tag{A2}$$

For the second set of terms, given i_0, for every arc $(i_0, j) \in E$, there are two cases:

- If $y_j = 0$, then by setting $y_{i_0} = 1$, we increase the objective function by

$$\frac{1}{1 + 1/v^2} \geq \frac{1}{2}$$

  Note that due to the choice of i_0, there must be at least one j_0 such that $y_{j_0} = 0$. This increment more than compensates the decrement in (A2).

- If $y_j = 1$, then by setting $y_{i_0} = 1$, we increase the objective function by

$$\frac{2v^2}{2v^2+1} - \frac{v^2}{v^2+1} > 0$$

So, we can increase the objective function value by setting $y_{i_0} = 1$, which is a contradiction to S being optimal.

3. We are left with showing that an optimal solution of J has the smallest possible number of variables y_i, $i = 1, \ldots, v$, set at $y_i = 1$. Let S be an optimal solution of J, and S' another optimal solution of J, but with fewer variables y_i set at one, i.e. $|S| > |S'|$. We define $Value_1(S)$ and $Value_2(S)$ (respectively, $Value_1(S')$, $Value_2(S')$) as the partial sum of the first $|E|$ terms of objective function value in (A1) and the last v terms when plugging in the corresponding values of the decision variables y.

Comparing $Value_2$(S) with $Value_2$(S'), give us

$$\begin{aligned} Value_2(S) - Value_2(S') &= |S|\frac{3}{2+1/v^2} + (n - |S|)\frac{2}{1+1/v^2} - |S'|\frac{3}{2+1/v^2} - (n - |S'|)\frac{2}{1+1/v^2} \\ &= (|S'| - |S|)\frac{v^4 - v^2}{2v^4 + 3v^2 + 1} \\ &< -\frac{1}{4}, \quad \forall v \geq 2 \end{aligned}$$

Regarding the first $|E|$ terms, the difference between the value of the summand $(k, j) \in S$ and the value of the summand $(k, j) \in S'$ is at most

$$\frac{2}{2+1/v^2} - \frac{1}{1+1/v^2} = \frac{2v^2}{2v^2+1} - \frac{v^2}{v^2+1} = \frac{v^2}{2v^4 + 3v^2 + 1}$$

Since the number of arcs verifies $|E| \leq v(v-1)/2$, then

$$\begin{aligned} Value_1(S) - Value_1(S') &\leq \frac{v(v-1)}{2}\frac{v^2}{2v^4+3v^2+1} \\ &= \frac{v^4-v^3}{4v^4+6v^2+2} \\ &< \frac{1}{4} \end{aligned}$$

So, $Value_1(S') + Value_2(S') > Value_1(S) + Value_2(S)$, which contradicts the fact that S is an optimal solution of instance J.

Hence, we have found a polynomial transformation from instance I to instance J, such that a solution of J implies a solution to I. $\square$

As an illustration of the polynomial transformation from an instance of minimum vertex cover to an instance of problem (A1), consider the graph $G = (V, E)$ in Figure A1. In this case, $v = 4$, and $|E| = 4$. In the corresponding instance of the 0–1 fractional programming problem with the structure defined in (A1) there are 5 variables and 8 summands defined as follows:

- The first $|E| = 4$ summands are:

$$\frac{y_1+y_2}{y_1+y_2+1/16} + \frac{y_1+y_3}{y_1+y_3+1/16} + \frac{y_2+y_3}{y_2+y_3+1/16} + \frac{y_3+y_4}{y_3+y_4+1/16}$$

- The last $v = 4$ summands are:

$$\frac{2y_5+y_1}{y_5+y_1+1/16} + \frac{2y_5+y_2}{y_5+y_2+1/16} + \frac{2y_5+y_3}{y_5+y_3+1/16} + \frac{2y_5+y_4}{y_5+y_4+1/16}$$

Maximizing the total sum with respect to $y_1, \ldots, y_5 \in \{0, 1\}$ gives an objective value of 10.47. The solution in Figure A1 corresponds to $y_2^* = y_3^* = y_5^* = 1$, and $y_1^* = y_4^* = 0$.

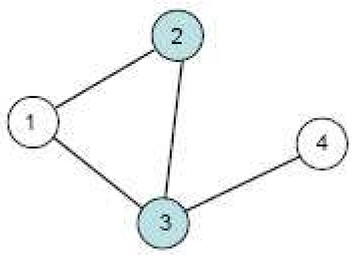

Figure A1: For this graph $G = (V, E)$, an instance of *minimum vertex cover* is given by the nodes 2 and 3. This is not the unique solution: the subset of nodes $\{1, 3\}$ is an alternative optimum.

We can now state the following corollary:

Corollary 1 *The associated decision problem to the optimization problem (8), i.e., "Given a constant $K > 0$, is there an assignment of variables $y \in \{0,1\}^n$ such that*

$$\sum_{j=1}^{n} w_j y_j \left(\sum_{l=1}^{L} \frac{\lambda_l v_{lj}}{\sum_{i \in C_l} v_{li} y_i + v_{l0}} \right) > K \quad ?" \tag{A3}$$

is NP-Complete.

Proof. The decision problem (A3) is NP, since given an instance of YES, it clearly takes polynomial time to check that the inequality is satisfied. The NP-Hard feature holds from Theorem 1. □.

A2 Supplement to Numerical Examples

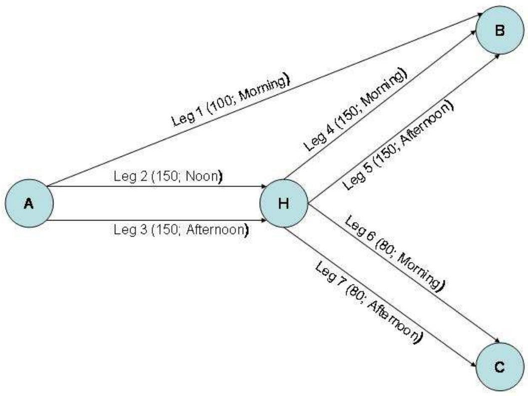

Figure A2: The airline Small Network instance.

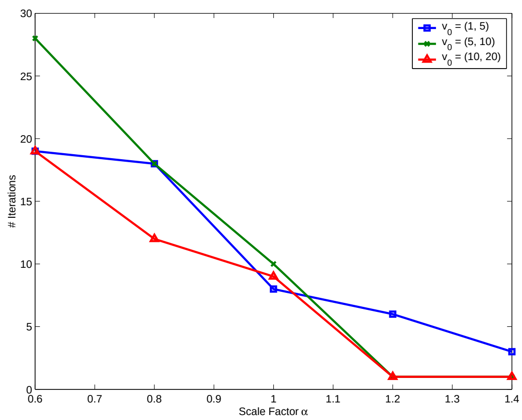

Figure A3: Number of iterations of CDLP as a function of α and v_0 for the Small Network example.

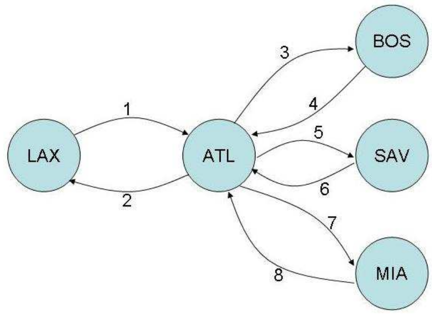

Figure A4: Hub-and-Spoke Network instance.

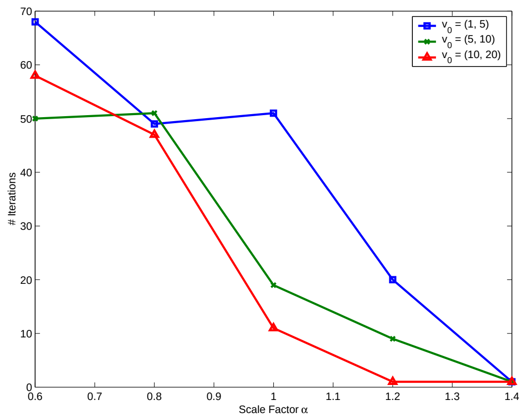

Figure A5: Number of iterations of CDLP as a function of α and v_0 for the Hub-and-Spoke Network example.

α	v_0	UB	DCOMP		DCOMP-0.5		CDLP		RCDLP		ROPT-0.01		ROPT-0.1		INDEP	
			Mean	%LF	Mean	%LF	Mean	%LF	Mean	%LF	Mean	%LF	Mean	%LF	Mean	%LF
0.6	(1,5,5,1)	56,884	55,948	98.60	55,872	97.82	54,177	94.60	53,783	94.42	51,632	95.83	49,468	93.48	49,794	97.41
	(1,10,5,1)	56,848	55,882	98.36	55,698	97.29	54,051	94.45	53,619	94.08	51,946	93.81	49,994	91.01	49,655	96.77
	(5,20,10,5)	53,819	51,326	94.59	51,338	94.23	50,058	92.94	50,476	93.44	48,602	97.35	46,274	95.05	46,246	92.53
0.8	(1,5,5,1)	71,936	69,533	95.78	69,163	94.79	68,105	94.69	68,641	95.11	66,336	94.57	64,490	92.75	60,346	94.14
	(1,10,5,1)	71,794	69,129	95.24	68,863	94.50	67,896	94.58	68,491	95.18	66,237	93.82	64,522	91.91	59,552	91.40
	(5,20,10,5)	61,868	60,147	90.48	60,222	90.49	59,073	90.82	59,289	90.83	56,724	91.98	55,842	91.16	53,044	81.94
1.0	(1,5,5,1)	79,155	76,954	95.65	77,096	95.91	75,726	94.96	75,996	94.89	77,106	95.36	76,136	94.36	66,224	85.38
	(1,10,5,1)	76,866	75,639	90.88	75,655	90.97	73,788	90.26	74,100	90.19	75,399	91.26	74,635	90.73	64,831	81.76
	(5,20,10,5)	63,255	62,775	78.09	62,792	78.15	62,702	78.41	62,541	78.41	62,040	79.94	61,827	79.96	56,203	72.44
1.2	(1,5,5,1)	80,371	79,817	84.28	79,818	84.24	79,666	84.33	79,698	84.32	79,834	84.58	79,774	84.53	68,970	76.84
	(1,10,5,1)	78,045	77,520	79.06	77,526	79.07	77,348	79.26	77,332	79.22	77,529	79.49	77,476	79.45	67,570	73.71
	(5,20,10,5)	63,296	63,111	67.52	63,113	67.52	62,491	68.88	62,677	68.87	62,422	70.20	62,293	70.20	58,543	65.88
1.4	(1,5,5,1)	81,066	80,408	73.08	80,376	72.83	80,362	72.75	80,362	72.75	80,439	73.27	80,421	73.22	71,418	70.60
	(1,10,5,1)	78,816	78,123	68.56	78,097	68.32	78,091	68.24	78,091	68.24	78,136	68.80	78,120	68.76	69,949	67.87
	(5,20,10,5)	63,337	63,211	60.54	63,212	60.54	62,553	62.04	62,775	62.00	62,822	62.82	62,734	62.85	60,732	60.82

Table A1: Revenue results for the Parallel Flights example.

Product	Legs	Class	Fare	Product	Legs	Class	Fare
1	1	H	1000	12	1	L	500
2	2	H	400	13	2	L	200
3	3	H	400	14	3	L	200
4	4	H	300	15	4	L	150
5	5	H	300	16	5	L	150
6	6	H	500	17	6	L	250
7	7	H	500	18	7	L	250
8	$\{2,4\}$	H	600	19	$\{2,4\}$	L	300
9	$\{3,5\}$	H	600	20	$\{3,5\}$	L	300
10	$\{2,6\}$	H	700	21	$\{2,6\}$	L	350
11	$\{3,7\}$	H	700	22	$\{3,7\}$	L	350

Table A2: Product definitions for the Small Network instance.

Segment	O-D	Consideration set	Pref. vector	λ_l	Description
1	$A \to B$	{1,8,9,12,19,20}	(10,8,8,6,4,4)	0.08	Price insensitive, early pref.
2	$A \to B$	{1,8,9,12,19,20}	(1,2,2,8,10,10)	0.2	Price sensitive
3	$A \to H$	{2,3,13,14}	(10,10,5,5)	0.05	Price insensitive
4	$A \to H$	{2,3,13,14}	(2,2,10,10)	0.2	Price sensitive
5	$H \to B$	{4,5,15,16}	(10,10,5,5)	0.1	Price insensitive
6	$H \to B$	{4,5,15,16}	(2,2,10,8)	0.15	Price sensitive, slight early pref.
7	$H \to C$	{6,7,17,18}	(10,8,5,5)	0.02	Price insensitive, slight early pref.
8	$H \to C$	{6,7,17,18}	(2,2,10,8)	0.05	Price sensitive
9	$A \to C$	{10,11,21,22}	(10,8,5,5)	0.02	Price insensitive, slight early pref.
10	$A \to C$	{10,11,21,22}	(2,2,10,10)	0.04	Price sensitive

Table A3: Segment definitions for the Small Network instance.

α	v_0	UB	DCOMP		DCOMP-0.5		CDLP		RCDLP		ROPT-0.01		ROPT-0.1		INDEP	
			Mean	%LF	Mean	%LF	Mean	%LF	Mean	%LF	Mean	%LF	Mean	%LF	Mean	%LF
	(1,5)	215,793	197,038	88.90	196,920	88.06	207,890	91.27	208,476	91.70	200,444	93.37	195,291	92.09	172,362	97.71
0.6	(5,10)	200,515	194,146	93.35	191,443	92.09	194,393	91.90	193,658	91.82	192,896	95.07	189,182	94.42	163,905	96.76
	(10,20)	170,137	167,866	92.68	167,902	92.79	164,089	91.45	164,296	91.39	166,919	93.07	165,516	92.47	151,801	92.48
	(1,5)	266,934	262,823	86.79	263,023	86.37	261,264	85.62	260,820	85.68	252,013	86.90	249,221	86.21	204,572	94.60
0.8	(5,10)	223,173	220,891	90.48	221,012	90.51	215,884	89.38	217,105	89.40	217,073	90.75	214,192	90.15	191,066	90.22
	(10,20)	188,574	186,219	85.29	185,969	85.27	184,182	84.86	184,289	84.92	186,325	85.61	185,841	85.26	172,246	84.09
	(1,5)	281,967	279,506	81.34	279,536	81.36	277,738	80.80	277,473	80.78	278,344	81.04	275,016	80.53	226,002	87.71
1.0	(5,10)	235,284	233,929	84.41	233,891	84.27	230,342	83.86	231,350	83.89	233,138	84.09	232,376	83.68	209,701	83.64
	(10,20)	192,038	191,646	76.10	191,623	76.05	190,283	76.34	190,393	76.34	191,727	75.89	191,627	75.70	188,058	76.73
	(1,5)	284,772	284,736	71.85	284,747	71.85	282,842	71.55	282,996	71.51	283,280	72.47	281,926	72.46	243,930	82.48
1.2	(5,10)	238,562	238,539	72.38	238,502	72.26	238,299	72.03	238,299	72.03	238,548	72.35	238,523	72.33	225,691	77.65
	(10,20)	192,373	192,530	65.86	192,524	65.87	192,511	65.88	192,511	65.88	192,532	65.87	192,526	65.87	192,416	65.80
	(1,5)	287,076	286,743	62.14	286,629	62.16	285,417	61.96	285,598	61.95	286,160	62.24	285,783	62.21	259,039	76.96
1.4	(5,10)	238,562	238,843	61.80	238,843	61.80	238,843	61.80	238,843	61.80	238,843	61.80	238,843	61.80	231,937	68.82
	(10,20)	192,373	192,541	56.48	192,541	56.48	192,541	56.48	192,541	56.48	192,541	56.48	192,541	56.48	192,468	56.42

Table A4: Revenue results for the Small Network example.

O-D Market	Legs	Revenue			
		Y	M	B	Q
ATLBOS/BOSATL	3/4	310	290	95	69
ATLLAX/LAXATL	2/1	455	391	142	122
ATLMIA/MIAATL	7/8	280	209	94	59
ATLSAV/SAVATL	5/6	159	140	64	49
BOSLAX/LAXBOS	4-2/1-3	575	380	159	139
BOSMIA/MIABOS	4-7/8-3	403	314	124	89
BOSSAV/SAVBOS	4-5/6-3	319	250	109	69
LAXMIA/MIALAX	1-7/8-2	477	239	139	119
LAXSAV/SAVLAX	1-5/6-2	502	450	154	134
MIASAV/SAVMIA	8-5/6-7	226	168	84	59

Table A5: Product definitions for the Hub-and-Spoke Network example.

Segment	C_l	v_l	λ_l	Segment	C_l	v_l	λ_l
ATL/BOS H	{1,2,3,4}	{6,7,9,10}	0.015	BOS/MIA H	{41,42,43,44}	{6,7,10,10}	0.008
ATL/BOS L	{3,4}	{8,10}	0.035	BOS/MIA L	{43,44}	{8,10}	0.03
BOS/ATL H	{5,6,7,8}	{6,7,9,10}	0.015	MIA/BOS H	{45,46,47,48}	{6,7,10,10}	0.008
BOS/ATL L	{7,8}	{8,10}	0.035	MIA/BOS L	{47,48}	{8,10}	0.03
ATL/LAX H	{9,10,11,12}	{5,6,9,10}	0.01	BOS/SAV H	{49,50,51,52}	{5,6,9,10}	0.01
ATL/LAX L	{11,12}	{10,10}	0.04	BOS/SAV L	{51,52}	{8,10}	0.035
LAX/ATL H	{13,14,15,16}	{5,6,9,10}	0.01	SAV/BOS H	{53,54,55,56}	{5,6,9,10}	0.01
LAX/ATL L	{15,16}	{10,10}	0.04	SAV/BOS L	{55,56}	{8,10}	0.035
ATL/MIA H	{17,18,19,20}	{5,5,10,10}	0.012	LAX/MIA H	{57,58,59,60}	{5,6,10,10}	0.012
ATL/MIA L	{19,20}	{8,10}	0.035	LAX/MIA L	{59,60}	{9,10}	0.028
MIA/ATL H	{21,22,23,24}	{5,5,10,10}	0.012	MIA/LAX H	{61,62,63,64}	{5,6,10,10}	0.012
MIA/ATL L	{23,24}	{8,10}	0.035	MIA/LAX L	{63,64}	{9,10}	0.028
ATL/SAV H	{25,26,27,28}	{4,5,8,9}	0.01	LAX/SAV H	{65,66,67,68}	{6,7,10,10}	0.016
ATL/SAV L	{27,28}	{7,10}	0.03	LAX/SAV L	{67,68}	{10,10}	0.03
SAV/ATL H	{29,30,31,32}	{4,5,8,9}	0.01	SAV/LAX H	{69,70,71,72}	{6,7,10,10}	0.016
SAV/ATL L	{31,32}	{7,10}	0.03	SAV/LAX L	{71,72}	{10,10}	0.03
BOS/LAX H	{33,34,35,36}	{5,5,7,10}	0.01	MIA/SAV H	{73,74,75,76}	{6,7,8,10}	0.01
BOS/LAX L	{35,36}	{9,10}	0.032	MIA/SAV L	{75,76}	{9,10}	0.025
LAX/BOS H	{37,38,39,40}	{5,5,7,10}	0.01	MIA/SAV H	{77,78,79,80}	{6,7,8,10}	0.01
LAX/BOS L	{39,40}	{9,10}	0.032	MIA/SAV L	{79,80}	{9,10}	0.025

Table A6: Segment definitions for Hub-and-Spoke Network example.

α	v_0	UB	DCOMP		DCOMP-0.5		CDLP		RCDLP		INDEP	
			Mean	%LF	Mean	%LF	Mean	%LF	Mean	%LF	Mean	%LF
	(1,5)	163,897	160,624	97.10	160,206	95.03	156,557	95.70	156,410	95.72	110,471	98.64
0.6	(5,10)	132,674	130,971	97.68	130,875	97.29	126,425	95.45	126,822	95.38	104,330	98.30
	(10,20)	111,897	110,314	97.61	110,209	96.93	106,688	95.53	106,879	95.45	96,661	97.85
	(1,5)	177,384	175,598	97.70	173,520	93.66	170,301	96.05	170,562	96.13	130,841	98.72
0.8	(5,10)	146,338	144,597	97.44	144,377	96.99	140,857	95.90	140,671	95.93	123,399	98.37
	(10,20)	122,464	121,062	96.24	120,985	96.14	117,621	96.03	117,654	96.07	114,012	97.53
	(1,5)	187,270	185,384	96.43	184,785	95.99	181,673	95.57	181,751	95.60	149,246	98.63
1.0	(5,10)	156,243	154,718	94.52	154,508	94.21	151,907	95.03	151,832	95.02	140,161	98.03
	(10,20)	128,386	127,343	91.65	127,255	91.88	125,811	92.27	125,883	92.27	126,091	92.09
	(1,5)	195,269	193,511	94.88	192,953	94.89	190,000	93.71	190,248	93.70	165,880	98.29
1.2	(5,10)	160,206	159,386	87.28	159,354	87.37	157,877	87.36	157,922	87.35	154,210	95.57
	(10,20)	128,448	128,336	78.36	128,336	78.36	128,336	78.36	128,336	78.36	128,361	78.38
	(1,5)	197,113	196,886	86.70	196,860	86.77	196,639	86.79	196,639	86.79	179,983	96.54
1.4	(5,10)	160,453	160,350	76.28	160,352	76.28	160,350	76.28	160,350	76.28	159,435	85.54
	(10,20)	128,448	128,336	68.22	128,336	68.22	128,336	68.22	128,336	68.22	128,363	68.24

Table A7: Simulation results for Hub-and-Spoke Network example.

α	v_0	UB	DCOMP		DCOMP-0.5		CDLP		RCDLP	
			Mean	%LF	Mean	%LF	Mean	%LF	Mean	%LF
0.6	(1,5)	162,627	160,202	96.74	160,016	94.73	154,356	95.43	155,185	95.51
0.8	(1,5)	177,128	175,442	97.62	174,043	94.23	168,914	96.19	170,277	96.09
	(5,10)	146,331	144,600	97.41	144,405	96.99	140,464	95.95	140,709	95.96
1.0	(1,5)	186,899	184,607	95.32	183,765	93.73	179,637	96.41	180,401	96.47
1.2	(1,5)	195,226	193,497	95.10	192,850	94.66	189,611	94.03	190,101	94.03

Table A8: Simulation results for Hub-and-Spoke Network example when the CDLP is solved approximately.

α	v_0	DCOMP	DCOMP-0.5	CDLP	RCDLP
0.6	(1,5)	0.26%	0.12%	1.41%	0.78%
0.8	(1,5)	0.09%	-0.30%	0.81%	0.17%
	(5,10)	0.00%	-0.02%	0.28%	-0.03%
1.0	(1,5)	0.42%	0.55%	1.12%	0.74%
1.2	(1,5)	0.01%	0.05%	0.20%	0.08%

Table A9: Expected suboptimality gaps when processing demand under the approximated CDLP outcome in the Hub-and-Spoke Network.