MANAGEMENT SCIENCE
DOI 10.1287/mnsc.1060.0646ec
pp. ec1–ec17

e-companion
ONLY AVAILABLE IN ELECTRONIC FORM

Electronic Companion—"Online Auction and List Price Revenue Management" by René Caldentey and Gustavo Vulcano, *Management Science* 2007, 53(5) 795–813.

Online Appendices

Appendix A. Proofs

The following lemma will be used a few times in this appendix.

Lemma A1. *Let $B(x)$ be a Poisson random variable with mean $x > 0$. Then, for a nonnegative integer n*

$$\frac{d}{dx}\mathbb{P}(B(x) \le n) = -\mathbb{P}(B(x) = n).$$

Proof.

$$\begin{aligned}\frac{d}{dx}\mathbb{P}(B(x) \le n) &= \sum_{k=0}^{n}\frac{d}{dx}\left(\frac{x^k\exp(-x)}{k!}\right) = \sum_{k=1}^{n}\frac{kx^{k-1}\exp(-x)}{k!} - \sum_{k=0}^{n}\frac{x^k\exp(-x)}{k!}\\ &= -\frac{x^n\exp(-x)}{n!} = -\mathbb{P}(B(x) = n). \quad \square\end{aligned}$$

Proof of Proposition 1. Using the fact that $P_H(v_\tau)$ is a random variable with positive mass at $v^{\mathrm{R}} = 0$ and the properties of conditional expectation, we have that

$$\begin{aligned}&\mathbb{E}[P_H(v_\tau) \mid P_H(v_\tau) < \hat{P}]\\ &\quad = v^{\mathrm{R}}\mathbb{P}(P_H(v_\tau) = v^{\mathrm{R}} \mid P_H(v_\tau) < \hat{P}) + \mathbb{E}[P_H(v_\tau) \mid v^{\mathrm{R}} < P_H(v_\tau) < \hat{P}]\mathbb{P}(P_H(v_\tau) > v^{\mathrm{R}} \mid P_H(v_\tau) < \hat{P})\\ &\quad = \left(\int_0^{\hat{P}} \mathbb{P}(P_H(v_\tau) \ge x \mid 0 < P_H(v_\tau) < \hat{P})\,\mathrm{d}x\right)\mathbb{P}(P_H(v_\tau) > 0 \mid P_H(v_\tau) < \hat{P})\\ &\quad = \int_0^{\hat{P}} \frac{\mathbb{P}(x \le P_H(v_\tau) < \hat{P})}{\mathbb{P}(P_H(v_\tau) < \hat{P})}\mathrm{d}x. \qquad \text{(EC.A1)}\end{aligned}$$

Combining this expression and (7), after some algebra we get the participation constraint:

$$\begin{aligned}\tau &\ge T - \frac{1}{w}\ln\left(\left[\hat{P}\mathbb{P}(P_H(v_\tau) < \hat{P}) - \int_0^{\hat{P}} \mathbb{P}(x \le P_H(v_\tau) < \hat{P})\,\mathrm{d}x\right]\frac{1}{v_\tau - \hat{P}} + 1\right)\\ &= T - \frac{1}{w}\ln\left(1 + \int_0^{\hat{P}} \frac{\mathbb{P}(P_H(v_\tau) < x)}{v_\tau - \hat{P}}\,\mathrm{d}x\right) \triangleq h_H(v_\tau),\end{aligned}$$

which proves (8). From this last inequality, it follows that the right-hand side $h_H(v_\tau)$ is the natural candidate for best-response strategy for buyer τ with valuation $v_\tau > \hat{P}$. However, for an arbitrary $H \in \mathcal{H}$, $h_H(v_\tau)$ is not guaranteed to be nonnegative, and so h_H is not necessarily a well-defined participation strategy in $\mathcal{H}$. Fortunately, note that we can correct this problem by simply setting the best-response strategy $\mathcal{R}(H)(v_\tau) = (h_H(v_\tau))^+$, which is equivalent to (9) and consistent with (7).

To prove the continuity of $\mathcal{R}(H)(v_\tau)$, first note that $\mathcal{R}(H)(v_\tau)$ is trivially continuous in $v_\tau \in [0, \hat{P}]$ as well as continuous in $v_\tau \in (\hat{P}, 1]$.[EC1] In addition, as $v_\tau \downarrow \hat{P}$ we have that $\mathcal{R}(H)(v_\tau) \to 0 = \mathcal{R}(H)(\hat{P})$. This observation follows by noting that

$$\text{for all } H \in \mathcal{H}, \quad \lim_{v_\tau \downarrow \hat{P}} \int_0^{\hat{P}} \frac{\mathbb{P}(P_H(v_\tau) < x)}{v_\tau - \hat{P}}\, \mathrm{d}x = +\infty,$$

because the numerator of the integrand is strictly positive and bounded away from zero. Therefore, for any $H \in \mathcal{H}$, the best-response strategy $\mathcal{R}(H)(v)$ is in fact continuous in $[0, 1]$. □

Proof of Lemma 1. First of all, note that $\mathcal{R}(H)(v) > 0$ if

$$1 + \int_0^{\hat{P}} \frac{\mathbb{P}(P_H(v_\tau) < x)}{v_\tau - \hat{P}}\, \mathrm{d}x < \exp(wT).$$

Because the left-hand side goes to infinite as $v_\tau \downarrow \hat{P}$, we can unambiguously define for every $H \in \mathcal{H}$

$$v_H \triangleq 1 \wedge \arg\min_{v \ge \hat{P}} \left\{ 1 + \int_0^{\hat{P}} \frac{\mathbb{P}(P_H(v) < x)}{v - \hat{P}}\, \mathrm{d}x \le \exp(wT) \right\}. \qquad \text{(EC.A2)}$$

Note that by construction, for any $H \in \mathcal{H}$ we must have $\mathcal{R}(H)(v) = 0$ for all $v \in [0, v_H]$. To obtain a lower bound on v_H independent of H, we can solve

$$\tilde{v} \triangleq \inf_{H \in \mathcal{H}} \{v_H\}. \qquad \text{(EC.A3)}$$

Unfortunately, this is not a straightforward optimization problem for which we can compute the optimal solution $\tilde{v}$. However, we can obtain a lower bound on $\tilde{v}$ by means of the following inequality.

$$\text{For all } H \in \mathcal{H}, \quad 1 + \int_0^{\hat{P}} \frac{\mathbb{P}(P_H(v) < x)}{v - \hat{P}}\, \mathrm{d}x \ge 1 + \int_0^{\hat{P}} \frac{\mathbb{P}(P_0(v) < x)}{v - \hat{P}}\, \mathrm{d}x,$$

where $P_0(v)$ stands for the auction price when $H = 0$. Because for $x < \hat{P}$, $\mathbb{P}(P_0(v) < x) = \mathbb{P}(B(\Lambda_0(x)) \le Q_0 - 1) = \mathbb{P}(B(\lambda T[1 - F(x)]) \le Q_0 - 1)$, we get a lower bound on $\tilde{v}$ solving for $\underline{v}$ in

$$1 + \int_0^{\hat{P}} \frac{\mathbb{P}(B(\lambda T[1 - F(x)]) \le Q_0 - 1)}{\underline{v} - \hat{P}}\, \mathrm{d}x = \exp(wT),$$

or equivalently,

$$\underline{v} = \hat{P} + \exp(-wT)\left(1 + \int_0^{\hat{P}} \mathbb{P}(B(\lambda T[1 - F(x)]) \le Q_0 - 1)\, \mathrm{d}x\right). \quad \square$$

Proof of Proposition 2. Note that we only need to prove the K-Lipschitz property of $\mathcal{R}(H)(v)$; the rest of the proposition follows directly from the definition of $\tilde{v}$.

To prove that $\mathcal{R}(H)(v)$ is K-Lipschitz continuous, we will make use of the following lemma (not hard to prove).

Lemma A2. *For arbitrary reals a, b, and c: $|[a-b]^+ - [a-c]^+| \le |b-c|$. For arbitrary nonnegative reals $x, y \ge 1$: $|\ln(x) - \ln(y)| \le |x - y|$.*

Recall from the definition of v_H that $\mathcal{R}(H)(v) = 0$ for all $v \in [0, v_H]$. Therefore, we can concentrate on proving the K-Lipschitz property on the interval $[v_H, 1]$. In this range, condition (9) implies that

$$\mathcal{R}(H)(v) = \left[T - \frac{1}{w}\ln(Z_H(v))\right]^+ \quad \text{where } Z_H(v) \triangleq 1 + \int_0^{\hat{P}} \frac{\mathbb{P}(P_H(v) < x)}{v - \hat{P}}\, \mathrm{d}x.$$

[EC1] This follows from the continuity of $\Lambda_H(x)$, which implies the continuity of $\mathbb{P}(P_H(v_\tau) < x)$ for $x \in [0, \hat{P}]$ (see Equation (5)).

Therefore, based on the previous lemma, we have that for arbitrary $v_1, v_2 \in [v_H, 1]$

$$\begin{aligned}|\mathscr{R}(H)(v_1) - \mathscr{R}(H)(v_2)| &= \left|\left[T - \frac{1}{w}\ln(Z_H(v_1))\right]^+ - \left[T - \frac{1}{w}\ln(Z_H(v_2))\right]^+\right| \\ &\le \frac{1}{w}|\ln(Z_H(v_1)) - \ln(Z_H(v_2))| \le \frac{1}{w}|Z_H(v_1) - Z_H(v_2)|.\end{aligned}$$

From the definition of $Z_H(v)$ and the fact that $\int_0^{\hat{P}} \mathbb{P}(P_H(v) < x)\,\mathrm{d}x$ is independent of v for $v \ge v_H > \hat{P}$ (because bidders with valuation in this range bid $b(v) = \hat{P}$ independent of v), we have that

$$\begin{aligned}|Z_H(v_1) - Z_H(v_2)| &= \left|\int_{v_1}^{v_2} \frac{\mathrm{d}}{\mathrm{d}v} Z_H(v)\,\mathrm{d}v\right| \\ &\le \left|\int_{v_1}^{v_2}\left[-\int_0^{\hat{P}} \frac{\mathbb{P}(P_H(v) < x)}{(v - \hat{P})^2}\,\mathrm{d}x\right]\mathrm{d}v\right| \le \left|\int_{v_1}^{v_2}\left[\frac{\hat{P}}{(\tilde{v} - \hat{P})^2}\right]\mathrm{d}v\right| \quad (\text{using } \tilde{v} \le v \le 1) \\ &= \left[\frac{\hat{P}}{(\tilde{v} - \hat{P})^2}\right]|v_1 - v_2|.\end{aligned}$$

We conclude that for arbitrary $v_1, v_2 \in [v_H, 1]$

$$|\mathscr{R}(H)(v_1) - \mathscr{R}(H)(v_2)| \le \frac{1}{w}\left[\frac{\hat{P}}{(\tilde{v} - \hat{P})^2}\right]|v_1 - v_2| \triangleq K|v_1 - v_2|.$$

The constant K is guaranteed to be finite because $\tilde{v} \ge \underline{v} > \hat{P}$. □

Proof of Theorem 1. To prove that $\mathscr{H}$ has the fixed-point property, we apply the Schauder-Tychonoff Fixed-Point Theorem (see Cheney 2001, Chapter 7 for details). For this, we need to show that $\mathscr{H}$ is a compact convex set. Convexity is immediate from the definition of $\mathscr{H}$. To check compactness, we apply the Arzelà-Ascoli Theorem II (Cheney 2001, Chapter 7), that is, we need to show that $\mathscr{H}$ is closed, bounded, and equicontinuous. Take a sequence $\{H^n\}_{n\ge 1}$ of strategies in $\mathscr{H}$ that converges pointwise to H. For all $v \in [0, \tilde{v}]$, $H^n(v) = 0$, and so $H(v) = 0$ as well. In addition, to verify the K-Lipschitz property of H, note that from Proposition 2, for $n \ge 1$ and $v_1, v_2 \in [0, 1]$,

$$|H^n(v_1) - H^n(v_2)| \le K|v_1 - v_2|.$$

By the continuity of the absolute value and the pointwise convergence of H^n to H, we conclude

$$|H(v_1) - H(v_2)| \le K|v_1 - v_2|,$$

which proves the closedness of $\mathscr{H}$. The boundedness of $\mathscr{H}$ follows from the fact $H(v) \in [0, T]$ for all $H \in \mathscr{H}$. Equicontinuity, on the other hand, follows directly from the fact that the elements of $\mathscr{H}$ are K-Lipschitz continuous. In fact, to prove equicontinuity we need to show that for $\epsilon > 0$ there is $\delta > 0$ such that

$$\text{For all } H \in \mathscr{H} \text{ and } v_1, v_2 \in \mathscr{V} \quad \text{such that } |v_1 - v_2| < \delta, \quad \text{then } |H(v_1) - H(v_2)| < \epsilon.$$

For this, take $\delta = \epsilon/K$ and use the K-Lipschitz continuity of $\mathscr{H}$ as follows:

$$\text{For all } H \in \mathscr{H} \text{ and } v_1, v_2 \in \mathscr{V} \quad \text{such that } |v_1 - v_2| < \delta, \quad |H(v_1) - H(v_2)| \le K|v_1 - v_2| < K\delta = \epsilon.$$

This proves that $\mathscr{H}$ has the fixed-point property.

We now prove that the best-response $\mathscr{R}$ mapping is continuous. Note that from the definitions of the mapping $\mathscr{R}$ and $\tilde{v}$, we need only to prove the result for the restriction of $\mathscr{R}$ to the interval $(\tilde{v}, 1]$. For the proof, we will require the following lemma.

Lemma A3. *Let a and b be two nonnegative reals and* $N \ge 1$ *and integer. Then, there is* $0 \le \beta(N) \le 1$ *such that*

$$\big|\mathbb{P}(B(b) \le N) - \mathbb{P}(B(a) \le N)\big| \le \beta(N)|b - a|.$$

Proof. Because $\mathbb{P}(B(a) \le N)$ is a continuous and differentiable function of a, we have that

$$\mathbb{P}(B(b) \le N) = \mathbb{P}(B(a) \le N) + \int_a^b \frac{\mathrm{d}}{\mathrm{d}x} \mathbb{P}(B(x) \le N)\,\mathrm{d}x,$$

which is straightforward to prove (using Lemma A1) that it is equivalent to

$$\mathbb{P}(B(b) \le N) = \mathbb{P}(B(a) \le N) - \int_a^b \mathbb{P}(B(x) = N)\,\mathrm{d}x.$$

Therefore,

$$\begin{aligned}\big|\mathbb{P}(B(b) \le N) - \mathbb{P}(B(a) \le N)\big| &= \int_{a\wedge b}^{a\vee b} \mathbb{P}(B(x) = N)\,\mathrm{d}x \\ &\le |b-a|\mathbb{P}(B(N) = N),\end{aligned}$$

where the inequality follows from the fact that $\mathbb{P}(B(x) = N)$ is maximized at $x = N$. Therefore, by setting

$$\beta(N) \triangleq \mathbb{P}(B(N) = N)$$

the proof of the lemma is completed. □

Based on Lemma A2, we have that for any $v \in (\tilde{v}, 1]$,

$$\begin{aligned}|\mathcal{R}(H)_v - \mathcal{R}(\widetilde{H})_v| &= \left|\left[T - \frac{1}{w}\ln\left(1 + \int_0^{\hat{P}} \frac{\mathbb{P}(P_H(v) < x)}{v - \hat{P}}\,\mathrm{d}x\right)\right]^+ - \left[T - \frac{1}{w}\ln\left(1 + \int_0^{\hat{P}} \frac{\mathbb{P}(P_{\widetilde{H}}(v) < x)}{v - \hat{P}}\,\mathrm{d}x\right)\right]^+\right| \\ &\le \frac{1}{w}\left(\int_0^{\hat{P}} \frac{\big|\mathbb{P}(P_H(v) < x) - \mathbb{P}(P_{\widetilde{H}}(v) < x)\big|}{v - \hat{P}}\,\mathrm{d}x\right).\end{aligned}$$

Now, by Lemma A3,

$$\begin{aligned}|\mathbb{P}(P_H(v) < x) - \mathbb{P}(P_{\widetilde{H}}(v) < x)| &= |\mathbb{P}(B(\Lambda_H(x)) \le Q_0 - 1) - \mathbb{P}(B(\Lambda_{\widetilde{H}}(x)) \le Q_0 - 1)| \\ &\le \beta(Q_0 - 1)|\Lambda_H(x) - \Lambda_{\widetilde{H}}(x)| = \beta(Q_0 - 1)\left|\lambda \int_x^1 (\widetilde{H}(y) - H(y))\,\mathrm{d}F(y)\right| \\ &\le \lambda\beta(Q_0 - 1)(1 - F(x))\|H - \widetilde{H}\|.\end{aligned}$$

Finally, from this inequality we get that for all $v \in (\tilde{v}, 1]$

$$\begin{aligned}|\mathcal{R}(H)_v - \mathcal{R}(\widetilde{H})_v| &\le \frac{\lambda\beta(Q_0 - 1)}{w}\left(\int_0^{\hat{P}} (1 - F(x))v - \hat{P}\,\mathrm{d}x\right)\|H - \widetilde{H}\| \\ &\le \frac{\lambda\beta(Q_0 - 1)}{w}\left(\int_0^{\hat{P}} \frac{(1 - F(x))}{\tilde{v} - \hat{P}}\,\mathrm{d}x\right)\|H - \widetilde{H}\|,\end{aligned}$$

where the second inequality follows from the fact that $v \in (\tilde{v}, 1]$. From this result, we conclude that $\mathcal{R}$ is continuous, which together with the fixed-point property of the set $\mathcal{H}$ guarantees the existence of an SPE. □

Proof of Proposition 3. Given the equilibrium H^*, we define $Z_{H^*}(v)$ by

$$Z_{H^*}(v) \triangleq 1 + \int_0^{\hat{P}} \frac{\mathbb{P}(P_{H^*}(v) < x)}{v - \hat{P}}\,\mathrm{d}x, \quad v \in [v_{H^*}, 1].$$

Note that in the range $[0, v_{H^*}]$ the function H^* is constant at zero.

Based on the definition of the best-response mapping $\mathcal{R}$, we can write the fixed-point condition $\mathcal{R}(H^*) = H^*$ satisfied by $H^*(v)$ as

$$H^*(v) = T - \frac{1}{w}\ln(Z_{H^*}(v)), \quad \text{for all } v \in [v_{H^*}, 1].$$

Taking the derivative with respect to v and using the fact that

$$\chi \triangleq \int_0^{\hat{P}} \mathbb{P}(P_{H^*}(v) < x)\,\mathrm{d}x$$

is independent of v for $v \geq v_{H^*} > \hat{P}$, we get that

$$\frac{\mathrm{d}}{\mathrm{d}v} H^*(v) = \frac{-1}{w}\frac{1}{Z_{H^*}(v)}\frac{\mathrm{d}}{\mathrm{d}v} Z_{H^*}(v) = \frac{\chi}{w}\left(\frac{1}{(v-\hat{P})(v-\hat{P}+\chi)}\right) > 0,$$

and we conclude that $H^*(v)$ is increasing in v for all $v \geq v_{H^*}$. Finally, taking a second derivative we get

$$\frac{\mathrm{d}^2}{\mathrm{d}v^2} H^*(v) = -\frac{\chi}{w}\left(\frac{2(v-\hat{P})+\chi}{(v-\hat{P})^2(v-\hat{P}+\chi)^2}\right) < 0,$$

and we conclude that $H^*(v)$ is concave in the range $v \in [v_{H^*}, 1]$. □

Proof of Theorem 2. We need the following preliminary result.

Lemma A4. *Let $B_i(\mu_n)$ be a sequence of i.i.d Poisson random variables with mean μ_n, and let y_n be an increasing sequence of nonnegative integers. For each n, define $S_{y_n} \triangleq \sum_{i=1}^{y_n} B_i(\mu_n)$. Suppose that $\lim_{n\to\infty} y_n = \infty$ and $\lim_{n\to\infty}\mu_n = \mu$. Then, the moment-generating function of the r.v. $Y_n \triangleq S_{y_n}/y_n$ converges to a constant $\exp(\theta\mu)$, and hence Y_n converges weakly to the constant μ.*

Proof of Lemma A4. First, note that $S_{y_n} \sim \text{Poisson}(y_n\mu_n)$. Using the moment-generating function for the Poisson,

$$\begin{aligned} \mathrm{E}\exp\left(\theta\frac{S_{y_n}}{y_n}\right) &= \mathrm{E}\exp\left(\frac{\theta}{y_n}S_{y_n}\right) = \exp(y_n\mu_n(e^{\theta/y_n}-1)) \\ &= \exp\left(y_n\mu_n\left(\frac{\theta}{y_n} + o(1/y_n)\right)\right) \\ &= \exp(\mu_n\theta) + o(1). \end{aligned}$$

Hence,

$$\lim_{n\to\infty} \mathrm{E}\exp(\theta S_{y_n}/y_n) \to \exp(\theta\mu),$$

the moment-generating function of the constant μ. This guarantees convergence in distribution.[EC2] □

We now prove the theorem. From the definitions of $\Lambda_H^n(x)$ and $\eta_H(x)$, and the relationship between λ^n and Q_0^n, we have that

$$\Lambda_H^n(x) = \frac{\eta_H(x)}{\rho^n} Q_0^n.$$

Now, let $\{B_i(\eta_H(x)(\rho^n)^{-1})\colon i = 1, \ldots, Q_0^n\}$ be a sequence of i.i.d Poisson r.v. with mean $\eta_H(x)(\rho^n)^{-1}$. It follows that $B(\Lambda_H^n(x))$ has the same distribution as the sum of the $B_i(\eta_H(x)(\rho^n)^{-1})$ from i equals 1 to Q_0^n. Therefore, for a given n,

$$\mathbb{P}(P_H^n(v) < x) = 1, \quad \text{if } x > \hat{P} \qquad \text{or} \tag{EC.A4}$$

$$\begin{aligned} \mathbb{P}(P_H^n(v) < x) &= \mathbb{P}(B(\Lambda_H^n(x)) \leq Q_0^n - \mathbb{1}(x \leq v)) \\ &= \mathbb{P}\left(\sum_{i=1}^{Q_0^n} B_i(\eta_H(x)(\rho^n)^{-1}) \leq Q_0^n - \mathbb{1}(x \leq v)\right) \\ &= \mathbb{P}\left(\frac{\sum_{i=1}^{Q_0^n} B_i(\eta_H(x)(\rho^n)^{-1})}{Q_0^n} \leq 1 - \frac{\mathbb{1}(x \leq v)}{Q_0^n}\right), \quad \text{if } x \leq \hat{P}. \end{aligned} \tag{EC.A5}$$

[EC2] See, for example, §30 in Billingsley (1995).

Taking the left-hand side of the last inequality inside the parentheses, we define

$$\mathscr{B}^n(x) \triangleq \frac{\sum_{i=1}^{Q_0^n} B_i(\eta_H(x)(\rho^n)^{-1})}{Q_0^n}.$$

From Lemma 4, $\mathscr{B}^n(x)$ converges in distribution to the constant $\eta_H(x)\rho^{-1}$. Moreover, it is clear that the right-hand side of the inequality in (EC.A5) converges to one. Therefore, by focusing on the continuity points, the distribution of $P_H^n(v)$ converges weakly to the distribution:

$$\mathbb{P}(P_H^\infty(v) < x) = \begin{cases} 1 & \text{if } x > \hat{P} \\ 1 & \text{if } \rho > \eta_H(x) \text{ and } x \le \hat{P} \\ 0 & \text{if } \rho < \eta_H(x) \text{ and } x \le \hat{P}. \end{cases} \tag{EC.A6}$$

This corresponds to the distribution of the constant $\min\{\hat{P}, \eta_H^{-1}(\rho)\}$ at its continuity points, and so $\mathbb{P}(P_H^n(v) < x) \Rightarrow \mathbb{P}(P_H^\infty(v) < x)$. Thus, $P_H^n(v) \Rightarrow P_H^\infty(v)$, for $P_H^\infty(v) = \min\{\hat{P}, \eta_H^{-1}(\rho)\}$. □

Proof of Theorem 3. From Theorem 2, we have that $P_{H^*} = \min\{v \in [0,1]: \eta_H(v) \le \rho\}$. Because the function $\eta_H(v)$ is monotonically decreasing, we conclude that $\eta_{H^*}(P_{H^*}) = \min\{\rho, \eta_{H^*}(0)\}$. This condition, together with Equation (13) and the equilibrium condition $H^* = \mathscr{R}(H^*)$, implies condition (14). The value of v_{H^*} and H^* follow from conditions (12) and (13), respectively. □

Proof of Proposition 4. Suppose that there exists a randomized equilibrium that is characterized by n different participation strategies $\{H_k(v) \in \mathscr{H}\}_{k=1}^n$ and n probability mappings $\{\gamma_k(v) \in [0,1]\}_{k=1}^n$, such that $\sum_{k=1}^n \gamma_k(v) = 1$ for all $v \in [0,1]$. In this equilibrium, a consumer with valuation $v \in [0,1]$ arriving at time $\tau(v) \in [0,T]$ will use the participation strategy $H_k(v)$ with probability $\gamma_k(v)$. Note that we only consider randomization in the participation decision (buy now or bid in the auction), because on the bidding side it is a dominant strategy to bid $b(v) = \min\{v, \hat{P}\}$.

To show that such a randomized equilibrium is not possible, let us take two participation strategies $H_i(v)$ and $H_j(v)$ such that there exists a $\tilde{v} \in [0,1]$, such that $H_i(\tilde{v}) < H_j(\tilde{v})$ and $\gamma_i(\tilde{v}) > 0$ and $\gamma_j(\tilde{v}) > 0$. Naturally, such a pair (i,j) and valuation $\tilde{v}$ must exist in order to have a randomized equilibrium. Also, note that $\tilde{v} > \hat{P}$ because $H_i(v) = H_j(v) = 0$ for $v \le \hat{P}$.

Consider a consumer with valuation $\tilde{v}$ arriving at time $\tau(\tilde{v}) \in (H_i(\tilde{v}), H_j(\tilde{v}))$. Because this consumer assigns positive probabilities to both participation strategies $H_i(v)$ and $H_j(v)$, he must be indifferent between the expected payoff generated by these two strategies.

- If he selects strategy $H_i(v)$, then he will purchase on the fixed-price channel (because $\tau(\tilde{v}) > H_i(\tilde{v})$), obtaining a payoff of $\tilde{v} - \hat{P}$.
- If he selects strategy $H_j(v)$, then he will enter the auction and bid $\hat{P}$. The corresponding expected payoff is

$$\mathbb{E}[e^{-w(T-\tau(\tilde{v}))}(\tilde{v} - \tilde{P}_A)],$$

where $\tilde{P}_A$ is the auction price given that consumer $\tilde{v}$ enters the auction and all other consumers use the randomized strategy $\{(H_k(v), \gamma_k(v))\}_{k=1}^n$.

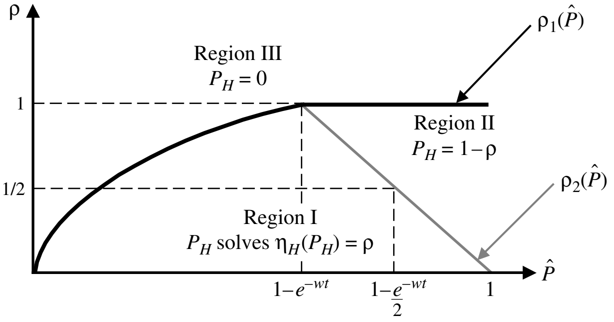
Hence, consumer $\tilde{v}$ will be willing to randomize between $H_i(v)$ and $H_j(v)$ only if

$$\tilde{v} - \hat{P} = \mathbb{E}[e^{-w(T-\tau(\tilde{v}))}(\tilde{v} - \tilde{P}_A)]. \tag{EC.A7}$$

Note that the left-hand side of this equality is independent of the arrival time $\tau(\tilde{v}) \in (H_i(\tilde{v}), H_j(\tilde{v}))$. On the other hand, it is not hard to see that $\tilde{P}_A$ is also independent of $\tau(\tilde{v})$, and so the right-hand side is a strictly increasing function of $\tau(\tilde{v}) \in (H_i(\tilde{v}), H_j(\tilde{v}))$. We conclude that the randomization condition (EC.A7) cannot be sustained for all $\tau(\tilde{v}) \in (H_i(\tilde{v}), H_j(\tilde{v}))$ unless $H_i(\tilde{v}) = H_j(\tilde{v})$. This argument shows that there cannot be a randomized equilibrium. (We note that the result in this Proposition 4 and its proof here are also valid for the dual-channel case.) □

Proof of Proposition 5. The proof uses extensively the notation and results of the uniform distribution example in §3.3 in the main paper. We suggest that the reader review this example before

Figure EC.A1 Auction Price P_{H^*} in the $(\rho, \hat{P})$ Space



going over the following proof. Also, because T is fixed in this proof, we will drop the dependence of $Q_{H^*}(\rho, T, \hat{P})$ and $P_{H^*}(\rho, T, \hat{P})$ on T.

First of all, we note that given the optimality condition $\rho < 1$, it follows that $Q_{H^*}(\rho, \hat{P}) = \rho$ for every T and $\hat{P}$. Hence, the seller's optimization problem reduces to maximizing the function

$$\max_{0\le\rho\le\bar{Q}/\lambda T} \{\lambda T \exp(-\alpha T)\rho P_{H^*}(\rho, \hat{P})\},$$

where $P_{H^*}(\rho, \hat{P})$ is the resulting auction price given ρ and $\hat{P}$. To simplify the exposition, we will use Figure EC.A1 as a guide to support the argument of the proof.

The figure depicts the value of the auction price P_{H^*} as a function of ρ and $\hat{P}$. The threshold functions ρ_1 and ρ_2 are derived from the definition of $\hat{P}_1(\rho, wT)$ and $\hat{P}_2(\rho, wT)$, respectively. Specifically, we have that

$$\rho_1(\hat{P}) = \frac{-1}{wT}\left[(1-(\hat{P}\wedge P_c))\ln(1-(\hat{P}\wedge P_c)) + (\hat{P}\wedge P_c)\ln\left(\frac{(\hat{P}\wedge P_c)}{\exp(wT)-1}\right)\right], \quad \text{and} \quad \rho_2(\hat{P}) = (1-\hat{P})\exp(wT),$$

where $P_c = 1-\exp(-wT)$ and $x \wedge y$ stands for $\min\{x, y\}$. Figure EC.A1 distinguishes three regions in the $(\rho, \hat{P})$ space, which are defined as follows:

$$\text{Region I} \triangleq \{(\rho, \hat{P}) \colon \rho \le \min\{\rho_1(\hat{P}), \rho_2(\hat{P})\}\}, \qquad \text{Region II} \triangleq \{(\rho, \hat{P}) \colon \rho_2(\hat{P}) \le \rho \le \rho_1(\hat{P})\}, \qquad \text{and}$$

$$\text{Region III} \triangleq \{(\rho, \hat{P}) \colon \rho_1(\hat{P}) \le \rho\}.$$

In Region III, the auction price is zero, $P_{H^*} = 0$, and so it never optimal for the seller to choose $\rho \ge \rho_1(\hat{P})$. Hence, the optimal ρ^* as a function of $\hat{P}$ must lie in Region I or in Region II. In order to characterize this optimal ρ^*, we will use the following intermediate result.

Lemma A5. *For any $\hat{P} \in [0, 1]$, the function $\rho P_{H^*}(\rho, \hat{P})$ is unimodal in $\rho \in [0, \rho_1(\hat{P})]$, and so it admits a unique local maximum.*

Proof of the Lemma. We divide the proof in to two cases.

Case 1. $\hat{P} \le 1-\exp(-wT)$. In this case, we will prove that the function $\rho P_{H^*}(\rho, \hat{P})$ is in fact concave, from which the unimodality follows directly. The value of $P_{H^*}(\rho, \hat{P})$ is the maximal root of the equation

$$\frac{1}{wT}\left[(1-P_{H^*})\ln(1-P_{H^*}) - (1-\hat{P})\ln(1-\hat{P}) - (\hat{P}-P_{H^*})\ln\left(\frac{\hat{P}-P_{H^*}}{\exp(wT)-1}\right)\right] = \rho. \tag{EC.A8}$$

From this condition we get P_{H^*} as a function of ρ. It is not hard to show that the left-hand side is a decreasing function of P_{H^*} in the range $P_{H^*} \in [\tilde{P}_0, \hat{P}]$, where

$$\tilde{P}_0 \triangleq \frac{(\hat{P}+1-\exp(wT))^+}{2-\exp(wT)}.$$

Hence, for any $\rho \in [0, \rho_1(\hat{P})]$, the solution $P_{H^*}(\rho, \hat{P})$ is the unique root of (EC.A8) in $[\tilde{P}_0, \hat{P}]$. Now, taking derivative of (EC.A8) with respect to ρ, we get that

$$\frac{\mathrm{d}P_{H^*}}{\mathrm{d}\rho} = \frac{wT}{\ln(\hat{P}-P_{H^*}) - \ln((\exp(wT)-1)(1-P_{H^*}))},$$

which is negative in $P_{H^*} \in [\tilde{P}_0, \hat{P}]$. Differentiating one more time, we get

$$\frac{\mathrm{d}^2 P_{H^*}}{\mathrm{d}\rho^2} = \frac{wT(1-\hat{P})}{(\hat{P}-P_{H^*})(1-P_{H^*})}\frac{\mathrm{d}P_{H^*}}{\mathrm{d}\rho},$$

which is also negative in $[\tilde{P}_0, \hat{P}]$.

The concavity of $\rho P_{H^*}(\rho, \hat{P})$ then follows from the fact that

$$\frac{\mathrm{d}^2 \rho P_{H^*}(\rho, \hat{P})}{\mathrm{d}\rho^2} = 2\frac{\mathrm{d}P_{H^*}}{\mathrm{d}\rho} + \rho\frac{\mathrm{d}^2 P_{H^*}}{\mathrm{d}\rho^2},$$

which is negative for any $\rho \in [0, \rho_1(\hat{P})]$

Case 2. $\hat{P} > 1-\exp(-wT)$. In this case, the function $\rho P_{H^*}(\rho, \hat{P})$ has two pieces, depending on whether $\rho \in [0, \rho_2(\hat{P})]$ or $\rho \in [\rho_2(\hat{P}), 1]$.

The arguments used in Case 1 extend in this case to the range $\rho \in [0, \rho_2(\hat{P})]$, and so $\rho P_{H^*}(\rho, \hat{P})$ is concave in this range. On the other hand, for $\rho \in [\rho_2(\hat{P}), 1]$ the revenue function is $\rho P_{H^*}(\rho, \hat{P}) = \rho(1-\rho)$, which is trivially concave. Hence, in order to show unimodality it suffices to show that the two pieces match smoothly at $\rho = \rho_2(\hat{P})$. This is equivalent to showing

$$\lim_{\rho\uparrow\rho_2(\hat{P})} P_{H^*} = 1-\rho_2(\hat{P}) \qquad \text{and} \qquad \lim_{\rho\uparrow\rho_2(\hat{P})} \frac{\mathrm{d}P_{H^*}}{\mathrm{d}\rho} = -1.$$

It is a matter of simple calculations to verify these two conditions, and we leave completing this final step to the reader. ☐

Based on this lemma, the rest of the proof of the proposition follows directly. In fact, because of the unimodality (as a function of ρ) of the revenue $\rho P_{H^*}(\rho, \hat{P})$ for every $\hat{P}$ there is a unique solution ρ^*.

For $\hat{P} \geq 1-\exp(-wT)/2$, it is not hard to show that the unconstrained solution is $\rho^* = 1/2$. This follows from the unimodality of the revenue function and the fact the revenue function is increasing at $\rho = \rho_2(\hat{P})$ if $\hat{P} \geq 1-\exp(-wT)/2$.

On the other hand, for $\hat{P} \leq 1-\exp(-wT)/2$ the revenue function is decreasing at $\rho = \rho_2(\hat{P})$, and so the optimal solution lies in Region I. In this region, the first-order condition is

$$\frac{\mathrm{d}\rho P_{H^*}(\rho, \hat{P})}{\mathrm{d}\rho} = 0 \implies P_{H^*} + \rho\frac{wT}{\ln(\hat{P}-P_{H^*}) - \ln((\exp(wT)-1)(1-P_{H^*}))} = 0.$$

We use condition (EC.A8) to replace ρ in terms of P_{H^*}. Rearranging, we get that the solution P_{H^*} satisfies

$$\left[\frac{(e^{wT}-1)(1-P_{H^*})}{\hat{P}-P_{H^*}}\right]^{1-2P_{H^*}} = \left[\frac{(e^{wT}-1)(1-\hat{P})}{\hat{P}-P_{H^*}}\right]^{1-\hat{P}},$$

which completes the proof. ☐

Proof of Proposition 6. From the asymptotic analysis in §4, it follows that at optimality $\rho \leq 1$ (or, equivalently, $Q_0 \leq \lambda T$). We will assume that this optimality condition is satisfied in the remainder of this proof.

Let us first show that at optimality $1-\hat{P}^* \leq \rho^* \leq (1-\hat{P}^*)\exp(wT)$.

Suppose that $\rho \leq 1-\hat{P}$. This corresponds to the case of *limited supply*, and in equilibrium all units are sold through the fixed-price channel, that is, $Q_{H^*}(\rho, T, \hat{P}) = 0$. The seller's revenue is given by

$$\lambda(1-\hat{P})\hat{P}\left(\frac{1-\exp(-\alpha\rho T/(1-\hat{P}))}{\alpha}\right).$$

It is not hard to see that in the region $\{(\rho, \hat{P})\colon \rho \leq 1-\hat{P} \text{ and } \rho \leq \bar{Q}/\lambda T\}$, this revenue function is maximized at $\rho_1 = \min\{1/2; \bar{Q}/\lambda T\}$, $\hat{P}_1 = 1-\rho_1$, and it is equal to

$$V_1 = \lambda\rho_1(1-\rho_1)\left(\frac{1-\exp(-\alpha T)}{\alpha}\right).$$

This proves that V_1 is a lower bound on the optimal seller's revenue and that at optimality $1-\hat{P}^* \leq \rho^*$.

Suppose now that $\rho \geq (1-\hat{P})\exp(wT)$. In this case, all units are sold in the auction in equilibrium. The seller's revenue is given by

$$\lambda T\rho(1-\rho)\exp(-\alpha T).$$

In the region $\{(\rho, \hat{P})\colon \rho \geq (1-\hat{P})\exp(wT) \text{ and } \rho \leq \bar{Q}/\lambda T\}$, this revenue function is maximized at $\rho = \rho_1$, and any $\hat{P}_2 \geq 1-\exp(-wT)\rho_2$ and equals

$$V_I = \lambda\rho_1(1-\rho_1)T\exp(-\alpha T).$$

Because $(1-\exp(-\alpha T))/\alpha \geq T\exp(-\alpha T)$ for all $T \geq 0$ and $\alpha \geq 0$, we conclude that $V_1 \geq V_I$. Therefore, it is never optimal to choose a solution $(\rho^*, \hat{P}^*)$ in the region $\rho \geq (1-\hat{P})\exp(wT)$.

We conclude that at optimality $1-\hat{P}^* \leq \rho^* \leq (1-\hat{P}^*)\exp(wT)$ and that the seller's optimal revenue V_D is bounded below by V_1.

Let us now derive the upper bound for V_D. For any ρ and $\hat{P}$, the seller's revenue can be bounded above by assuming that every buyer with valuation greater than $\hat{P}$ buys the product in the fixed-price channel, and the remaining $\lambda T[\rho-(1-\hat{P})]$ units are sold in the auction at a price $P_H = 1-\rho$. Note that the optimality condition $1-\hat{P}^* \leq \rho^*$ guarantees the nonnegativity of the number of units sold in the auction. The corresponding upper bound on the seller's revenue is given by

$$\bar{V}(\hat{P}, \rho) = \lambda\hat{P}(1-\hat{P})\left(\frac{1-\exp(-\alpha T)}{\alpha}\right) + \lambda(1-\rho)(\hat{P}-(1-\rho))T\exp(-\alpha T).$$

In order to maximize $\bar{V}(\hat{P}, \rho)$, let us first fix ρ and optimize over $\hat{P}$ under the constraint $\hat{P} \geq 1-\rho$. It follows that the optimal $\hat{P}(\rho)$, as a function of ρ, satisfies

$$\hat{P}(\rho) = \begin{cases} 1-\rho & \text{if } \rho \leq \dfrac{1-\delta}{2-\delta} \\[2ex] \dfrac{1}{2}(1+(1-\rho)\delta) & \text{if } \rho \geq \dfrac{1-\delta}{2-\delta}. \end{cases}$$

Recall that $\delta = \alpha T\exp(-\alpha T)/(1-\exp(-\alpha T))$. From this solution, $\hat{P}(\rho)$, we recover the two cases in the proposition.

Case 1. Suppose $\bar{Q}/\lambda T \leq (1-\delta)/(2-\delta)$; then any feasible ρ also satisfies $\rho \leq (1-\delta)/(2-\delta)$ and the upper bound $\bar{V}$ equals

$$\bar{V}(\hat{P}(\rho), \rho) = \lambda\rho(1-\rho)\left(\frac{1-\exp(-\alpha T)}{\alpha}\right).$$

Because $(1-\delta)/(2-\delta) \leq 1/2$, it follows that $\bar{V}$ is maximized at $\rho = \bar{Q}/\lambda T$ and $\bar{V} = V_1$ because $\rho_1 = \bar{Q}/\lambda T$ in this case. Because the upper and lower bounds are equal, we conclude that $V_D = V_1$.

Case 2. Suppose $\bar{Q}/\lambda T \geq (1-\delta)/(2-\delta)$. Replacing the value of $\hat{P}(\rho) = (1/2)(1+(1-\rho)\delta)$ in $\bar{V}(\hat{P}, \rho)$ and maximizing over ρ, we get that the optimal solution is

$$\rho_2 = \min\left\{\frac{\bar{Q}}{\lambda T}; \frac{3-\delta}{4-\delta}\right\} \qquad \text{and} \qquad P_2 = \frac{1}{2}[1+(1-\rho_2)\delta]$$

and the upper bound $\bar{V}$ equals

$$V_2 = \frac{\lambda T\exp(-\alpha T)}{\delta}(P_2^2 - (1-\rho_2)^2\delta).$$

To verify that the bounds are asymptotically tight, we first note that as $w \downarrow 0$ the optimality condition $1-\hat{P}^* \leq \rho^* \leq (1-\hat{P}^*)\exp(wT)$ implies that $\rho^* = 1-\hat{P}^*$ and so $V_D = V_1$. On the other hand, as $w \to \infty$, all buyers with valuation greater than $\hat{P}$ will buy the product on the fixed-price channel, and so $V_D = V_2$.

Finally, let us derive the upper bound on T^*. Suppose the seller chooses a time T so that the condition

$$\frac{\bar{Q}}{\lambda T} \leq \frac{1-\delta}{2-\delta}$$

is satisfied. Then, because the left-hand side decreases with T and the right-hand side increases with T, the inequality will also hold for any $T' \geq T$.

According to Case 1 in Proposition 6, this inequality implies that the seller's revenue is given by

$$V_D = \frac{\lambda T \exp(-\alpha T)}{\delta} \frac{\bar{Q}}{\lambda T}\left(1 - \frac{\bar{Q}}{\lambda T}\right) = \frac{\bar{Q}(1-\exp(-\alpha T))(\lambda T - \bar{Q})}{\alpha \lambda T^2}.$$

Taking a first derivative with respect to T, and after some manipulations, we get

$$\frac{\mathrm{d}}{\mathrm{d}T} V_D = \frac{\bar{Q}}{\alpha T^2}\left[\frac{\bar{Q}}{\lambda T}(2 - 2\exp(-\alpha T) - \alpha T \exp(-\alpha T)) - (1 - \exp(-\alpha T) - \alpha T \exp(-\alpha T))\right].$$

It is not hard to show that under the assumption $\bar{Q}/\lambda T \le (1-\delta)/(2-\delta)$, the derivative above is nonpositive. Hence, the seller will never choose T so that $\bar{Q}/\lambda T \le (1-\delta)/(2-\delta)$ is satisfied. This observation, together with the discussion in the first paragraph of this proof, implies that an upper bound for T is given by the unique nonnegative solution of

$$\frac{\bar{Q}}{\lambda T} = \frac{1-\delta}{2-\delta} \qquad \text{or, equivalently,} \qquad \frac{\bar{Q}}{\lambda T} = \frac{1 - \exp(-\alpha T) - \alpha T \exp(-\alpha T)}{2 - 2\exp(-\alpha T) - \alpha T \exp(-\alpha T)}.$$

The uniqueness follows because of the (opposed) monotonicity in T of the left- and right-hand sides of the equality. ☐

Appendix B. Dual Channel with Static List Price

This appendix is an extended version of §4 in the main paper. It contains a detailed mathematical description of the dual-channel model, including some numerical examples that complement the discussion in the main paper.

Recall that in this dual channel, a monopolistic seller has Q_0 units to sell through two different channels: a list price channel, in which she sets a constant list price $\hat{P}$ that will be kept during the whole horizon of length T; and the auction that will take place at the end, with the remaining Q_T units. We also recall that (because of the finite number of units in the market) the optimal bidding strategy for bidders is $b(v) = v$ (see §4 in the main paper for more details).

EC.B1. Characterization of a Symmetric Participation Equilibrium $H(v)$

Suppose $\hat{P}$ has been selected. We restrict buyers' participation strategies to the set

$$\mathcal{H} = \{H \in \mathcal{D}\colon [0,1] \to [0,T] \text{ such that } H(v) = 0, \text{ for all } v \in [0, \hat{P}]\},$$

and we characterize the decision of a buyer by means of a threshold function $H \in \mathcal{H}$, such that a buyer arriving at time t with valuation v_t will join the auction only if $H(v_t) \le t$. Because we have assumed that arriving buyers do not get any information regarding number of units sold or outstanding bids, the participation strategy $H(v_t)$ does not depend explicitly on these quantities. That is, at any time $t \in [0, T]$, buyer t knows only the initial quantity Q_0.

We analyze the buyer's problem at his arriving time in order to compute $\mathcal{R}(H)$, the best-response participation strategy given that all other buyers use the strategy H. Define the random variable $P_H(v)$ as follows:

$P_H(v)$ is the (random) auction price given that (i) there is a v-buyer that has joined the auction, (ii) all other buyers use the participation strategy H, and (iii) the (random) number Q_T of items left for the auction is equal to the difference between the initial value Q_0 and the number of items purchased through the fixed-price channel during $[0, T]$. We also use the convention $P_H(v) = 1$ if $Q_T = 0$.

As we will see shortly, an SPE in this dual-channel case depends on the probability distribution of $P_H(v)$ and the value of $\mathbb{P}(Q_\tau > 0 \mid H)$, the probability that at time τ there are still unsold units given that buyers use the participation strategy H. In order to compute these quantities, we follow an approach similar to the one developed for the single auction channel: We fix H, and in addition we define

$$\Lambda_{H+}(\tau) \triangleq \lambda T \eta_{H+}(\tau), \quad \text{for } \eta_{H+}(\tau) \triangleq \int_0^1 \min\left\{\frac{\tau}{T}, \frac{H(v)}{T}\right\} \mathrm{d}F(v), \qquad \text{and}$$

$$\Lambda_{H-}(x) \triangleq \lambda T \eta_{H-}(x), \quad \text{for } \eta_{H-}(x) \triangleq \int_x^1 \frac{H(v)}{T}\, \mathrm{d}F(v) = \bar{F}(x) - \eta_H(x),$$

where $\bar{F}(x)$ stands for the tail distribution of the valuations. Note that $\Lambda_{H+}(\tau)$ is the average number of buyers selecting the fixed-price channel during $[0,\tau]$, and $\eta_{H+}(\tau)$ is the corresponding fraction of arrivals in this category. Because buyers arrive according to a Poisson process of rate λ, it follows that

$$\mathbb{P}(Q_\tau > 0 \mid H) = \mathbb{P}(B(\Lambda_{H+}(\tau)) \le Q_0 - 1) = \sum_{k=0}^{Q_0-1} \frac{(\Lambda_{H+}(\tau))^k \exp(-\Lambda_{H+}(\tau))}{k!}. \tag{EC.B1}$$

On the other hand, $\Lambda_{H-}(x)$ represents the average number of fixed-price buyers (i.e., those below the threshold H) with valuation greater than or equal to x, and $\eta_{H-}(x)$ is the fraction of arrivals with valuation greater than or equal to x who go for the fixed-price channel.

The random variable $B(\Lambda_{H-}(x))$ represents the number of fixed-price buyers with valuation greater than or equal to x. Again, given that customers arrive according to a Poisson process with rate λ, we have that $B(\Lambda_{H-}(x))$ has a Poisson distribution with mean $\Lambda_{H-}(x)$. One important member of this family of random variables is $B(\Lambda_{H-}(0))$, which represents the total number of buyers that have selected the fixed-price channel. Therefore, we can define the number of units left for the auction as $Q_T = (Q_0 - B(\Lambda_{H-}(0)))^+$. From this observation and condition (5), we obtain

$$\begin{aligned}
\mathbb{P}(P_H(v) < x) &= \sum_{k=1}^{Q_0} \mathbb{P}(B(\Lambda_H(x)) + \mathbb{1}(x \le v) \le k)\mathbb{P}(Q_T = k) \\
&= \sum_{k=1}^{Q_0} \sum_{n=0}^{k-\mathbb{1}(x\le v)} \frac{(\Lambda_H(x))^n \exp(-\Lambda_H(x))}{n!} \frac{(\Lambda_{H-}(0))^{Q_0-k} \exp(-\Lambda_{H-}(0))}{(Q_0-k)!} \\
&= \left[\sum_{k=0}^{Q_0-1} \frac{(\Lambda_H(x) + \Lambda_{H-}(0))^k}{k!} + \mathbb{1}(x > v) \frac{(\Lambda_H(x) + \Lambda_{H-}(0))^{Q_0} - (\Lambda_{H-}(0))^{Q_0}}{Q_0!}\right] \\
&\quad \cdot \exp(-(\Lambda_H(x) + \Lambda_{H-}(0))).
\end{aligned}$$

We note that for $x \le v$, the distribution of $P_H(v)$ reduces to

$$\begin{aligned}
\mathbb{P}(P_H(v) < x) &= \sum_{k=0}^{Q_0-1} \frac{(\Lambda_H(x) + \Lambda_{H-}(0))^k}{k!} \exp(-(\Lambda_H(x) + \Lambda_{H-}(0))) \\
&= \mathbb{P}(B(\Lambda_H(x) + \Lambda_{H-}(0)) \le Q_0 - 1).
\end{aligned} \tag{EC.B2}$$

To get some intuition about this condition (EC.B2), note that $\Lambda_H(x) + \Lambda_{H-}(0)$ represents the average number of buyers that either enter the auction bidding more than x (first summand) or buy the object directly from the fixed-price channel (second summand).

We are now ready to characterize the best-response mapping $\mathcal{R}$ in this dual-channel case. Consider buyer τ arriving at time τ with valuation v_τ. If $v_\tau \le \hat{P}$, then the auction is his only profitable channel, and so he enters the auction independently of τ. On the other hand, if $v_\tau > \hat{P}$, then both channels are potentially profitable. If he decides to buy a unit through the fixed-price channel, his expected utility is zero if $Q_\tau = 0$ (that is, there are no units left) or equals $u(\tau, \tau, v_\tau - \hat{P})$ if $Q_\tau > 0$. Thus, the expected utility if he selects the fixed-price channel is given by $(v_\tau - \hat{P})\mathbb{P}(Q_\tau > 0 \mid H)$. On the other hand, if buyer τ decides to bid and gets one object, then his utility is $u(\tau, T, v_\tau - P_H(v_\tau))$, and zero otherwise.[EC3] Therefore, buyer τ enters the auction if his expected utility from bidding exceeds his expected utility from the fixed-price channel. From the exponentially discounted utility function (1) that we consider, this participation condition is equivalent to

$$\exp(-w(T-\tau))(v_\tau - \mathbb{E}[P_H(v_\tau) \mid P_H(v_\tau) < v_\tau]) \Pr(P_H(v_\tau) < v_\tau) \ge (v_\tau - \hat{P})\mathbb{P}(Q_\tau > 0 \mid H),$$

which we can rewrite for the case $v_\tau > \hat{P}$ in the more convenient form (see Equation (EC.A1)):

$$\frac{1}{v_\tau - \hat{P}} \int_0^{v_\tau} \mathbb{P}(P_H(v_\tau) < x)\, \mathrm{d}x \ge \exp(w(T-\tau))\mathbb{P}(Q_\tau > 0 \mid H). \tag{EC.B3}$$

[EC3] Note that in this dual-channel setting, the seller clears $\min\{Q_0, B(\lambda T)\}$ units through both channels by time T.

Condition (EC.B1) implies that for every $H \in \mathcal{H}$ the function $\mathbb{P}(Q_\tau > 0 \mid H)$ is continuous and nonincreasing in $\tau \in [0, T]$. Therefore, the function $\mathcal{F}(H)(\tau) \triangleq \exp(w(T-\tau))\mathbb{P}(Q_\tau > 0 \mid H)$ is monotonically decreasing in τ and admits a continuous decreasing inverse function $\mathcal{F}(H)^{-1}$ in the domain $[\mathbb{P}(Q_T > 0 \mid H), \exp(wT)]$. We find it convenient to (continuously) extend this domain of $\mathcal{F}(H)^{-1}$ to the entire $\mathbb{R}^+$ as follows:

$$\mathcal{F}(H)^{-1}(x) = T, \quad x \in [0, \mathbb{P}(Q_T > 0 \mid H)] \qquad \text{and} \qquad \mathcal{F}(H)^{-1}(x) = 0, \quad x \geq \exp(wT).$$

Although a closed-form expression for $\mathcal{F}(H)^{-1}$ is not available, its existence is all that we need to establish the following result.

Proposition B1. *In the dual-channel case, for any strategy $H \in \mathcal{H}$, the corresponding best-response participation strategy $\mathcal{R}(H) \in \mathcal{H}$ is continuous and satisfies*

$$\mathcal{R}(H)(v_\tau) = \begin{cases} 0 & \text{if } v_\tau \in [0, \hat{P}] \\ \mathcal{F}(H)^{-1}\left(\dfrac{\int_0^{v_\tau} \mathbb{P}(P_H(v_\tau) < x)\,dx}{v_\tau - \hat{P}}\right) & \text{if } v_\tau \in (\hat{P}, 1]. \end{cases}$$

The proof of the proposition is omitted because it follows directly from the participation condition (EC.B3) and the extended definition of $\mathcal{F}(H)^{-1}$ above. Only the continuity of $\mathcal{R}(H)(v_\tau)$ at $v_\tau = \hat{P}$, as it is required by the condition $\mathcal{R}(H) \in \mathcal{H}$, deserves some attention. For this, note that for all $H \in \mathcal{H}$ we have that

$$\lim_{v_\tau \downarrow \hat{P}} \frac{\int_0^{v_\tau} \mathbb{P}(P_H(v_\tau) < x)\,\mathrm{d}x}{v_\tau - \hat{P}} \to +\infty.$$

Continuity at $\hat{P}$ now follows from the fact that $\mathcal{F}(H)^{-1}(x) = 0$ for all $x \geq \exp(wT)$. Using a similar argument, we also note that for every $H \in \mathcal{H}$ there is a $v_H > \hat{P}$ such that $\mathcal{R}(H)(v) = 0$ for all $v \in [\hat{P}, v_H]$. As in Equation (EC.A3), we define

$$\tilde{v} \triangleq \inf_{H \in \mathcal{H}} \{v_H\}.$$

We can get a lower bound on $\tilde{v}$ from the fact that

$$\frac{\int_0^{v_\tau} \mathbb{P}(P_H(v_\tau) < x)\,\mathrm{d}x}{v_\tau - \hat{P}} \geq \frac{\int_0^{v_\tau} \mathbb{P}(B(\lambda T(1 - F(\hat{P}))) \leq Q_0 - 1)\,\mathrm{d}x}{v_\tau - \hat{P}} = \frac{v_\tau \mathbb{P}(B(\lambda T(1 - F(\hat{P}))) \leq Q_0 - 1)}{v_\tau - \hat{P}}.$$

The lower bound $\underline{v}$ is obtained by solving

$$\frac{v_\tau \mathbb{P}(B(\lambda T(1 - F(\hat{P}))) \leq Q_0 - 1)}{v_\tau - \hat{P}} = \exp(wT),$$

that is,

$$\underline{v} = \frac{\exp(wT)\hat{P}}{\exp(wT) - \mathbb{P}(B(\lambda T(1 - F(\hat{P}))) \leq Q_0 - 1)}.$$

As in the single-channel case, the existence of $\tilde{v} > \hat{P}$ guarantees that the best-response strategy $\mathcal{R}(H)$ is K-Lipschitz continuous for an appropriate constant K. Therefore, we can redefine the space of strategies to be

$$\mathcal{H} \triangleq \{H\colon [0,1] \to [0,T] \text{ s.t. } H \text{ is } K\text{-Lipschitz continuous and } H(v) = 0 \text{ in } v \in [0, \tilde{v}]\}.$$

The following result formalizes this claim and proves the existence of a symmetric participation equilibrium (SPE) for this dual-channel case.

Theorem B1. *For the exponential utility function* (1) *and for all $H \in \mathcal{H}$, there is a positive constant K* (*independent of H*) *such that the best-response strategy $\mathcal{R}(H)(v)$ is a K-Lipschitz continuous function that satisfies $\mathcal{R}(H)(v) = 0$ for all $v \in [0, \tilde{v}]$. In addition, the best-response mapping $\mathcal{R}$ is continuous in $\mathcal{H}$ equipped with the uniform norm, and so a symmetric equilibrium always exists in $\mathcal{H}$.*

The proof of this theorem can be found at the end of this appendix.

Again, as in the end of §3.2, we point out the existence of v_{H^*} and its implication: There is always a range of buyer valuations above the list price $\hat{P}$, such that those buyers will join the auction regardless of their arrival time.

EC.B2. Asymptotic Analysis

In this section, we analyze the limiting regime for the dual-channel setting when both the initial number of units Q_0 and the arrival rate λ grow proportionally large (see Equation (10) in the main paper). The auction price—for instance, n given that a bidder with valuation v enters the auction—is $P_H^n(v)$, and the final random number of units to auction is Q_T^n. The next theorem characterizes the asymptotic regime:

Theorem B2. *Suppose that the participation strategy $H(v)$ and the static price $\hat{P}$ are given. Then, in the limit as $n \to \infty$:*

(i) *The rescaled number of units Q_T^n/n to sell through the auction converges weakly to a constant $Q_T \triangleq (Q_0 - \lambda T \eta_{H-}(0))^+$.*

(ii) *If a final auction takes place (i.e., $Q_T > 0$), its price $P_H^n(v)$ converges weakly to a constant $P_H^\infty \triangleq \min\{v \in [0,1]\colon \eta_H(v) \le \rho - \eta_{H-}(0)\}$, where $\rho = Q_0/\lambda T$.*

The proof of the theorem can be found at the end of this appendix.

Following the argument in §3.3, in order to determine the value of the auction price P_H and the corresponding participation strategy $H(v)$, we have to impose the equilibrium condition $\mathscr{R}(H^*) = H^*$. Because $\mathscr{R}(H)(v) = 0$ for all $v < \hat{P}$, we must have $H^*(v) = 0$ in $v \in [0, \hat{P})$. In other words, buyers with valuation smaller than the posted price $\hat{P}$ have no other choice but entering the auction. To describe the behavior of $H^*(v)$ in $v \in [\hat{P}, 1]$, we need to distinguish two cases:

Case 1. Suppose that the initial supply of units is *limited* in the sense that $\rho \le 1 - F(\hat{P})$. In this situation, buyers with valuation greater than $\hat{P}$ have no incentive to enter the auction because the auction price is guaranteed to be greater than or equal to $\hat{P}$. Therefore, the resulting participation strategy is $H^*(v) = T$ for all $v \ge \hat{P}$, and the auction never takes place because all the units will be bought at the posted price (i.e., $Q_T = 0$).

We note that $H^*(v) = T\mathbb{1}(v \ge \hat{P})$ is not the only SPE in this case. In fact, let us define $\tau^* \triangleq T\rho(1 - F(\hat{P}))^{-1}$. Then, any H of the form $H(v) = (\tau^* + h(v))\mathbb{1}(v \ge \hat{P})$ for an arbitrary nonnegative and bounded function $h(v) \le T - \tau^*$ is an SPE. In fact, for such an H the initial Q_0 units will be depleted by time τ^* (i.e., $Q_{\tau^*} = 0$, because $\tau^* \lambda (1 - F(\hat{P})) = Q_0$). Therefore, any buyer arriving after τ^* will never get a unit, and so he becomes indifferent between the two channels.

Case 2. Suppose that initial supply is *abundant* in the sense that $\rho > 1 - F(\hat{P})$. In this case, $Q_T > 0$, and some buyers with valuation smaller than $\hat{P}$ get units through the auction. It is not hard to see that in this case the auction price is given by $P_{H^*} = F^{-1}(1-\rho)$. Therefore, buyer τ arriving at time τ with valuation $v_\tau \ge \hat{P}$ enters the auction only if $v_\tau - \hat{P} \le \exp(-w(T-\tau))(v_\tau - P_{H^*})$. We conclude that in this *abundant* case the unique SPE $H^*(v)$ is given by:

$$H^*(v) = \begin{cases} 0 & \text{if } v \in [0, v_{H^*}] \\ T - \dfrac{1}{w}\ln\left(\dfrac{v - P_{H^*}}{v - \hat{P}}\right) & \text{if } v \in [v_{H^*}, 1] \end{cases} \quad \text{where } v_{H^*} = \min\left\{\frac{\hat{P}\exp(wT) - P_{H^*}}{\exp(wT) - 1}, 1\right\}.$$

Figure EC.B1 compares the optimal asymptotic strategy $H^*(\infty, \rho)$ to four optimal participation strategies $H^*(Q_0, \rho)$ ($Q_0 = 1, 5, 10, 20$) for the case of $\rho = 0.5$. As in the single-channel case, the asymptotic strategy is almost identical to the optimal strategy for values of Q_0 greater than 10 units. The table on the right compares the expected value of the auction price P_H and the approximation P_H^{Approx} obtained using the asymptotic participation strategy $H^*(\infty, 0.5)$. Similarly to the single auction channel, the asymptotic approximation is very accurate even for small values of Q_0.

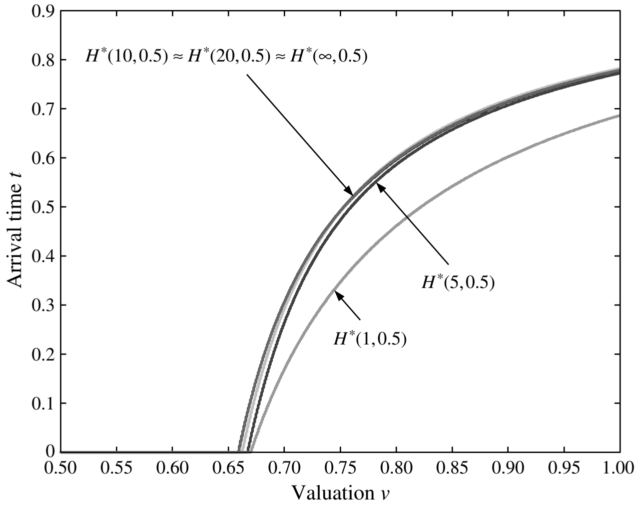
We conclude this appendix by specializing the asymptotic results to the case of uniformly distributed valuations.

Examples

Uniform Distribution Case

Suppose buyers' valuations are uniformly distributed in $[0,1]$. Under this assumption, we will characterize the auction price P_{H^*} as well as the number of units sold in the auction Q_{H^*}, the cumulative

Figure EC.B1 Asymptotic Approximation for the Case with Valuations Uniformly Distributed in $[0, 1]$, **and** $T = w = 1$, $\hat{P} = 0.6$, **and** $\rho = 0.5$



Q_0	P_H	P_H^{Approx}	Error (%)
1	0.2793	0.2834	1.428
5	0.4175	0.4184	0.213
10	0.4539	0.4540	0.034
20	0.4760	0.4760	0.002

Note. In this case, the asymptotic price is $P_{H^*}^{\infty} = 1 - \rho = 0.5$.

number of units sold in the fixed-price channel during $[0, T]$ Q_{F^*}, and the corresponding rate $\lambda_{F^*}(t)$ at which these units are sold. In the asymptotic regime under consideration we have that

$$Q_{F^*} = \int_0^T \lambda_{F^*}(t)\,\mathrm{d}t.$$

Depending on the values of ρ and $\hat{P}$, we distinguish two cases.

Limited-Supply Case. $\rho \le 1 - \hat{P}$
In this case, all units are depleted through the fixed-price channel, that is, $Q_{F^*} = Q_0$ and $Q_{H^*} = 0$. In addition, the fixed-price channel demand rate satisfies

$$\lambda_{F^*}(t) = \lambda(1 - \hat{P})\mathbb{1}(t \le \tau^*), \quad t \in [0, T],$$

where $\tau^* = Q_0/(\lambda(1 - \hat{P})) \le T$ is the time at which all Q_0 units are sold.

Abundant-Supply Case. $\rho > 1 - \hat{P}$
In this case, there is a positive number of units that are sold through the auction at a price $P_{H^*} = (1 - \rho)^+$. The number of units auctioned and sold through the fixed-price channel are

$$Q_{H^*} = \lambda T \eta_{H^*}(P_{H^*}) \qquad \text{and} \qquad Q_{F^*} = \lambda T(1 - \eta_{H^*}(P_{H^*})),$$

respectively, where

$$\eta_{H^*}(P_{H^*}) = \int_{P_{H^*}}^1 \left(1 - \left[1 - \frac{1}{wT}\ln\left(\frac{v - (1-\rho)^+}{v - \hat{P}}\right)\right]^+\right)\mathrm{d}v.$$

In this case, the fixed-price channel demand rate satisfies

$$\begin{aligned}\lambda_{F^*}(t) &= \lambda H^{*-1}(t)\\ &= \lambda\left[1 - \frac{\hat{P}\exp(w(T-t)) - (1-\rho)^+}{\exp(w(T-t)) - 1}\right]^+.\end{aligned} \tag{EC.B4}$$

Proofs

Proof of Theorem B1. We first prove K-Lipschitz continuity of $\mathcal{R}(H)(v)$ in $(\tilde{v}, 1]$. Observe that due to the shape of the function $\mathcal{F}_H^{-1}(x)$ (flat at T in the range $[0, \mathbb{P}(Q_T > 0 \mid H)]$, and flat at zero for $x \geq \exp(wT)$), it is enough to prove this property in the range $(\mathbb{P}(Q_T > 0 \mid H), \exp(wT))$, where $\mathcal{R}(H)(v)$ is differentiable. Let us define

$$Z_H(v) \triangleq \frac{1}{v - \hat{P}} \int_0^v \mathbb{P}(P_H(v) < x)\,\mathrm{d}x.$$

Now, for any pair $v_1, v_2 \in (\tilde{v}, 1]$ we have that

$$\begin{aligned} |\mathcal{R}(H)(v_1) - \mathcal{R}(H)(v_2)| &= |\mathcal{F}_H^{-1}(Z_H(v_1)) - \mathcal{F}_H^{-1}(Z_H(v_2))| \\ &= \left| \int_{Z_H(v_1)}^{Z_H(v_2)} \frac{\mathrm{d}}{\mathrm{d}x} \mathcal{F}_H^{-1}(x)\,\mathrm{d}x \right| = \left| \int_{Z_H(v_1)}^{Z_H(v_2)} \left(\frac{\mathrm{d}}{\mathrm{d}\tau} \mathcal{F}_H(\tau) \right)^{-1} \bigg|_{\tau = \mathcal{F}_H^{-1}(x)} \mathrm{d}x \right|. \end{aligned}$$

Note that the differentiability of $\mathcal{F}_H^{-1}(x)$ follows from the fact that the function $\mathcal{F}_H(\tau)$ is differentiable. In fact, from Lemma A1 we have that

$$\frac{\mathrm{d}}{\mathrm{d}\tau} \mathcal{F}_H(\tau) = -\exp(w(T-\tau)) \left[w\mathbb{P}(Q_\tau > 0 \mid H) + \mathbb{P}(B(\Lambda_{H+}(\tau)) = Q_0 - 1) \frac{\mathrm{d}}{\mathrm{d}\tau} \Lambda_{H+}(\tau) \right],$$

where

$$\frac{\mathrm{d}}{\mathrm{d}\tau} \Lambda_{H+}(\tau) = \lambda \frac{\mathrm{d}}{\mathrm{d}\tau} \int_0^1 \min\{\tau, H(v)\}\,\mathrm{d}F(v) = \lambda \int_0^1 \mathbb{1}(\tau \leq H(v))\,\mathrm{d}F(v).$$

Using the fact that $\mathbb{P}(Q_\tau > 0 \mid H) \geq \mathbb{P}(Q_T > 0 \mid H) \geq \mathbb{P}(B(\lambda T) \leq Q_0 - 1)$, that $0 \leq (\mathrm{d}/\mathrm{d}\tau)\Lambda_{H+}(\tau) \leq \lambda$, and that $1 \leq \exp(w(T-\tau)) \leq \exp(wT)$, we get that

$$w\mathbb{P}(B(\lambda T) \leq Q_0 - 1) \leq \left| \frac{\mathrm{d}}{\mathrm{d}\tau} \mathcal{F}_H(\tau) \right| \leq \exp(wT)[w + \lambda],$$

and so

$$\begin{aligned} |\mathcal{R}(H)(v_1) - \mathcal{R}(H)(v_2)| &= |\mathcal{F}_H^{-1}(Z_H(v_1)) - \mathcal{F}_H^{-1}(Z_H(v_2))| \\ &=\leq (w\mathbb{P}(B(\lambda T) \leq Q_0 - 1))^{-1} |Z_H(v_1) - Z_H(v_2)|. \end{aligned} \qquad \text{(EC.B5)}$$

K-Lipschitz continuity follows now, combining this inequality and the following:

$$\begin{aligned} |Z_H(v_1) - Z_H(v_2)| &= \left| \int_{v_1}^{v_2} \frac{\mathrm{d}}{\mathrm{d}v} Z_H(v)\,\mathrm{d}v \right| = \left| \int_{v_1}^{v_2} \left[\frac{\mathbb{P}(P_H(v) < v)}{v - \hat{P}} - \int_0^v \frac{\mathbb{P}(P_H(v) < x)}{(v - \hat{P})^2}\,\mathrm{d}x \right] \mathrm{d}v \right| \\ &\leq | \int_{v_1}^{v_2} \left[\frac{1}{\tilde{v} - \hat{P}} + \frac{1}{(\tilde{v} - \hat{P})^2} \right] \mathrm{d}x \bigg| = \left(\frac{1 + \tilde{v} - \hat{P}}{(\tilde{v} - \hat{P})^2} \right) |v_1 - v_2|. \end{aligned}$$

The constant K equals $(w\mathbb{P}(B(\lambda T) \leq Q_0 - 1))^{-1}((1 + \tilde{v} - \hat{P})/(\tilde{v} - \hat{P})^2)$, and it is well defined because $\tilde{v} \geq \underline{v} > \hat{P}$.

To prove the continuity in $\mathcal{H}$ of the mapping $\mathcal{R}$, we first note that the mapping $\mathcal{F}(H)$ is continuous in $\mathcal{H}$. In fact,

$$\begin{aligned} |\mathcal{F}(H)(\tau) - \mathcal{F}(\tilde{H})(\tau)| &= \exp(w(T-\tau)) |\mathbb{P}(Q_\tau > 0 | H) - \mathbb{P}(Q_\tau > 0 | \tilde{H})| \\ &= \exp(w(T-\tau)) |\mathbb{P}(B(\Lambda_{H+}(\tau)) \leq Q_0 - 1) - \mathbb{P}(B(\Lambda_{\tilde{H}+}(\tau)) \leq Q_0 - 1)| \\ &\leq \exp(w(T-\tau)) \mathbb{P}(B(Q_0 - 1) = Q_0 - 1) |\Lambda_{H+}(\tau) - \Lambda_{\tilde{H}+}(\tau)| \\ &\leq \lambda \exp(w(T-\tau)) \mathbb{P}(B(Q_0 - 1) = Q_0 - 1) \|H - \tilde{H}\| \triangleq K_{\mathcal{F}} \|H - \tilde{H}\|, \end{aligned}$$

where the first inequality follows from Lemma A3, and the second one follows from the definition of $\Lambda_{H+}(\tau)$ and the property $|\min\{\tau, a\} - \min\{\tau, b\}| \leq |a - b|$. The continuity of the mapping $\mathcal{R}(H) = \mathcal{F}(H)^{-1}$ follows now from

$$\begin{aligned} |\mathcal{R}(H)(v) - \mathcal{R}(\tilde{H})(v)| &= |\mathcal{F}(H)^{-1}(Z_H(v)) - \mathcal{F}(\tilde{H})^{-1}(Z_{\tilde{H}}(v))| \\ &\leq |\mathcal{F}(H)^{-1}(Z_H(v)) - \mathcal{F}(\tilde{H})^{-1}(Z_H(v))| + |\mathcal{F}(\tilde{H})^{-1}(Z_H(v)) - \mathcal{F}(\tilde{H})^{-1}(Z_{\tilde{H}}(v))|. \end{aligned} \qquad \text{(EC.B6)}$$

Regarding the first term in (EC.B6), from condition (EC.B5), we have that

$$|\mathcal{F}(\widetilde{H})^{-1}(Z_H(v)) - \mathcal{F}(\widetilde{H})^{-1}(Z_{\widetilde{H}}(v))| \le (w\mathbb{P}(B(\lambda T) \le Q_0 - 1))^{-1}|Z_H(v) - Z_{\widetilde{H}}(v)|.$$

As in the proof of Theorem 1, we can prove that

$$\left|Z_H(v) - Z_{\widetilde{H}}(v)\right| \le K_Z\|H - \widetilde{H}\|$$

for an appropriate constant K_Z.

Now we focus on the second term in (EC.B6). Without loss of generality, suppose $\mathcal{F}(H)^{-1}(Z_H(v)) \le \mathcal{F}(\widetilde{H})^{-1}(Z_H(v))$. Using the continuity of $\mathcal{F}$ in $\mathcal{H}$ that we just proved, it follows that

$$\mathcal{F}(\widetilde{H})(\mathcal{F}(H)^{-1}(Z_H(v))) \le Z_H(v) + K_{\mathcal{F}}\|H - \widetilde{H}\|.$$

Applying $\mathcal{F}(\widetilde{H})^{-1}$ in both sides, and given that $\mathcal{F}(\widetilde{H})^{-1}(v)$ is nonincreasing in v, we have that

$$\mathcal{F}(H)^{-1}(Z_H(v)) = \mathcal{F}(\widetilde{H})^{-1}(\mathcal{F}(\widetilde{H})(\mathcal{F}(H)^{-1}(Z_H(v)))) \ge \mathcal{F}(\widetilde{H})^{-1}(Z_H(v) + K_{\mathcal{F}}\|H - \widetilde{H}\|).$$

From the assumption $\mathcal{F}(H)^{-1}(Z_H(v)) \le \mathcal{F}(\widetilde{H})^{-1}(Z_H(v))$ we get

$$\begin{aligned}|\mathcal{F}(H)^{-1}(Z_H(v)) - \mathcal{F}(\widetilde{H})^{-1}(Z_H(v))| &\le |\mathcal{F}(\widetilde{H})^{-1}(Z_H(v) + K_{\mathcal{F}}\|H - \widetilde{H}\|) - \mathcal{F}(\widetilde{H})^{-1}(Z_H(v))| \\ &\le (w\mathbb{P}(B(\lambda T) \le Q_0 - 1))^{-1}K_{\mathcal{F}}\|H - \widetilde{H}\|,\end{aligned}$$

where the second inequality follows from condition (EC.B5) above.

Therefore, using the bounds for the two absolute terms in (EC.B6), we conclude that

$$|\mathcal{R}(H)(v) - \mathcal{R}(\widetilde{H})(v)| \le (w\mathbb{P}(B(\lambda T) \le Q_0 - 1))^{-1}(K_Z + K_{\mathcal{F}})\|H - \widetilde{H}\|,$$

which proves the continuity of $\mathcal{R}$ in $\mathcal{H}$.

Finally, as in Theorem 1, the existence of a symmetric equilibrium follows again from the Schauder-Tychonoff Fixed-Point Theorem. □

Proof of Theorem B2. Recall that $B(\Lambda^n_{H-}(x))$ is a Poisson r.v. with mean $\Lambda^n_{H-}(x)$. To prove (i), we start by rewriting $\Lambda^n_{H-}(x)$ as

$$\Lambda^n_{H-}(x) = \frac{\eta_{H-}(x)}{\rho^n}Q^n_0.$$

Let $\{B_i(\eta_{H-}(x)(\rho^n)^{-1})\colon i = 1, \ldots, Q^n_0\}$ be a sequence of i.i.d Poisson random variables with mean $\eta_{H-}(x)(\rho^n)^{-1}$. The random variable $B(\Lambda^n_{H-}(x))$ has the same distribution as the sum of the $B_i(\eta_{H-}(x)(\rho^n)^{-1})$, $1 \le i \le Q^n_0$. For a fixed $0 \le \alpha \le 1$,

$$\begin{aligned}\mathbb{P}(Q^n_T \ge \alpha Q^n_0) &= \mathbb{P}(B(\Lambda^n_{H-}(0)) \le Q^n_0(1-\alpha)) \\ &= \mathbb{P}\left(\sum_{i=1}^{Q^n_0} B_i(\eta_{H-}(0)(\rho^n)^{-1}) \le Q^n_0(1-\alpha)\right) \\ &= \mathbb{P}\left(\frac{\sum_{i=1}^{Q^n_0} B_i(\eta_{H-}(0)(\rho^n)^{-1})}{Q^n_0} \le 1-\alpha\right),\end{aligned}$$

where the first equality follows from the fact that all the units put into the auction are the remaining ones from the list price channel. Let

$$\mathcal{B}^n_-(0) \triangleq \frac{\sum_{i=1}^{Q^n_0} B_i(\eta_{H-}(0)(\rho^n)^{-1})}{Q^n_0}.$$

From Lemma A4, $\mathcal{B}^n_-(0)$ converges in distribution to the constant $\eta_{H-}(0)\rho^{-1}$. Given that for n sufficiently large $Q^n_0 = nQ_0 + o(n)$, then by focusing on the continuity points, the tail distribution of Q^n_T/n converges weakly to the tail distribution:

$$\bar{F}_{Q_T}(\alpha Q_0) = \begin{cases} 1 & \text{if } \eta_{H-}(0) < \rho(1-\alpha) \\ \\ 0 & \text{if } \eta_{H-}(0) > \rho(1-\alpha). \end{cases}$$

In other words, the first case corresponds to $\alpha Q_0 < Q_0 - \lambda T \eta_{H-}(0)$, that is, there are more units available in the auction channel than the requested αQ_0; the second case is the opposite. This is the distribution of the constant $(Q_0 - \lambda T \eta_{H-}(0))^+$ at its continuity points, and so $Q_T^n/n \Rightarrow Q_T$.

For part (ii), we have:

$$\begin{aligned}\mathbb{P}(P_H^n(v) < x) &= \mathbb{P}(B(\Lambda_H^n(x)) \le Q_T^n - \mathbb{1}(x \le v)) \\ &= \mathbb{P}\bigg(\sum_{i=1}^{Q_0^n} B_i(\eta_H(x)(\rho^n)^{-1}) \le Q_T^n - \mathbb{1}(x \le v)\bigg) \\ &= \mathbb{P}\bigg(\frac{\sum_{i=1}^{Q_0^n} B_i(\eta_H(x)(\rho^n)^{-1})}{Q_0^n} \le \frac{Q_T^n - \mathbb{1}(x \le v)}{Q_0^n}\bigg). \qquad \text{(EC.B7)}\end{aligned}$$

A similar argument to the one above shows that as $n \to \infty$,

$$\mathscr{B}^n(x) \triangleq \frac{\sum_{i=1}^{Q_0^n} B_i(\eta_H(x)(\rho^n)^{-1})}{Q_0^n} \Rightarrow \eta_H(x)\rho^{-1}.$$

Regarding the right-hand side in (EC.B7), from part (i) if a final auction occurs, for n large enough, we have $Q_T^n \approx Q_0^n - \lambda^n T \eta_{H-}(0)$. Then, as $n \to \infty$,

$$\frac{Q_T^n - \mathbb{1}(x \le v)}{Q_0^n} \to 1 - \frac{\eta_{H-}(0)}{\rho}.$$

By focusing on the continuity points, the distribution of $P_H^n(v)$ converges weakly to the distribution:

$$\mathbb{P}(P_H^\infty(v) < x) = \begin{cases} 1 & \text{if } \eta_H(x) < \rho - \eta_{H-}(0) \\ 0 & \text{if } \eta_H(x) > \rho - \eta_{H-}(0). \end{cases}$$

This corresponds to the distribution of the constant $P_H^\infty = \min\{v \in [0,1]\colon \eta_H(v) \le \rho - \eta_{H-}(0)\}$ at its continuity points, and so $P_H^n(v) \Rightarrow P_H^\infty$. $\square$

References

See references list in the main paper.

Billingsley, P. 1995. *Probability and Measure*. John Wiley and Sons, New York.