Optimal Dynamic Auctions for Revenue Management

Gustavo Vulcano Garrett van Ryzin Costis Maglaras *

July 24, 2002

A Online Appendix

Proof of Lemma 1

We prove that $\Delta V_t(x)$ is decreasing in x by induction on t. For $t = 0$ the theorem trivially holds since $V_0(x) = 0$ for all x. For period $t-1$, the inductive hypothesis (IH) is that $\Delta V_{t-1}(x) \geq \Delta V_{t-1}(x+1)$. We will then show that if IH holds, then $\Delta V_t(x)$ is decreasing as well.

To do so, fix the number of buyers n_t and consider a given realization $v = (v_1, \ldots, v_{n_t})$ of buyers's valuations. Define the inner maximized value in (3) in terms of k as follows (see equation (7)):

$$H_t(x, n_t, v) = \max_{0 \leq k \leq x} \{R(k) + V_{t-1}(x-k)\}, \tag{A.1}$$

and define the difference function

$$\Delta H_t(x, n_t, v) = H_t(x, n_t, v) - H_t(x-1, n_t, v),$$

Note that for random N_t and v,

$$\Delta V_t(x) = E_{N_t, v}[\Delta H_t(x, N_t, v)]$$

Thus, it suffices to establish that $\Delta H_t(x, n_t, v)$ is decreasing in x to prove that $\Delta V_t(x)$ is decreasing in x. For notational simplicity, we henceforth suppress the arguments n_t, v in $\Delta H_t(x, n_t, v)$ and simply use $\Delta H_t(x)$.

Using (A.1) and Lemma 2 (which holds because $\Delta V_{t-1}(x)$ is decreasing in x by IH), we make the following two observations (denoted Observations 1 and 2):

1. If $k_t^*(x+1) = k_t^*(x)$, then $\Delta H_t(x+1) = \Delta V_{t-1}(x - k_t^*(x) + 1)$
2. If $k_t^*(x+1) = k_t^*(x) + 1$, then

$$\Delta H_t(x+1) = \Delta R(k_t^*(x)+1) \quad \text{and} \quad \Delta R(k_t^*(x)+1) > \Delta V_{t-1}(x - k_t^*(x) + 1)$$

*All authors are with the Graduate School of Business, Columbia University, New York. E-mails: {gjv9, gjv1, cm479}@columbia.edu, respectively.

Consider now $\Delta H_t(x+1)$ and $\Delta H_t(x)$. Given the different values that $k_t^*(x)$ and $k^*(x+1)$ can take by Lemma 2, there are four cases to analyze.

Case 1: $k_t^*(x+1) = k_t^*(x) = k_t^*(x-1) \equiv k^*$

In this scenario,

$$\begin{aligned} \Delta H_t(x) &= \Delta V_{t-1}(x-1-k^*+1) \qquad \text{(by Obs. 1)} \\ &= \Delta V_{t-1}(x-k^*) \\ &\geq \Delta V_{t-1}(x-k^*+1) \qquad \text{(by IH)} \\ &= \Delta H_t(x+1) \qquad \text{(by Obs. 1)} \end{aligned}$$

Then, $\Delta H_t(x) \geq \Delta H_t(x+1)$.

Case 2: $k_t^*(x+1) = k_t^*(x) > k_t^*(x-1)$

From Lemma 2, $k_t^*(x+1) = k_t^*(x) = k_t^*(x-1)+1$. Thus,

$$\begin{aligned} \Delta H_t(x) &= \Delta R(k_t^*(x-1)+1) \qquad \text{(by Obs. 2)} \\ &> \Delta V_{t-1}(x-1-k_t^*(x-1)+1) \qquad \text{(by optimality of } k^*(\cdot)) \\ &= \Delta V_{t-1}(x-k_t^*(x-1)) \\ &= \Delta V_{t-1}(x-k_t^*(x)+1) \\ &= \Delta H_t(x+1) \qquad \text{(by Obs. 1)} \end{aligned}$$

Then, $\Delta H_t(x) > \Delta H_t(x+1)$.

Case 3: $k_t^*(x+1) > k_t^*(x) > k_t^*(x-1)$

From Lemma 2, $k_t^*(x+1) = k_t^*(x)+1$ and $k_t^*(x) = k_t^*(x-1)+1$. Then,

$$\begin{aligned} \Delta H_t(x) &= \Delta R(k_t^*(x-1)+1) \qquad \text{(by Obs. 2)} \\ &\geq \Delta R(k_t^*(x-1)+2) \\ &= \Delta R(k_t^*(x)+1) \\ &= \Delta H_t(x+1) \qquad \text{(by Obs. 2)} \end{aligned}$$

Therefore, $\Delta H_t(x) \geq \Delta H_t(x+1)$.

Case 4: $k_t^*(x+1) > k_t^*(x) = k_t^*(x-1)$

Note that $k_t^*(x+1) = k_t^*(x)+1$. So,

$$\begin{aligned} \Delta H_t(x) &= \Delta V_{t-1}(x-1-k_t^*(x-1)+1) \qquad \text{(by obs. 1)} \\ &= \Delta V_{t-1}(x-k_t^*(x-1)) \\ &= \Delta V_{t-1}(x-k_t^*(x)) \\ &= \Delta V_{t-1}(x-[k_t^*(x)+1]+1) \\ &\geq \Delta R(k_t^*(x)+1) \qquad \text{(by Theorem 1 applied to IH)} \\ &= \Delta H_t(x+1) \qquad \text{(by Obs. 2)} \end{aligned}$$

Then, $\Delta H_t(x) \geq \Delta H_t(x+1)$.

Thus, $\Delta H_t(x)$ and hence $\Delta V_t(x)$ is decreasing as well, which completes the induction proof.

We next show that $\Delta V_t(x)$ is increasing in t. As above, it suffices to prove the statement for a particular realization n_t, v using (8).

Following the notation introduced in (A.1), and using Observations 1 and 2 above, we get that

$$\Delta H_t(x) \geq \Delta V_{t-1}(x - k^*(x)),$$

and from the fact that $\Delta V_{t-1}(\cdot)$ is decreasing, $\Delta H_t(x) \geq \Delta V_{t-1}(x)$. The expectation operator will then preserve this property. $\square$

Proof of Lemma 2

We start from the LHS inequality. If $k_t^*(x) = 0$, the statement immediately holds true. If $k_t^*(x) > 0$, by Theorem 1, $\Delta V_{t-1}(x - k_t^*(x) + 1) \geq \Delta V_{t-1}(x + 1 - k_t^*(x) + 1)$, and $k_t^*(x)$ is a feasible allocation quantity when the inventory is $x + 1$. The optimal allocation $k_t^*(x+1)$ may be higher, however, due to the additional unit in stock. Hence, $k_t^*(x+1) \geq k_t^*(x)$.

For the RHS inequality, suppose by contradiction that $k_t^*(x+1) > k_t^*(x)+1$. By the optimality of $k_t^*(x+1)$, we know that $\Delta R(j) > \Delta V_{t-1}(x+1-j+1)$, for $j = 1, 2, \ldots, k_t^*(x+1)$. Using the fact that $k_t^*(x+1) \geq k_t^*(x) + 2$ and setting $j = k_t^*(x) + 2$,

$$\Delta R(k_t^*(x) + 2) > \Delta V_{t-1}(x + 1 - [k_t^*(x) + 2] + 1) = \Delta V_{t-1}(x - [k_t^*(x) + 1] + 1).$$

The LHS is decreasing in its argument, which implies that

$$\Delta R(k_t^*(x) + 1) \geq \Delta R(k_t^*(x) + 2) > \Delta V_{t-1}(x - [k_t^*(x) + 1] + 1)$$

However, this contradicts the optimality of $k_t^*(x)$. $\square$

Proof of Footnote 6

Redefine the profit function for buyer i who bids $b \notin \mathcal{D}_t(x)$ as $\Pi_i(b, v_i) = P(b)(v_i - \hat{B}(b))$. Differentiating with respect to b we get

$$\begin{aligned}
\frac{\partial \Pi_i}{\partial b}(b, v_i) &= v_i \frac{\mathrm{d}P(b)}{\mathrm{d}b} - \frac{\mathrm{d}}{\mathrm{d}b}[P(b)\hat{B}(b)] \\
&= v_i \frac{\mathrm{d}P(b)}{\mathrm{d}b} - b\,\frac{\mathrm{d}P(b)}{\mathrm{d}b} \qquad \text{(by (17))} \\
&= (v_i - b)\,\frac{\mathrm{d}P(b)}{\mathrm{d}b}.
\end{aligned}$$

Since $P(\cdot)$ is strictly increasing, $\frac{\partial \Pi_i}{\partial b}(b, v_i) = 0$ if and only if $b = v_i$. Moreover, the derivative is positive to its left and negative to its right, so $b = v_i$ is a maximum.

At points of discontinuity, we use (19) to show that $P(v_i)(v_i - B(v_i)) = P(v_i)(v_i - \underline{b}) > P(b)(v_i - b)$, $\forall b \in (\underline{b}, \overline{b}]$. $\square$

Proof of Proposition 1

Take first a T-period setting, and a particular realization $(n_t, v^t), t = 1, \ldots, T$ of the valuations across all the periods. For the same realization of valuations, consider now a $2T$-period setting, in such a way that the private values of period t in the T-problem map to the values in periods $2t-1$ and $2t$ of this $2T$-period version. Essentially, we are splitting each period in two to get the $2T$-problem, and any such map can be considered.

We proceed by induction on the number t of remaining rounds to complete the sequence of auctions. Let $V_t^T(x)$ be the revenue that the seller expects to collect in the last t rounds left in the T-period problem, $1 \leq t \leq T$. So, for $t = 1$, $V_1^T(x) \geq V_2^{2T}(x)$, that is, if we only have one period to go for a particular realization of values, we can not do worse than having the same values split in two periods: the single period rank of them will be optimal.

The inductive hypothesis is $V_t^T(x) \geq V_{2t}^{2T}(x)$. Assume now that we have $t+1$ periods to go in the T-problem, and let compare revenues $V_{t+1}^T(x)$ and $V_{2t+2}^{2T}(x)$. For the first of these $t+1$ remaining periods in the T-problem, we can argue as in the one-period to-go case, and verify that its revenue is no lower than the first two periods of the $2t+2$ periods to-go in the $2T$-problem. For the remaining t and $2t$ periods respectively, the IH holds, and hence $V_{t+1}^T(x) \geq V_{2t+2}^{2T}(x)$.

Since the argument is valid for any particular instance of the valuations, it also holds when taking expectation and hence, $V^T(x) \geq V^{2T}(x)$. $\square$

Proof of Proposition 3

Define the empirical distribution by $F^n(v) = \frac{1}{N^n} \sum_{i=1}^{N^n} \mathbf{I}\{v_i \leq v\}$. It is well known (see Durrett[15, Thm. 1.7.4]) that F^n converges uniformly to F as $n \to \infty$. Specifically, $\sup_v |F^n(v) - F(v)| \to 0$, almost surely. By definition $F^n(v_{(k^n)}) = 1 - k^n/N^n$. Using uniform convergence we have that

$$|F^n(v_{(k^n)}) - F(v_{(k^n)})| \to 0 \quad \text{and} \quad 1 - k^n/N^n \to 1 - k/N,$$

which implies that $F(v_{(k^n)}) \to 1 - k/N$. By the continuity of F it follows that $v_{(k^n)} \to F^{-1}(1-k/N)$. Next, we analyze the asymptotic behavior of $x^{k^n,N^n}(v)$. Since $k \leq N$, we can assume that n is sufficiently large such that $k^n \leq N^n$. Rewriting $x^{k^n,N^n}(v)$ as

$$x^{k^n,N^n}(v) = \max\{i \leq \min\{k^n, N^n\} : v_{(i)} > v^*\} = \sum_{i=1}^{N^n} \mathbf{I}\{v_i > \max(v^*, v_{(k^n)})\},$$

and using the fact that F^n converges to F we get that $\frac{1}{n} x^{k^n,N^n}(v) \to N\bar{F}(v_\infty)$, where $v_\infty = \max\{v^*, F^{-1}(1-k/N)\}$. If $v^* > F^{-1}(1-k/N)$ or equivalently $k > N\bar{F}(v^*)$, then $N\bar{F}(v_\infty) = N\bar{F}(v^*)$; otherwise, $N\bar{F}(v_\infty) = k$.

Similarly, $F^n(v_{(x^{k^n,N^n}(v)+1)}) = 1 - (x^{k^n,N^n}(v)+1)/N^n \to 1 - \bar{F}(v_\infty)$ and $|F^n(v_{(x^{k^n,N^n}(v)+1)}) - F(v_{(x^{k^n,N^n}(v)+1)})| \to 0$. This implies that $v_{(x^{k^n,N^n}(v)+1)} \to v_\infty$, and in turn that $\pi^{k^n,N^n}(v) \to v_\infty$, almost surely as $n \to \infty$. Combining these results we complete the proof. $\square$