Stockholm School of Economics

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Contract Theory

Final Exam

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Please try to keep your answers as short and concise as possible. You can refer to the results in the lecture notes and problem sets without restating the respective proofs. Good luck!

Problem 1 (35 points) This problem deals with the bilateral trade model discussed in class. The setting is the same as in the lecture notes, except that the valuations θ_1 and θ_2 are both drawn from a uniform distribution on $[0, 1]$. The seller's and buyer's utility functions are $\theta_1 x_1 + t_1$ and $\theta_2 x_2 + t_2$, respectively.

1. **(10 points)** Derive the AGV mechanism for this problem (the mechanism consists of a vector of (successful) decision rules $(x_1(\hat{\theta}), x_2(\hat{\theta}))$ and a vector of transfer functions $(t_1(\hat{\theta}), t_2(\hat{\theta}))$). Hint: If you determine $t_1(\hat{\theta})$ and $t_2(\hat{\theta})$, remember that the values of $x_1(\hat{\theta})$ and $x_2(\hat{\theta})$ in the integrals depend on the variables of integration θ_1 and θ_2. As a consequence, you'll have to change the lower integral limits (i.e. replace 0 with something else). Note that the integrals in $t_1(\hat{\theta})$ and $t_2(\hat{\theta})$ can be easily solved.

2. **(8 points)** Show that in the above AGV mechanism, truthtelling is a Bayesian equilibrium. Hint: Unlike in theorem 15, derive the FOC's for the seller's and buyer's problem. In either case, verify that truthtelling is a global maximum.

3. **(10 points)** Let us now introduce IIR constraints. As you know from the lecture notes, this means that the seller's expected utility must exceed θ_1 (since she can always keep the object) and the buyer's expected utility must exceed 0. Check whether in the above AGV mechanism, IIR is satisfied for all types of sellers and buyers. If not, state explicitly for which types of sellers and/or buyers IIR is violated.

4. **(2 points)** Does the result obtained in 3 surprise you? Argue.

5. **(5 points)** In the bilateral trade setting, do there exist

(a) SCRs that are TIDS (note the "D"), ex-post efficient, and IIR?

(b) SCRs that are TIBS, ex-post efficient, and ex-post individually rational?

(c) SCRs that are TIBS, IIR, successful and budget-balanced ex ante?

Problem 2 (30 points) This problem studies adverse selection in procurement and is based on problem set 3. In question 4, we obtained the following results:

- Optimal effort as a function of θ is characterized by the principal's FOC

$$\psi'\left(e^*\left(\theta\right)\right) = 1 - \frac{\lambda}{(1+\lambda)}\psi''\left(e^*\left(\theta\right)\right)\frac{\Pi\left(\theta\right)}{\pi\left(\theta\right)}. \tag{2.1}$$

- Optimal effort is nonincreasing, i.e. $e^{*\prime}\left(\theta\right) \leq 0$.

- The optimal transfer function is

$$t^*\left(\theta\right) = \psi\left(e^*\left(\theta\right)\right) + \int_{\theta}^{\overline{\theta}} \psi'\left(e^*\left(\eta\right)\right) d\eta. \tag{2.2}$$

Upon truthfully announcing its type, the firm receives a cost target $C^*\left(\theta\right)$ (recall the discussion prior to question 4 in the problem set). From $e^{*\prime}\left(\theta\right) \leq 0$, it then follows that $C^*\left(\theta\right) = \theta - e^*\left(\theta\right)$ is strictly increasing in θ. Hence, we can compute the inverse of $C^*\left(\theta\right)$ as $C^{*-1}\left(C\right) = \theta \Leftrightarrow C = C^*\left(\theta\right)$, where $C^{*-1}\left(C\right)$ is the type that optimally receives the cost target C.

Inserting $C^{*-1}\left(C\right)$ for θ in (2.2) yields the optimal transfer as a function of the realized cost level C:

$$T^*\left(C\right) \equiv t^*\left(C^{*-1}\left(C\right)\right) = \psi\left(e^*\left(C^{*-1}\left(C\right)\right)\right) + U^*\left(C^{*-1}\left(C\right)\right),$$

where $U^*\left(C^{*-1}\left(C\right)\right) = \int_{C^{*-1}(C)}^{\overline{\theta}} \psi'\left(e^*\left(\eta\right)\right) d\eta$.

1. **(1 point)** The transfer function $T^*\left(C\right)$ is no longer based on the firm's announced type, but on the observed cost level C. What is the technical term for $T^*\left(C\right)$? Hint: $C = C^*\left(\theta\right)$ is a strategy for type θ.

2. **(10 points)** Show that the function $T^*\left(C\right)$ is decreasing and convex in C. Hint: Note that by the inverse function theorem, $\frac{dC^{*-1}(C)}{dC} = 1/\frac{dC^*(\theta)}{d\theta}$, where $C^*\left(\theta\right) = \theta - e^*\left(\theta\right)$. Moreover, recall that $\psi'\left(\cdot\right) > 0$ and $\psi''\left(\cdot\right) > 0$. Also, you must know how to differentiate w.r.t. the lower limit of an integral.

Since $T^*(C)$ is convex, it can be approximated by the family of its tangents. Therefore, instead of offering the firm the transfer function $T^*(C)$, the government can offer a menu of compensation schemes that are linear in realized cost C. You can think of this menu as follows: The government asks the firm to announce a type. If the firm announces $\hat{\theta}$, it receives the compensation scheme

$$t(\hat{\theta}, C) = t^*(\hat{\theta}) - \psi'(e^*(\hat{\theta}))(C - C^*(\hat{\theta})), \tag{2.3}$$

which is linear in C. As you can see in the diagram, (2.3) is just the tangent of $T^*(\cdot)$ in point $C = C^*(\hat{\theta})$. Subsequently, the firm selects an effort level e which leads to production cost $C = \theta - e$ and effort cost $\psi(e)$. Thus, when being offered the menu of compensation schemes, the firm's choice variables are $\hat{\theta}$ and e, while under $T^*(C)$, the firm's choice variable is e.

3. **(8 points)** Show that when facing the menu of linear contracts (2.3), the firm announces the truth and selects the same effort as under $T^*(C)$, i.e. $\hat{\theta} = \theta$ and $e = e^*(\theta)$. You may disregard the firm's second-order conditions (they are satisfied!). Hint: In (2.3), substitute $\theta - e$ and $\hat{\theta} - e^*(\hat{\theta})$ for C and $C^*(\hat{\theta})$, respectively.

4. **(8 points)** From question 3, it follows that each type selects the same transfer/cost-pair whether it faces $T^*(C)$ or the menu of its tangents $t(\hat{\theta}, C)$. You are now asked to show this graphically.

(a) Write down the equation for the firm's indifference curves in the (t, C)-space (recall that the firm's utility is $t - \psi(\theta - C(\theta))$. Show that the slope of the indifference curves coincides with the slope of $T^*(C)$ at $C = C^*(\theta)$.

(b) The result in (a) implies that after the firm selects a compensation scheme (a tangent) from the menu, it produces exactly at the level $C = C^*(\theta)$ where the tangent touches the curve $T^*(C)$. Show this in the diagram by drawing the firm's indifference curve that goes through the point of optimality for each of the three tangents.

The linear structure of $t(\hat{\theta}, C)$ becomes more clear when we write (2.3) as

$$t(\theta, C) = \alpha(\theta) - \beta(\theta) C, \tag{2.4}$$

where $\alpha(\theta) = t^*(\theta) + \psi'(e^*(\theta)) C^*(\theta)$, $\beta(\theta) = \psi'(e^*(\theta))$, and $\theta = \hat{\theta}$ (since we know from question 3 that the firm announces the truth).

5. **(3 points)** Let us examine the menu (2.4) in more detail.

   (a) How does β vary with θ? Interpret your result.

   (b) What is the slope β for the most efficient type $\theta = \underline{\theta}$? Again, provide some intuition. Hint: Use (2.1).

Problem 3 (35 points) This problem studies moral hazard when the principal is restricted to linear sharing rules. Consider the following setting:

- The principal is risk neutral and maximizes expected profits. The agent is risk averse with utility function $U(s(x), a) = -e^{-r(s(x)-c(a))}$, where the constant r denotes the agent's coefficient of risk aversion. The agent's cost of action is $c(a) = \frac{K}{2}a^2$ with $K > 0$.
- Output is normally distributed with mean a and variance σ^2. Hence, $\pi(x, a) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x-a}{\sigma}\right)^2}$.
- The agent's reservation wage is 0 and his reservation utility is $-e^0$.
- The principal is restricted to sharing rules of the form $s(x) = \alpha + \beta(x)$.

In this framework, the principal's problem is to find optimal values for a, α, and β. Let us begin with the first-best. The first-best sharing rule provides the agent with full insurance, i.e. $\beta^*_{FB} = 0$ and $s^*_{FB}(x) = \alpha^*_{FB}$.

1. **(4 points)** Determine α^*_{FB} and a^*_{FB} explicitly. In addition, show that the principal's first-order condition is sufficient for a global maximum. Hint: When setting up the principal's first-best problem, you may already use the (equilibrium) results that $s^*_{FB}(x) = \alpha^*_{FB}$ and that IR holds with equality.

Consider now the principal's second-best problem. A nice feature of the combination normal distribution/exponential utility is that we can explicitly compute the second-best solutions a^*, α^*, and β^*. The trick is to work with the agent's certainty equivalent (CE) instead of his expected utility (CE is defined as the sure payment that yields the agent the same utility as the random payment $s(x) = \alpha + \beta(x)$ when he chooses action a). By definition,

$$E[U(s(x), a)] \equiv \int_{-\infty}^{\infty} -e^{-r\left(\alpha+\beta x-\frac{K}{2}a^2\right)} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x-a}{\sigma}\right)^2} dx \equiv -e^{-rCE},$$

which can be rearranged as

$$\frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} -e^{-r\beta x-\frac{1}{2}\left(\frac{x-a}{\sigma}\right)^2} dx \equiv -e^{-r\left(CE-\alpha+\frac{K}{2}a^2\right)}. \tag{3.1}$$

2. **(10 points)** Show that in (3.1),

$$CE = \alpha + \beta a - \frac{K}{2}a^2 - \frac{r}{2}\beta^2\sigma^2. \tag{3.2}$$

   Hint: You must use a technique known as "completing the square". More concretely, you'll have to complete the square for $-r\beta x - \frac{1}{2}\left(\frac{x-a}{\sigma}\right)^2$. What you then get is the product of a non-stochastic term $-e^{\cdots}$ and some integral $\int_{-\infty}^{\infty} ...dx$, where the integrand is the density function of a normally distributed random variable. Of course, the value of the integral is 1. If you don't know how to complete a square, see the attached copies from Sydsaeter and Hammond's textbook.

3. **(2 points)** Since $E[x, a] = a$, the first three terms in (3.2) represent the agent's expected net income from taking action a. What does the last term in (3.2) represent?

4. **(10 points)** Derive the optimal values a^*, α^*, and β^*. Proceed as follows:

   (a) Replace IC with the agent's FOC. Prove that the FOA is valid. Hint: Note that $\max_a -e^{-f(a)}$ is equivalent to $\max_a f(a)$.

(b) Solve FOC and IR for α^* and β^* and insert the results in the principal's objective function (you can again use the fact that IR must be satisfied with equality). Subsequently, solve for the remaining parameter a^*.

(c) Prove that the principal's FOC is sufficient for a global maximum.

5. **(4 points)** Interpret the second-best solution.

   (a) What are the solution values a^*, α^*, and β^* when either $r = 0$ or $\sigma^2 = 0$?

   (b) How does the incentive intensity β^* vary with r and σ^2?

   (c) How does the agent's effort a^* vary with r and σ^2?

   (d) Compare a^* with a^*_{FB}.

   In (a)-(d), provide some intuition for your results whenever possible.

So far, we have simply assumed that the principal is restricted to linear sharing rules. An interesting and still ongoing area of research is to find explanations *why* the principal would want to use linear sharing rules.

6. **(5 points)** Suppose the monthly pay of an insurance agent depends on the number of insurance contracts sold at the end of the month. His employer, the insurance company has the choice between a linear compensation scheme and a bonus scheme (where a bonus is paid if and only if the number of contracts sold exceeds a certain level). Each day, the agent meets a fixed number of potential customers, and the insurance company wants him to expend the same amount of effort every day and with every customer (the agent's effort determines the probability that a sales talk leads to the signing of a contract). Argue why the insurance company should use a linear compensation scheme rather than a bonus scheme.