Risk Choice under High-Water Marks

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I solve in closed form for the optimal dynamic risk choice of a fund manager who is compensated with a high-water mark contract. The optimal risk choice depends on the ratio of the fund’s assets under management to its high-water mark. If the manager’s outside option value is low, investors’ termination policy is strict, or management fees are high, then negative returns induce the manager into “derisking.” Otherwise, he engages in “gambling.” Having the option to walk away increases risk taking, though in many cases exercise is never optimal. In particular, leaving to restart at a proportionally smaller fund is always suboptimal. (JEL G01, G11, G23)

High-water mark (HWM) contracts are the predominant compensation contract for managers in the hedge fund industry. Under this contract, performance fees are only paid on returns made in excess of the maximum cumulative return, the high-water mark. Hence, any previous losses must be recovered before further performance fees apply. In contrast, mutual funds are prohibited by law from having such an asymmetric performance bonus and are usually compensated with a fixed management fee.

What is the manager’s optimal risk-taking strategy under such incentives? This is both an interesting portfolio-choice question and a question that has potentially important ramifications. Hedge funds are a key group of financial intermediaries. In 2013 U.S. hedge funds had more than $1.47 trillion in net assets (i.e., equity), similar to the total equity of U.S. banks, and more than one trillion dollars in borrowings. Moreover, hedge funds are widely viewed as sophisticated and are thought to play an important role as arbitrageurs and...
liquidity providers. Therefore, knowing how the HWM impacts their optimal risk-taking strategy may be important for understanding the dynamics of trading and prices in markets in which they are influential. In addition, other types of compensation structures implicitly embed a HWM-like mechanism, so insights about optimal risk choice under a HWM may extend beyond the hedge fund setting.

In this paper I provide an answer to this question by deriving the optimal risk-taking policy of a manager who faces a HWM contract. I model a manager who is risk neutral and can freely trade a risky strategy that earns positive expected excess returns, as well as a riskless money market account. By varying the fund’s leverage, the manager dynamically controls the fund’s overall risk. The model also includes other important features of a hedge fund manager’s environment, such as performance-based termination, investor withdrawals, management fees, and the option to walk away, which interact with the HWM to determine the manager’s optimal risk-taking policy.

I exhibit the optimal policy in closed form. It shows that the manager’s optimal risk choice is a function of the ratio of the fund’s assets to its high-water mark. Although the manager is risk neutral, the solution shows that the incentives induce him to behave as if he is risk averse, with his effective risk aversion varying with the ratio of the fund’s value to its HWM. The resulting optimal risk-taking dynamics take one of two possible forms, depending on the parameters of the manager’s environment: (1) “derisking,” whereby negative returns make the manager increasingly risk averse, inducing him to reduce risk taking continuously as the fund’s value falls further below the HWM, and (2) “gambling,” where the manager instead increases risk taking as the fund’s value falls further below the HWM. In either case, the dynamic is nonlinear, with risk-taking remaining fairly constant near the HWM and then changing more rapidly as the fund’s value falls further below the HWM.

Two important parameters of the manager’s environment are investors’ termination policy and the value of the manager’s outside option. I assume that investors terminate the manager if negative returns reduce the fund’s value down to a given fraction of the HWM, called the termination point. Upon termination the manager receives his outside option, the value of which is proportional to the size of the fund. Other important parameters include the Sharpe ratio of the risky strategy, the rate of investor withdrawals from the fund, the performance fee, and the management fee.

The optimal policy balances a trade-off that the manager faces in choosing the level of risk. On one hand, he would like to reduce the expected waiting time to receive his next payout, which occurs either when he gets back to the HWM and receives additional performance fees, or when he hits the termination point and receives his outside option. Increasing risk reduces this expected waiting time. On the other hand, the manager would much rather reach the HWM than the termination point, and greater risk taking reduces the probability of this happening.
The ratio of the manager’s value function at the termination point and the HWM is an important factor in determining the optimal balance between these considerations. If the manager’s outside option value is low relative to his continuation utility at the HWM, then he takes lower risk to reduce the probability of termination. A second important factor is investors’ termination policy, as measured by the distance between the termination point and the HWM. A termination point close to the HWM increases the likelihood of termination for any given risk choice and therefore induces lower optimal risk taking, whereas one that is farther away allows for a higher optimal level of risk. I derive comparative statics for these parameters that show that the optimal risk-taking policy is decreasing in the termination point and increasing in the manager’s outside option at any ratio of the fund’s value to its HWM.

The comparative statics further show that the impact of these parameters on risk taking is larger when the fund’s value is further below the HWM. The result is that the optimal level of risk taking depends on the ratio of the fund’s value to the fund’s HWM, and this leads to the two possible risk-taking dynamics described above. If the outside option value is low and the termination point is close to the the HWM, then the optimal policy tilts the risk-choice trade-off toward avoiding termination. The manager then optimally reduces risk taking as the fund’s value falls further below the HWM (i.e., “derisking”). Alternatively, if the outside option is relatively high and the termination point is far away from the HWM, then the optimal policy tilts toward the desire to reduce the waiting time to payout. The manager then optimally increases risk taking as the fund’s value falls further below the HWM. I call this second dynamic “gambling” because, similar to a gambler, the manager takes greater risk after suffering losses.

Interestingly, the functional form for the manager’s effective risk aversion closely resembles the external habits preferences of Campbell and Cochrane (1999), but with the surplus consumption ratio replaced by the ratio of the fund’s value to the fund’s HWM. Whereas this functional form is assumed in the external habits model, in this paper it arises endogenously.

I further analyze the impact on optimal risk taking of investor’s rate of withdrawals, the risky strategy’s Sharpe ratio, and the management fee. A higher rate of investor withdrawals induces higher risk taking because it decreases the fund’s growth rate and hence reduces the tempering influence of future continuation values on risk taking. The impact of a change in the Sharpe ratio depends on the level of the Sharpe ratio and is nonmonotonic. An increase in the management fee has two opposing effects: one due to its drag on fund growth and the second to the manager’s increased desire to protect higher management fees from termination. The net effect can be nonmonotonic and can depend on the ratio of the fund’s value to the fund’s HWM.

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3 Panageas and Westerfield (2009) also contains such a result.
I analyze two further extensions of the main model. In the first I solve for optimal risk choice when losses in excess of a given threshold trigger an increase in fund withdrawals. I show that the optimal policy takes a cautiously low level of risk when the fund is above the threshold, and increases risk taking to a high level when the fund falls below the threshold.

In the second extension I give the manager the option to walk away from the fund, and I analyze the optimal decision to walk away and its impact on optimal risk choice. I show that if walking away from the fund at some point is optimal for the manager, then having the option to do so increases risk taking. However, I also prove that for a class of reasonable outside option assumptions walk-away is never optimal, and hence risk taking is unaffected. This class includes managers whose outside option value is linear in the size of their fund. An implication is that walk-away is suboptimal for a manager whose outside option is to restart at a new fund with assets equal to a fixed fraction of his current fund’s.

The main antecedent to this paper is Panageas and Westerfield (2009) (henceforth PW), who are the first to provide closed-form results for the HWM risk-choice problem. Like PW, I solve for the risk-choice of an indefinitely tenured, risk-neutral manager who faces a HWM. PW show that their manager optimally acts like a CRRA investor with a fixed risk-aversion of less than one and emphasize that despite the (call) option-like nature of the HWM payout, the risk-neutral manager does not put an unbounded weight on the risky asset. Their result provides an important underpinning for the results in this paper. However, the framework in PW does not incorporate a number of important considerations that fundamentally impact the nature of the manager’s optimal risk-taking behavior, including termination, the outside payoff, management fees, or the walk-away option. Incorporating these considerations reveals a rich class of possible optimal risk-taking dynamics. The constant risk choice found by PW then appears as a unique special case of the general solution. Moreover, the set of endogenous manager “risk aversions” expands to include all positive values, rather than just values less than one, as is in PW. Though taking these factors into account substantially increases the complexity of the problem, I am still able to present closed-form solutions that illustrate how the different factors impact the manager’s optimal risk choice policy.

This paper is also closely related to Goetzmann, Ingersoll, and Ross (2003) (henceforth GIR). GIR solve for the present value of the hedge fund manager’s fees and investors’ claim under a high-water mark contract, when the risk-choice of the manager is exogenously fixed at a constant level. Their framework incorporates several important features, including termination, which is triggered when fund wealth falls to a given percentage of the HWM, investor withdrawals, and management fees. The model in this paper also incorporates these features, while solving for the manager’s optimal risk choice. On the technical side, this paper shares some features with Browne (1997, 2000). Browne (1997, 2000) solves a set of control problems in which...
the objective is to maximize an expected discounted reward from attaining a goal.

Carpenter (2000) solves for the risk choice of a risk-averse manager who faces a finite horizon and is compensated with a single call option. The HWM contract resembles a series of call options with an indefinite horizon, where an option that is exercised is replaced with a new one that has a higher strike price (the new HWM). In Carpenter’s model, the risk choice of the manager goes to infinity as the asset value decreases to zero. I show that with a HWM and an indefinite horizon both derisking and gambling dynamics are possible. Dai and Sundaresan (2010) model the option of investors and counterparties to withdraw funding from hedge funds following losses. They show that these options reduce the manager’s optimal risk choice and the expected return to investors. In contemporaneous work, Lan, Wang, and Yang (2013) numerically solve the problem of a HWM-compensated hedge fund manager whose alpha-generating strategy suffers from decreasing returns to scale.

Recent empirical studies find that hedge funds actively change their level of risk over time and document evidence consistent with derisking dynamics. Patton and Ramadorai (2013) find that hedge funds have time-varying risk exposures and tend to reduce risk exposures and retreat to cash in response to significant negative market returns or increases in volatility. They attribute this variation in risk exposures to active changes in portfolio weights. Aragon and Nanda (2013) find that hedge funds that have a HWM provision do not appear to increase risk following losses. They also document evidence of antitournament behavior among hedge funds with high-water marks, as funds that are underwater at the beginning of the year are more likely to have lower return volatility in the second half. Buraschi, Kosowski, and Sritrakul (2014) analyze funds that have experienced large deviations from their HWM and show that funds that are further below the HWM have larger subsequent decreases in return volatility.

During the “quant crisis” of August 2007, losses suffered by hedge funds that specialized in quantitative equity appear to have induced them into a simultaneous and sharp derisking (Pedersen 2009; Khandani and Lo 2011). Ben-David, Franzoni, and Moussawi (2013) document that, in contrast to mutual funds, hedge funds significantly cut their equity positions in the third and fourth quarters of 2007 and 2008. Only half of this reduction is explained by redemptions, whereas funds’ returns strongly predict their selling, consistent with loss-induced derisking. Moreover, consistent with a desire to actively reduce risk, they find that funds were more likely to close out positions in high-volatility stocks, both long and short, and unwind value and momentum strategies. Ang, Gorovyy, and van Inwegen (2011) document that aggregate

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4 Important early work on hedge funds includes Fung and Hsieh (1997), Ackermann, McEnally, and Ravenscraft (1999), Brown, Goetzmann, and Ibbotson (1999), and Brown, Goetzmann, and Park (2001). Recent works include Agrawal, Daniel, and Nanda (2009), Fung et al. (2008), and Laidler (2011).
hedge fund leverage decreased significantly from mid-2007 until early 2009, whereas banks’ market leverage increased strongly over this period.

This paper also relates to the literature on frictions in financial intermediation and their impact on asset prices. For example, He and Krishnamurthy (2013) develop a model in which the risk-bearing capacity of the financial sector drives risk premiums and asset prices. By solving for the optimal dynamic risk choice of hedge funds, and highlighting the potential for nonlinear derisking dynamics, I show that hedge funds’ incentives could have reduced hedge funds’ willingness to take risk following the substantial negative returns that were realized before and during the onset of the financial crisis.

This paper proceeds as follows. Section 1 lays out the general model. Section 2 solves for the manager’s value function and optimal risk choice, and provides comparative statics, for a baseline version of the model with no management fees and a constant rate of withdrawals. Section 3 extends the solution to incorporate management fees. Section 4 analyzes another model extension in which losses trigger an increase in investor withdrawals. Section 5 analyzes the manager’s optimal walk-away decision and its impact on risk choice. Section 6 concludes. Proofs are left to the Appendix, and the Internet Appendix contains some additional results.

1. Model

A risk-neutral manager allocates a portfolio between a risky strategy and a money market account. The price of the money market account evolves according to

$$\frac{dP_{0,t}}{P_{0,t}} = r dt,$$

where $r$ is the fixed interest rate earned by the money market account. The manager can go long or short the money market account. The price of the risky strategy evolves according to

$$\frac{dP_{1,t}}{P_{1,t}} = \mu dt + \sigma dB_t,$$

where $\mu > r$ and $\sigma > 0$ are constant and $B_t$ is a one-dimensional Brownian motion. I do not take a stand on the source of the risky strategy’s expected return because I study how the manager responds to this investment opportunity and not why it exists. The strategy’s positive expected excess return could simply reflect an equilibrium risk premium, or it could represent risk-adjusted excess return ("alpha") that the manager generates through proprietary skill\(^5\), \(^6\) Note

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\(^5\) This would be the case for a manager who is properly benchmarked, so that he does not get compensated for taking systematic risks. In practice, some funds are mandated to remain "beta- (or market-) neutral," with the risk-free rate ("cash") as their benchmark return.

\(^6\) Under no arbitrage, the existence of a positive risk-adjusted return requires market incompleteness. The set of traded securities must be insufficient for the manager to hedge out all of the risky-strategy variance, so that $\sigma > 0$. 

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that in the absence of any constraints a risk-neutral investor will normally take infinite long-short positions to exploit such a positive premium. However, consistent with PW, I show that under HWM compensation this is not the case, because the manager actually behaves as if he is risk averse.

The manager chooses the fraction, \( \pi_t \), of fund wealth, \( W_t \), to invest in the risky strategy at time \( t \). The remaining \( 1 - \pi_t \) then goes into the money market account. The only restriction imposed on the position that the manager can take in the risky strategy and money market account is to rule out doubling strategies, by requiring that \( \int_0^T (\pi_t W_t)^2 dt < \infty \), a.s., for all \( T < \infty \).

Like GIR and PW, I model the fund as being invested in by a group of investors with the same HWM. This eliminates the need to keep track of different high-water marks for each investor. This will arise in practice if all investors currently in the fund begin investing at the same time or if inflows into the fund only occur at the HWM. This assumption does not place any restriction on withdrawals, which I model as occurring at a continuous rate \( \phi_t \).

To avoid having to keep track of different HWMs, I do not model inflows. The HWM, denoted \( H_t \), is the highest level that the net assets of the fund have reached, subject to some adjustments. As in GIR, the HWM is adjusted down for time-\( t \) withdrawals and adjusted up at a contractually set rate, that I set equal to \( r \). Adjusting the HWM up at the rate \( r \) implies that the manager does not earn performance fees for earning the risk-free rate on funds (it is a form of benchmarking). A final adjustment to the HWM accounts for the management fee paid to the manager that is taken out of fund wealth at a rate \( m \). A si st h e case in practice (and in GIR), the three adjustments to \( H_t \) (for \( \phi_t \), \( r \), and \( m \)) are proportional, and hence the ratio \( W_t / H_t \) is unaltered by these adjustments.

When \( W_t < H_t \) and the fund is not reaching a new high, these are the only adjustments made to \( H_t \). Hence, in this region \( H_t \) evolves deterministically, following \( dH_t = (r - \phi_t - m) H_t dt \).

The other region of importance is when the fund is at the high-water mark, \( W_t = H_t \). When the fund’s wealth increases from \( W_t = H_t \) to \( W_t = H_t + \varepsilon \), a performance fee of \( k\varepsilon \) is paid, fund wealth is reduced by \( k\varepsilon \), and the HWM is reset to \( H + \varepsilon \). It will be convenient to have a notation for just these “\( \varepsilon \)” increases. Let these increases be denoted by \( dH^\varepsilon \), so that overall the HWM evolves as follows:

\[
dH_t = dt^\varepsilon + (r - \phi_t - m) H_t dt
d(1)
\]

Using the manager’s choice of \( \pi_t \), the outflow rate \( \phi_t \), the management fee \( m \), and the returns on the money market account and risky strategy, we can now write the law of motion for fund wealth as

\[
\frac{dW_t}{W_t} = r dt + \pi_t(\mu - r) dt + \pi_t \sigma dB_t - (\phi_t + m) dt - kdH^\varepsilon_t.
\]

Moreover, it is implicit that investors are not able to identify (and hence eliminate) the mispricing themselves, perhaps because they do not possess the manager’s information or skill. Together, these forms of incompleteness allow the mispricing to exist in equilibrium.
Note that the last term is due to the payout of the performance fee and appears only when the HWM, $H_t$, is reset.

Because the manager’s problem is homogenous in $H_t$, it is convenient to write this problem in terms of the ratio of the fund current’s wealth to its HWM,

$$X_t = \frac{W_t}{H_r}.$$ 

Taking differentials gives $dX_t = \frac{dW_t}{H_t} - \frac{W_t}{H_t^2} dH_t$. Substituting in (1) and (2) then gives the law of motion for this variable:

$$dX_t = X_t \pi_t (\mu - r) dt + X_t \pi_t \sigma dB_t - (1 + k) dH_t / H_t.$$  (3)

The last term is again zero everywhere, except at the the boundary $\{X_t = 1\}$, where the performance fee is paid out, $dW_t = -kdH_t$, and the HWM is reset, $dH_t = dH_t$. 

Termination of the manager may occur for two reasons. The first is if fund wealth drops to some “low” proportion $C$ of the high-water mark (i.e., $X_t = C$), at which point investors lose confidence in the manager. Following GIR and PW, I also allow for an exogenous random termination of the fund that is assumed to be Poisson with intensity $\lambda$. This can represent the possibility of a liquidity shock for investors that induces liquidation of the fund. 

Upon termination, the manager receives his outside option, which has value $V_r$. This can be the value to the manager of starting or managing a new fund, or the value of his leisure. It is plausible that the manager of a large fund should have a more valuable outside opportunity than the manager of a small fund. To capture this, I let that the manager’s outside payoff be equal to a fraction of his value function at the HWM ($X_t = 1$). Formally, let $V_t = V(X_t, H_t)$ denote the manager’s value function at the given values of $X_t$ and $H_t$. I let $V_r$ be given by

$$V_r = gV(X_t = 1, H_t), \quad 0 \leq g < 1,$$  (4)

where $g$ is the parameter governing the outside payoff value. Note that because $g < 1$, the manager’s outside payoff is smaller than his value function at the HWM. This condition is necessary because otherwise the manager is always better off outside the fund (i.e., his participation constraint is violated). 

I show below that the value function is in fact homogeneous in $H_t$. Hence, (4) implies that the outside payoff is simply proportional to $H_t$. This means that as the fund grows in size, so does the outside payoff. An implication is that the relative importance of the outside payoff does not diminish as the fund grows over time. 

We can now formally represent the manager’s objective. Define $\tau$ to be a stopping time that is equal to the termination time of the manager, or $\infty$ if he

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7 Section 2.2.3 lets $V_r$ be proportional to fund wealth $W_t$ instead, and shows that optimal risk choice remains very similar qualitatively and quantitatively.
is never terminated. Because the manager is risk neutral, his objective is to maximize the expected discounted value of his payoffs,

$$V_t = \max_{\pi_t} E \left[ \int_t^\tau e^{-\rho(t-s)} (m W_s + k d H_s^\varepsilon) + e^{-\rho(\tau-t)} V_{\tau} \right],$$

where $\rho$ is the manager’s time-discount rate. The manager maximizes this objective by choosing the optimal dynamic policy for $\pi_t$, the fraction of fund wealth invested in the risky strategy.

2. Solution

I begin by deriving the Hamilton-Jacobi-Bellman (HJB) equation satisfied by the solution to the manager’s problem. I then present the solution for $V_t$ and analyze its components. I derive the corresponding optimal choice of $\pi_t$ and present comparative statics results for the parameters of the manager’s environment. I also look at the solution to some useful special cases of the model.

Let $V_t = V(X_t, H_t)$ denote the value function of the manager, given by the solution to (5). The state vector is taken to be $(X_t, H_t)$. For $t < \tau$, the process $e^{-\rho t} V_t + \int_0^t e^{-\rho s} (m W_s + k d H_s^\varepsilon)$ is a martingale under the maximizing choice of $\pi_t$. It therefore satisfies the following HJB equation:

$$0 = - (\rho + \lambda) V_t + m X_t H_t + \lambda V_{X_t} + \sup_{\pi_t} \left\{ V_{XX_t} \pi_t (\mu - r) + \frac{1}{2} V_{XX_t} \pi_t^2 \sigma^2 \right\}$$

$$+ V_{X_t} (1 + k) d H_s^\varepsilon H_t + V_{H_t} (r - \phi_t - m) + k d H_s^\varepsilon,$$

where the last three terms are nonzero only at the boundary $\{X_t = 1\}$.

2.1 Baseline model solution

In this section I focus on a baseline model that makes two simplifications to the general HJB equation (6). The first is to set the management fee to zero, $m = 0$. The second is to assume that $\phi_t$ is a constant $\phi$. These assumptions simplify the exposition of the main results on optimal risk-taking dynamics, but do not significantly affect their qualitative properties. In subsequent sections, I relax both assumptions. Section 3 extends the solution to incorporate positive management fees, whereas Section 4 extends the model to allow $\phi$ to vary, depending on the fund’s performance.

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8 If the manager is risk averse, then his personal pricing measure will in general depend on his outside wealth and income stream and their size relative to the fund. If payouts from the fund are an important source of income for the manager, then the manager will be more cautious in his portfolio choice in order to decrease the volatility of the fund’s performance and hence also his income stream. I conjecture that solution to the manager’s optimal policy will remain qualitatively similar in this case, but leave this problem to future work.
Assuming that $V_X \geq 0$ and $V_{XX} < 0$, we can use the first-order condition to find the optimizing value of $\pi$ in (6),

$$\pi^*_t = -\frac{(\mu - r)}{\sigma^2 V_{XX}}.$$  \hfill (7)

Substituting in (7) and (4) for $\pi^*_t$ and $V_t$, respectively, in (6), setting $m = 0$, and simplifying gives

$$0 = -(\rho + \lambda) V + \lambda g V(X_t = 1, H_t) - \omega \frac{V^2_{XX}}{V_{XX}} + VH(r - \phi)$$  \hfill (8)

where $\omega$ is defined as

$$\omega = \frac{1}{2} S R^2 = \frac{1}{2} \frac{(\mu - r)^2}{\sigma^2}.$$  \hfill (9)

I conjecture that

$$V(X_t, H_t) = \beta_1 H_t G(X_t)$$  \hfill (10)

where the coefficient $\beta_1$ is a normalization constant chosen so that $G(1) = 1$. The homogeneity of $V_t$ in $H_t$ is a consequence of the scale invariance of the problem. The following proposition verifies this conjecture and gives $G$ and $\beta_1$.

Proposition 1. Assume that $\rho + \lambda - r + \phi > 0$ and $\beta_1 > 0$ (defined below). Then $V_t$ is given by (10) with

$$G(X_t) = \left(\frac{X_t - D_0}{D_1}\right)^{\eta} + D_2,$$  \hfill (11)

$$\beta_1 = \frac{k}{G_X(1)(1+k) - 1}$$

where

$$\eta = \frac{\rho + \lambda - r + \phi}{\omega + \rho + \lambda - r + \phi}$$

$$D_2 = \frac{\lambda g}{\rho + \lambda - r + \phi}$$

and, if $g \geq D_2$, then

$$D_0 = C - \frac{1 - C}{(1 - D_2)^{1/\eta}} = (g - D_2)^{1/\eta}$$

$$D_1 = \frac{(1 - C)}{(1 - D_2)^{1/\eta} - (g - D_2)^{1/\eta}}$$

9 This assumption is verified by the solution, as shown below.
otherwise, \( g < D_2 \), and

\[
D_0 = C \\
D_1 = \frac{(1 - C)}{(1 - D_2)^{1/\eta}}.
\]

The solution shows that \( V_t \) has several parts. The main part is the function \( G(X_t) \). It consists of an increasing, power function of \( X_t \) and the constant \( D_2 \). Note that because \( \omega > 0 \) and \( \rho + \lambda - r + \phi > 0 \), the exponent \( \eta \) has a value \( 0 < \eta < 1 \). Hence, \( G(X_t) \) (and therefore also \( V(X_t) \)) is concave. Together with the constants \( D_0 \) and \( D_1 \), the constant \( \eta \) determines the level and dynamics of the manager’s risk taking, as discussed below.

Next, consider the constant \( D_2 \) in \( G(X_t) \). When (10) is substituted into (9), it gives rise to a term \( \beta_1 H_t D_2 \) in the expression for the manager’s value function. This term has a useful economic interpretation: it is the discounted value of the future payments that would accrue to the manager if he were to shut down any further risk taking (by setting \( \pi_s = 0 \) for \( s \geq t \)). I therefore refer to this payoff as the risk-shutdown payoff. Because setting \( \pi_t = 0 \) implies \( dX_t = 0 \), under risk shutdown neither the HWM nor the termination boundary are ever reached. The manager then only earns a payout upon investors’ exogenous liquidation, as the numerator of \( D_2 \) indicates (\( \lambda g \)). Section 3 shows that this expression expands to include management fees when these are positive, because the manager receives these fees even under risk shutdown.

The denominator of \( D_2 \) gives the effective “discount rate” for these payoffs, which is the sum of four parameters. Three of these discount the stream of future payouts: the manager’s time-discount factor \( (\rho) \), the stochastic termination intensity \( (\lambda) \), and the rate of outflows from the fund \( (\phi) \). The remaining parameter, the contractual growth rate of the HWM \( (r) \), acts like a negative discount rate because it increases future payments. For the risk-shutdown payoff to be finite, the effective discount rate given by the sum of these four parameters must be positive, as Proposition 1 assumes.

Proposition 1 shows that the expressions for the constants \( D_0 \) and \( D_1 \) in \( G(X_t) \) depend on the relation between the outside payoff parameter \( g \) and the risk-shutdown payoff \( D_2 \). There is an intuitive explanation for this. When the outside payoff value is larger than the risk-shutdown payoff \( (g \geq D_2) \), the fund manager maintains positive risk taking even as fund wealth approaches the termination point. This means that the fund may cross the termination point, and therefore the outside payoff determines the value function at the termination boundary. In the absence of a management fee, the condition \( g \geq D_2 \) reduces to \( \rho - r + \phi \geq 0 \), because this inequality implies that the manager prefers receiving the outside payoff immediately, rather than shutting down risk and waiting for exogenous liquidation.

Alternatively, if the risk-shutdown payoff is greater than the outside payoff \( (g < D_2) \), then the manager shuts down risk taking when the fund reaches the termination point \( C \). Hence, the boundary condition for the value function
becomes $\pi^*_t(C) = 0$. Termination is then avoided, and the manager realizes the risk-shutdown payoff instead of the outside payoff. A look at the solution in this case shows that indeed $G(C) = D_2$, and hence the value function equals the risk-shutdown payoff at the termination boundary ($X_t = C$).

Recall that the constant $\beta_1$ is chosen so that $G(1) = 1$, and hence the value function is equal to $\beta_1 H_t$ at the HWM ($X_t = 1$). This means that $\beta_1$ captures the discounted value of the stream of future performance fees paid to the manager per unit of $H_t$. The scaling is multiplicative due to the homogeneity of the manager’s problem in $H_t$. The expression $G_X(1)(1+k)$ in the denominator of $\beta_1$ acts like a discount rate. It captures the impact that decreasing $X_t$ at the HWM (due to resetting of the HWM and payout of the performance fee) has on the expected value of the stream of future performance payouts. The assumption that $\beta_1 > 0$, which is equivalent to $G_X(1)(1+k) > 1$, ensures that the discounted sum of future payouts is finite.

2.2 Risk choice

The optimal risk choice of the manager follows by substituting the solution for $V(X_t, H_t)$ in Proposition 1 into Equation (7) for $\pi^*_t$.

**Proposition 2.** The manager’s optimal dynamic risk choice in the baseline model is given by

$$
\pi^*_t = \frac{1}{(1 - \eta)(X_t - 1)} \frac{\mu - r}{\sigma^2}.
$$

One way to view this result is that it gives the same weight in the risky asset as would be chosen by a myopic risk-averse investor with relative-risk aversion equal to $(1 - \eta) \frac{X_t}{X_t - 1}$. Note that, except for the case $D_0 = 0$, the manager’s risk choice and effective risk aversion depend dynamically on $X_t$. Inspection of (12) reveals that the risk-taking dynamics fall into one of two main cases, depending on the endogenous value of $D_0$.

**Corollary 1.** The relationship between the manager’s optimal risk choice ($\pi^*_t$) and the ratio of the fund’s assets to its HWM ($X_t$) takes one of two main forms, depending on the sign of $D_0$:

1. **Derisking:** If $D_0 > 0$, then the manager reduces risk taking as the ratio of the fund’s assets to its HWM falls.
2. **Gambling:** If $D_0 < 0$, then the manager increases risk taking as the ratio of the fund’s assets to its HWM falls.

When $D_0 = 0$, risk taking is invariant to the ratio of the fund’s assets to its HWM.

Consider first the case $D_0 > 0$. As $X_t$ decreases from its maximum value of one (at the HWM), the manager’s effective risk aversion increases and he
Figure 1 plots the manager’s effective risk aversion \((1 - \eta) \frac{X_t}{D_0}\) (solid line, left axis), and \(\pi^*_t(X_t)\) (dashed line, right axis) against \(X_t\) (the ratio of fund wealth to the HWM). The parameters for the plot are \(C = 0.6, g = 0.35, \phi = 0.03, \rho = 0.03, \lambda = 0, m = m_H = 0, k = 0.2, \mu = 0.07, \tau = 0.01,\) and \(\sigma = 0.16.\)

reduces risk taking (decreases \(\pi^*_t\)). Figure 1 illustrates this relationship for a manager with \(D_0 > 0\). As I show in the figure, the manager’s effective risk aversion (solid line, left axis) increases, and risk taking (dashed line, right axis) decreases as the ratio \(X_t\) of the fund wealth’s to its HWM falls. The effective risk aversion rises from a value of near one at the HWM \((X_t = 1)\) to around ten near the termination point \((X_t = C)\). As the plot makes clear, the impact of losses on risk aversion and risk choice can be strongly nonlinear. Indeed, the elasticity of the manager’s effective risk aversion with respect to \(X_t\) is equal to \(\frac{D_0}{X_t(X_t - D_0)}\). Hence, when \(D_0 > 0\) risk aversion increases, and risk taking decreases, at an increasing rate as losses push fund wealth further below the HWM. Note that the maximum effective risk aversion across all possible values of \(D_0\) is unbounded, and corresponds to a manager who has \(D_0 = C\). Such a manager engages in risk shutdown as the fund approaches the termination point, which is why his effective risk aversion rises without bound.

Interestingly, the functional form and effective risk-aversion dynamics for \(D_0 > 0\) strongly resemble those of the external habits process proposed by Campbell and Cochrane (1999). In Campbell and Cochrane’s (1999) habits specification, the local coefficient of risk aversion is (in their notation) \(\gamma \frac{C_t}{H_t} \left( \frac{C_t}{H_t} - 1 \right)\). The similarity can be seen by mapping \(\gamma\) to \((1 - \eta)\) and \(C_t/H_t\) to \(X_t/D_0\). Hence, the ratio \(X_t\) of fund wealth to the HWM drives variation in the manager’s effective risk aversion similar to the way in which \(C_t/H_t\) drives variation in the habits local risk aversion. Both processes are in a sense “countercyclical.” In the case of the manager, negative returns cause a decrease in \(X_t\) and a corresponding increase in effective risk aversion, whereas under
habits, negative consumption shocks decrease $C_t/H_t$ and increase local risk aversion. Moreover, the two risk aversions have similar nonlinear dynamics. Though there are many similarities, there is also an intriguing difference between habits preferences and the manager’s effective risk aversion. The functional form of habits preferences is assumed exogenously by Campbell and Cochrane (1999), in order to capture empirical features of asset prices. In contrast, the manager’s effective risk-aversion arises endogenously in response to the HWM-based portfolio-choice problem. Moreover, whereas variation in habits risk-aversion is due to consumption shocks, the manager’s effective risk aversion is driven directly by asset returns.

Now consider the case in which $D_0 < 0$. In this case, the manager actually increases risk taking as the fund falls further below the HWM. I refer to this as “gambling” because the manager responds to a series of negative outcomes by becoming increasingly risk tolerant and ratcheting up risk. I illustrate this in Figure 2 which shows sample plots of $\pi^*_t$ for each of the following cases: $D_0 < 0$, $D_0 > 0$, and $D_0 = 0$. The dash-dot line corresponds to $D_0 < 0$. Note how $\pi^*_t$ increases as $X_t$ decreases, whereas the opposite holds for the case of $D_0 > 0$ (dashed line).

What explains the sign of $D_0$, and hence the difference in risk-taking dynamics? The expression for $D_0$ in Proposition shows that $D_0$ summarizes the joint impact of several factors on the manager’s risk choice, including the termination point ($C$), the outside payoff ($g$), the risk-shutdown payoff ($D_2$),

Figure 2
Outside payoff and risk choice
Figure plots $\pi^*_t(X_t)$ for three cases, illustrating the possible relationships between the optimal risk choice $\pi^*_t$ and $X_t$: (1) $D_0=0$ (solid line), (2) $D_0 > 0$ (dashed line), and (3) $D_0 < 0$ (dash-dot line).
and the curvature coefficient \( \eta \). The next section provides a comparative statics analysis of each factor on the manager’s risk taking. The value of \( D_0 \) captures the combined impact of these factors, determining the dynamics of the manager’s risk taking.

Finally, in the case of \( D_0 = 0 \), the manager has a constant effective risk aversion, so his risk taking is invariant to \( X_t \). This is illustrated by the solid line in Figure 2. Panageas and Westerfield’s (2009) result is a special case of this outcome, as explain in Section 2.2.2.

Which of the two main dynamics is more likely in practice? The empirical evidence previously appears to favor the derisking dynamics, and I therefore focus more on it below. Nevertheless, the evidence does not rule out the possibility that both types of dynamics would be displayed across different funds/managers.

2.2.1 Comparative statics. The following Proposition provides comparative statics for the manager’s risk taking in several important parameters.

**Proposition 3.** The following obtain in the baseline model:

1. \( \frac{d\pi^*}{dC} < 0 \). Moreover, \( \frac{d\pi^*}{dC} \propto -\frac{1}{X_t} \).
2. \( \frac{d\pi^*}{dg} > 0 \) if \( g \geq D_2 \). Moreover, \( \frac{d\pi^*}{dg} \propto \frac{1}{X_t} \).
   
   If \( g < D_2 \), then \( \frac{d\pi^*}{dg} = 0 \).
3. \( \lim_{\omega \to 0} \pi^*_t \to \infty \) and \( \lim_{\omega \to \infty} \pi^*_t \to \infty \).

   Furthermore, \( \frac{d\pi^*}{d\omega} < 0 \) if \( \omega < \rho + \lambda + \phi - r \).
4. \( \frac{d\pi^*}{d\phi} > 0 \).

Point one says that the manager’s risk taking is everywhere decreasing in the termination point \( C \). Point two says that risk taking is everywhere increasing in the outside payoff \( g \), as long as risk shutdown is ruled out. If risk-shutdown does obtain, then the outside payoff value does not affect risk taking at the margin.

These two results are the outcome of a trade-off the manager faces in choosing the optimal level of risk. On one hand, the manager wants to reduce the expected waiting time to receive his next payout, either at the HWM or at the termination point. Increasing risk reduces this expected waiting time. On the other hand, greater risk taking decreases the probability of reaching the HWM, where his utility is higher than at the termination point. An increase in the termination point \( C \) increases the likelihood of termination for any risk-choice policy, and therefore induces a lower optimal level of risk taking. This is captured by a higher value of \( D_0 \). In contrast, an increase in the outside payoff \( g \) means that
termination is relatively less bad, so the optimal policy takes greater risk in order to reduce the expected waiting time to payout. This is captured by a lower value of $D_0$.

To analyze this trade-off further, I define two stopping times. Let $\tau_H$ denote the first time $s > t$ that fund wealth reaches the current HWM, and $\infty$ if termination occurs before the current HWM is reached. Second, let $\tau_C$ be the termination time if $\tau_H = \infty$, and $\infty$ if the current HWM is reached before termination. For the sake of simplicity, let $\phi = \lambda = r = 0$. Then, by the principle of optimality, the manager’s value function can be written as

$$V(X_t, H_t) = \max_{\pi_s} E\left[ e^{-\rho(\tau_H - t)} V(1, H_t) + e^{-\rho(\tau_C - t)} V(C, H_t) \right].$$

Given $V(C, H_t)$ and $V(1, H_t)$, the optimal policy must solve

$$\arg\max_{\pi_s} E\left[ e^{-\rho(\tau_H - t)} \right] + E\left[ e^{-\rho(\tau_C - t)} \right] V(C, H_t) V(1, H_t) = \arg\max_{\pi_s} E\left[ e^{-\rho(\tau_H - t)} | \tau_H < \infty \right] p(\tau_H < \infty)$$

$$+ E\left[ e^{-\rho(\tau_C - t)} | \tau_H = \infty \right] (1 - p(\tau_H < \infty)) V(C, H_t) V(1, H_t).$$

(13)

The top line shows that the maximization solved by the optimal policy effectively minimizes the weighted sum expected time until a payout is received, either by getting the continuation utility $V(1, H_t)$ at the HWM, or the outside option $V(C, H_t)$ at termination. The weight put on reaching the HWM is higher because the utility there is greater. The bottom line expands this expression to explicitly show the probability that the fund reaches the HWM, $p(\tau_H < \infty)$.

Equation (13) highlights the two opposing effects caused by increasing risk choice $\pi^*(X)$. The first effect is that increasing risk taking reduces the expected time to payout. This is illustrated by the top two plots in Figure 3. The left panel shows that increasing risk decreases the (certainty-equivalent) expected time to reach the HWM, $(-\rho)^{-1} \ln E_s\left[ e^{-\rho(\tau_H - t)} | \tau_H < \infty \right]$, conditional on no termination. The right panel shows that increasing risk also decreases the expected time to reach the termination point, $(-\rho)^{-1} \ln E_s\left[ e^{-\rho(\tau_C - t)} | \tau_H = \infty \right]$, conditional on not getting to the HWM. The panels plot these two quantities as a function of $X_t$ for values of $g$ of 0.35 (solid line), 0.63 (dashed line), and 0.80 (dotted line). They show that as $g$ is increased, and risk taking rises, there is a decrease in both the expected time to the HWM, and to the termination point, for every value of $X_t$.

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10 To incorporate positive values of $\phi$, $\lambda$, and $r$ in what follows, replace $\rho$ in the expressions with $\rho + \phi + \lambda - r$.

11 The details of the calculations of these quantities are provided in the Internet Appendix.
Figure 3 Probability of reaching the HWM and expected time to the boundaries

Figure 3 shows a comparison of the optimal risk-taking strategies that correspond to three values of the outside option, \( g = 0.35 \) (solid line), \( g = 0.63 \) (dashed line), and \( g = 0.80 \) (dotted line). The top left panel plots \((-\rho)^{-1} \ln E[e^{-\rho \tau_H} \mid \tau_H < \infty]\), the certainty-equivalent expected time to reach the current HWM conditional on no termination. The right panel plots \((-\rho)^{-1} \ln E[e^{-\rho \tau_C} \mid \tau_C < \infty]\), the certainty-equivalent expected time to termination conditional on not reaching the current HWM. The bottom panel plots \( P(\tau_H < \infty) \), the probability of reaching the current HWM prior to termination. The parameters used to generate the plots are \( \phi = 0, \lambda = 0, m = m_H = 0, k = 0.2, \mu = 0.06, r = 0.0, \) and \( \sigma = 0.16 \).

The opposing effect is that increasing risk reduces the probability of reaching the HWM prior to termination. This is illustrated in the bottom panel in Figure 3. It plots the probability of reaching the current HWM prior to termination, \( p(\tau_H < \infty) \), and shows that as risk taking rises this probability decreases for every value of \( X_t \).

The optimal risk choice equates the marginal benefit of increasing risk, due to a decreased expected time to payout, with its marginal cost, due to a decreased probability of reaching the HWM. Now consider an increase in the outside option value. This increases the term \( V(C, H_t) / V(1, H_t) \) in Equation (13), decreasing the cost of failing to reach the HWM. It is then optimal for the manager to take greater risk. Next, consider an increase in the termination point. Within the context of Equation (13), this can be viewed like a decrease in the outside option. To see this, let \( C' > C \) be the increased termination point, and let \( V', \pi'_s, \tau'_H, \) and \( \tau'_C \) be the value function, optimal risk choice, and stopping times corresponding to \( C' \). The policy \( \pi'_s \) must solve the corresponding version.
of Equation (13), but now on the interval \([C', 1]\). The original policy \(\pi_s\) can also be viewed as solving Equation (13) on \([C', 1]\) but with the last term replaced by \(V(C', H_t)/V(1, H_t)\), because \(V(C', H_t)\) gives the value function at \(C'\), and with the stopping times adjusted accordingly. Now, since \(V(C', H_t) > V(C, H_t)\), we have

\[
\frac{V(C', H_t)}{V(1, H_t)} > \frac{V(C, H_t)}{V(1, H_t)} = g = \frac{V'(C', H_t)}{V'(1, H_t)}.
\]

Therefore, increasing the termination point to \(C'\) is similar to decreasing the relative value of the outside option (from \(V(C', H_t)/V(1, H_t)\) to \(g\)). Hence, optimal risk taking decreases.

Proposition 3 also shows that the impact of a change in the outside option or termination point on \(\pi^*(X_t)\) is inversely proportional to \(X_t\). The reason is that as \(X_t\) decreases (and the likelihood of termination rises), the location of the termination point, and the value of the termination payoff, become increasingly important factors in the optimal risk-choice trade-off. Hence, when \(C\) is increased or \(g\) decreased, risk taking decreases much more quickly at low values of \(X_t\). The result is derisking dynamics and \(D_0 > 0\). Conversely, if \(C\) is decreased or \(g\) is increased sufficiently, then risk taking rises at low values of \(X_t\), until gambling dynamics results, and \(D_0 < 0\). At the border between these cases, risk taking remains constant and \(D_0 = 0\).

The third point of Proposition 3 shows that the impact of the (squared) Sharpe ratio \(\omega\) of the risky strategy on risk taking is nonmonotonic. Varying \(\omega\) changes two quantities in the optimal risk-choice expression (12): the quantity \((\mu - r)/\sigma^2\) and the endogenously determined constant \(\eta\). The interaction of these two changes produces the non-monotonicity. As shown in the Appendix, \(\pi^*_t\) can be rewritten

\[
\pi^*_t = \sqrt{\frac{\sigma}{2}} \left( \frac{\sqrt{\sigma} + \rho + \lambda - r + \phi}{\sqrt{\omega}} \right) \frac{X_t - D_0}{X_t}.
\]

This expression shows that \(\pi^*_t\) gets large for both large and small \(\omega\). The result that \(\pi^*_t\) gets large for large \(\omega\) is consistent with the standard portfolio choice intuition that an improved investment opportunity increases the weight in the risky asset. The surprising result that \(\pi^*_t\) also gets larger for smaller \(\omega\) was first shown in \(PW\) and continues to hold true here. As \(PW\) point out, effective “risk aversion” is endogenously determined and depends on the importance of the manager’s continuation value. As \(\omega\) becomes smaller, the decrease in effective risk aversion dominates the decreased attractiveness of the investment opportunity, so that eventually \(\pi^*_t\) actually increases. In fact, as the Sharpe ratio goes to zero, risk taking increases without bound.

Point four of Proposition 3 looks at the impact of an increase in the withdrawal rate. By decreasing the rate of fund growth and the future size of the fund, an increase in the withdrawal rate \(\phi\) diminishes the importance of the continuation value in the manager’s problem. Because this continuation value acts to attenuate the manager’s incentive for risk taking, a decrease in its value
implies greater risk taking by the manager. Figure 4 illustrates a comparison of $\pi^*(X_t)$ for three values of $\phi$, with the solid line representing the lowest rate of outflow and the dash-dot line the highest.

2.2.2 Special cases. It is useful to examine two special cases of the general solution. The first is the model of Panageas and Westerfield (2009). Their model does not have termination and hence implicitly sets $C = 0$. They also do not explicitly consider an outside payoff; hence, their model implicitly sets $g = 0$. Substituting in these values gives $D_2 = 0, D_0 = 0, D_1 = 1$, and hence

$$G(X_t) = X_t^g \quad \pi^*_t = \frac{1}{1-\eta} \frac{\mu-r}{\sigma^2} \quad \beta_1 = \frac{k}{\eta(1+k)-1}. \quad (14)$$

Note that the resulting effective risk aversion is just $1-\eta$, which is independent of $X_t$. The manager in PW therefore behaves exactly as would a CRRA investor with this same risk aversion of less than one. We can actually attain the same risk choice and value function with a termination point $C > 0$ by simultaneously increasing the outside payoff to $g = C^g$. This works because increasing both the outside payoff and termination point has opposing effects on risk choice that exactly cancel out.

---

12 Like the baseline model, PW’s model also does not have a management fee.
The second special case is $C > 0$ and $g = 0$. In this case there is termination, but the manager has a zero outside payoff. Substituting into the expressions gives $D_2 = 0$, $D_0 = C$, $D_1 = 1 - C$, and

$$G(X_t) = \left( \frac{X_t - C}{1 - C} \right)^n \pi_t^* = \frac{1}{(1 - \eta)(\frac{X_t}{X_t - C})} \frac{\mu - r_f}{\sigma^2} \beta_1 = \frac{k}{\eta(1 - C)^{-1}(1 + k)^{-1}}.$$ (15)

The manager’s effective risk aversion is now given by $(1 - \eta)(\frac{X_t}{X_t - C})$. As $X_t \to C$ effective risk aversion approaches infinity and the manager engages in risk shutdown ($\pi_t^* \to 0$). Note that the term $(1 - C)^{-1}$, which appears in the denominator of $G(X)$, and increases risk aversion, also appears in the denominator of $\beta_1$. There it increases the effective discounting of future cash flows that is captured by $\beta_1$ and hence decreases $V_t$.

### 2.2.3 Robustness.

Equation (4) implicitly sets the outside payoff to be proportional to the fund’s HWM $H_t$. An alternative possibility is to let the outside payoff be proportional to the fund’s current wealth $W_t$,

$$V_t = g V(1, C^{-1} W_t).$$ (16)

Note that the two specifications coincide when $W_t = C H_t$ (i.e., at the termination point), but that under (16) the outside option value is sensitive to variation in the fund’s wealth away from the HWM. The disadvantage of (16) is that it does not permit a closed-form solution to the manager’s problem.

To assess whether this change has a significant impact on optimal risk choice, Appendix A.4 solves the model under (16). Note that if fund liquidation occurs only at the termination point $C$, then this change actually makes no difference because the two specifications become equivalent. However, if there is a positive probability of Poisson liquidation of the fund away from the termination point, then there is a difference between the specifications. Appendix A.4 shows, however, that this difference is small, and the corresponding optimal risk choices are very similar.

### 2.3 Performance fee

Next, I examine the impact of the incentive fee, $k$, on the value function and risk choice of the manager. Differentiating $V$ with respect to $k$ gives

$$V_k(X_t, H_t) = \frac{G_X(1) - 1}{[G_X(1)(1 + k) - 1]^2} H_t G(X_t).$$

Therefore, the sign of the derivative depends on whether $G_X(1) \geq 1$. As noted earlier, $G_X(1)$ acts like a discount rate. When $G_X(1) < 1$, we see that the manager prefers a lower performance fee. As can be seen from (14), this is always the case in Panageas and Westerfield’s (2009) model. As PW explain, the manager in their model prefers to defer payments to the future in order
to maximize the growth rate of the fund, while a higher performance fee represents a drag on the fund’s growth rate. In contrast, when $G_X(1) > 1$, the manager instead prefers a higher performance fee. This holds for the manager represented in (15) whenever $C > 1 - \eta$, as $G_X(1) = \eta(1 - C)^{-1}$. In this case, due to the possibility of performance-based termination, the benefit of earlier payment outweighs the cost imposed by a higher drag on fund growth.

When the manager receives a performance payment of $k$, the value of $X_t$ is reduced by $k/H_t$, and the manager’s utility decreases at the rate $V_X(1)$. The net impact on the manager’s utility is

$$k - \frac{k}{H} V_X(1).$$

Equation (17) shows that the net impact of increasing the performance payment on the manager’s utility is positive if and only if $V_X(1)/H_t < 1$. Substituting in for $V_X(1)$, I find that this condition is equivalent to $G_X(1) > 1$.

The value of $G_X(1)$ can be written as

$$G_X(1) = \eta \frac{1 - D_2}{1 - D_0}.$$ 

Hence, changes in parameters that increase $D_0$ imply that the manager prefers a higher performance fee. These changes also tend to increase the manager’s effective risk aversion because his effective risk aversion is closely tied to $D_0$. For instance, higher values of the termination point $C$ make the manager more risk averse, and also imply a preference for higher performance fees. $G_X(1)$ is also increasing in $\eta$. Recall that this endogenous parameter depends on the parameters $\rho$ (impatience) and $\phi$ (the rate of withdrawals), which control how highly the manager discounts his continuation value. Increases in these parameters make the manager discount the future more strongly, and he therefore prefers a higher performance fee, even at the expense of a higher fund growth rate.

2.3.1 Impact on risk choice. Under the baseline model a change in the performance fee has no impact on the optimal risk choice, as shown in the following lemma.

**Lemma 1.** Under the baseline model

$$\frac{d\pi^*_t}{dk} = 0.$$ 

Under the baseline model, $k$ does not impact any of the constants that enter into $\pi^*_t$, and hence changing it has no effect on risk choice (though as shown above the value function is impacted). The reason is that holding $g$ (the ratio of the value function at the termination point and HWM) fixed, the performance fee has no independent role in the optimal risk-choice trade-off faced by the
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This is no longer true with positive management fees, as shown in Section 3.

3. Management Fees

I now extend the model solution to include management fees. When \( m > 0 \), Equation (6) (the manager’s HJB equation) does not admit an analytical solution. Nevertheless, by using an approximation, I am able to obtain closed-form expressions that extend Proposition 1 to include management fees. These expressions highlight the intuition for how the management fee impacts the manager’s optimal policy. Moreover, I show that they are numerically very similar to the exact solution, and explain why this is the case.

To approximate the management fee, I replace the management fee payout \( mW_t \) with the expression \( mH_t \), where the parameter \( m_H \) is the approximate management fee payout rate. In other words, I set the payout to the manager to be a constant fraction of the HWM rather than fund wealth. Substituting this approximation into Equation (6) and the expression (4) for \( V_t \) gives

\[
0 = - (\rho + \lambda) V_t + m_H H_t + \lambda g V(X_t = 1, H_t) + \sup_{\pi_t} \left\{ V_X X_t \pi_t (\mu - r) + \frac{1}{2} V_{XX} X_t^2 \pi_t^2 \sigma_t^2 \right\} + V_H H_t (r - \phi_t - m) + kdH \epsilon - V_X (1+k) \frac{dH^\epsilon}{H_t} + V_H dH^\epsilon.
\]

This is the only change I make to the model. In particular, I continue to assume that, as under the exact management fee, funds are withdrawn from fund wealth at the rate \( m \), and the HWM is appropriately adjusted downwards at this rate. Note that this means that the payout to the manager and the withdrawals from the fund are in general not the same, unless \( m_H = m = 0 \), in which case (18) reduces back to the baseline model.

Although the exact and approximate management fees differ locally, they both grow at the same rate as fund wealth increases and new high-water marks are reached. Hence, in a “global” sense the approximate management fee payout tracks the exact fee quite well, and leads to an optimal policy that is very close to the one obtained under the exact fee.

One way to reconcile the difference between the money paid to the manager and the money taken out of the fund is to imagine investors having a separate money-market account set aside that they use to make up this difference, either by withdrawing or depositing funds as necessary. Note that whether investors pay the manager more or less than they withdraw depends on the value of \( m_H \).

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13 It may be reasonable instead to think of varying \( g \) with the performance fee. In that case, Proposition 1 would imply that the manager’s risk taking does vary in \( k \). For example, suppose that the value of the outside option (not its ratio with the value function at the HWM) is held fixed as \( k \) is increased. Then \( g \) would vary inversely with the value function at the HWM, and hence risk taking and the value function would respond with opposite signs to changes in \( k \).
In particular, if $m_H \leq m \times C$, then investors always pay the manager less than they withdraw. This case is arguably realistic because in practice management fees are often intended to pay for the fund’s operating expenses (e.g., research), whereas performance fees are meant to serve as compensation.\footnote{The setting in which all of the management fees go to expenses is captured by setting $m > 0$ and $m_H = 0$, in which case the approximation becomes exact.}

The following proposition solves Equation (18) for the value function and optimal risk choice of the manager under the approximate management fee. It represents the solution as an extension of the results in Propositions 1 and 2.

**Proposition 4.** Assume that $\phi + \lambda - r + \phi + m > 0$, and $\beta_1 > 0$. Then the solution $V_t$ to Equation (18) is (again) given by (9) and the expressions in Proposition 1 but with $\eta$ and $D_2$ replaced by

\[
\eta = \frac{\rho + \lambda - r + \phi + m}{\omega + \rho + \lambda - r + \phi + m},
\]

\[
D_2 = \frac{m H \beta_1^{-1} + \lambda g}{\rho + \lambda - r + \phi + m}.
\]

Furthermore, the manager’s optimal risk choice continues to be given by Equation (12) of Proposition 2.

As the proposition shows, much of Proposition 1 carries over unchanged. The main difference is in the expression for $D_2$. In particular, the numerator of $D_2$ expands to incorporate into the risk-shutdown payoff the discounted value of the management fees that would accrue to the manager under risk-shutdown. The denominator, which gives the effective “discount rate” for this stream of payouts, increases by $m$ to account for outflows due to the management fee. Note that the two possible cases for $D_0$ and $D_1$ are also unchanged from Proposition 1. However, which case attains depends on the value of $m_H$, as a higher management fee makes risk-shutdown more attractive.

### 3.1 Impact on risk choice

Proposition 4 also shows that the form of the optimal risk-taking policy given in Proposition 2 carries over unchanged to positive management fees. Thus, the possible optimal risk-taking dynamics are the same as in Corollary 1. Hence, management fees do not change the main qualitative properties of optimal risk-taking dynamics given under the baseline model. Nevertheless, management fees do change the numerical values of the optimal risk-taking coefficients (i.e., $D_0$) and hence do impact risk choice.

Increasing the management fee has two opposing effects on the manager’s problem. The first is a “withdrawal effect;” the payout of the management fee acts like a withdrawal from the fund, reducing the growth rate of assets.
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...under management. The second is a “payout effect;” the manager benefits from receiving the stream of management fees (in contrast with withdrawals). The two effects are directly tied to the two parameters, \( m \) and \( m_H \), respectively, which control the (approximate) management fee.

The expressions given in Proposition 4 show that an increase in \( m \), the rate at which wealth is withdrawn to pay the management fees, acts exactly like an increase in the withdrawal rate \( \phi \). So, from Proposition 3 the manager’s optimal risk choice is increasing in \( m \). In contrast, an increase in \( m_H \), the rate at which management fees are paid to the manager, raises the value of the stream of future management fees. This is captured by an increase in \( D_2 \), causing the manager to decrease risk taking in order to reduce the probability of losing the stream of management fees to termination.15

The net effect of these two forces on risk choice depends on the specific parameter values, as well as \( X_t \). Indeed the sign of the net effect can change with \( X_t \). I show an example of this in the top panel of Figure 5. Figure 5 plots risk choice for three levels of management fees, \( m = m_H = 0 \) (solid line), \( m = m_H = 0.02 \) (dashed line), and \( m = m_H = 0.04 \) (dash-dot line). For the given parameters, risk choice increases in the management fee when the fund is close to the HWM (\( X_t \) near one) but decreases in the management fee when the fund is close to termination (\( X_t \) near \( C \)). The reason for this is that the risk of termination, and hence the risk of losing the stream of management fees, becomes increasingly important as the fund approaches the termination point. Therefore, the “payout effect” dominates near the termination point, whereas the “withdrawal effect” dominates near the HWM.16 The bottom panel of Figure 5 plots the corresponding value functions. Under the given parameters the value function is increasing everywhere in the management fee.

The results in Proposition 3 appear unchanged for \( m > 0, m_H > 0 \), though analytical expressions are now complicated. The main difference is that management fees introduce an additional consideration into the manager’s risk-choice decision: the value of the stream of management fees relative to the total value function at the HWM. A higher relative value of management fees makes avoiding termination, with its consequent loss of management fees, more important, and therefore leads to decreased risk taking.17 This consideration reinforces the results in Proposition 3 for \( C \) and \( g \)–since an increase in the termination point reduces the value function–and therefore

---

15 This is the case as long as the manager is not already engaging in risk shutdown \( \xi \geq D_2 \). If the manager is already engaging in risk shutdown, then there is no risk of termination, and hence a small change in \( m_H \) has no effect on risk taking.

16 This is not an artifact of the management fee approximation. The same result is found for the exact solution to the manager’s problem (found numerically) under a nonzero management fee.

17 The additional term in the analysis relative to Proposition 4 is \( m_H \beta_1 \). Because \( \beta_1 = V(X = 1, H_t = 1) \), this added term captures the value of management fees relative to the total value function at the HWM. An increase in this term (because of a higher \( m_H \) or a decrease in \( V(X = 1) \)) increases \( D_2 \), which in turn raises the manager’s effective risk aversion and reduces his risk taking.
Figure 5
Management fee comparative static
Figure 5 plots a comparison of \( \pi^*(X_t) \) (top panel) and \( V(X_t) \) (bottom panel) for \( m = m_H = 0 \) (solid line), \( m = m_H = 0.02 \) (dashed line), and \( m = m_H = 0.04 \) (dash-dot line). The remaining parameters are \( C = 0.55, g = 0.35, \lambda = 0, \phi = 0.05, \mu = 0.03, \kappa = 0.20, \mu = 0.07, \rho = 0.01, \) and \( \sigma = 0.16. \)

makes management fees relatively more important, whereas an increase in the outside payoff increases the value function, and therefore makes management fees less important. I illustrate this in Figure 6 by plotting risk choice for \( m = m_H = 2\% \) for several values of the termination point \( C. \) It shows that, as in Proposition 3, an increase in \( C \) decreases risk taking everywhere.
Figure 6
Termination point comparative static with \( m = 2\% \)
Figure 6 plots \( \pi^* (X_t) \) for \( C = 0.5 \) (solid line), \( C = 0.55 \) (dashed line), and \( C = 0.6 \) (dash-dot line) for \( m = m_H = 0.02 \), \( g = 0.35 \), \( k = 0 \), \( \phi = 0.03 \), \( \lambda = 0.20 \), \( \mu = 0.07 \), \( r = 0.01 \), and \( \sigma = 0.16 \).

For the outflow rate \( \phi \), the management fee creates a countervailing force relative to Proposition 3. Because outflows decrease the value function, they make management fees relatively more important. This makes the manager more cautious, especially near the termination point, and counters the increase in risk choice highlighted in Proposition 3. The net effect can be complicated. In Figure 7, I plot a comparative static for \( \phi \) with \( m = m_H = 2\% \). The figure shows that under the given parameters an increase in \( \phi \) continues to increase risk taking as in Proposition 3 but decreasingly so as \( X_t \) approaches the termination point.

3.1.1 Interaction with performance fee. With a positive management fee a change in the performance fee does impact optimal risk taking, unless the manager is engaging in risk shutdown (\( g < D_2 \)).

Lemma 2. When \( m_H > 0 \) a change in the performance fee \( k \) has the following impact on the manager’s optimal risk choice:

\[
\text{sign} \left( \frac{d\pi^*_t}{dk} \right) = \text{sign} \left( V_k (1, H) \right) \quad \text{if} \quad g \geq D_2 \\
\frac{d\pi^*_t}{dk} = 0 \quad \text{if} \quad g < D_2.
\]

The response of risk choice to an increase in the performance fee depends on its impact on the manager’s value function. An increase in the value function...
reduces the relative importance of management fees in the manager’s overall utility, and this induces greater risk taking. Although raising risk comes at the cost of an increased probability of losing the management fees to termination, this is justified by the fees’ decreased relative importance. Alternatively, if the manager’s value function decreases, then higher performance fees induce lower risk taking. Finally, if the manager engages in risk shutdown ($g < D_2$), then his optimal policy is (locally) fixed and does not respond to small changes in the performance fee.

3.2 Robustness

One may ask whether the management fee approximation has a large impact on the optimal risk choice results. In this section I show that there is little qualitative or quantitative difference between the optimal policies obtained under the exact and approximate solutions at conventional levels of the management fee.

To evaluate the approximation, I compare the optimal risk-taking policies under the exact management fee payout $m W_t$ (Equation 6) and the approximate payout $m_H H_t$ (Equation 18). Because an analytical solution is unavailable under the exact management fee, I solve the model numerically. Figure 8 presents the comparison of $\pi^*(X_t)$ under the exact fee (solid line) and the approximate fee (dashed line). I set $m = 2\%$ for the exact management fee, and $m = m_H = 2\%$ for the approximate management fee, because this value
Figure 8
π∗(X) under exact and approximate management fees

Figure 8 shows π∗(Xt) when the management fee is proportional to fund wealth, and when the management fee is proportional to the high-water mark, as in the benchmark model. The solid (blue) line shows π∗(Xt) when the management fee is exactly 2% of fund wealth. The dashed (red) line is for m_H = m = 2%. The remaining parameters used to generate the plots are C = 0.6, g = 0.35, φ = 0.03, ρ = 0.03, λ = k = 0.2, μ = 0.07, r = 0.01, and σ = 0.16.

is standard. The other parameters are kept fixed. Alternative values of the management fee result in a very similar comparison.

The figure shows that the two optimal risk-taking policies are very similar, both qualitatively and quantitatively, with the derisking dynamics clearly present in both. The small difference between them is that the slope of π∗(Xt) is a bit flatter under the exact management fee. The reason for this is that the exact management fee reduces the payout stream as fund wealth falls. Hence, at low values of Xt, the "payout effect" is smaller under the exact management fee, and π∗(Xt) resembles the risk-choice policy for an approximate management fee with m_H < 2%. This makes π∗(Xt) higher at low Xt values and results in a flatter plot. In a sense, the exact fee has a more attenuated impact on risk taking than does the approximate fee. Overall though, the difference in risk taking between the two management fees is very small.

There are two main reasons why the management fee approximation works well. First, whereas the approximate and exact fees are locally different, globally they track each other quite closely. This is because as the fund grows and reaches new highs, the approximate and exact management fees increase in tandem, and indeed equal each other whenever the fund is at the HWM. Second, the approximation only affects one part of the management fee, the rate of payment received by the manager, whereas the rate of withdrawals from the fund (m_Wt) is the same under the two specifications.
4. Performance-Based Withdrawals

In the baseline model, the rate of withdrawals from the fund is constant and does not respond to the fund’s performance. It is plausible, however, that an accumulation of negative returns that causes fund wealth to fall far below the HWM would lead to an increased rate of withdrawal from the fund, as investors lose confidence in the manager. In this section I examine the impact that such a pattern in withdrawals has on the optimal risk choice dynamics of the manager.

To that end, let $\phi_1$ and $\phi_2$ be two rates of withdrawal, with $\phi_1 < \phi_2$. I assume that when $X_t \in (\mathcal{X}, 1]$, investor’s rate of withdrawal is $\phi_1$, but that it increases to $\phi_2$ when $X_t \in [C, \mathcal{X}]$. Refining this partition to include more than two rates of withdrawal is straightforward.

Within each rate-of-withdrawal region, the HJB equation and value function have the same form as in Equations (8) and (9). Let $G_i(X_t)$ and $G_j(X_t)$ denote the $G$ function corresponding to $\phi_1$ and $\phi_2$, respectively, and define coefficients $\eta_i$, $D_{0,i}$ and $D_{1,i}$ analagously for each $G_i$ function, $i = 1, 2$. The equation for $\eta$ in Proposition 4 shows that $\phi_1 < \phi_2$ implies that $\eta_1 < \eta_2$. The impact of $\eta$ on the effective risk aversion in Equation (12) for $\pi^*_t$ suggests that an increase in the withdrawal rate should induce greater risk taking by the manager. However, to see if such an effect indeed holds, we must also incorporate the solutions for $D_{0,i}$ and $D_{1,i}$ into the optimal risk-choice. This following Proposition confirms that this is the case.

**Proposition 5.** Let the rate of withdrawals be given by $\phi_1$ for $X_t \in (\mathcal{X}, 1]$ and by $\phi_2$ for $X_t \in [C, \mathcal{X}]$, with $\phi_1 < \phi_2$. Then

$$\lim_{X_t \uparrow \mathcal{X}} \pi^*_t = \frac{1 - \eta_2}{\eta_2} \frac{\eta_1}{1 - \eta_1} < 1.$$  

The Proposition shows that the increase in the rate of withdrawals induces a jump up in the manager’s risk-choice. The extent of this increase depends on the relative magnitudes of $\eta_1$ and $\eta_2$. An increase in outflows raises the rate at which fund size decreases over time. This shortens the effective horizon of the manager by diminishing the importance of continuation values that occur further in the future. Because the desire to attain a high continuation value attenuates the manager’s risk taking, the result is increased risk taking.

In Figure 9 I plot a comparison of $\pi^*(X_t)$ (top panel) and $G(X_t)$ (bottom panel) for the case in which the withdrawal rate increases as $X_t$ falls (solid line) versus the case in which the withdrawal rate is constant (dashed-line). The parameters used imply that $D_{0,i} > 0$, so losses induce the manager to reduce risk taking. Hence, within reach $\phi_i$ region, risk-choice monotonically decreases as $X_t$ decreases. However, the top panel shows that there is a jump up in the manager’s risk taking at the boundary between the regions, where the decrease in $X_t$ triggers the increase in the rate of withdrawal.
Figure 9
Performance-sensitive withdrawals
The top panel of Figure 9 also displays a second interesting difference relative to the constant withdrawal case: the manager chooses a lower level of risk taking in the upper region even though the withdrawal rate there is the same. That is, the potential of a future increase in the withdrawal rate makes the
manager more “risk-averse,” even when the fund is far away from the actual switch in withdrawal rates. This occurs because the manager wants to decrease the chance of entering the lower region. The function $G(X)$ (bottom panel) is lower everywhere for the manager who faces the increase in the withdrawal rate, including at the point (marked by the circle) at which the switch in withdrawals takes place. The same relation holds for the value function $V(X)$.

4.1 Gates and lock-ups

Figure 9 shows that even when the fund is doing well, the risk of an increase in future withdrawals reduces the manager’s willingness to take risk. This increase in the manager’s cautiousness may be suboptimal from investors’ point of view. Moreover, individual investors do not fully internalize the impact of their future withdrawals on the manager’s risk taking and other investors’ welfare. Therefore, there is a negative externality associated with the threat of future withdrawals: each individual investor finds increasing withdrawals optimal when the fund suffers negative returns, but this imposes a negative externality on other investors by changing the ex ante risk-taking incentives of the manager.

This problem provides one reason for “gates”, a provision common in the hedge fund industry which allows the manager to restrict the rate of withdrawals from the fund. In the notation of Proposition 5, gates restrict the value of $\phi_2$, the rate of withdrawals following “bad” performance. A commonly given reason for such a restriction is to limit redemptions from a fund whose holdings are costly to liquidate (see, e.g., Aragon 2007). However, a second reason, illustrated by Proposition 5 and Figure 9, is that by capping $\phi_2$, a gate mitigates the impact of performance-sensitive withdrawals on the manager’s ex ante risk-taking. This second reason can be important even if the fund invests in liquid assets.

A second common type of hedge-fund provision that limits withdrawals is called a lockup. This is a period, usually between one and two years from initial investment in the fund, during which investor withdrawals are prohibited. An important function of a lockup is that it protects the fund from having to sell illiquid assets during the lockup period. A second effect of a lockup is that it increases the manager’s effective risk tolerance (Proposition 2) by giving him temporary immunity from termination (akin to reducing the termination point $C$) (Aragon 2007). This finding that lockups are more common among funds with shorter track records. The model can explain this if managers with short track records have lower outside option values, or investors are quicker to withdraw from newer funds following losses, thereby amplifying the associated externality. Dai and Sundaresan’s (2014) work contains additional discussion of the impact of redemption restrictions on a hedge fund’s investment decision.

18 In addition to preventing a full termination of the fund, the lockup also prevents an increase in withdrawals because individual investors cannot withdraw funds during the lockup period. Hence, like a gate, the lockup mitigates the impact of performance-sensitive withdrawals on the manager’s ex ante risk-taking incentives.
5. Optimal Walk-Away

In this section I introduce the manager’s option to voluntarily leave the fund ("walk away") and receive his outside payoff before he is terminated. I then examine the optimal walk-away decision and its implications for risk choice.

Let $C^*$ be the value of $X_t$ at which it is optimal for the manager to walk away, following the convention that he walks away only when walking away is strictly preferable to staying in the fund. If walk-away is never optimal, let $C^*$ be any value less than $C$. Note that $C^*$ will never be equal to $C$, because at $X_t = C$ the manager is either indifferent to walk-away, or in the case of risk-shutdown, actually better off staying in the fund. Next, let $\pi^*_{\text{walk}}(X_t)$ denote the optimal risk choice of the manager when he has the option to walk away. The following proposition shows how the walk-away option can affect optimal risk choice.

**Lemma 3.** If walk-away is optimal ($C^* > C$), then having the walk-away option increases the manager’s optimal risk choice everywhere, that is, $\pi^*_{\text{walk}}(X_t) > \pi^*(X_t)$ for all $X_t \in [C^*, 1]$.

If walk-away is not optimal, then risk choice is unaffected by the walk-away option. The intuition for Lemma 3 is the following. If walk-away is optimal, then it must increase the manager’s value function at $X_t = C^*$. The impact of this is similar to increasing the outside payoff of the manager. As shown in Proposition 3, this induces the manager to increase risk taking in order to reduce the expected time to payout. Hence, when having the walk-away option strictly increases the payoff to the manager at $X_t = C^*$, then greater risk taking is also induced.

Next, I examine the optimal walk-away decision itself. Proposition 6 below shows that under the benchmark model walking away is actually never optimal for the manager. In fact, the proposition proves a more general result. To state this result, I generalize the outside payoff function, $V(X_t, H_t)$, so that it may depend on both $X_t$ and $H_t$. The outside payoff at termination ($X_t = C$) is then given by $V(C, H_t)$. Proposition 6 shows that walk-away is not optimal for any $V(X_t, H_t)$ that is a linear combination of $H_t$ and fund wealth $W_t$ (which equals $X_t H_t$), except in an uninteresting case in which walk-away is optimal immediately at the HWM.

**Proposition 6.** Suppose the outside payoff function takes the form

\[ V(X_t, H_t) = a H_t + b (X_t - C) H_t, \]

where $a > 0, b > 0$. Then either walk-away is not optimal, or it is optimal immediately at the HWM ($C^* = 1$).

---

19 Any value $0 \leq C^* < C$ will do because in this case termination always occurs prior to walk-away.

20 For the sake of simplicity, I also set this to be the outside payoff in case of Poisson liquidation.
The $C^* = 1$ case is uninteresting because we would never actually observe the manager managing the fund (in other words, the outside payoff violates the manager’s participation constraint). The benchmark model, which has $V_t = gV(X_t = 1, H_t) = g\beta_1 H_t$, corresponds to setting $a = g\beta_1$ and $b = 0$ in the proposition. Hence, we see that under the benchmark model walk-away is never optimal.

The Proposition further encompasses outside payoffs that are proportional to the fund’s current wealth. A plausible scenario is that the outside payoff arises from the manager’s option to “start over” at a new fund, with assets under management a fraction $\alpha < 1$ of his current fund’s (at the time of walk-away). Using $W_t = X_t H_t$, this can be expressed as

$$V(X_t, H_t) = \frac{V(X_t = 1, \alpha X_t H_t)}{\alpha < 1} = \alpha\beta_1 X_t H_t = \frac{\alpha\beta_1 W_t}{\alpha < 1}.$$

Hence, this outside payoff corresponds to setting $a = bC$ and $b = \alpha\beta_1$ in the proposition. It follows that the manager should never walk away to restart at a proportionally smaller fund.²¹

Note, however, that even though Proposition 6 applies to a large set of outside payoff scenarios, in general walking away can be optimal for an arbitrary outside payoff function. For example, walking away can be optimal if the outside payoff function has a discontinuous drop at a point $C + \epsilon$ near the termination point $C$,

$$\lim_{X \uparrow C + \epsilon} V(X, H_t) < V(C + \epsilon, H_t).$$

If $\epsilon$ is sufficiently small, then the manager is better off walking away at $X_t = C + \epsilon$ than continuing in the fund, otherwise the discrete drop in payoff is incorporated into his value function.²²

Why is walking away suboptimal? Suppose, for example, that a fund suffers large losses and is far below the HWM. Then the manager expects to wait a while until the fund again reaches the HWM and he receives further performance fees. Could it not be optimal to walk away in this situation?

Consider the case of the benchmark model. Figure 10 helps to explain the answer to this question. Here, I plot the manager’s normalized value function

²¹ If we let $a > 1$, then the manager can walk away to manage a bigger fund. This pathological case corresponds to the situation in which walk-away is optimal immediately at the HWM.

²² Technically, we also need to check that the risk-shutdown payoff is not so large that the manager prefers to shut down risk at $C + \epsilon$. If the manager does not shut down risk, then by continuity a sufficiently small $\epsilon$ implies that the manager’s indirect utility at $C + \epsilon$ is arbitrarily close to $\mathbb{E}(C, H)$. Because this is worse than $\mathbb{E}(C + \epsilon, H)$, it is better for the manager to walk away at $C + \epsilon$. 
Figure 10

The suboptimality of walk-away

Figure 10 plots a comparison of $G(X_t)$ (top panel) and $\pi^*(X_t)$ (bottom panel) for decreasing values of the walk-away point $C_w$. The solid line corresponds to the highest value of $C_w$ and the dash-dot line to the lowest value. The plots show how decreasing $C_w$ increases $G(X_t)$ and $\pi^*(X_t)$ everywhere. The parameters are $C_w = 0.6$ (solid line), $C_w = 0.55$ (dashed line), $C_w = 0.50$ (dash-dot line), and $g = 0.35$, $\rho = 0.03$, $\phi = 0.11$, $\lambda = 0$, $m = m_H = 0$, $\beta = 0.2$, $\mu = 0.07$, $r = 0.01$, and $\sigma = 0.16$.

$G(X_t)$ (top panel) and optimal risk choice (bottom panel) under the benchmark model for different possible values of the walk-away point. Because these values are not the optimal walk-away point, I denote them by $C_w$ rather than $C^*$. Consider the solid line, which corresponds to the highest value of $C_w$.
(i.e., earliest walk away). Because the value function equals the outside payoff at this value of $C_w$ and is downwards sloping, walking away at this point, rather than staying in the fund, may appear best for the manager. This would indeed be true if the manager were required to maintain the same risk-taking policy regardless of his choice of $C_w$. Yet, this is not the case. I show in the bottom panel of Figure 10 that for a lower value of $C_w$ (later walk-away) the optimal risk-taking policy is higher everywhere. In conjunction with the lower walk-away point, this change in policy raises the value function everywhere. Hence, rather than walk away, delaying walk-away and simultaneously increase risk taking everywhere is better for the manager. Because this is always true, walking away is never optimal.

This reasoning suggests that walk-away could become optimal if the manager faces a constraint in taking risk. This turns out to be true, as shown below. When the manager faces binding risk-taking constraints, the option to walk-away can be valuable.

5.1 Risk limits and walk-away

Assume that under the benchmark model the manager also faces the following position limit,

$$\pi_t \leq \pi. \tag{19}$$

Such a limit could arise because of margin constraints, or it may be imposed by outside investors. Following from the proof of Proposition 6, if the constraint does not bind at a given point, then walk-away at that point cannot be optimal. The following proposition shows that if the constraint does bind, walk-away may be optimal.

**Proposition 7.** Let $C_w > C$. If the position limit (19) binds on all of $[C_w, 1]$, that is, $\pi^*(X_t) = \pi \quad \forall X_t \in [C_w, 1]$; and $C_w$ solves Equation (A14) in the Appendix, then $C_w$ is the optimal walk-away point (i.e., $C^* = C_w$).

The proof is provided in the Appendix. It solves for the condition under which $C_w$ is an optimal walk-away point for the case in which the constraint (19) binds on all of $[C_w, 1]$. This case can arise for a manager who, in the absence of the position limit, would follow derisking dynamics (i.e., $D_0 > 0$). For such a manager, if the constraint binds at $C_w$, then it also binds for all $X_t > C_w$. Walking away can be valuable in this case because the constraint limits the manager’s ability to increase his value function by taking risk. Walking away can then be optimal following a series of negative returns.

Consistent with this intuition, the following lemma shows that a higher outside payoff induces earlier walk-away.

**Lemma 4.** The optimal walk-away point in Proposition 7 is increasing in the outside payoff,

$$\frac{dC^*}{dg} > 0.$$
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A higher outside payoff results in earlier optimal walk-away because it raises the opportunity cost of staying in the fund for the constrained manager. A second case in which a constrained manager can find walk-away to be optimal is if he follows gambling dynamics. This case is left to the Internet Appendix.

6. Conclusion

High-water mark contracts are the predominant compensation structure for managers in the hedge fund industry. This paper seeks to understand the optimal dynamic risk-taking strategy of a hedge fund manager who is compensated under such a contract. This is both an interesting portfolio-choice question, and one with potentially important ramifications for the willingness of hedge funds to bear risk in their role as arbitrageurs and liquidity providers, especially in times of crises. High-water mark mechanisms are also implicit in other types of compensation structures, so insights from this question extend beyond hedge funds. An example is a corporate manager whose is paid performance bonuses based on record earnings or stock price and whose choice of projects influences the firm’s level of risk.

This paper provides a closed-form solution to this problem. The solution shows that the optimal risk choice is a function of the ratio of the fund’s assets under management to its high-water mark. Even though the manager is risk neutral, the incentives induce him to act as if he is risk-averse, with his effective risk-aversion a function of the ratio of the fund’s value to its HWM. The resulting optimal-risk taking dynamics take one of two possible forms: “derisking,” whereby the manager continuously reduces risk as the fund falls further below its HWM, or “gambling,” where the manager instead increases risk as the fund falls further below its HWM. Which optimal dynamic arises depends on important parameters of the manager’s environment, including investors’ termination policy, the manager’s outside option, the Sharpe ratio of the manager’s risky strategy, investors rate of withdrawals, and the management fee. In either case the dynamic is highly nonlinear, so that risk-taking becomes much more sensitive following a large negative return. This result may help explain patterns of derisking displayed by hedge funds in the “quant crisis” in 2007 and at the onset of the financial crisis in 2008.

The paper further examines the manager’s optimal walk-away decision and its impact on risk taking. I show that if the manager finds it optimal at some point to exercise his walk-away option, then having the right to do so increases his risk taking. However, for a class of reasonable outside option assumptions, including outside options that are linear in the fund’s size, walk-away is actually never optimal and hence risk-choice is unaffected. This result can be reversed if there are binding limits on risk taking, which can make walk-away optimal.

Though the HWM contract is very prevalent, whether it is optimal is an open question. Even though this is not the question addressed by this paper, it is worth mentioning several characteristics of the HWM which seem quite
reasonable. First, the inclusion of a performance fee may be important in inducing the manager to share proprietary knowledge (or make a high “effort”), because it ties his compensation closely to his performance. In contrast, a pure management-fee compensation is not very sensitive to the fund’s performance. Second, the HWM, together with an indefinite horizon, produces concavity in the manager’s value function. As Panageas and Westerfield (2009) first showed, this mitigates the manager’s incentive to risk shift, so that even a risk-neutral investor avoids unlimited risk taking.

Third, the HWM implies that the investor can never pay more performance fees than the maximum cumulative gain of the fund, a possibility under alternative performance fee arrangements. For example, suppose the manager is instead paid a tiny fraction of every positive return made by the fund. Then no matter how small the fraction, the investor would soon have nothing left, since the fund return process has unbounded positive variation. Finally, under a HWM contract investors only pay the manager performance fees when they are doing particularly “well” (i.e., increasing cumulative gains) and do not pay at all when realized returns are negative. In this sense the fund’s performance hedges the payment to the manager, making the payment smaller on a marginal-utility-weighted basis.

Appendix

A. Derivations and Proofs

A.1 Proof of Propositions 1, 2, and 4

I prove Proposition 4 because Propositions 1 and 2 are the special cases in which the management fee (approximation) is zero, \( m_H = m = 0 \).

The top line of (18) holds throughout, whereas the second line applies only at \( X = 1 \). The solution for \( V_t \) must satisfy both equations separately. Consider first the second line. Substituting in the conjecture (9) shows that at \( X = 1 \) the following condition must hold:

\[ k - \beta_1 G_X(1)(1+k) + \beta_1 G(1) = 0. \]

Solving for \( \beta_1 \) and applying the normalization \( G(1) = 1 \) gives

\[ \beta_1 = \frac{k}{G_X(1)(1+k) - 1}. \] (A1)

Because \( V_i > 0 \), as is clear from (9), and because \( G(X_t) > 0 \), we must have \( \beta_1 > 0 \). This ensures that \( V_i \) is finite. A zero or negative value implies that \( V_i \) is infinite as \( \beta_1 \) represents the value of an infinite sum.

The first line of (18) must hold for all \( X_t \). Substituting the conjecture (9) for \( V_t \) into this ordinary differential equation shows that it has two parts. The first part consists of constants, and the other part is terms in \( X_t \). Because the terms must sum to zero for all \( X_t \), so must each individual part. Setting the sum of the constants to zero gives

\[ D_2 = \frac{m_H \beta_1^{-1} + \lambda g}{\rho + \lambda - r + \phi + m}. \]

By assumption the denominator of \( D_2, \rho + \lambda - r + \phi + m, \) is positive, and therefore so is \( D_2 \). This is required as \( D_2 \) represents the value of an infinite sum of positive terms and would otherwise be...
infinite. Collecting the \( X_t \) terms, factoring out \( \left( \frac{X_t - D_0}{D_1} \right) \eta \), and setting the sum of the terms to zero shows that \( \eta \) is given by
\[
\eta = \frac{\rho + \lambda - r + \phi + m}{\omega + \rho + \lambda - r + \phi + m}.
\] (A2)

Because \( \mu - r > 0 \), then \( \omega > 0 \) and hence \( 0 < \eta < 1 \). Therefore, \( V_{XX} < 0 \), verifying the assumed concavity of \( V \).

We must also impose the boundary conditions, which give the values for \( D_0 \) and \( D_1 \). The first boundary condition, which is at the HWM, is
\[
G(1) = 1.
\] (A3)

There are two possible cases for the second boundary condition, depending on whether the optimal policy dictates that the fund manager continue to take risk at the termination boundary (case 1) or whether he reduces risk taking to zero at this point (case 2).

Case 1: If the manager continues to take risk at the termination boundary \( (X_t = C) \), then the diffusive component of \( dX_t \) is nonzero (Equation 3) and the fund may cross the termination point \( C \). The second boundary condition is therefore \( V(C, H_t) = V_t = gV(X_t = 1, H_t) \). Substituting into this expression gives
\[
G(C) = g.
\] (A4)

Substituting the conjecture (10) for \( G(X) \) into (A3) and (A4) and solving for the two unknown coefficients \( D_0 \) and \( D_1 \) gives
\[
D_0 = C - \frac{1 - C}{(1 - D_2)^{1/\eta} - (g - D_2)^{1/\eta}} (g - D_2)^{1/\eta}
\]
\[
D_1 = \frac{(1 - C)}{(1 - D_2)^{1/\eta} - (g - D_2)^{1/\eta}}.
\] (A5)

Case 2: Alternatively, if the optimal policy for the manager dictates reducing risk taking to zero at \( X_t = C \) and avoiding termination, then we have the following alternative lower boundary condition
\[
\pi^*_t(C) = 0.
\] (A6)

Equation (A5) replaces (A4) because the manager avoids termination, and therefore the outside payoff is never realized. Equation (A6) gives \( \pi^*_t \), which is obtained by substituting \( V(X_t) \) into (4). Inspection of (A6) shows that it implies
\[
D_0 = C.
\] (A7)

From (A6) it then follows that
\[
D_1 = \frac{(1 - C)}{(1 - D_2)^{1/\eta}}.
\] (A8)

Note that these values for \( D_0 \) and \( D_1 \) imply that \( G(C) = D_2 \) in this case. It follows from this that \( \rho_t, H_t, D_2 \) is in fact the value of the risk-shutdown payoff. Note, furthermore, that the values of \( D_0 \) and \( D_1 \) for case 2 can be obtained as a special version of case 1 by setting the outside payoff value \( g \) equal to \( D_2 \).

To determine which case corresponds to the solution, compare \( g \) and \( D_2 \). If \( g \geq D_2 \), then \( V(C) \) for case 1 is greater than for case 2, and therefore \( V(X_t) \) under case 1 is the value function. Alternatively, if \( g < D_2 \), then the opposite holds true, and \( V(X_t) \) determined under case 2 is the value function. As an aside, note also that one could express the two boundary conditions more succinctly with the following single condition
\[
G(C) = \max(g, D_2).
\]

These steps confirm that \( V_t \) given by (6) satisfies (A5) and the appropriate boundary conditions. It follows from a standard martingale verification argument that \( V_t \) is indeed the value function
and that the optimal policy for \( \pi \) is given by (7) (see, e.g., Browne 1997; Oksendal 2003; Panageas and Westerfield 2009).

Finally, substituting Equations (9) and then (10) into Equation (7), and simplifying, gives Equation (12).

A.2 Proof of Proposition 3

Comparative static for \( C \).

Note that the term

\[
\frac{(g - D_2)^{1/\eta}}{(1 - D_2)^{1/\eta} - (g - D_2)^{1/\eta}},
\]

which appears in the expression for \( D_0 \) is positive because \( 0 < g - D_2 < 1 - D_2 \) and \( \eta > 0 \). The sum of the two terms in \( C \) in the expression for \( D_0 \) is positive, and hence

\[
\frac{dD_0}{dC} > 0.
\]

From Equation (12) for \( \pi^* \)

\[
\frac{d\pi^*_t}{dD_0} - \frac{1}{X_t} \frac{1}{1 - \eta} \frac{\mu - r}{\sigma^*} = \frac{d\pi^*_t}{dD_0} < 0,
\]

and putting these together shows that

\[
\frac{d\pi^*_t}{dC} - \frac{d\pi^*_t}{dD_0} \frac{dD_0}{dC} \propto -\frac{1}{X_t} < 0.
\]

Note, moreover, that from (A8) and \( 1 - C > 0 \) we get that \( D_0 \leq C \), while \( D_0 = C \) when \( g = D_2 \).

Comparative static for \( g \). Note that

\[
g - D_2 = \frac{\rho - r + \phi}{\rho + \lambda - r + \phi} g.
\]

The assumption \( g > D_2 \) implies that \( \rho - r + \phi > 0 \), and hence \( g - D_2 \) is increasing in \( g \). Next, note that

\[
\frac{g - D_2}{1 - D_2} = \frac{(\rho - r + \phi)g}{\rho + \lambda - r + \phi - \lambda g}
\]

which is also increasing in \( g \). Taken together, these two observations show that the second term in \( D_0 \) is increasing in \( g \), and because this term is subtracted, we have

\[
\frac{dD_0}{dg} < 0.
\]

Putting this together with (A9),

\[
\frac{d\pi^*_t}{dg} = \frac{d\pi^*_t}{dD_0} \frac{dD_0}{dg} \propto -\frac{1}{X_t} > 0.
\]

Comparative static for \( \omega \). A change in \( \omega \) has both a direct effect on \( \pi^*_t \) via the term \( (\mu - r)/\sigma \) and an indirect via a change in the value of \( \eta \). To see the net effect, note that

\[
\frac{1}{1 - \eta} \frac{\mu - r}{\sigma^*} = \frac{\sqrt{2}}{\sigma} \left( \sqrt{\frac{\rho + \lambda - r + \phi}{\omega}} \right).
\]
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Hence,

$$\pi^*_t = \frac{\sqrt{2}}{\sigma} \left( \sqrt{\omega} \right) \left( \frac{\rho + \lambda - r + \phi}{\omega} \right) \frac{X_t - D_0}{X_t} \quad (A10)$$

Above we show that $D_0 \leq C$, whereas $X_t \geq C$. Hence, the term $(X_t - D_0)/X_t$ is bounded from below by 0 and from above by 1. On the other hand, the first term becomes unboundedly large as $\omega \to \infty$ and as $\omega \to 0$. Therefore,

$$\lim_{\omega \to 0} \pi^*_t = \infty$$
$$\lim_{\omega \to \infty} \pi^*_t$$

Taking the derivative of this first term with respect to $\omega$ gives

$$-\frac{1}{\sqrt{2}\sigma} \left( \frac{\rho + \lambda - r - \omega}{\omega^{1/2}} \right) = \begin{cases} < 0 & \text{if } \omega < \rho + \lambda + \phi - r \\ > 0 & \text{if } \omega > \rho + \lambda + \phi - r. \end{cases}$$

Next, note that

$$\frac{d}{d\omega} \left( \frac{X_t - D_0}{X_t} \right) = -\frac{1}{X_t} \frac{dD_0}{d\eta} \frac{d\eta}{d\omega},$$

and by inspection $\frac{d\eta}{d\omega} < 0$. Moreover,

$$\text{sgn} \left( \frac{dD_0}{d\eta} \right) = \text{sgn} \left[ \frac{d}{d\eta} \left( \frac{1 - D_2}{g - D_2} \right) \right] < 0$$

because $1 - D_2 > g - D_2 > 0$. Therefore,

$$\frac{d}{d\omega} \left( \frac{X_t - D_0}{X_t} \right) < 0.$$

Therefore, $\frac{d\pi^*_t}{d\omega} < 0$ for “small” $\omega$, including for all $\omega < \rho + \lambda + \phi - r$.

Comparative static for $\phi$. The impact of $\phi$ on $\pi^*_t$ works through the change in $\eta$ and the change in $D_0$,

$$\frac{d\pi^*_t}{d\phi} \propto \frac{1}{(1 - \eta)^2} \frac{d\eta}{d\phi} \left( \frac{X_t - D_0}{X_t} \right) = \frac{1}{1 - \eta} \frac{1}{X_t} \left( \frac{dD_0}{d\eta} \frac{d\eta}{d\phi} + \frac{dD_0}{dD_2} \frac{d\eta}{d\phi} \right).$$

It is easy to see that $\frac{d\eta}{d\phi} > 0$. Also, as shown above, $\frac{dD_0}{d\eta} < 0$. Finally, a straightforward differentiation shows that $\frac{dD_0}{d\eta} > 0$ and $\frac{d\eta}{d\phi} < 0$. Hence, all the terms complement each other, so that $\frac{d\pi^*_t}{d\phi} > 0$.

A.3 Proof of Lemmas 1 and 2

The value of $k$ affects $\pi^*_t$ indirectly by changing the value of $D_2$, which in turn can affect the value of $D_0$. This occurs as $k$ is a determinant of $\beta_1$ (Equation 11), which is an input into $D_2$. However, when $m_H = 0$, then $\beta_1$ drops out of $D_2$, and hence $\pi^*_t$ no longer depends on the value of $k$. This proves Lemma 1.

When $g < D_2$ (risk-shutdown), then $D_1$ does not depend on the value of $D_2$. Hence, in this case $\pi^*_t$ does not vary (locally) with $k$, proving the $g < D_2$ case of Lemma 2.

To prove the $g \geq D_2$ case of Lemma 2, note that when $m_H > 0$ and $g \geq D_2$, then $\text{sign}(V_t(1, H_t)) = \text{sign}(dD_1/dk)$. Following from the expression for $D_2$, $\text{sign}(dD_2/dk) = -\text{sign}(dD_1/dk)$. Moreover, as shown in the proof of Proposition 3, $\text{sign}(d\pi^*_t/dk) = -\text{sign}(dD_2/dk)$. Putting these together gives the result.
A.4 Risk Choice with a wealth-proportional outside payoff

I solve the model with an outside payoff \( V_t \) that is directly proportional to fund wealth and compare it to the baseline model, where the outside payoff is proportional to \( H_t \). Note that the two specifications only differ if there is liquidation of the fund away from the termination point. Otherwise, termination only occurs at \( X_t = C \), and the two specifications are equivalent because the baseline model’s outside payoff of \( gV(X_t = 1, H_t) \) can be rewritten as \( gV(X_t = 1, C^{-1}W_t) \). Hence, unless the Poisson liquidation intensity \( \lambda \) is positive, the two specifications are equivalent.

I solve the model with the outside payoff given by

\[
V_t = \left( g + g_1 C^{-1}(X_t - C) \right) \beta_1 H_t,
\]

where \( g_1 \) is an additional parameter. This is a generalized expression that captures both the baseline model’s outside payoff and a wealth-proportional payoff. When \( g_1 > 0 \), the outside payoff is increasing in fund wealth \( W_t \). If \( g_1 = g \), then \( V_t = g C^{-1} \beta_1 W_t \) (which equals \( gV(X_t = 1, C^{-1}W_t) \) by the homogeneity of \( V \) in its second argument), and the outside payoff is directly proportional to fund wealth. When \( g_1 = 0 \), the outside payoff is the same as in the baseline model. Under any case, the value at the termination boundary \( C \) is \( g \beta_1 H_t \), the same as under the baseline model.

Figure A1 plots a comparison of \( \pi^*(X_t) \) for the model when the outside payoff is directly proportional to fund wealth (solid blue line), and under the baseline model (dashed red line). I set \( \lambda = 0.05 \), so that there is Poisson-based liquidation away from the termination boundary. The plot shows that optimal risk choice \( \pi^*(X_t) \) is qualitatively and quantitatively similar across

\[
\pi^*(X) \text{ vs. } X
\]

Figure A1

\( \pi^*(X_t) \) under an outside payoff proportional to fund wealth and to HWM

Figure A1 shows \( \pi^*(X_t) \) (left panel) when the outside payoff \( V_t \) is proportional to fund wealth and when it is proportional to the high-water mark. The solid (blue) line corresponds to an outside payoff \( V_t = gV(X_t = 1, C^{-1}W_t) \), whereas the dashed (red) line is for the baseline model \( V_t = gV(X_t = 1, H_t) \). The parameters used to generate the plots are: \( g = 0.35, C = 0.6, \phi = 0.03, \mu = 0.03, \lambda = 0.05, \xi = 0.2, \mu = 0.07, \tau = 0.01, \) and \( \sigma = 0.16 \).
Within each rate-of-withdrawal region, the HJB equation and value function are the same as in Equations (8) and (9). I therefore divide the solution for $G_i(X)$ into two pieces, $G_1(X)$ and $G_2(X)$, corresponding to the two outflow rates, $\phi_1$ and $\phi_2$, on their respective regions. Each solution for $G_i(X)$ maintains the same form as in (10). As a result, we now have four boundary conditions. Two are as before: $G_2(C)=g_0$ and $G_1(1)=1$. The two additional boundary conditions are (1) $G_1(X)=G_2(X)$ and (2) $\frac{dG_1(X)}{dX}=\frac{dG_2(X)}{dX}$, which match the value and first derivative, respectively, of the value function at the boundary of the two rate-of-withdrawal regions. I assume for simplicity that $m=0$ and $\lambda=0$, so that $D_{2,1}=0$. Incorporating $D_{2,1} > 0$ is straightforward, though cumbersome.

Solving for the $D_{0,1}$ and $D_{1,1}$ in terms of the values of $G_1$ and $G_2$ at the boundary point $X^*$ gives

$$D_{0,1}=\frac{X-G_1(X)}{1-G_1(X)}$$

$$D_{1,1}=\frac{1-X}{1-G_1(X)}$$

$$D_{0,2}=\frac{C G_2(X) \eta_1 - X g_0}{G_2(X) \eta_1 - g_0}$$

$$D_{1,2}=\frac{X-C}{G_2(X) \eta_1 - g_0}. $$

By the value-matching condition, $G_1(X)=G_2(X)$. Denote this common value by $G$. Substituting the value-matching condition into the derivative-matching condition and simplifying gives the following result:

$$\frac{X-D_{0,1}}{X-D_{0,2}} = \frac{\eta_1}{\eta_2}. $$

(A11)

Substituting Equation (A11) into $\lim_{X \to \lambda} \pi_t^*/\pi_t^* \bigg|_{X=\lambda}$ then gives

$$\lim_{X \to \lambda} \frac{\pi_t^*}{\pi_t^*} = \frac{(1-\eta_2)(X-D_{0,1})}{(1-\eta_1)(X-D_{0,2})}$$

$$= \frac{1-\eta_2}{\eta_2} - \frac{1-\eta_1}{\eta_1} < 1. $$

where the inequality follows from the fact that $\phi_1 < \phi_2$ implies that $\eta_1 < \eta_2$.

Finally, to solve for the value of $G$ substitute the solutions for the $D_{0,1}$ into Equation (A11) and rearrange to get the following equation solved by $G$:

$$\left(\frac{1}{X} - 1\right) \frac{\eta_2}{\eta_1} - \frac{\eta_1}{\eta_1} \left(\frac{g_0}{\eta_1} \frac{1}{X} - \frac{1}{\eta_1}\right) = 0.$$
The solution to this equation is 

\[ V(X_t, H_t) = \pi(X_t) \] for all \( X_t \in [C^*, 1] \).

A.7 Proof of Proposition 6

Assume, by way of contradiction, that \( C^* \in (C, 1) \). Then the smooth-pasting condition must hold at \( C^* \). Substituting this into the HJB equation and subsequently check whether the assumption of a binding constraint holds. The conjectured solution for \( V(X_t, H_t) \) remains \( \pi(X_t) \), as does the expression \( \beta \) for \( \beta \).

However, because the constraint on risk choice is binding, the equation satisfied by \( G(X_t) \) is now given by

\[ 0 = -\rho X_t + X_t \pi(X_t) + \frac{1}{2} \sigma^2 X_t^2, \]

The solution to this equation is

\[ G(X_t) = A_1 X_t^{\gamma_1} + A_2 X_t^{\gamma_2}, \]

where \( \gamma_1, \gamma_2 \) solve the quadratic equation

\[ \frac{1}{2} \gamma_1^2 \sigma^2 + \gamma_1 - \frac{1}{2} \sigma^2 + \pi(\mu - r) - (\rho - r + \phi) = 0. \]

Let \( \gamma_1 \) denote the negative root and \( \gamma_2 \) the positive root. The boundary conditions, \( G(1) = 1 \) and \( G(C_{\infty}) = g_0 \), give the following solutions for the \( A_i \):

\[ A_1 = \frac{C_{\infty}^\gamma - C^*^\gamma}{C_{\infty}^\gamma - C^*^\gamma}, \quad A_2 = 1 - A_1. \]

For \( C^* \) to be the optimal walk-away point, it must satisfy the smooth-pasting condition

\[ G(X^*_w) = 0. \]

Straightforward but tedious substitution of \( G \) and the \( A \) coefficients into this condition gives the following equation,

\[ \frac{\gamma_2 C^*_w - \gamma_1 C^*_{\infty} - \gamma_1 - \gamma_2}{\gamma_2 - \gamma_1} = \frac{1}{\beta}, \]

that determines the unique value of \( C^*_w \). Walk-away is optimal only if \( C^*_w \geq C \), because otherwise termination occurs at \( X_t = C \) and the conjecture does not represent the solution to the value function.
Finally, we need to check that the position limit \[ C^* \] indeed binds on \([C^*, 1]\). We can do this by verifying that the Lagrange multiplier on the constraint \( C^* \) in the manager’s problem, given by \( V_X(\mu - r) + \pi X^2 - V_{XX} \), is positive.

The Internet Appendix provides an example of a case in which walk-away is optimal for a constrained manager.

A.8.1 Proof of Lemma 4. It follows from \( \gamma_1 < 0, \gamma_2 > 0 \) and \( C^* < 1 \) that the derivative of the left-hand term in Equation (A14) is negative. Hence, this left-hand term is decreasing in \( C^* \). Because the right-hand term of (A14) is decreasing in \( g, \frac{dC^*}{dg} > 0 \).

References


