The Role of Payout Horizon in Determining the Risks and Returns of Claims to Aggregate Cash Flows

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Abstract

This paper examines the implications of payout horizon for the prices of aggregate cash-flows. The interaction of two long-run forces — a long-run risk in consumption and aggregate dividends, and a cointegration relationship between consumption and aggregate dividends — leads to non-monotonic relationships between a payout’s horizon and its cash flow risks, discount rate risks, and risk premia. These relationships with payout horizon are presented as term structures of risk sensitivities and return premia for so-called zero-coupon equity strips. Analytic expressions are derived for these term structures and examined to see how the interaction of long-run risk and cointegration determine risk and expected return. It is found that differences in payout horizon can result in significant differences in mean returns, with long-run strips earning the lowest risk premia and ‘intermediate’ term strips earning the highest premia. Based on this result, the paper then considers the possibility that differences in firms’ payout horizon can account for a value premium within this long-run risks framework. Significant differences in mean returns can arise. However, when payout horizon is the sole difference between firms, the model is unable to account for the failure of the CAPM when calibrated to match first and second moments of aggregate consumption, dividends, and the market.

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1 Introduction

The long-run risks literature posits that a small but persistent, predictable component in dividend growth imbues long horizon cash flows with substantial risk. With increasing payout horizon comes increased risk. In Bansal and Yaron (2004), this long-run component in consumption and dividend growth is the key to accounting for the high equity premium. Yet, there is another potentially important long-run effect to take into account — a cointegration relationship between consumption and dividends. Bansal and Yaron (2006) provide evidence that aggregate dividends share of aggregate consumption is covariance stationary, leading them to impose (unit) cointegration between aggregate dividends and consumption in their model. They show that at long horizons, the inclusion of this relationship leads to important differences in their empirical results, and they also note some theoretical implications of this restriction for their model.

The cointegration restriction has important economic content. If consumption and aggregate dividends are (unit) cointegrated and aggregate dividends' share of consumption is covariance stationary, then at long horizons their respective growth rates will converge. This means that at sufficiently long horizons, the aggregate consumption and dividend payout should carry a similar risk. Yet in the short to intermediate run, dividends are riskier than consumption. The combination of short and long run effects implies that there is no longer a monotonic relation between dividend payout horizon and risk. This corresponds to the intuition that — though investors may find aggregate dividend growth to be quite uncertain at short to intermediate horizons — in the long-run, aggregate uncertainty stems mostly from the trend in aggregate consumption. The resulting differences in asset prices can be significant. As Bansal and Yaron (2006) note, the addition of the cointegration restriction greatly reduces the volatility of the market return inside their model, leading them to make changes to their model's parameters from the ones used in Bansal and Yaron (2004).

This paper examines in detail the relationship between a payout claim’s horizon and its risk and return. It turns out that the relationship between an aggregate dividend claim’s payout horizon and its growth risks, volatility risks, risk premia, and CAPM betas is no longer monotonically increasing. Instead, the general pattern is that risk and expected return rise sharply in the short to intermediate term and then decrease towards an asymptotic long-run level. The result is that payout horizon is of first-order importance for the pricing of claims that are closely tied to the aggregate dividend stream. A model of firms as such claims is discussed in the paper.

The analysis in this paper is closely related to the strand of literature dealing with the asset pricing implications of cash flow duration. I most closely follow the analysis of Lettau and Wachter (2006), who develop a model for the value premium based on differences in firms’ cash flow timing. Since the methods and results in this paper are in some ways comparable to Lettau and Wachter
(2006), a short description of their paper is helpful. In the Lettau Wachter model, firms are portfolios of claims to fractions of future aggregate dividends. A result of their model is that firms that are overweight in long horizon cash flows endogenously receive low risk premia and high valuation ratios (growth firms), while firms with cash flows weighted toward the short run endogenously receive high risk premia and low valuation ratios (value firms). Since payout horizon is the determining factor in their firms’ risks, Lettau and Wachter find it natural to start their analysis by deriving the price of a claim to the aggregate dividend occurring at a fixed time in the future. Such a claim is referred to as a ‘zero coupon equity strip’ with the given maturity. In Lettau and Wachter’s model, long horizon zero-coupon strips are less risky than short and intermediate term strips. This results mainly from two properties of the model. The first is a large negative correlation between the model’s (immediate) dividend growth shock and the innovation to long term dividend growth. As a result, sensitivity to long term growth acts as a hedge to short-run dividend shocks. The hedge increases with horizon, so long-term strips are the most hedged. The second important property of the model is that discount rate innovations have a zero price of risk. This is important since long maturity strips are generally more sensitive to discount rate changes. Thus, while discount rate innovations are an important source of return volatility for long term strips’, they do not increase the strips’ risk premia. The combined effect of these two properties of the model is that long horizon equity strips carry a lower risk premium, i.e. long-horizon equity is less risky.

This paper is also part of a recent group of papers that use a long-run risks framework to account for cross-sectional differences in returns (such as the value premium) while simultaneously matching aggregate time-series moments. Kiku (2006) builds on the empirical work of Bansal, Dittmar, and Lundblad (2005) and focuses directly on the per-share dividend growth processes of the extreme value, extreme growth, and market portfolios. Similar to Bansal Dittmar and Lundblad’s estimation of cash flow betas, she estimates the different sensitivities of the three dividend processes to the predictable component in consumption growth. She also calibrates the correlations among the dividend innovations and between these innovations and the consumption innovation. Having fully specified the models for these three assets, she solves for the equilibrium asset pricing moments. She then argues that the model moments are consistent with the data, matching the value premium, the equity premium, the risk free rate, and the failure of the CAPM.

Since the model in Kiku (2006) deals with per-share dividend streams and makes no connection to aggregate dividends, there is no cointegration relationship with aggregate consumption. It is not clear in her model that a cointegration restriction should apply or how one would model it. Nevertheless, there is the potential that a cointegration relationship between per-share dividends and consumption would play an important part. This is the point made empirically by Bansal, Dittmar, and Kiku (2006). They argue that empirically there is strong evidence of cointegration between consumption and the per-share dividends of size and book-to-market sorted portfolios.
Moreover, they provide evidence that the cointegration relationship is the dominating factor in determining the cross-section of risk premia in the short-run and that its importance increases further with the investment horizon.

Finally, Croce, Lettau, and Ludvigson (2006) follow Lettau and Wachter (2006) in focusing on payout horizon as the determining factor of risk and risk premia. They also model firms as portfolios of fractional claims to the aggregate dividend stream, with growth firms overweight in long horizon claims and value firms weighted towards short and intermediate claims. The drivers of their model however, are quite different. In their model the rep agent has incomplete information and must solve a filtering problem to estimate the long-run component in consumption and dividend growth. Zero-coupon equity prices, derived using the knowledge of only the filtered consumption and dividend growth processes, are then riskier for short maturities than long maturities, so that a value premium arises.

This paper follows several of the steps outlined from Lettau and Wachter (2006). However, unlike Lettau and Wachter (2006), the model is set within a long-run risks general equilibrium similar to Bansal and Yaron (2004), with the additional restriction that aggregate dividends and consumption are cointegrated, as in Bansal and Yaron (2006). Therefore, before we can price dividend strips in the model, we must first solve for the equilibrium pricing kernel from the assumed preferences and the model’s exogenous processes for consumption growth and volatility. This is done in Section 2. The solution for the pricing kernel pins down the prices of all risks, so that we have less freedom to choose them than do Lettau and Wachter. Having solved for the pricing kernel, the paper then solves for the equilibrium risk-free rate and the price of the aggregate market. It then analyzes how the cointegration restriction’s interaction with long-run risk is fundamental in determining the market’s asset pricing properties.

In Section 3, the paper derives closed-form analytic expressions for the prices of aggregate dividend strips. The analytic expressions allow an in-depth analysis of the term structure of strips’ risks and risk premia and highlight the role of payout horizon. Based on the shape of the term structure of risk premia, I consider modeling firms as portfolios of aggregate dividend strips. Such an approach could produce a premium on firms that are overweight in short and intermediate run strips relative to those that overweight long horizon strips. However Section 4 derives analytical expressions for strips’ CAPM betas and alphas to show that for a wide range of feasible parameterizations of the model, risk premia are well accounted for by the CAPM. Moreover, it shows that even as the balance between cash flow risks and discount rate risks is varied in the model, the CAPM betas of the strips remain closely tied to their risk premia, so that the CAPM continues to hold very well. Section 5 then concludes.
2 The Model

The model (and notation) follows Bansal and Yaron (2004, 2006). As in Bansal and Yaron (BY), we consider a representative agent framework with Epstein-Zin preferences. Thus, the gross return $R_{i,t+1}$ on an asset $i$ must satisfy the asset pricing restriction

$$E_t[\delta^\theta G_{t+1} R_{a,t+1}^{-(1-\theta)} R_{i,t+1}] = 1$$

(1)

where $G_{t+1}$ is the gross growth rate of aggregate consumption and $R_{a,t+1}$ is the unobservable gross return on the asset that is the aggregate consumption claim. The parameter $\delta$ is the time discount factor, $\theta \equiv \frac{1-\gamma}{1-\psi}$, where $\gamma$ is the coefficient of relative risk aversion and $\psi$ is the IES parameter.

We derive approximate analytical expressions of most of the key features of our model. The analytical expressions enable us to pinpoint the factors governing risk sensitivities and associated returns. We begin by approximating returns via the standard Campbell-Shiller (1988) log-linearization. We do this first for the aggregate consumption claim. Let $P_{a,t}$ denote the price of this claim and $C_t$ its payoff. Then

$$R_{a,t+1} = \frac{P_{a,t+1} + C_{t+1}}{P_{a,t}} = G_{t+1} \frac{(P_{a,t+1} + 1)}{C_t}$$

With lowercase variables denoting log quantities, we set $r_{a,t+1} = \log R_{a,t+1}$, $z_t = \log(\frac{P_{a,t+1}}{C_t})$, and $g_t = \log(G_t)$. The Campbell-Shiller log-linearization of $R_{a,t+1}$ is then given by

$$r_{a,t+1} = \kappa_0 + \kappa_1 z_{t+1} - z_t + g_{t+1}$$

(2)

where $\kappa_0$ and $\kappa_1$ are constants that come from the linearization. Thus, conditional on the time $t$ state, the (log) return is a linear combination of the price-consumption ratio and the (log) growth rate in consumption. To ‘solve’ for the return process we specify a process for consumption and solve for the functional form of the price-consumption ratio, $z_t$ as a function of state variables. As in BY, the processes have a convenient conditional log-normal form that, in conjunction with the log-linearity of (2), makes it possible to get analytical solutions. The solutions are only approximate since (2) is an approximation for the true $r_{a,t+1}$.

The exogenous processes of our economy are very similar to those of Bansal and Yaron (2006)

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1. The linearization assumes $z_t$ is a covariance stationary process and ‘linearizes’ around its unconditional mean, $\bar{z}$, giving $\kappa_1 = \frac{\bar{z}}{\bar{z}^2 + 1}$ and $\kappa_0 = \log(\bar{z} + 1) - \kappa_1 \bar{z}$
and are as follows:

\[
\begin{align*}
\Delta c_{t+1} &= g_{t+1} = \mu_c + x_t + \sigma_t \eta_t \\
x_{t+1} &= \rho x_t + \sigma_t \varphi e_{t+1} \\
\sigma^2_{t+1} &= \sigma^2 + \nu(\sigma_t^2 - \sigma^2) + \sigma_w w_{t+1} \\
d_{t+1} - c_{t+1} &= \mu_{d-c} + s_{t+1} \\
s_{t+1} &= \rho_s s_t + \phi_{sx} x_t + \varphi_d \sigma_t u_{t+1}
\end{align*}
\]

and \(v_{t+1} = (\eta_{t+1}, c_{t+1}, w_{t+1}, u_{t+1})' \) iid \( \sim N(0, \Omega) \) with

\[
\Omega = \begin{pmatrix}
1 & \rho_{\eta e} & 0 & \rho_{\eta u} \\
\rho_{\eta e} & 1 & 0 & \rho_{e u} \\
0 & 0 & 1 & 0 \\
\rho_{\eta u} & \rho_{e u} & 0 & 1 \\
\end{pmatrix}
\]

As in BY, we model aggregate consumption and aggregate dividends as separate processes. The difference of the two includes most prominently labor income, but also all other sources of income. Also like BY, the (log) consumption growth process has a small, predictable long-run component (4) that follows an AR(1) process. This component is the key source of ‘long-run’ risks. (6) describes the cointegration relationship between the levels of the aggregate consumption and dividend processes. We defer discussion of this important restriction and of the dividend process until later. The volatility of the time \( t + 1 \) innovations to the log processes is given by \( \sigma_t \), governed by (5). \( \sigma_t \) determines the volatility of all processes’ innovations other than its own. The innovations are potentially correlated, unlike BY(2004) and even BY(2006)\(^2\). The exception is the innovation \( w_{t+1} \) to \( \sigma_{t+1} \), which is uncorrelated with the other three innovations.

### 2.1 The Aggregate Consumption Claim

Given the consumption and volatility processes, we solve for \( z_t \) as in BY. We conjecture that 

\[ z_t = A_0 + A_1 x_t + A_2 \sigma_t^2 \]

and then solve for \( A_0, A_1 \) and \( A_2 \) so that \( r_{a,t+1} \) meets the restriction imposed by the Euler condition (1). To do this it is easiest to consider (1) in log form

\[ E_t[\exp(\theta \ln \delta - \frac{\theta}{\psi} g_{t+1} + (\theta - 1) r_{a,t+1} + r_{i,t+1})] = 1 \]

and then set \( r_{i,t+1} = r_{a,t+1} \). Substituting (2) for \( r_{a,t+1} \) and solving for the \( A \) coefficients, one obtains:

\[ A_1 = \frac{1 - \frac{1}{\varphi}}{1 - \kappa_1 \rho} \]

\(^2\)In BY(2006) only \( \rho_{\eta u} \neq 0 \)
\[ A_2 = (1 - \gamma)(1 - \frac{1}{\psi}) \left[ 1 + 2 \frac{\kappa_1 \varphi_e}{1 - \kappa_1 \rho} \rho_{\eta e} + \left( \frac{\kappa_1 \varphi_e}{1 - \kappa_1 \rho} \right)^2 \right] \frac{2}{2(1 - \kappa_1 \nu)} \]  

(10)

Further details and the expression for \( A_0 \) are given in A.1.

We note that throughout we will follow Bansal and Yaron (2004) and the other papers mentioned above in taking \( \gamma > 1 \) and \( \frac{1}{\psi} < 1 \). This means that \( A_1 > 0 \), so \( z_t \) is increasing in \( x_t \), and \( A_2 < 0 \), so that \( z_t \) is decreasing in \( \sigma_t \).

Having solved for \( z_t \) and thus for \( r_a \), we can solve for the consumption claim’s conditional risk premia, variance and also the risk-free rate, \( r_f \). To that end, consider the log of the pricing kernel, \( m_{t+1} = \ln M_{t+1} \), and the log return on asset \( i \), \( r_{i,t+1} \). It is easy to show that if \( m_{t+1} \) and \( r_{i,t+1} \) are conditionally jointly normal then the conditional premium on asset \( i \) is given by

\[ E_t(r_{i,t+1} - r_{f,t}) + \frac{1}{2} \text{var}_t(r_{i,t+1}) = -\text{cov}_t(m_{t+1}, r_{i,t+1}) \]  

(11)

The term \( \frac{1}{2} \text{var}_t(r_{i,t+1}) \) is a Jensen’s inequality correction, which results from using continuously compounded returns instead of simple returns. We will consider this correction to be part of the conditional risk premium for an asset. From (11) we see that to find the conditional premium on an asset, we need its conditional covariance with \( m_{t+1} \). Since we now have an expression for \( r_{a,t+1} \) in terms of the exogenous processes, we can now easily find \( r_{a,t+1} - E_t(r_{a,t+1}) \), the innovation in \( r_{a,t+1} \), in terms of the exogenous innovations:

\[ r_{a,t+1} - E_t(r_{a,t+1}) = \sigma_t \eta_{t+1} + B \sigma_t \varepsilon_{t+1} + A_2 \kappa_1 \sigma_w \omega_{t+1} \]  

(12)

where \( B = \kappa_1 A_1 \varphi_e \). Also, since \( m_{t+1} = \theta \ln \delta - \frac{\theta}{\psi} g_{t+1} + (\theta - 1)r_{a,t+1} \), we can substitute in our solution for \( r_a \), and obtain an expression for \( m_{t+1} \) in terms of only the exogenous processes and constants. It is then easy to derive an expression for \( m_{t+1} - E_t(m_{t+1}) \), the innovation in the (log) pricing kernel:

\[ m_{t+1} - E_t(m_{t+1}) = \lambda_{m,n} \sigma_t \eta_{t+1} - \lambda_{m,e} \sigma_t \varepsilon_{t+1} - \lambda_{m,w} \sigma_w \omega_{t+1} \]  

(13)

with \( \lambda_{m,n} = \left( -\frac{\theta}{\psi} + (\theta - 1) \right) = -\gamma \), \( \lambda_{m,e} = (1 - \theta)B \), and \( \lambda_{m,w} = (1 - \theta)\kappa_1 A_2 \). To get the conditional premium on \( r_a \), we now set \( r_{i,t+1} = r_{a,t+1} \) in (11) and use (12) and (13) to determine the covariance between the innovations in the pricing kernel and \( r_a \). For simplicity let \( \rho_{\eta e} = 0 \). Then the result is:

\[ E_t(r_{a,t+1} - r_{f,t}) + \frac{1}{2} \text{var}_t(r_{a,t+1}) = -\lambda_{m,n} \sigma_t^2 + \lambda_{m,e} B \sigma_t^2 + \lambda_{m,w} A_2 \kappa_1 \sigma_w^2 \]  

(14)

The full expression is given in A.2.
2.2 The Risk-free Rate

The expressions for the risk-free rate $r_{f,t}$ and its unconditional variance are derived in A.4. There are a few important things to note. Fluctuations in the risk-free rate are driven by two factors; the (long-term) growth component, $x_t$, and the stochastic volatility, $\sigma_t^2$. First, the usual substitution effect implies that the risk-free rate increases with $x_t$. Since we assume Epstein-Zin preferences, the magnitude of the substitution is controlled by $\psi$, the elasticity of intertemporal substitution. As $\psi$ increases, the agent is less averse to intertemporal substitution and so $x_t$ shocks result in smaller fluctuations in the risk-free rate. This is clear from A.4.2. Secondly, variations in volatility also cause variations in the risk-free rate. One reason is the usual precautionary savings motive, which implies that increases in volatility lead to a decrease in the risk-free rate. For Epstein-Zin preferences there is an additional effect. From (14) we can see that the volatility term controls the conditional premium on $r_a$. Since $r_a$ enters in the pricing kernel, there is an additional channel for volatility fluctuations to cause fluctuations in the risk-free rate.

2.3 The Dividend Process

We call the aggregate dividend claim “the market” and denote its time $t$ return by $r_{m,t+1}$. (6) shows that (log) aggregate consumption and (log) aggregate dividends are (unit) cointegrated. Their difference is the covariance stationary process $s_t$ that appears in (7). We are interested in the process implied by (6) and (7) for dividend growth, $\Delta d_{t+1}$. Taking first differences of both sides of (6), we have $\Delta d_{t+1} = \Delta s_{t+1} + \Delta c_{t+1}$. In order to get an expression for $\Delta s_{t+1}$, subtract $s_t$ from both sides of (7). $\Delta c_{t+1}$ is given by (3). Substituting in both of these we get:

$$\Delta d_{t+1} = \mu_c + (1 + \phi_{sx})x_t + (\rho_s - 1)s_t + \sigma_t \eta_{t+1} + \varphi_d \sigma_t u_{t+1} \quad (15)$$

The cointegration relationship between aggregate dividends and consumption is a potentially important, empirically plausible, and intuitive restriction that does not appear in much long-run risks literature. Bansal and Yaron (2006) present evidence that aggregate dividends and consumption are cointegrated and that aggregate dividends’ share of consumption is covariance stationary. The cointegration restriction is imposed here in the same way as Bansal and Yaron (2006). The form of the restriction implies that (log) aggregate dividends and (log) consumption have the same deterministic trend and that (log) aggregate dividends have a unit sensitivity to the stochastic trend in (log) consumption. Thus, aggregate dividends (not in logs) tends towards a fixed proportion of aggregate consumption, with $s_t$, the cointegration residual, determining the deviation from that fixed proportion. The restriction results in the ‘error-correction’ term $(\rho_s - 1)s_t$ in (15), which influences future dividend growth rates. Since $\rho_s < 1$ if $s_t$ is covariance stationary, $(\rho_s - 1) < 0$, and will decrease dividend growth if the dividend-to-consumption ratio
is higher than its mean \((s_t > 0)\), and increase dividend growth if the opposite is true. Moreover, the amount of correction scales linearly with the magnitude of the deviation, \(s_t\), from the mean.

The connection between the dividend-consumption ratio and long-run expected dividend growth is similar to the one in Lettau and Wachter (2006), though it comes about for different reasons. The mechanism is important to both models so a comparison is useful. Lettau and Wachter use a result from Lettau and Ludvigson (2005), which shows that if consumption growth follows a random walk and if the consumption-dividend ratio is stationary, then the predictable component of dividend growth can be identified with the consumption-dividend ratio up to an additive and multiplicative constant. In the data, they find that there is a strong, negative correlation (-0.83) between dividend innovations and innovations to the consumption-dividend ratio. They therefore set this as their model’s correlation between dividend innovations and the innovations to the predictable component of dividend growth. The result is that positive dividend innovations are strongly associated with decreases in long-term dividend growth rates. Although consumption is not modeled, the idea is that a positive dividend innovation is generally not completely offset by the consumption innovation, so that there is a negative innovation to the consumption-dividend ratio. Since the consumption-dividend ratio is assumed to be mean-reverting, a decrease in it points to lower expectations for future dividend growth. In the model of this paper, a positive innovation to \(d_t\) that is not fully offset by the innovation to \(c_t\) results in a positive innovation to \(s_t\), which is essentially the dividend-consumption ratio. As described above, an increase in \(s_t\) decreases \((1 + \phi_{sx})x_t + (\rho_s - 1)s_t\), the predictable component of dividend growth. Note that this is the case even when \(\rho_{uu}\), the correlation between dividend innovations and \(x_t\) innovations, is 0. So while there are similar phenomena in both papers, they arise for very different reasons.

The parameter \(\rho_s\), which we call the ‘persistence’ of the cointegration relationship, determines how tightly the relationship between dividends and consumption is maintained. If \(\rho_s\) is close to 0, the relationship is very tight as \(s_t\) remains close to 0. Thus, average dividend growth will remain close to average consumption growth, even over ‘short’ periods. On the other hand, if \(\rho_s\) is close to 1, average dividend growth may diverge from average consumption growth for even ‘long’ periods. In the limiting case of \(\rho_s = 1\), \(s_t\) has a unit root and dividends are no longer cointegrated with consumption. In this case, \(\Delta d_{t+1} = \mu_c + (1 + \phi_{sx})x_t + \sigma_t \eta_{t+1} + \varphi_d \sigma_t u_{t+1}\), which is very similar to the form of the (per-share) dividend process in Bansal Yaron (2004). In this case, dividends and consumption still have the same deterministic trend. However, they are not cointegrated, so that eventually the dividend-to-consumption ratio will approach 0 or diverge to infinity.

The effects of the cointegration relationship between the dividend and consumption processes will show up prominently in the pricing of dividend claims, discussed below. In the model, the aggregate dividend process is (substantially) more volatile than consumption in the short run, as is the case empirically. This is due to the parameter \(\phi_{sx}\), a leverage parameter controlling the
sensitivity of dividend growth to long-run expected growth movements in consumption, and to $\varphi_d$, which controls dividends’ innovation volatility. Intuitively, this means that short horizon dividend payouts are much more volatile than short-run consumption payouts. In the long-run, however, the cointegration restriction forces dividend growth to converge to consumption growth, so that the two types of payouts have a similar distribution for long enough horizons. At the intermediate horizon, the pattern of dividend payout distributions can be complicated, as we discuss below. The point here is that the cointegration restriction is key to determining the relationship between a cash flow’s horizon and its sensitivity to economic shocks, and thus also the volatility of its distribution.

2.4 The Market Return

We solve for the (approximate) market return process much as we solved for the (approximate) return on the consumption claim. Letting $z_{m,t}$ denote the log price-dividend ratio for the market, we log-linearize the expression for the (log) market return around the unconditional mean of $z_{m,t}$:

$$r_{m,t+1} = \kappa_{0,m} + \kappa_{1,m}z_{m,t+1} - z_{m,t} + \Delta d_{t+1}$$

(16)

where $\kappa_{0,m}$ and $\kappa_{1,m}$ are the constants that come from the linearization. As with $z_t$, we now conjecture and verify an expression for $z_{m,t}$ in terms of the state variables. The combination of log-linearized returns with conditionally, jointly normal exogenous processes means that, as before, $z_{m,t}$ should be an affine combination of the relevant state variables. The difference is that $s_t$ is now included as a state variable since it governs $\Delta d_{t+1}$. Therefore:

$$z_{m,t} = A_{0,d} + A_{1,d}x_t + A_{2,d}\sigma_t^2 + A_{3,d}s_t$$

(17)

Proceeding as in the case of $r_a$, we substitute the expression for $z_{m,t}$ into (16) to get an expression for $r_m$ in terms of the exogenous processes, and then set $r_i = r_m$ in (8). We then solve for the $A$ coefficients so that (8) holds. If we let the correlations $\rho_{\eta e}, \rho_{\eta u}, \rho_{eu}$ be zero for simplicity and omit $A_{0,d}$, we obtain (in the order in which they must be solved):

$$A_{3,d} = \frac{\rho_s - 1}{1 - \kappa_{1,m}\rho_s}$$

(18)

$$A_{1,d} = \frac{1 - \frac{1}{\varphi} + \phi_s (1 - \kappa_{1,m})}{1 - \kappa_{1,m}\rho}$$

(19)

$$A_{2,d} = \frac{(1 - \theta)A_2(1 - \kappa_{1,\nu}) + \frac{1}{2}[H_1 + H_2]}{1 - \kappa_{1,m}\nu}$$

(20)

where $H_1 = (1 - \gamma)^2 + (\varphi_d \frac{1 - \kappa_{1,m}}{1 - \kappa_{1,m}\rho_s})^2$ and $H_2 = \left[\left(\theta - 1\right)\kappa_{1,\nu}A_1 + \kappa_{1,m}A_{1,d}\varphi_e\right]^2$. The full expressions, including $A_{0,d}$, are given in B.1.

$^1$Recall that $\lambda_{m,n} = -\gamma$, $(1 + \lambda_{m,n}) = (1 - \gamma)$
The $A$ coefficients are the elasticities of the market’s price-divided ratio with respect to the three state variables. From these elasticities we see the effects of the cointegration restriction and also its interaction with the long-run risks component. Consider first the ‘new’ $A_{3,d}$ term that shows up in the price-dividend ratio $z_{m,t}$. Since $\rho_s < 1$, we see that it is negative. This means that as $s_t$ increases, ceteris paribus, the valuation ratio is lower. This follows our earlier reasoning, as a large positive deviation from the long term dividend to consumption ratio implies lower future dividend growth (via a large, negative ‘error-correction’ term in dividend growth). This is known to the representative investor and therefore leads to a lower valuation ratio. The term $A_{3,d}$ can be interpreted as the ‘ex-dividend’ value of a one unit increase in dividends — the present value of future decreases in dividends that results from a unit increase in the (log) dividend-consumption differential. With $\kappa_{1,m}$ as the time discount factor, this present value is equal to the infinite sum

$$(\rho_s - 1) + \kappa_{1,m}(\rho_s - 1)\rho_s + \kappa_{1,m}^2(\rho_s - 1)\rho_s^2 + \cdots = (\rho_s - 1) \sum_{n=0}^{\infty} (\kappa_{1,m}\rho_s)^n$$

which is exactly $A_{3,d}$. The ‘cum-dividend’ value of a one unit increase in dividends (with consumption remaining constant) is then $1 + \kappa_{1,m}A_{3,d} = \frac{1}{1 - \kappa_{1,m}\rho_s}$ (the multiplication by $\kappa_{1,m}$ accounts for one period of discounting). This term appears prominently in the expressions for $A_{1,d}$, $A_{2,d}$, and $H_1$. In $A_{1,d}$, it multiplies the leverage parameter $\phi_{sx}$ for $x_t$. An increase of one unit in $x_t$ increases the expectation for dividend growth over consumption growth in period $t + j$ by $\phi_{sx} \times \rho^{j-1}$. This increase has a present value of $\kappa_{1,m}^{-1} \times \frac{1}{1 - \kappa_{1,m}\rho_s} \times \phi_{sx}\rho^{j-1}$. The sum of all these present values is $\frac{\phi_{sx}(1 - \kappa_{1,m}\rho_s)}{1 - \kappa_{1,m}\rho_s}$, which is precisely the term that appears in $A_{1,d}$. The remaining part of $A_{1,d}$: $\frac{1 - \kappa_{1,m}^{-1} - \phi_{sx} \rho^{j-1}}{1 - \kappa_{1,m}\rho_s}$, accounts for the increase in present value due to expected dividend growth that is matched by expected consumption growth.

The analysis of $A_{1,d}$ demonstrates that the cointegration restriction interacts with the long-run risks component $x_t$. As $\rho_s$ is decreased and the cointegration restriction is ‘tightened’, the elasticity of the price-dividend ratio to $x_t$ decreases. Thus cointegration ‘reverses’ some of the risk induced by the $x_t$ component in long-run risk models. Moreover, we can see from the $A$ coefficients that the price-dividend ratio is very sensitive to the degree of cointegration persistence. Consider a typical magnitude for $\frac{1 - \kappa_{1,m}^{-1}}{1 - \kappa_{1,m}\rho_s} = \frac{1}{\kappa_{1,m}\rho_s} - 1$. Usually, $\kappa_{1,m} \approx 0.996$ (at a monthly time interval), which implies that $\frac{1}{\kappa_{1,m}} \approx 1.004$ and so $\frac{1}{\kappa_{1,m}\rho_s} - 1 \approx 0.004(1 - \rho_s)$. Now it can be seen that, even for a very ‘loose’ cointegration restriction (high values of $\rho_s$), an unmatched (by consumption growth) unit increase in dividend growth has a relatively low ‘cum-dividend’ present value. For example, with no cointegration ($\rho_s = 1$), the present value is exactly 1, while with $\rho_s = 0.996$ (a high value), it is approximately 0.5. This directly influences the expression for the market return innovation, $r_{m,t+1} - E_t(\theta_{m,t+1})$, which we can derive by substituting the market’s $A$ coefficients
into (16) to obtain the market return in terms of the state variables. What results is:
\[
  r_{m,t+1} - E_t(r_{m,t+1}) =
  \sigma \eta_{t+1} + \kappa_{1,m} A_{1,d} \varphi_e \sigma_t e_{t+1} + \kappa_{1,m} A_{2,d} \sigma_w w_{t+1} + \left( \kappa_{1,m} A_{3,d} + 1 \right) \varphi_d \sigma_t u_{t+1}
\]

(21)

Thus, the sensitivity of return news to the dividend innovation is given by \(1 - \kappa_{1,m}\). This sensitivity decreases with \(\rho_s\) from its maximum possible value of 1, the case of no cointegration, for the reasons just discussed. Basically, a positive dividend innovation today is not a net benefit, since it means that one can expect lower future dividend growth. The magnitude of the net benefit is precisely captured by the term \(\kappa_{1,m} A_{3,d} + 1\).

For convenience we will use the following ‘beta’ notation to denote the sensitivity of the market return innovation to innovations in the state variables:

\[
  \beta_{m,e} = \kappa_{1,m} A_{1,d} \varphi_e, \quad \beta_{m,w} = \kappa_{1,m} A_{2,d}, \quad \beta_{m,u} = (\kappa_{1,m} A_{3,d} + 1) \varphi_d
\]

in which case we can rewrite (21) as:

\[
  r_{m,t+1} - E_t(r_{m,t+1}) = \sigma \eta_{t+1} + \beta_{m,e} \sigma_t e_{t+1} + \beta_{m,w} \sigma_w w_{t+1} + \beta_{m,u} \sigma_t u_{t+1}
\]

(22)

### 2.4.1 Return Premia and Variance

Since we have the market return innovation we can easily obtain expressions for the conditional equity premium and conditional market variance, much as we did for \(r_a\). From (21) we obtain:

\[
  \text{var}_t(r_{m,t+1}) = \beta_{m,w}^2 \sigma_w^2 + \left\{ 1 + \beta_{m,e}^2 + \beta_{m,u}^2 + 2 \beta_{m,u} \rho_{w,u} + 2 \beta_{m,e} \rho_{e,w} + 2 \beta_{m,e} \beta_{m,u} \rho_{e,u} \right\} \sigma_t^2
\]

(23)

To get the premium we set \(r_i = r_m\) in (11) and again use (13) to get:

\[
  E_t(r_{m,t+1} - r_{f,t}) + 0.5 \text{var}_t(r_{m,t+1}) =
  (\gamma + \lambda_{m,e} \rho_{e,w}) \sigma_t^2 + \beta_{m,e} (\lambda_{m,e} + \gamma \rho_{e,w}) \sigma_t^2 + \beta_{m,w} \lambda_{m,w} \sigma_w^2 + \beta_{m,u} (\gamma \rho_{u} + \lambda_{m,u} \rho_{e,u}) \sigma_t^2
\]

(24)

This expression is written as the sum over the four innovations \((\eta, e, w, u)\), of the market’s exposure to an innovation (its innovation \(\beta\), weighted by the total compensation for exposure to that risk. Note that, if the three correlations were 0, the compensation for \(u\) risk would be 0 and the last term would drop out, as is the case in Bansal and Yaron (2004). It is important to recognize that this has little bearing on the asset pricing importance of the cointegration relationship since it still enters directly and indirectly into the other \(A_d\) coefficients.

Finally, an expression for the unconditional variance of the market return is derived in B.2. The unconditional variance differs from the conditional variance because of variation in the risk-free rate and in the risk premium, \(E_t(r_{m,t+1} - r_{f,t})\). By (23), (24) we can see that there is variation in the risk premium only if there is heteroscedasticity \((\sigma_t^2 \neq \text{constant}, \ i.e. \ \sigma_w^2 \neq 0)\).
2.5 Calibrated Output

We calibrate the model by choosing the parameters of the consumption and dividend processes to match annual moments in the data, as in Bansal and Yaron (2004, 2006) and Kiku (2006). The estimates for consumption and dividend moments we use are taken directly from Bansal and Yaron (2006), Tables 1 and 10. As in Bansal and Yaron (2004, 2006) and Kiku (2006), the model simulation assumes the agent’s decision interval is one month and the model’s output is then aggregated to an annual frequency. The simulation uses the (approximate) analytic expressions derived above to get price-dividend ratios and the risk-free rate from the simulated values of the state variables (the exogenous processes). The discussion will focus on the economy under a parametrization we call parametrization I, though we use other parameterizations for comparison.

Table 1 presents the simulated output for the economy under parametrization I, which is given below the table. The consumption process in this economy is very similar to that of Bansal and Yaron (2004) and Kiku (2006). However, the dividend process is quite different. The specification in (15) of $\Delta d$ is for the growth in aggregate dividends, as is the case in Bansal and Yaron (2006), rather than the usual per-share cash dividends. This is consistent with the approach of Lettau and Wachter (2006), since their firms’ cash flows are modeled as a fraction of aggregate dividends. It also means that our dividend series includes the effects of share repurchases, which add significant volatility to dividend growth. Bansal and Yaron (2006) calculate that the volatility of per-share cash dividend growth is approximately 13%, whereas the volatility of aggregate dividend growth is over 23%. Moreover, aggregate dividend growth includes growth in the scale of the market via share issuance. Valuation of the market requires that issuance be accounted for. We adopt the approach of Bansal and Yaron (2006) and account for growth in scale by subtracting from (15) the average rate of issuance, denoted by $\chi$. Table 1 presents the moments for this ‘adjusted’ dividend process, which differs from the aggregate series only in its mean growth rate.

Table 1 shows that at the calibrated parameters, the model does a good job of matching the first two moments of aggregate consumption and dividends. As was just discussed, an important feature is the high dividend growth volatility, which is actually higher than the market return volatility. This is an atypical problem for an asset pricing model, since models normally focus on per-share cash dividend growth, which is less volatile than the market return (i.e. there is excess volatility). As discussed above, the cointegration restriction implies that return innovations have a fractional (and possibly quite low) sensitivity to dividend innovations. From this point of view, the challenge for the model is to match what appears to be a relatively high return volatility. The table shows that at parametrization I, the model does a fairly good job of matching both the market volatility and the market risk premium. The last two lines of the table give the first two moments of the market’s value-dividend ratio. Notably, the model manages to match the high volatility of the market’s valuation ratio, an aspect not captured well in Bansal and Yaron (2004).
Finally, for comparison we present model output for two other parameterizations. To illustrate the sensitivity of the model to the cointegration restriction, Table 2 shows the model output for parametrization II, where $\rho_s$ is increased to 1 (i.e. no cointegration) from its parametrization I value of 0.994. Note that the value of 0.994 can already be considered a ‘loose’ cointegration restriction. With $\rho_s = 0.994$, the ‘half-life’ of a dividend innovation is 115 months. Even so, the change in $\rho_s$ has a large impact on asset pricing moments while dividends are almost unchanged. Dividend growth volatility increases only very slightly, but the return volatility increases by almost 11% and the risk premium increases by more than 5%. Note also that the volatility of the v-d ratio has decreased. This is due to the decrease in the volatility of expected dividend growth (decreased growth predictability) in the absence of the cointegration restriction. Lastly, Table 3 shows model output for parametrization III, where $\rho_s = 1$ again, but the parametrization also tries to match the market risk premium and return volatility. Since return innovations are now fully exposed to dividend innovations, they tend to be more volatile than dividends, and the model has difficulty matching both volatilities.

3 Term Structures of Equity Risk and Return

We now consider claims on the aggregate dividend stream. The claim we first consider is an entitlement to the aggregate dividend payout at a fixed time in the future. By analogy with the traditional term structure literature, we refer to such a claim as a zero-coupon equity strip. As with zero-coupon bonds, the payoff date of the zero-coupon equity strip shall be called its ‘maturity’ and we will talk about short-duration versus long-duration zero-coupon strips. As mentioned earlier, the notion of zero-coupon equity is central to Lettau and Wachter (2006) and Croce, Lettau, and Ludvigson (2006), who model a ‘firm’ in the economy as a portfolio of zero-coupon equity strips. Such a ‘firm’ should really be thought of as one share in a firm that is part of a collection of similar firms. For example, such a collection may be an industry, a sector, or the set of firms that are in the same stage of the life-cycle. Like these two papers, we will focus particularly on the cash-flow timing/payout horizon of the firms as their distinguishing characteristic. The use of the zero-coupon equity concept allows a great deal of flexibility in modeling the timing of firms’ cash-flows. This is useful for answering questions about how firms’ cash flow timing is related to sensitivity to aggregate risks and to risk premium. However, first one must answer these questions at the level of a cash flow, i.e. at the level of zero-coupon equity strips. We proceed by finding an analytical expression for the price of zero-coupon equity within our model. The solution reveals the sensitivities of a strip to aggregate risks. We can then solve for return variance, risk premium and other asset pricing properties. Since we are decomposing
the stream of aggregate dividends into a term structure of equity strips, we present the asset
pricing results as term structures of premia, variance, and so on.

3.1 An Analytical Solution for Prices

Let $P_{n,t}$ denote the time $t$ price of a zero-coupon equity strip that pays out in $n$ periods. By
definition the payout at a maturity of zero is $D_t$, the aggregate dividend stream at time $t$, so
$P_{0,t} = D_t$. For $n \geq 1$, the next period payoff is $P_{n-1,t+1}$, so this period’s price is
$P_{n,t} = E_t[M_{t+1}P_{n-1,t+1}]$ where $M_{t+1}$ denotes the pricing kernel ($m_{t+1} = \ln M_{t+1}$) at time $t+1$. We normalize the price
by the level of dividends to get the pricing condition in terms of price-payoff ratios (valuation
ratios):

$$E_t \left[ M_{t+1} \frac{P_{n-1,t+1}}{D_{t+1}} \frac{D_{t+1}}{D_t} \right] = \frac{P_{n,t}}{D_t}$$  \hspace{1cm} (25)

Since both $M_{t+1}$ and $\frac{D_{t+1}}{D_t}$ are conditionally log-normal, it is natural to guess that $\frac{P_{n-1,t+1}}{D_{t+1}}$ may
also be conditionally log-normal. Following this reasoning, and using the fact that the exogenous
processes (state variables) are conditionally log-normal, we conjecture that $\frac{P_{n,t}}{D_t}$ is exponential
affine in the state variables:

$$\frac{P_{n,t}}{D_t} = \exp \left( A_0^c(n) + A_1^c(n)x_t + A_2^c(n)\sigma_t^2 + A_3^c(n)s_t \right)$$  \hspace{1cm} (26)

The boundary condition

$$\frac{P_{0,t}}{D_t} = 1 \quad \text{for all values of } x_t, \sigma_t^2, s_t$$

implies that $A_0^c(0) = A_0^c(1) = A_0^c(2) = A_0^c(3) = 0$. For $n \geq 1$ we verify the conjecture by substituting (26) into (25) and solving for the $A^c$ coefficients. If again, for simplicity, we let the correlations $\rho_{xe}$, $\rho_{pu}$, $\rho_{eu}$ be zero and omit $A_0^c$, we get (in the order in which they must be solved):

$$A_0^c(n) = \rho_s A_0^c(n-1) + (\rho_s - 1)$$

$$A_1^c(n) = \rho_s^2 - 1$$

$$A_2^c(n) = \rho A_1^c(n-1) + \left( 1 - \frac{1}{\psi_0^+} \right) + (1 + A_3^c(n-1))\phi_{sx} \dagger \dagger$$

$$A_3^c(n) = A_2^c(n-1)\nu + (1 - \theta)A_2(1 - \kappa\nu)$$

$$+ \frac{1}{2} \left( (1 - \gamma)^2 + (-\lambda_{m,e} + A_1^c(n-1)\varphi_e)^2 + \left( (1 + A_3^c(n-1))\varphi_d \right)^2 \right)$$  \hspace{1cm} (29)

The full expressions, including $A_0^c$, are given in C.1. The expressions are recursive; the $A^c(n)$
are expressed in terms of the $A^c(n-1)$, starting from the base case $A^c(0)$. The base case satisfies

$$\dagger \dagger \quad A_3^c(n) = \begin{cases} \frac{(1 - \frac{1}{\psi_0^+}) + \phi_{sx}\rho_s (\rho_s - 1) (\rho_s^{-1} - \rho^{n-1})}{(1 - \rho_s)(\rho - \rho_s)} & \text{if } \rho \neq \rho_s \\
\frac{(1 - \frac{1}{\psi_0^+}) + \phi_{sx}(\rho_s^{-1} - \rho(n-1))}{1 - \rho_s} & \text{if } \rho = \rho_s \end{cases}$$
the boundary condition, so the conjecture satisfies (25) for all \( n \geq 0 \) and is thus verified. Note that including the three \( \rho \) correlations has no effect on the \( A_1^c \) and \( A_3^c \) expressions, so they are equal to their ‘full’ counterparts in C.1.

Figure 1 presents the term structure of \( A_1^c \) for parametrization I. The resulting ‘hump’ shape, which in this case peaks around 90 months, is typical for \( \rho_s < 1 \), but very different from the \( \rho_s = 1 \) (no cointegration) case presented in Figure 2. From this viewpoint, \( \rho_s = 1 \) is actually a very special “knife-edge” case. When \( \rho_s = 1 \), sensitivity to \( x_t \) increases monotonically in the payout horizon toward its asymptotic value. Since sensitivity to \( x_t \) is the main driver of consumption risk in the long-run risks model, and is further amplified for equity by the leverage parameter \( \phi_{sx} \), both consumption and equity payout risk increase with the horizon. Accounting for the cointegration relationship causes the risks of dividend payouts to eventually converge, in the very long run, to those of consumption payouts. Loosely speaking, this should cause dividends’ \( x_t \) risk to decrease at some point as it converges to consumption’s level of \( x_t \) risk. This reasoning is confirmed by Figure 3, in which the term structure of \( x_t \) sensitivity is plotted for parametrization I against two boundary cases: (1) the case of no cointegration and (2) the case of a ‘consumption-like’ claim that has no leverage (\( \phi_{sx} = 0 \)) to long-run expected growth movements in the economy.

We give some intuition for (27) and (28). The exponential affine form of the price-dividend ratio shows that \( A_3^c(n) \) is the the change in \( \ln(\frac{P_{n,t}}{D_t}) \) caused by a unit increase in \( s_t \). If \( \rho_s < 1 \) (cointegration is imposed), a unit increase in \( s_t \) lowers the value of a dividend strip because it decreases expected dividend growth. Specifically, from (15) and (7) we see that a unit increase in \( s_t \) lowers expected (log) dividend growth for the current period by \( (\rho_s - 1) \) and raises the expected value of \( s_{t+1} \) by \( \rho_s \). Thus, a unit increase in \( s_t \) changes \( \ln(\frac{P_{n,t}}{D_t}) \) by \( (\rho_s - 1) + \rho_s A_3^c(n - 1) \), which is what (27) says. Now consider the change in \( \ln(\frac{P_{n,t}}{D_t}) \) caused by a unit increase in \( x_t \). The resulting increase in the current period dividend growth has two parts. One part is a \( \phi_{sx} \) units increase in dividend growth that is not offset by consumption growth (the levered part of dividend growth). This part also results in a \( \phi_{sx} \) units increase in \( s_{t+1} \). Thus, its net contribution is \( \phi_{sx}(1 + A_3^c(n - 1)) \). The remaining one unit increase in current period dividend growth is offset by consumption growth. This adds \( (1 - \frac{1}{\psi}) \) net units to \( \ln(\frac{P_{n,t}}{D_t}) \). Finally, the unit increase in \( x_t \) implies an expected \( \rho \) units increase in \( x_{t+1} \), which then adds \( \rho A_1^c(n - 1) \). Putting these together, we get (28). Since the net contribution of levered dividend growth, \( \phi_{sx}(1 + A_3^c(n - 1)) \), decreases with horizon towards 0, \( A_1^c(n) \) first increases and then decreases with the horizon, resulting in its humped shape term structure.

Finally, Figure 4 presents a comparison of the term structure of \( A_2^c(n) \) for parametrization I (on the left) and with \( \rho_s = 1 \) (on the right). Since \( \sigma_t^2 \) controls variation in the risk premium, a strip’s sensitivity to \( \sigma_t^2 \) innovations is a function of its level of risk. The major source of a dividend’s riskiness in the long-run framework is sensitivity to \( x_t \). Thus, \( \sigma_t^2 \) risk is closely tied to
$x_t$ risk, causing the term structure of $A^2_c(n)$ to inherit its basic form from $A^1_c(n)$, as evidenced by Figure 4 and (29).

### 3.2 Risks and Risk Premia

The $A^c(n)$ coefficients are the elasticities of the price of the equity strip that matures in $n$ periods with respect to shocks to the state variables (the sources of aggregate risk). If the coefficient $A^c_i(n)$ varies greatly with $n$, then the sensitivity of an equity strip to aggregate risk $i$ will vary greatly with its maturity. In other words, the term structure of $A^c_i(n)$ will be important in determining the risk of a zero-coupon equity strip. Below we show that the $A^c(n)$ allow a precise decomposition, by risk source, of equity strips’ returns and their term structures of risk premia, variance, market covariance/CAPM beta, and CAPM alpha.

#### 3.2.1 Zero-Coupon Return Innovations and Conditional Variance

Let $r^{n}_{c,t+1}$ denote the continuously compounded return from time $t$ to time $t+1$ on the equity strip that matures in $n$ periods. We have:

$$r^{n}_{c,t+1} = \ln \left( \frac{P_{n-1,t+1}}{P_{n,t}} \right) = \ln \left( \frac{D_{t+1}}{D_t} \frac{P_{n-1,t+1}}{P_{n,t} D_t} \right)$$

$$= \Delta d_{t+1} + \ln \frac{P_{n-1,t+1}}{D_{t+1}} - \ln \frac{P_{n,t}}{D_t}$$

$$\Rightarrow r^{n}_{c,t+1} = \Delta d_{t+1} + A^c_0(n-1) - A^c_0(n) + A^c_1(n-1)x_{t+1} - A^c_1(n)x_t$$

$$+ A^c_2(n-1)\sigma^2_{t+1} - A^c_2(n)\sigma^2_t + A^c_3(n-1)s_{t+1} - A^c_3(n)s_t$$

Note that in this case we don’t need to apply a log-linearization since zero-coupon equity pays no dividend. The innovation in the zero-coupon equity return is:

The term $1 + A^c_3(n-1)$, which scales the dividend innovation $u_{t+1}$, is closely related to the term $1 + \kappa_{1,m}A^c_3$ for the market return innovation. The underlying intuition is the same. Consider a positive unit innovation in dividend growth in that growth is not matched by consumption growth. The value of this increase in dividend growth is offset by decreases in expected future dividend growth due to the cointegration of dividends and consumption. Since $1 + A^c_3(n-1) = \rho_s^{n-1}$, no cointegration means that $1 + A^c_3(n-1) \equiv 1$, and returns are fully exposed to dividend innovations. This is always the case for $n = 1$, since there is no ‘future’ to consider for the 1-period ahead zero-coupon strip. For $\rho_s < 1$, the exposure of the return innovation to dividend innovations decays exponentially. For a value for $\rho_s$ of 0.99, the ‘half-life’ of a dividend innovation is about 70 months (model periods). Thus, short-duration equity strips will be much more sensitive to the relatively high
volatility dividend innovations than long-duration strips. Under most parameterizations of the model, dividend innovations will be a significant source of return volatility and covariation with the market, for short-duration strips, but the effect will be minor for very long-duration zero-coupon equity. Since the zero-coupon equity strips add up to the market, $1 + \kappa_{1,m} A_{3,d}$ is a value-weighted average of the $1 + A_3^1(n - 1)$. If we set $\rho_s = 0.99$ and $\kappa_{1,m} \approx 0.996$, then $1 + \kappa_{1,m} A_{3,d} \approx 0.29$, so from this perspective the market would have the same sensitivity as a 123 month zero-coupon equity strip. For comparison, we can let $\rho_s \to 1$ and find, in the limit of no cointegration, the analogous ‘duration’ of the market. The result is that without any cointegration restriction, the market has the same sensitivity as a $\frac{\kappa_{1,m}}{1 - \kappa_{1,m}} \approx 249$ month zero-coupon strip.

We can also obtain an expression for the conditional variance of a zero-coupon return. Simply take conditional variances of both sides of (30). This expression is given in C.2.

### 3.2.2 Zero Coupon Risk Premia

To get the risk premia on the zero coupon strip of maturity $n$, let $r_i = r_{c,i}$ in (11) and use (30) and (13) to obtain:

$$E_t(r_{c,t+1} - r_{f,t}) + \frac{1}{2} \text{var}_t(r_{c,t+1}) = (\gamma + \lambda_m \rho_y) \sigma_t^2 + A_1^c (n - 1) \varphi_e (\gamma \rho_y + \lambda_{m,e}) \sigma_t^2$$

$$+ A_2^c (n - 1) \lambda_{m,w} \sigma_w^2 + (1 + A_3^e(n - 1)) \varphi_d (\gamma \rho_u + \lambda_{m,e} \rho_u) \sigma_t^2$$

(31)

Like its market counterpart, the risk premia is written here as the sum over the four innovations $(\eta, e, w, u)$, of the zero coupon strip’s exposure to an innovation weighted by the total compensation for exposure to that risk. Simulated excess returns for the whole set of strips are presented in Figure 5. The left panel shows the results for parametrization I, and parametrization III is on the right. The simulated term structures are very close to the analytical ones one obtains by plotting the excess returns from (31). Consider first parametrization I, where the cointegration restriction is imposed. Again we observe a ‘hump-shaped’ curve. Excess returns rise steeply and peak around 100 months at 9-10%. They then decrease steadily to approximately 3% around 480 months. By comparison, the excess returns for parametrization III are simply monotonically increasing with the horizon of the strip. The intuition is simple. If ones knows that dividend growth must converge to consumption growth in the long-run, then the dividend ‘news’ that matters most for long-horizon strips is news about long-run consumption growth. However, the vast majority of ‘news’ is short to intermediate term in nature. This news matters greatly for short to intermediate term strips, making them the riskiest kind of strip. However, if there is no such long-term relationship between dividends and consumption, then long-run strips are just as exposed to dividend news as their shorter-run counterparts, so that their cumulative risk is greatest.

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3The limit must be taken since the expression is indeterminate for $\rho_s = 1$
Thus, accounting for cointegration has a large impact on the term structure of risk premia. The interaction of long-run growth movements in the economy and the cointegration relationship results in a more complicated, non-monotonic relationship between payout horizon and risk premia than the simple relationship implicit in the baseline model of Bansal and Yaron (2004).

4 CAPM Betas and Alphas

Now consider the implications for ‘firms’, modeled as portfolios of zero-coupon equity strips, as in Lettau and Wachter (2006) and Croce, Lettau, and Ludvigson (2006). The excess return on a firm is the value-weighted average of the excess returns on the strips. Hence, in the parametrization I economy, firms overweight in ‘intermediate term’ strips would endogenously have high risk premia, while firms overweight in long-term strips would endogenously receive low risk premia. This phenomenon is accentuated as $\rho_s$ is decreased and the cointegration restriction is tightened. For example, when $\rho_s = 0.985$, the peak excess return occurs around 60 months, and the subsequent decline in excess returns is steeper. In this case the highest premia firms would be overweight in even shorter horizon strips, while the lowest premia firms would remain those overweight in the longest horizon cash flows.

Following the procedure of Lettau and Wachter (2006), we could create firms and then sort them into portfolios, obtaining a high return premium ‘value’ portfolio and a low premium ‘growth’ portfolio. Payout horizon alone would then account for a cross-sectional difference in mean returns, with growth firms having a longer payout horizon than value firms, as is the case in Lettau and Wachter (2006). Thus, payout horizon can play an important role in producing a cross-section of mean returns in this model. However, it turns out that if payout horizon is the sole difference between firms, then it is very difficult, if not impossible, to calibrate the model properly without also having the CAPM work well within the model. Therefore, we will not pursue an explanation for the value premium within the model. Instead we look at why the CAPM continues to work well within the model even as we choose parameterizations that shift the composition of risk from mostly ‘cash-flow’ risk to mostly ‘discount-rate’ risk.

4.1 Decomposition of Betas and Alphas

To begin analyzing the performance of the CAPM within the model, we derive an analytical formula for the one-period conditional market covariance of a zero-coupon equity strip. We use this expression, in conjunction with our analytical expressions for strips’ premia and the market’s premia, to calculate strips’ conditional CAPM betas and alphas. Like the expressions we derived above for premia, the betas and alphas are decomposed into a sum of the contributions of each of
the four basic innovations \((\eta, e, w, u)\). This decomposition will let us see the relative contributions of each of the risk sensitivities (the \(A^c\) coefficients) in determining CAPM betas and alphas and how these contributions change with the parametrization of the economy.

To get an expression for a zero-coupon strip’s conditional CAPM beta, we find the conditional covariance of its excess return with the market’s. Note that, \(\text{cov}_t(r^T_{c,t+1} - r_{f,t}, r_{m,t+1} - r_{f,t}) = \text{cov}_t(r^T_{c,t+1} - E_t(r^T_{c,t+1}), r_{m,t+1} - E_t(r_{m,t+1}))\) and we can substitute in (22) and (30), obtaining:

\[
\text{cov}_t(r^T_{c,t+1} - E_t(r^T_{c,t+1}), r_{m,t+1} - E_t(r_{m,t+1})) = \left(1 + \beta_{m,e} \rho_{\eta e} + \beta_{m,u} \varphi_d \rho_{\eta u}\right) \sigma^2_t + A^1_t(n - 1) \varphi_e \left(\rho_{\eta e} + \beta_{m,e} + \beta_{m,u} \varphi_d \rho_{\eta u}\right) \sigma^2_t + A^2_t(n - 1) \beta_{m,w} \sigma^2_w + (1 + A^3_t(n - 1)) \varphi_d \left(\rho_{\eta u} + \beta_{m,e} \rho_{eu} + \beta_{m,u} \varphi_d\right) \sigma^2_t
\]

Like the expressions above for risk premia, (32) is written as the sum of the contributions of the four innovations. The first contribution is the same across all maturities as a result of a uniform unit sensitivity to \(\eta\) innovations. The other contributions vary with maturity, reflecting the term structures of the \(A^c\) coefficients. To get the conditional CAPM beta and its decomposition, we simply divide through by \(\text{var}_t(r_{m,t+1})\), given by (23). Figure 6 and Figure 7 show the term structures of premia and CAPM betas and their decompositions by innovation contribution for parametrization I.\(^4\) Figure 6 shows that the two main drivers of risk, innovations to \(x_t\) and \(\sigma^2_t\), are roughly equal in their determination of strips’ risk premia. These innovations can be thought of as the model’s main sources of ‘cash-flow’ risk and ‘discount-rate’ risk, respectively, since \(x_t\) drives variation in growth rates and \(\sigma^2_t\) drives variation in risk premia. Thus, cash-flow and discount-rate risks hold roughly equal sway in determining risk premia under parametrization I. Figure 7 shows roughly the same relative importance for the two risk drivers in determining strips’ CAPM betas. The main difference between the decomposition of betas and premia is that \(u\) innovations play a minor role in determining risk premia, but account for much of strips’ market covariation, especially at the short maturities. The reason is that the \(u\) innovation is ‘idiosyncratic’ to the market, and has no effect on the aggregate wealth return, \(r_a\). Were it not for the correlation between \(u\) and the consumption growth innovation, \(\eta\), it would bear a zero price of risk. At the same time, \(\varphi_d\) is large, so \(u\) innovations account for most of the volatility in dividend growth, and therefore also a substantial portion of the return volatility of short maturity strips. As the horizon increases, cointegration causes strips’ exposure to the idiosyncratic dividend innovations to quickly decrease, and their relative contribution to diminish, leaving predominantly the contributions of the remaining three innovations.

To see how well the CAPM accounts for strips’ premia and for the contributions of each source

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\(^4\)To calculate the term structures of premia and betas, the volatility, \(\sigma^2_t\), was set to \(\sigma^2\), its unconditional mean. The resulting term structures are very close to the ones obtained from a simulation of the economy.
of risk, we obtain the decomposition of CAPM alphas by innovation contribution. To find it, we simply multiply the CAPM beta decomposition by the conditional market premium (24), and subtract the result from the zero-coupon premia decomposition (31). The resulting decomposition summarizes the contribution of each source of risk to a potential failure of the CAPM in the model. Figure 8 presents this CAPM alpha decomposition for parametrization I. The figure shows that, at all but the shortest horizons, these (annual) CAPM alphas are relatively small in magnitude and also smaller than the alphas observed empirically for most of the value/growth portfolios. It also shows that the two main drivers of risk premia, $e$ and $w$, contribute little to the alphas, and that their contributions are offset by the idiosyncratic innovation $u$. Since $u$ contributes little to risk premia, but accounts for a lot of volatility, its contribution to alpha is negative. Thus, the CAPM captures the risk premia on strips’ quite well and would likely do even better at the firm level. We conclude that under parametrization I, differences in mean returns due to differences in payout horizon are captured fairly well by market betas.

4.2 Cash Flow Risks vs. Discount Rate Risks

Our decomposition analysis is similar to the approach adopted by Campbell and Vuolteenaho (2004) in their explanation of the value premium. They decompose value/growth portfolios’ market betas into a cash-flow news sensitivity and a discount-rate news sensitivity. Within an Epstein-Zin setting, they provide an argument for why a portfolio’s risk premium decomposition should weight the cash-flow news sensitivity much more than the discount-rate news sensitivity (by a ratio equal to the relative risk aversion). They then present empirical evidence arguing that cash-flow news sensitivity makes a relatively small contribution to the market beta decomposition. Thus, cash-flow news sensitivity figures prominently in their risk-premium decomposition, but discount rate sensitivity dominates their market beta decomposition. The result is a failure of the CAPM betas to explain portfolios’ risk premia, since premia are due mostly to cash-flow news sensitivity.\textsuperscript{5} Their model seems to do well in explaining the cross-section, yet has difficulty in accounting for the high level of the equity premium. With Campbell and Vuolteenaho’s argument in mind, we argue that it is very difficult, if not impossible, to find a calibrated parametrization of our economy that matches the equity premium and also leads to their kind of decomposition for strips’ risk premia and market betas.\textsuperscript{6}

We look at the premia, beta, and alpha decompositions for two very different parameterizations

\textsuperscript{5}This is only true in the second half of their sample. In their pre-1963 data, cash-flow sensitivity also makes a significant contribution to market beta, so that the CAPM appears to work well.

\textsuperscript{6}The Lettau and Wachter (2006) model is based on a similar decomposition; discount rate news carries a zero price of risk, though it accounts for much of assets’ return variation, while sensitivity to dividend news completely determines an assets’ premia. In Lettau and Wachter (2006), the price of dividend risk is exogenously given, as is the zero price of discount rate risk.
of the model, both of which do a decent job of matching the first two moments of aggregate consumptions, dividends, and the market return. The first is parametrization IV, which increases the importance of cash-flow risks in premia compared to parametrization I. This is achieved mainly by increasing the market’s leverage to long-term movements in consumption, $\phi_{sx}$, and decreasing the magnitude of volatility shocks, $\sigma_w$. The top panel of Figure 9 shows the resulting decomposition of strips’ premia. At all maturities, strips’ premia are due mostly to sensitivity to $x_t$ shocks (cash-flow news) while sensitivity to $\sigma^2_t$ shocks (discount-rate news) plays a small role that increases with payout horizon. Thus, discount-rate sensitivity mostly matters for long-horizon equity. The middle panel of Figure 9 shows the resulting CAPM beta decomposition. The only significant difference between the beta and premia decompositions is that $u$ shocks contribute to betas, but not premia, since $\rho_{wu} = 0$ in parametrization IV. The beta decomposition shows that, in making sensitivity to cash-flow news ($x_t$ shocks) the main determinant of premia, we have also made it the main determinant of CAPM betas. The net result is the term structure of alphas shown in the bottom panel of the figure. The contribution of cash flow sensitivity to alpha is always positive, but also very small. The same is true for discount-rate sensitivity. As before, $u$ innovations account for a lot of short maturity covariation, so their alpha contribution is quite negative for short-horizon strips, but declines with the horizon. Overall, the alphas are small in magnitude so that the CAPM appears to hold rather well.

The second parametrization is parametrization V, which increases the importance of discount-rate risks for CAPM betas compared to parametrization I. It achieves this by significantly increasing $\sigma_w$, the magnitude of volatility shocks. It also decreases $\rho_s$ in order to ‘tighten’ the cointegration restriction, thereby keeping the level of risk premia approximately the same, as shown in the top panel of Figure 10. The middle panel of the figure shows that for CAPM betas, sensitivity to $w$ shocks (discount-rate news) is the main determinant, while sensitivity to cash-flow news makes only a secondary contribution. The top panel, showing the risk premia decomposition, looks almost identical. This shows that, in making discount-rate news the main driver of volatility, we have also made the sensitivity to it the main determinant of risk premia. The net result is the term structure of alphas shown in the bottom panel of the figure. Now the contribution of discount-rate sensitivity to alphas is negative, but its magnitude remains very small (note the units for the figure). Indeed, though the shape of the term structure of alphas is very different than before, the magnitude of the alphas remains small, and the CAPM continues to work well.

Thus, we see that changing the balance between cash-flow risks and discount rate risks does not drive a wedge between the main driver of volatility/covariation and the primary determinant of cross-sectional differences in risk premia. Although the paper presents only three parameterizations of the model, this result appears to be robust across a wide variety of possible changes in the model’s parameters. In any case, the large zero-coupon equity alphas obtained in Lettau and Wachter (2006) and Croce, Lettau, Ludvigson (2006) do not appear to be feasible within
this model. This does not mean that the model is incompatible with a value premium. It means, however, that the model of firms must be enriched. For example, as in Lettau and Wachter (2006), we have thought of a firm as a portfolio of aggregate dividend strips, i.e. a portfolio of fractional shares of the aggregate dividend. The firm’s fractional shares of the aggregate dividend are deterministically known at a point in time. Making the share process stochastic in nature is a realistic expansion of the model for firms which may lead to a failure of the CAPM and thus also a value premium. This is left to further work.

5 Conclusion

This paper examines the role of payout horizon in determining the risks and risk premia on claims to aggregate cash flows. The setting is an equilibrium with both long-run consumption-dividend growth risk and a cointegration relationship between consumption and aggregate dividends. The paper finds that the interaction of these two long-run forces leads to a hump-shaped relationship between payout horizon and exposures to aggregate risks. The ‘intermediate’ horizon at which risks peak grows longer as the cointegration restriction is relaxed, and in the ‘knife-edge’ case where cointegration is eliminated entirely, risk increases monotonically in the payout horizon. Cointegration is an economically meaningful restriction which is especially natural to impose at the aggregate level. In this paper’s model it corresponds to the intuition that — though investors may find aggregate dividend growth to be quite uncertain at short to intermediate horizons — in the long-run, aggregate uncertainty stems mostly from the trend in aggregate consumption. The analysis in this paper of the term structure of aggregate dividend strips’ prices and risk sensitivities demonstrates this interaction between long run risks and cointegration.

The paper is also at the intersection of two recent strands of the literature that focus on either differences in payout horizon or exposure to long-run risk as an explanation of the value premium. Combining these two features, the paper establishes that in a long-run risks equilibrium, differences in payout horizon can result in significant differences in assets’ mean returns. Modeling firms as portfolios of aggregate dividend strips, the paper then asks whether this payout horizon is sufficient to establish a value premium. When the model is required to match the moments of consumption, dividends, and the market return, the answer is apparently no. The reason is that, within the model, the CAPM appears to do a good job of capturing the risk premia of strips, and by extension the risk premia on firms as well. Even as the relative importance of cash-flow risk and discount-rate risk is changed within the model, the success of the CAPM continues to hold. This is because there is always a close link between the driver of dividend strips’ return volatility and risk premia. Therefore, it seems unlikely that there is a configuration of the model where discount-rate sensitivity is the main determinant of strips’ volatility, but cash-flow risk is
the main determinant of risk premia. Thus, the model of firms as deterministically constructed portfolios of dividend strips must be enriched, if the model is to explain the value premium.
References


A Appendix: Aggregate Consumption Claim

A.1 Price-Consumption Ratio Coefficients

Substituting (2) for \( r_{a,t+1} \) in (8) gives

\[
E_t[\exp(\theta \ln \delta - \frac{\theta}{\psi} g_{t+1} + \theta (\kappa_0 + \kappa_1 z_{t+1} - z_t + g_{t+1})] = 1
\]

Since \( g_{t+1} \) is conditionally normal, if \( z_{t+1} \) is also conditionally normal then the exponent is normal and we can easily evaluate the conditional expectation. Since \( x_t \) and \( \sigma_t \) are conditionally normal, the conjectured form \( A_0 + A_1 x_{t+1} + A_2 \sigma_{t+1} \) for \( z_{t+1} \) is a straightforward way to achieve the conditionally normality of \( z_{t+1} \). Now we substitute in this conjecture for \( z_{t+1} \), evaluate the conditional expectation and solve for the \( A \) coefficients so that the Euler condition holds for \( r_a \).

The resulting \( A_1 \) and \( A_2 \) are given in the text, and \( A_0 \) is given by:

\[
A_0 = \frac{\ln \delta + (1 - \frac{1}{\psi}) \mu_c + \kappa_0 + \kappa_1 A_2 \sigma^2(1 - \nu) + \frac{1}{2} \theta (\kappa_1 A_2 \sigma_w)^2}{1 - \kappa_1}
\]  

(A.1.1)

A.2 Risk Premia of the Aggregate Wealth Claim

The full expression for the risk premia of the aggregate wealth claim is given by:

\[
E_t(r_{a,t+1} - r_t) + \frac{1}{2} \text{var}_t(r_{a,t+1}) = -\lambda_{m,n} \sigma_t^2 + \lambda_{m,e} B \sigma_t^2 + \lambda_{m,w} A_2 \kappa_1 \sigma_w^2 - \lambda_{m,n} B \rho_{\eta e} \sigma_t^2 + \lambda_{m,e} \rho_{\eta e} \sigma_t^2 \]  

(A.2.1)

A.3 Conditional Variances

Using (12) and (13) we can get expressions for the conditional variances of the pricing kernel and of the return on the aggregate consumption claim:

\[
\text{var}_t(m_{t+1}) = (\lambda_{m,n}^2 + \lambda_{m,e}^2) \sigma_t^2 + \lambda_{m,w}^2 \sigma_w^2 - 2 \lambda_{m,n} \lambda_{m,e} \rho_{\eta e} \sigma_t^2 \]  

(A.3.1)

\[
\text{var}_t(r_{a,t+1}) = (1 + B^2) \sigma_t^2 + (A_2 \kappa_1)^2 \sigma_w^2 + 2 B \rho_{\eta e} \sigma_t^2 \]  

(A.3.2)

A.4 The Risk-free Rate

To solve for the risk-free rate, we set \( r_{i,t+1} = r_{f,t} \) in the pricing condition \( E_t[\exp(m_{t+1} + r_{i,t})] = 1 \) and evaluate the expectation by using the conditional normality of \( m_{t+1} \) and the fact that \( r_{f,t} \) is known at time \( t \). Then, taking logs of both sides gives:

\[
r_{f,t} = -E_t[m_{t+1}] - \frac{1}{2} \text{var}_t(m_{t+1})
\]  

25
Subtracting \((1 - \theta) r_{f,t}\) from both sides and dividing by \(\theta\) then gives:

\[
r_{f,t} = -\frac{1}{\theta} E_t[m_{t+1}] - \frac{(1 - \theta)}{\theta} r_{f,t} - \frac{1}{2\theta} \text{var}_t(m_{t+1})
\]

and \(E_t[r_{a,t+1} - r_{f,t}]\) can be determined from (A.2.1) and (A.3.2) while (A.3.1) gives \(\text{var}_t(m_{t+1})\).

If we make these substitutions and then take the variance of both sides of the resulting equation, we find that \(\text{var}(r_{f,t})\) can be written as follows:

\[
\text{var}(r_{f,t}) = \left(\frac{1}{\psi}\right)^2 \text{var}(x_t) + \left\{\frac{1 - \theta}{\theta} Q_1 - Q_2 \frac{1}{2\theta}\right\}^2 \text{var}(\sigma^2_t)
\]

with \(Q_1 = -\lambda_{m,n} + (1 - \theta)B^2 - 0.5(1 + B^2) - \lambda_{m,n}B\rho_{qe} + \lambda_{m,e}\rho_{qe} - B\rho_{qe}\) and \(Q_2 = \lambda_{m,n}^2 + \lambda_{m,e}^2 - 2\lambda_{m,n}\lambda_{m,e}\rho_{qe}\).

Note that \(\text{var}(x_t)\) and \(\text{var}(\sigma^2_t)\) in this expression are unconditional variances. We compute the unconditional covariance matrix for our processes below, but note that the correlation structure implies that \(\text{cov}(x_t, \sigma^2_t) = 0\), so that there is no covariance term in the expression above.

\section*{B Appendix: The Market}

\subsection*{B.1 Market Price-Dividend Coefficients}

As before, we substitute the conjectured expression for \(z_{m,t}\) into (16) to get an expression for \(r_m\) in terms of the exogenous processes, and then set \(r_i = r_m\) in (8). We then solve for the \(A\) coefficients so that (8) holds. As before, we use the conditional joint normality of the exogenous processes to evaluate the conditional expectation. The expectation must hold for all values of the state variables and this restriction produces four equations in the four \(A\) coefficients. The resulting solutions are as follows:

\[
A_{3,d} = \frac{\rho_s - 1}{1 - \kappa_{1,m}\rho_s}
\]

\[
A_{1,d} = \frac{1 - \frac{1}{\psi} + \phi_{sx} \left(\frac{1 - \kappa_{1,m}}{1 - \kappa_{1,m}\rho_s}\right)}{1 - \kappa_{1,m}\rho_s}
\]

\[
A_{2,d} = \left[\begin{array}{c}
(1 - \theta) A_2(1 - \kappa_1 \nu) + \frac{1}{2} [H_1 + H_2] \\
+ (\kappa_{1,m} A_{1,d} + (\theta - 1) \kappa_1 A_1)(1 + \lambda_{m,n}) \varphi_{e}\rho_{qe} \\
+ (\kappa_{1,m} A_{1,d} + (\theta - 1) \kappa_1 A_1)(\frac{1 - \kappa_{1,m}}{1 - \kappa_{1,m}\rho_s}) \varphi_e \varphi_d \rho_{eu}
\end{array}\right]
\]

\[
A_{2,d} = \frac{1 - \kappa_{1,m}\nu}{1 - \kappa_{1,m}\nu}
\]
The sum of the last two terms. First, note that \( \var(\kappa_1) \) where we have used the iid property of the innovations to get the term in brackets. As noted \( \kappa \)

\[ A_{0,d} = \left[ \frac{\theta \ln \delta + \mu_c \lambda_{m,n} \gamma + (\theta - 1) [\kappa_0 + (\kappa_1 - 1) A_0] + \kappa_{0,m} + \mu_c + \kappa_{1,m} A_{2,d} \sigma^2 (1 - \nu) + (\theta - 1) \kappa_1 A_2 \sigma^2 (1 - \nu)}{1 - \kappa_{1,m}} \right] \tag{B.1.4} \]

where \( H_1 = (1 - \gamma) + (\var \left[ \frac{1 - \kappa_{1,m}}{1 - \kappa_{1,m} \rho_s} \right] )^2 + 2 \rho u (1 - \gamma) \left( \var \left[ \frac{1 - \kappa_{1,m}}{1 - \kappa_{1,m} \rho_s} \right] \right) \) and \( H_2 = \left[ (\theta - 1) \kappa_1 A_1 + \kappa_{1,m} A_{1,d} \right] \var(\nu \kappa_1, m) \]

**B.2 Unconditional Variance of the Market Return**

To get the unconditional variance, start with the solution for \( r_{m,t+1} \) in terms of the \( A \) coefficients and the state variables and then de-mean it. To get the mean \( E[r_{m,t+1}] \), we use the unconditional means of the state variables, \( E[x_t] = 0 \), \( E[\sigma_t^2] = \sigma^2 \), and \( E[s_t] = 0 \). The result is:

\[ r_{m,t+1} - E(r_{m,t+1}) = \sigma_t \eta_{t+1} + \frac{x_t}{\psi} + \beta_{m,e} \sigma_t e_{t+1} + \beta_{m,w} \sigma_t w_{t+1} + A_{2,d} (\nu \kappa_1 - 1) (\sigma_t^2 - \sigma^2) + \beta_{m,u} \sigma_t u_{t+1} \tag{B.2.1} \]

Note here that the \( x_t \) term simplifies to \( \frac{1}{\psi} \). Perhaps more surprisingly, the \( s_t \) term completely drops out. This reflects the, perhaps surprising, fact that the level of \( s_t \) does not show up in the expression for \( r_{m,t+1} \), though the existence of cointegration has a great effect on the \( r_m \) process. Now we can square both sides of (B.2.1) and take unconditional means to get:

\[ \var(r_m) = \var(x_t) \psi^2 + \left\{ 1 + \beta_{m,e}^2 + \beta_{m,u}^2 + 2 \rho u \beta_{m,e} \beta_{m,u} + 2 \rho u \beta_{m,e} + 2 \rho u \beta_{m,u} \right\} \sigma^2 + \left[ A_{2,d} (\nu \kappa_1 - 1) \right] \var(\sigma_t^2) + \beta_{m,w}^2 \sigma_w^2 \tag{B.2.2} \]

where we have used the iid property of the innovations to get the term in brackets. As noted earlier, \( \var(x_t) \) and \( \var(\sigma_t^2) \) are *unconditional* variances. The computation of the unconditional distribution for the processes is discussed in a later section, though we note that \( \cov(x_t, \sigma_t^2) = 0 \), so that there is no covariance term. We can simplify this expression a little by approximating the sum of the last two terms. First, note that \( \var(\sigma_t^2) = \frac{\sigma_w^2}{\nu - 1 + \nu} \). Next, rewrite \( (\nu \kappa_1 - 1) \) as \( \kappa_{1,m} (\nu - 1 + \nu) \) and approximate it by \( \kappa_{1,m} (\nu - 1) \). Since \( \kappa_{1,m} A_{2,d} = \beta_{m,w} \), we have

\[ \left[ A_{2,d} (\nu \kappa_1 - 1) \right] \var(\sigma_t^2) \approx (\beta_{m,w} \sigma_w^2) (\nu - 1)^2 \approx \beta_{m,w} \sigma_w^2 \frac{(1 - \nu)}{1 + \nu} \]

\[ \Rightarrow \left[ A_{2,d} (\nu \kappa_1 - 1) \right] \var(\sigma_t^2) + \beta_{m,w}^2 \sigma_w^2 \approx \beta_{m,w}^2 \sigma_w^2 \frac{(1 - \nu)}{1 + \nu} + 1 \]

\[ \text{Recall that } \lambda_{m,n} = -\gamma, (1 + \lambda_{m,n}) = (1 - \gamma) \]
But \( \nu \) will generally be close to 1 to model persistent volatility, so another approximation gives
\[
\beta^2_{m,w} \sigma^2_w \left( \frac{1-\nu}{1+\nu} + 1 \right) \approx \beta^2_{m,w} \sigma^2_w. \tag{B.2.2}
\]
(B.2.2) can then be written in an ‘all-beta’ form as:
\[
\text{var}(r_m) = \frac{\text{var}(x_t)}{\psi^2} + \left\{ 1 + \beta^2_{m,c} + \beta^2_{m,u} + 2\rho_{eu}\beta_{m,c}\beta_{m,u} + 2\rho_{se}\beta_{m,c} + 2\rho_{se}\beta_{m,u} \right\} \sigma^2 + \beta^2_{m,w} \sigma^2_w \tag{B.2.3}
\]
Comparing with (23), we can see that the expressions are quite similar. The difference is the inclusion of the term \( \frac{\text{var}(x_t)}{\psi^2} \), which comes from variation in the risk-free rate (see (A.4.2)).

B.3 The Unconditional Distribution

It is useful to think of the model’s dynamics as a VAR(1) system with heteroscedastic innovations. More specifically, let
\[
\mathbf{y}_{t+1} = \begin{bmatrix}
\Delta c_{t+1} - \mu_c \\
x_{t+1} \\
\sigma^2_{t+1} - \sigma^2 \\
s_{t+1} \\
\Delta d_{t+1} - \mu_c
\end{bmatrix}
\]
so that \( \mathbb{E}[\mathbf{y}_{t+1}] = 0 \) and then \( \mathbf{y}_{t+1} = F\mathbf{y}_t + G_t v_{t+1} \) where
\[
F = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 \\
0 & \rho & 0 & 0 & 0 \\
0 & 0 & \nu & 0 & 0 \\
0 & \phi_{se} & 0 & \rho_s & 0 \\
0 & 1 + \phi_{se} & 0 & \rho_s - 1 & 0
\end{bmatrix}
\]
and where
\[
G_t = \begin{bmatrix}
\sigma_t & 0 & 0 & 0 \\
0 & \sigma_t \varphi_c & 0 & 0 \\
0 & 0 & \sigma_w & 0 \\
0 & 0 & 0 & \varphi_d \sigma_t \\
\sigma_t & 0 & 0 & \varphi_d \sigma_t
\end{bmatrix}
\]
and \(v_{t+1}\) is defined as above. If we let \(G\) denote the resulting matrix when \(\sigma_t\) is fixed at \(\sigma\) in \(G_t\), then we can express \(\Sigma = E[y_t'y_t']\) (the unconditional var/cov matrix) as follows:

\[
Sigma = E[y_{t+1}'y_{t+1}'] \\
\Rightarrow \Sigma = F\Sigma F' + E[G_tv_{t+1}'v_{t+1}'G_t'] \\
\Rightarrow \Sigma = F\Sigma F' + G\Omega G'
\]

\[
\Rightarrow vec(\Sigma) = \{F \otimes F\}vec(\Sigma) + (G \otimes G)vec(\Omega) \\
\Rightarrow vec(\Sigma) = [I - (F \otimes F)]^{-1}(G \otimes G)vec(\Omega)
\]

The third equality follows from the second because the innovation \(w_{t+1}\) (which drives changes in \(\sigma_t\)) is uncorrelated with the other innovations, leading to \(E[G_tv_{t+1}'v_{t+1}'G_t'] = GE[v_{t+1}'v_{t+1}']G\).

### C Appendix: Zero-Coupon Equity Strips

#### C.1 Pricing Coefficients/Elasticities

As mentioned in the main text, for \(n \geq 1\) we substitute (26) into (25) and solve via the method of undetermined coefficients. The resulting four equations in the four \(A^c\) coefficients produce the following recursive solutions (in the order in which they must be solved):

\[
A_0^c(n) = \rho A_0^c(n-1) + (\rho_s - 1)
\]

\[
\Rightarrow A_0^c(n) = \rho_s n - 1
\]  

\[
A_1^c(n) = \rho A_1^c(n-1) + (1 - \frac{1}{\psi}) + (1 + A_0^c(n-1))\phi_{sx} \tag{C.1.1}
\]

\[
A_2^c(n) = A_2^c(n-1)\nu + (1 - \theta)A_2(1 - \kappa_1\nu) \\
+ \frac{1}{2}\left\{ (1 - \gamma)^2 + (-\lambda_{m,e} + A_0^c(n-1)\varphi_e)^2 + \left[ 1 + A_0^c(n-1)\varphi_d \right]^2 \\
\right.
\]

\[
\quad + 2\rho_{\eta e}(1 - \gamma)(1 + A_2^c(n-1))\varphi_d + 2\rho_{\eta e}(-\lambda_{m,e} + A_1^c(n-1)\varphi_e)(1 - \gamma) \\
\quad + 2\rho_{\mu e}(-\lambda_{m,e} + A_1^c(n-1)\varphi_e)(1 + A_2^c(n-1))\varphi_d \right\} \tag{C.1.3}
\]

\[
A_0^c(n) = A_0^c(n-1) + \theta \ln \delta + \mu_c \lambda_m \varphi_0 + (\theta - 1)(\kappa_0 + (\kappa_1 - 1)A_0) \\
+ \mu_c + (\theta - 1)\kappa_1 A_2\sigma^2(1 - \nu) + A_2^c(n-1)\sigma^2(1 - \nu) \\
+ \frac{1}{2}(\theta - 1)\kappa_1 A_2 + A_2^c(n-1)\varphi_d^2 \tag{C.1.4}
\]

\[
\Rightarrow A_i^c(n) = \begin{cases} 
\frac{(1 - \frac{1}{\psi})(1 - \rho_s)(\psi - \rho_s) + \phi_{sx}\rho_s(1 - \rho)(\rho^{n-1} - \rho_s^{n-1})}{(1 - \rho)(\psi - \rho_s)} & \text{if } \rho \neq \rho_s \\
\frac{(1 - \frac{1}{\psi})(1 - \rho_s) + \phi_{sx}\rho_s^{n-1}(1 - \rho)(n-2)}{1 - \rho} & \text{if } \rho = \rho_s 
\end{cases}
\]

\[
A_i^c(n) \Rightarrow A_i^c(n) \tag{C.1.2}
\]

\[
\]
C.2 Zero Coupon Conditional Variance

Taking the conditional variance of both sides of (30) we easily obtain the conditional variance of $r_{c,t+1}^n$:

\[
\text{var}_t(r_{c,t+1}^n) = \sigma_t^2 + (A_1(n-1)\varphi_e)^2\sigma_t^2 + (A_2(n-1))^2\sigma_w^2 + ((1 + A_3(n-1))\varphi_d)^2\sigma_t^2 \\
+ 2\rho_\eta e A_1(n-1)\varphi_e\sigma_t^2 + 2\rho_\epsilon u (1 + A_3(n-1))A_1(n-1)\varphi_d\varphi_e\sigma_t^2 \\
+ 2\rho_\eta u (1 + A_3(n-1))\varphi_d\sigma_t^2
\]  
(C.2.1)
### Table 1: Data and Model Output

<table>
<thead>
<tr>
<th></th>
<th>Data: Bansal-Yaron (2006)</th>
<th>Model</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Estimate</td>
<td>Std. Error</td>
</tr>
<tr>
<td>$E[\Delta c]$</td>
<td>1.96</td>
<td>0.32</td>
</tr>
<tr>
<td>$\sigma(\Delta c)$</td>
<td>2.20</td>
<td>0.45</td>
</tr>
<tr>
<td>$E[\Delta d - \chi]$</td>
<td>0.76</td>
<td>1.47</td>
</tr>
<tr>
<td>$\sigma(\Delta d - \chi)$</td>
<td>23.11</td>
<td>3.54</td>
</tr>
<tr>
<td>$E[R_m - R_f]$</td>
<td>7.62</td>
<td>1.86</td>
</tr>
<tr>
<td>$\sigma(R_m)$</td>
<td>19.9</td>
<td>2.52</td>
</tr>
<tr>
<td>$E[R_f]$</td>
<td>0.85</td>
<td>0.40</td>
</tr>
<tr>
<td>$\sigma(R_f)$</td>
<td>1.22</td>
<td>0.33</td>
</tr>
<tr>
<td>$E[v - d]$</td>
<td>3.04</td>
<td>0.09</td>
</tr>
<tr>
<td>$\sigma(v - d)$</td>
<td>0.34</td>
<td>0.04</td>
</tr>
</tbody>
</table>

The data is taken directly from Table 1 and Table 10 in Bansal and Yaron (2006). The Model Output is for parametrization I: $\rho_s = 0.994$, $\rho = 0.980$, $\sigma = 0.0048$, $\phi_{sx} = 4.8$, $\varphi_e = 0.037$, $\varphi_d = 10.5$, $\nu = 0.983$, $\sigma_w = 7e - 006$, $\gamma = 12$, $\psi = 1.5$, $\mu_c = 0.0016$, $\delta = 0.999$, $\chi = 0.0006$, $\rho_{nu} = 0.35$, $\rho_{ne} = 0$, $\rho_{eu} = 0$.
Table 2: Model Output: Parametrization II ($\rho_s = 1$)

<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$E[\Delta c]$</td>
<td>1.94</td>
<td></td>
</tr>
<tr>
<td>$\sigma(\Delta c)$</td>
<td>2.39</td>
<td></td>
</tr>
<tr>
<td>$E[\Delta d - \chi]$</td>
<td>1.04</td>
<td></td>
</tr>
<tr>
<td>$\sigma(\Delta d - \chi)$</td>
<td>23.60</td>
<td></td>
</tr>
<tr>
<td>$E[R_m - R_f]$</td>
<td>12.13</td>
<td></td>
</tr>
<tr>
<td>$\sigma(R_m)$</td>
<td>34.70</td>
<td></td>
</tr>
<tr>
<td>$E[R_f]$</td>
<td>1.65</td>
<td></td>
</tr>
<tr>
<td>$\sigma(R_f)$</td>
<td>0.95</td>
<td></td>
</tr>
<tr>
<td>$E[v - d]$</td>
<td>2.59</td>
<td></td>
</tr>
<tr>
<td>$\sigma(v - d)$</td>
<td>0.29</td>
<td></td>
</tr>
</tbody>
</table>

Model Output for parametrization II: $\rho_s = 1$, $\rho = 0.980$, $\sigma = 0.0048$, $\phi_{sx} = 4.8$, $\varphi_e = 0.037$, $\varphi_d = 10.5$, $\nu = 0.983$, $\sigma_w = 7e - 006$, $\gamma = 12$, $\psi = 1.5$, $\mu_c = 0.0016$, $\delta = 0.999$, $\chi = 0.0006$, $\rho_{\eta u} = 0.35$, $\rho_{\eta e} = 0$, $\rho_{eu} = 0$

Table 3: Model Output: No Co-integration Restriction

<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$E[\Delta c]$</td>
<td>1.94</td>
<td></td>
</tr>
<tr>
<td>$\sigma(\Delta c)$</td>
<td>2.28</td>
<td></td>
</tr>
<tr>
<td>$E[\Delta d - \chi]$</td>
<td>1.05</td>
<td></td>
</tr>
<tr>
<td>$\sigma(\Delta d - \chi)$</td>
<td>19.04</td>
<td></td>
</tr>
<tr>
<td>$E[R_m - R_f]$</td>
<td>7.16</td>
<td></td>
</tr>
<tr>
<td>$\sigma(R_m)$</td>
<td>24.31</td>
<td></td>
</tr>
<tr>
<td>$E[R_f]$</td>
<td>1.75</td>
<td></td>
</tr>
<tr>
<td>$\sigma(R_f)$</td>
<td>0.90</td>
<td></td>
</tr>
<tr>
<td>$E[v - d]$</td>
<td>2.97</td>
<td></td>
</tr>
<tr>
<td>$\sigma(v - d)$</td>
<td>0.19</td>
<td></td>
</tr>
</tbody>
</table>

Model Output for parametrization III: $\rho_s = 1$, $\rho = 0.980$, $\sigma = 0.0048$, $\phi_{sx} = 2.6$, $\varphi_e = 0.037$, $\varphi_d = 9$, $\nu = 0.983$, $\sigma_w = 6e - 006$, $\gamma = 12$, $\psi = 1.5$, $\mu_c = 0.0016$, $\delta = 0.999$, $\chi = 0.0006$, $\rho_{\eta u} = 0.35$, $\rho_{\eta e} = 0$, $\rho_{eu} = 0$
Figure 1: Term structure of $A_1^c$ coefficients for parametrization I

Note: $A_1^c(n) = \rho A_1^c(n-1) + \rho_\psi^{-1} \phi_{sx} + (1 - \frac{1}{\psi})$

Notes: The figure displays the term structure of $A_1^c$, the sensitivity of zero-coupon equity to $x_t$, under the equilibrium of parametrization I: $\rho_s = 0.994$, $\rho = 0.980$, $\sigma = 0.0048$, $\phi_{sx} = 4.8$, $\varphi_e = 0.037$, $\varphi_d = 10.5$, $\nu = 0.983$, $\sigma_w = 7e-006$, $\gamma = 12$, $\psi = 1.5$, $\mu_c = 0.0016$, $\delta = 0.999$, $\chi = 0.0006$, $\rho_{nu} = 0.35$, $\rho_{ne} = 0$, $\rho_{eu} = 0$
Figure 2: Term structure of $A_1^c$ coefficients for parametrization II ($\rho_s = 1$)

Note: $A_1^c(n) = \rho A_1^c(n-1) + \rho_s^{n-1} \phi_{sx} + (1 - \frac{1}{\psi})$

Notes: The figure displays the term structure of $A_1^c$, the sensitivity of zero-coupon equity to $x_t$, under the equilibrium of parametrization II ($\rho_s = 1$): $\rho_s = 1, \rho = 0.980, \sigma = 0.0048, \phi_{sx} = 4.8, \varphi_c = 0.037, \varphi_d = 10.5, \nu = 0.983, \sigma_w = 7e-006, \gamma = 12, \psi = 1.5, \mu_c = 0.0016, \delta = 0.999, \chi = 0.0006, \rho_{nu} = 0.35, \rho_{\eta e} = 0, \rho_{cu} = 0$
Note: $A_1^c(n) = \rho A_1^c(n-1) + \rho_s^{n-1} \phi_{sx} + (1 - \frac{1}{\phi_s})$

Notes: The figure displays a comparison of term structures of $A_1^c$. The sold line is for parametrization I. The dashed line forming the upper bound is for parametrization I with $\rho_s = 1$ (no cointegration). The lower bound is for a 'consumption-like' dividend stream where $\phi_{sx} = 0$. Parametrization I is: $\rho_s = 0.994$, $\rho = 0.980$, $\sigma = 0.0048$, $\phi_{sx} = 4.8$, $\varphi_e = 0.037$, $\varphi_d = 10.5$, $\nu = 0.983$, $\sigma_w = 7e^{-0.006}$, $\gamma = 12$, $\psi = 1.5$, $\mu_c = 0.0016$, $\delta = 0.999$, $\chi = 0.0006$, $\rho_{uu} = 0.35$, $\rho_{ue} = 0$, $\rho_{eu} = 0$. 
Figure 4: Term structures of $-1 \times A_2^c$ coefficients for parametrization I and II

Notes: The figure displays the term structure of $A_2^c$, the sensitivity of zero-coupon equity to $\sigma_t$, under the equilibria of parametrization I (on the left) and II (on the right). Parametrization I is: $\rho_s = 0.994$, $\rho = 0.980$, $\sigma = 0.0048$, $\phi_{sz} = 4.8$, $\varphi_e = 0.037$, $\varphi_d = 10.5$, $\nu = 0.983$, $\sigma_w = 7e^{-0.006}$, $\gamma = 12$, $\psi = 1.5$, $\mu_c = 0.0016$, $\delta = 0.999$, $\chi = 0.0006$, $\rho_{qu} = 0.35$, $\rho_{qe} = 0$, $\rho_{cu} = 0$. The only difference in parametrization II is that $\rho_s = 1$. 
Notes: The figure displays the term structure of (unconditional) excess returns for the set of zero coupon strips. The excess returns are taken from the simulation of the economy under parametrization I and III. Parametrization I is: \( \rho_s = 0.994, \rho = 0.980, \sigma = 0.0048, \phi_{sx} = 4.8, \phi_e = 0.037, \phi_d = 10.5, \nu = 0.983, \sigma_w = 7e - 006, \gamma = 12, \psi = 1.5, \mu_c = 0.0016, \delta = 0.999, \chi = 0.0006, \rho_{nu} = 0.35, \rho_{ne} = 0, \rho_{eu} = 0. \) Parametrization III is: \( \rho_s = 1, \rho = 0.980, \sigma = 0.0048, \phi_{sx} = 2.6, \phi_e = 0.037, \phi_d = 9, \nu = 0.983, \sigma_w = 6e - 006, \gamma = 12, \psi = 1.5, \mu_c = 0.0016, \delta = 0.999, \chi = 0.0006, \rho_{nu} = 0.35, \rho_{ne} = 0, \rho_{eu} = 0. \)
Figure 6: Decomposition of conditional risk premia by innovation sensitivity for parametrization I

Notes: The figure displays the term structure of strips’ conditional risk premia, decomposed into their innovation contributions, for parametrization I. To calculate premia in the figure, the volatility, \( \sigma_t^2 \), was set to \( \sigma^2 \), its unconditional mean. Parametrization I is: \( \rho_s = 0.994 \), \( \rho = 0.980 \), \( \sigma = 0.0048 \), \( \phi_{sx} = 4.8 \), \( \phi_e = 0.037 \), \( \varphi_d = 10.5 \), \( \nu = 0.983 \), \( \sigma_w = 7e - 006 \), \( \gamma = 12 \), \( \psi = 1.5 \), \( \mu_c = 0.0016 \), \( \delta = 0.999 \), \( \chi = 0.0006 \), \( \rho_{yu} = 0.35 \), \( \rho_{qe} = 0 \), \( \rho_{eu} = 0 \).
Figure 7: Decomposition of the conditional CAPM betas by innovation sensitivity for parametrization I

Notes: The figure displays the term structure of strips’ conditional CAPM betas, decomposed into their innovation contributions, for parametrization I. To calculate the betas in the figure, the volatility, $\sigma_t^2$, was set to $\sigma^2$, its unconditional mean. Parametrization I is: $\rho_s = 0.994$, $\rho = 0.980$, $\sigma = 0.0048$, $\phi_{sx} = 4.8$, $\varphi_e = 0.037$, $\varphi_d = 10.5$, $\nu = 0.983$, $\sigma_w = 7e-006$, $\gamma = 12$, $\psi = 1.5$, $\mu_c = 0.0016$, $\delta = 0.999$, $\chi = 0.0006$, $\rho_{nu} = 0.35$, $\rho_{qe} = 0$, $\rho_{cu} = 0$. 
Figure 8: Decomposition of the conditional CAPM alphas by innovation sensitivity for parametrization I

Notes: The figure displays the term structure of strips’ conditional CAPM alphas, decomposed into innovation contributions, for parametrization I. To calculate the alphas in the figure, the volatility, \( \sigma_t^2 \), was set to \( \sigma^2 \), its unconditional mean. Parametrization I is: \( \rho_s = 0.994, \rho = 0.980, \sigma = 0.0048, \phi_{sx} = 4.8, \varphi_e = 0.037, \varphi_d = 10.5, \nu = 0.983, \sigma_w = 7e-006, \gamma = 12, \psi = 1.5, \mu_c = 0.0016, \delta = 0.999, \chi = 0.0006, \rho_{gw} = 0.35, \rho_{gc} = 0, \rho_{eu} = 0. \)
Notes: The figure displays the decomposition of the term structures of strips’ risk premia, conditional beta, and conditional alpha under the equilibrium of parametrization IV: $\rho_s = 0.995,$ $\rho = 0.972,$ $\sigma = 0.007,$ $\phi_{sx} = 7,$ $\varphi_e = 0.044,$ $\varphi_d = 6,$ $\nu = 0.987,$ $\sigma_w = 3e - 006,$ $\gamma = 12,$ $\psi = 1.6,$ $\mu_c = 0.0016,$ $\delta = 0.999,$ $\chi = 0.0006,$ $\rho_{\eta w} = 0,$ $\rho_{\eta e} = 0,$ $\rho_{\eta u} = 0$
Notes: The figure displays the decomposition of the term structures of strips’ risk premia, conditional beta, and conditional alpha under the equilibrium of parametrization V: $\rho_s = 0.985$, $\rho = 0.972$, $\sigma = 0.0053$, $\phi_{xx} = 4$, $\varphi_e = 0.044$, $\varphi_d = 7$, $\nu = 0.98$, $\sigma_w = 1.2e-005$, $\gamma = 12$, $\psi = 1.6$, $\mu_c = 0.0015$, $\delta = 0.999$, $\rho_{qu} = 0.5$, $\rho_{ee} = 0$, $\rho_{xx} = 0.73$