Online Appendix for "Behavioral Equilibrium in Economies with Adverse Selection"

A. A Class of Games with Monotone Selection

This section presents a setup that arises frequently in adverse selection settings and provides conditions on the primitives such that the properties that define a game with monotone selection are satisfied.

The following definitions and results (in addition to those in the print appendix) are used: A correspondence $\phi : \mathcal{T} \longrightarrow \mathcal{X}$ is *increasing in the strong set order* if when $t > t'$, then for each $x \in \phi(t)$ and $y \in \phi(t')$, $\sup(x, y) \in \phi(t)$ and $\inf(x, y) \in \phi(t')$. A function $f : \mathcal{X} \longrightarrow \mathbb{R}$ is *supermodular* if for all $x, y \in \mathcal{X}$, $f(\inf(x, y)) + f(\sup(x, y)) \geq f(x) + f(y)$. A function $g : \mathcal{X} \times \mathcal{T} \longrightarrow \mathbb{R}$ has *increasing differences* in its arguments (x, t) if $g(x, t) - g(x, t')$ is nondecreasing in x for all $t \geq t'$. A function with increasing differences in (x, t) is also single-crossing in (x, t), but the reverse need not hold.

MCS2. (Susan Athey, 1998) Let $u : \mathcal{A} \times \mathcal{V} \longrightarrow \mathbb{R}$ be a function where $\mathcal{A} \subset R^K$ and $\mathcal{V} \subset R^K$ is the support of a vector of affiliated random variables S. Let $\Phi : \mathcal{Z} \longrightarrow \mathcal{S}$ be a correspondence that is nondecreasing in the strong set order. Define $U(a, z) \equiv E\left[u(a, V) \mid V \in \Phi(z)\right]$. If $u(a, \cdot)$ is nondecreasing in v, then $U(a, \cdot)$ is nondecreasing in z. If u is supermodular in (a, v), then U has increasing differences in (a, z).

Consider the framework in Section II where payoffs are $u_i^*(a_i, x)$ when $(a_{-i}, y) \in \Phi_i(a_i)$ and zero otherwise, where $v = (x, y) \in \mathcal{X} \times \mathcal{Y}$ represents payoff uncertainty, $a_i \in \mathcal{A}_i \subset \mathbb{R}$ is player i's action, and $\mathcal{A}_i$ is nonempty and finite. The interpretation is that there are two possible outcomes, each of which occurs depending on players' actions and a random variable Y.[1] In addition, suppose that attention is restricted to nondecreasing strategies.

Given action a_i and the profile of opponents' strategies α_{-i}, let $\varphi_i(a_i; s_i, \alpha_{-i})$ denote the probability that the non-zero outcome occurs, conditional on s_i. Suppose, for simplicity, that this probability is never zero.[2] In addition, consider the following assumptions on the economic environment and on players' beliefs.

Assumptions on fundamentals. **F1**. (X, Y, S) are affiliated; **F2**. u_i^* is nondecreasing in x for all i; **F3**. u_i^* is supermodular in (a_i, x) for all i; **F4**. for all i: $\Phi_i(a_i') \subset \Phi_i(a_i)$ whenever $a_i' \leq a_i$; i.e. the probability of the non-zero outcome is nondecreasing in a player's own action; **F5**. Φ_i is nondecreasing in the strong set order for all i.

[1] At the expense of additional notation, the setting can be slightly generalized to encompass, e.g., the k-th unit auction and the effort game discussed in Section III.

[2] As discussed in Section I, if the zero outcome occurs with probability one, then a player is likely to receive no feedback about X, so that multiple conjectures may be entertained.

Assumptions on beliefs. B1. players only receive feedback about their own payoff; **B2**. every player i has correct beliefs about ρ_i for every a_i, given the equilibrium strategies of other players; **B3**. every player i believes Y is independent of (X, A_{-i}); **B4.** sophisticated players know MSP.[3]

Theorem A. The previous environment is a game with monotone selection when F1-F5 and B1-B4 hold.

As the proof makes clear, B1-B3 imply existence of a unique naive profit function, adding F1, F2, and F5 implies MSP, and further adding F3-F4 implies action-belief complementarity.

Proof of Theorem A. By B2, beliefs about $\varphi_i(a_i; s_i, \alpha_{-i})$ are correct for any a_i, so consider beliefs about $Eu_i^*(a_i, X)$ when a_i^* is played. Since only payoffs are revealed (B1), it follows that players observe the exact realization of X if and only if $(\alpha_{-i}(s_{-i}), y) \in \Phi_i(a_i^*)$, and get no feedback otherwise. In addition, player i believes that this conditional expectation does not depend on their action a_i^* since: i) she believes Y is independent of X (B3), and ii) she is naive, so she does not know that opponents' actions might be correlated with X. Therefore, naive-consistency requires that beliefs about $Eu_i^*(a_i, X)$ be given by the conditional expectation

$$\nu_i(a_i; a_i^*, s_i) \equiv E\left(u_i^*(a_i, X) \mid (\alpha_{-i}(S_{-i}), Y) \in \Phi_i(a_i^*),\ S_i = s_i\right),$$

establishing uniqueness of a naive profit function $\pi_i^N = \varphi_i(a_i; s_i, \alpha_{-i}) \times \nu_i(a_i; a_i^*, s_i)$.

Since (X, Y, S) are affiliated (F1), u_i^* is nondecreasing in x (F2), and Φ_i is nondecreasing in the strong set order (F5), it follows from (MCS2) and from the assumption that α is nondecreasing that $\nu_i(a_i; a_i^*, s_i)$ is nondecreasing in a_i^*, so that MSP holds. Since (X, Y, S) are affiliated (F1), u_i^* is supermodular in (a_i, x) (F3), and Φ_i is nondecreasing in the strong set order (F5), it follows from (MCS2) and from the assumption that α is nondecreasing that $\nu_i(a_i; a_i^*, s_i)$ has increasing differences in (a_i, a_i^*). Since in addition $\varphi_i(a_i; s_i, \alpha_{-i})$ is nonnegative and nondecreasing in a_i (F4), it then follows that π_i^N has increasing differences in (a_i, a_i^*), implying that it is single-crossing in (a_i, a_i^*). Together with (B4), the properties that define a game with monotone selection in Section III are then established. ■

B. General definition of naive-consistency

The definition of naive-consistency in the general case where partial marginal feedback may be revealed requires players to have a belief over the conditional probability of an element of, say, $\mathcal{V}$, conditional on having observed that the true realization belongs to a subset of the elements of $\mathcal{V}$.

[3] As usual, B2-B4 can be captured by placing restrictions on feasible beliefs. B2 requires players, e.g., to know the demand/supply they face (i.e. willingness to pay in the population), but not necessarily the relationship between willingness to pay and the types of potential customers. In some settings (e.g. Section I), B2 automatically holds when players receive feedback about each others' actions.

Let $G_{s_i} = G_{s_i}^{A_{-i}} \times G_{s_i}^{V}$, where $G_{s_i}^{A_{-i}} \in \Delta(\mathcal{A}_{-i})$ and $G_{s_i}^{V} \in \Delta(\mathcal{V})$ are the beliefs of player i with signal s_i regarding others' actions and payoff uncertainty, respectively. Feedback $\gamma_i^{\mathcal{V}}$ results in a probability distribution over the set of subsets of $\mathcal{V}$, $2^V \equiv \{V^* : V^* \subset \mathcal{V}\}$. The true probability distribution over 2^V depends on the true probability distribution $G_{s_i,\alpha_{-i}}^0$, and is denoted by $G_{s_i,\alpha_{-i}}^{0,2^V}$. Similarly, $G_{s_i,\alpha_{-i}}^{0,2^{A_{-i}}}$ is the true probability over $2^{A_{-i}}$. Naive-consistency requires player i's belief over this probability distribution to be correct. The novelty in this extension is to determine how player i believes that a subset $V^* \subset 2^V$ comes to be observed. In general, it must be true that

$$\textstyle\sum_{v \in V^*} P_{s_i}^V(V^* \mid v) \times G_{s_i}^V(v)$$

is the probability that player i with signal s_i assigns to feedback V^*, where $P_{s_i}(V^* \mid v)$ is the probability she assigns to observing V^* whenever the true realization is v (and similarly for beliefs about subsets of $\mathcal{A}_{-i}$). While it is possible to allow P_{s_i} to be determined endogenously in equilibrium (together with G_{s_i}), this would amount to allowing the player to have a theory, and therefore to be possibly aware, of selection in the data. To capture naivete, the definition of naive-consistency requires players to believe that the process that generates outcomes is independent from the process that determines the information obtained about marginal outcomes. Formally, naive players believe that for every $V^* \in 2^V$, $P_{s_i}(V^* \mid v) = P_{s_i}^{V^*}$ for every $v \in V^*$, and zero otherwise. In words, the probability of observing a subset of $\mathcal{V}$ is the same irrespective of which element of that subset has actually been realized. It is straightforward to show that the above restrictions on beliefs are equivalent to the following definition.

Definition (naive consistency). *A belief $G_{s_i} \in G_i$ of player i with signal s_i is γ_i-naive-consistent for (a_i, α_{-i}) if the distribution of $\gamma_i^{\mathcal{U}}(a_i, A_{-i}, V)$ is the same whether the distribution over (A_{-i}, V) is given by G_{s_i} or by $G_{s_i,\alpha_{-i}}^0$, and where $G_{s_i} = G_{s_i}^{A_{-i}} \times G_{s_i}^V$ and*

$$G_{s_i}^{A_{-i}}(a_{-i}) = \sum_{\{A_{-i}^* \subset \mathcal{A}_{-i} : a_{-i} \in A_{-i}^*; G_{s_i,\alpha_{-i}}^{0,2^{A_{-i}}}(A_{-i}^*) > 0\}} G_{s_i,\alpha_{-i}}^{0,2^{A_{-i}}}(A_{-i}^*) \times \frac{G_{s_i}^{A_{-i}}(a_{-i})}{\sum_{\{a'_{-i} \in A_{-i}^*\}} G_{s_i}^{A_{-i}}(a'_{-i})} \tag{1}$$

for all $a_{-i} \in A_{-i}$, and

$$G_{s_i}^V(v) = \sum_{\{V^* \subset \mathcal{V} : v \in V^*; G_{s_i,\alpha_{-i}}^{0,2^V}(V^*) > 0\}} G_{s_i,\alpha_{-i}}^{0,2^V}(V^*) \times \frac{G_{s_i}^V(v)}{\sum_{\{v' \in V^*\}} G_{s_i}^V(v')} \tag{2}$$

for all $v \in V$.

Example (consistency vs naive-consistency). Consider a two-player game where player 1 with signal s_i can only choose action a while player 2 can choose an action in $\mathcal{A}_2 = \{a_1, a_2, a_3\}$ and the set of payoff uncertainty is $\mathcal{V} = \{v_1, v_2, v_3\}$. Suppose player 1 faces the distribution $G_{s_i,\alpha_{-i}}^0$ over

$\mathcal{A}_2 \times \mathcal{V}$ given in Figure Aa, and player 1's feedback can be characterized by the following partition over the set of states, depicted in Figure Ab is

$$\mathcal{P}_1 = \left\{ \{\omega_1\}, \{\omega_2\}, \{\omega_3\}, \{\omega_4\}, \{\omega_5, \omega_6\}, \{\omega_7, \omega_8, \omega_9\} \right\}.$$

(in the set-up, feedback can be more general – if it arises from a partition, as in this example, one constructs $\gamma_i^{\mathcal{A}_{-i}}$ and $\gamma_i^{\mathcal{V}}$ in the obvious way; e.g. $\gamma_i^{\mathcal{A}_{-i}}(\omega)$ is a singleton for each ω, $\gamma_i^{\mathcal{V}}(\omega_5) = \{v_2, v_3\}$, etc.)

Consistency requires player 1 to have correct beliefs about the probability of each of the previous information sets, so that her belief satisfies $G_{s_i}^S(\omega) = G_{s_i,\alpha_{-i}}^0(\omega)$ for $\omega \in \{\omega_1, \omega_2, \omega_3, \omega_4\}$, $G_{s_i}^S(a_2, v_2) + G_{s_i}^S(a_2, v_3) = 4/18$, and $\sum_{i=1}^3 G_{s_i}^S(a_3, v_i) = 6/18$, where S stands for a sophisticated player.

In contrast, a naive player learns about V and A_2 independently. Since $\gamma_i^{\mathcal{A}_{-i}}$ reveals the actions of other players, player 1 has correct beliefs about the marginal probability of player 2's actions, i.e. $G_{s_i}^{A_{-i}}(a) = 6/18$ for all $a \in \mathcal{A}_2$. Regarding $\mathcal{V}$, equation (2) implies the following system of equations,

$$\begin{aligned}
G_{s_i}^V(v_1) &= \frac{5}{18} + \frac{6}{18} \times \frac{G_{s_i}^V(v_1)}{\sum_{i=1}^3 G_{s_i}^V(v_i)} \\
G_{s_i}^V(v_2) &= \frac{2}{18} + \frac{4}{18} \times \frac{G_{s_i}^V(v_2)}{G_{s_i}^V(v_2) + G_{s_i}^V(v_3)} + \frac{6}{18} \times \frac{G_{s_i}^V(v_2)}{\sum_{i=1}^3 G_{s_i}^V(v_i)} \\
G_{s_i}^V(v_2) &= \frac{1}{18} + \frac{4}{18} \times \frac{G_{s_i}^V(v_3)}{G_{s_i}^V(v_2) + G_{s_i}^V(v_3)} + \frac{6}{18} \times \frac{G_{s_i}^V(v_3)}{\sum_{i=1}^3 G_{s_i}^V(v_i)},
\end{aligned}$$

which has a unique solution $G_{s_i}^V(v_1) = 15/36$, $G_{s_i}^V(v_2) = 14/36$, $G_{s_i}^V(v_3) = 7/36$. If naive-consistent beliefs exist, they are then given by the product of the marginals, $G_{s_i}^N = G_{s_i}^{A_{-i}} \times G_{s_i}^V$, where N stands for naive.

To check for existence, it remains to check whether the distribution of $\gamma_i^{\mathcal{U}}(a_i, A_{-i}, V)$ implied by $G_{s_i}^N$ is correct. Consider the payoffs in Figure Ac, where $\gamma_i^{\mathcal{U}}(\omega) = \{1\}$ for $\omega \in \{\omega_1, \omega_4\}$, $\gamma_i^{\mathcal{U}}(\omega) = \{2\}$ for $\omega \in \{\omega_2, \omega_3, \omega_5, \omega_6\}$, and $\gamma_i^{\mathcal{U}}(\omega) = \{0\}$ for $\omega \in \{\omega_7, \omega_8, \omega_9\}$. Naive-consistency requires $G_{s_i}^N(a_1, v_1) + G_{s_i}^N(a_2, v_1) = 5/18$, $\sum\limits_{a \in \{a_1, a_2\}} G_{s_i}^N(a, v_2) + G_{s_i}^N(a, v_3) = 7/18$, $\sum\limits_{v \in V_2} G_{s_i}^N(a_3, v) = 6/18$, and it can be checked that these conditions hold. $\square$

Next, I provide a critical sufficient condition for existence of naive-consistent beliefs. Given feedback γ_i, for each a_i we can construct a partition over $\Omega_{a_i} \equiv \left\{(a_i', a_{-i}', v') : a_i' = a_i\right\}$, which is denoted $\mathcal{P}_{i,a_i}^V$, in the following way. Let $X_{a_i} \equiv \{\omega \in \Omega_{a_i} : \gamma_i^{\mathcal{V}}(\omega) = \mathcal{V}\}$. If for all $\omega \in X_{a_i}$ and $\omega' \in \Omega_{a_i}/X_{a_i}$, $\gamma_i^{\mathcal{A}_{-i}}(\omega) \cap \gamma_i^{\mathcal{A}_{-i}}(\omega') \neq \varnothing$, then let X_{a_i} be an element of $\mathcal{P}_{i,a_i}^V$. In words, X_{a_i} is the set of states where no information is observed about V – we require this to be an element of the partition if the player has a correct belief over the probability of this event (since then she can throw this information away, and condition her beliefs about each element of $\mathcal{V}$ on the probability of obtaining some information), which will be true if the additional condition above holds (since it guarantees that she will have correct beliefs over the set of others' actions associated with no feedback about V). Next, consider the remaining states of Ω_{a_i} (either Ω_{a_i} or Ω_{a_i}/X_{a_i}, depending on whether the above condition holds). Partition these remaining states in the following way: (i) ω and ω' belong to the same element of $\mathcal{P}^{V}_{i,a_i}$ if $\gamma^{\mathcal{V}}_i(\omega) \cap \gamma^{\mathcal{V}}_i(\omega') \neq \varnothing$; (ii) for each ω that belongs to an element with more than one state, there exists $\omega' \neq \omega$ that belongs to the same element as ω and $\gamma^{\mathcal{V}}_i(\omega) \cap \gamma^{\mathcal{V}}_i(\omega') \neq \varnothing$. This procedure results in a unique partition $\mathcal{P}^{\mathcal{V}}_{i,a_i}$. Similarly, we can define a partition over Ω_{a_i} by interchanging $\gamma^{\mathcal{V}}_i$ and $\gamma^{\mathcal{A}_{-i}}_i$, which is denoted $\mathcal{P}^{\mathcal{A}_{-i}}_{i,a_i}$.

Next, let $\mathcal{I}_i$ be the set of all elements that belong to $\mathcal{P}^{\mathcal{V}}_{i,a_i}$ and $\mathcal{P}^{\mathcal{A}_{-i}}_{i,a_i}$, for all $a_i \in \mathcal{A}_i$. In addition, let $\mathbb{P}$ denote the set of all partitions of Ω that can be formed using the elements in $\mathcal{I}_i$.

Finally, let $\mathcal{P}^{\mathcal{U}}_{i,a_i}$ denote the partition of Ω_{a_i} where two states ω and ω' belong to the same element of $\mathcal{P}^{\mathcal{U}}_{i,a_i}$ if and only if $\gamma^{\mathcal{U}}_i(\omega) = \gamma^{\mathcal{U}}_i(\omega')$. Denote the union of all elements in $\mathcal{P}^{\mathcal{U}}_{i,a_i}$, for all a_i, by $\mathcal{P}^{\mathcal{U}}_i$ (which is a partition of Ω).

Theorem B (Existence of naive-consistent beliefs) If $\mathcal{P}^{\mathcal{U}}_i$ is coarser[4] than some partition in $\mathbb{P}$, then naive-consistent beliefs exist for every (a_i, α_{-i}).

Proof. Existence follows by establishing that (i) naive-consistent beliefs are obtained endogenously, and by a fixed point argument there always exists a belief that satisfies (1) and (2);[5] and (ii) conditions (1) and (2) imply that naive-consistent beliefs over the probability of observing the payoff feedback associated with any element of $\mathcal{I}_i$ is correct. I now prove claim (ii) above. Suppose beliefs are naive-consistent. Fix an element of $\mathcal{I}_i$ – for concreteness, fix a_i and fix an element that belongs to $\mathcal{P}^{\mathcal{V}}_{i,a_i}$ (the proof is similar for an element of $\mathcal{P}^{\mathcal{A}_{-i}}_{i,a_i}$). There are two cases to consider. First, suppose the element is X_{a_i}, as defined above. Let $\mathcal{A}^*_{-i} = \cup_{\omega \in X} \gamma^{\mathcal{A}_{-i}}_i(\omega)$, and note that since X belongs to $\mathcal{P}^{\mathcal{V}}_{i,a_i}$, the complement of $\mathcal{A}^*_{-i}$ is $(\mathcal{A}^*_{-i})^c = \cup_{\omega \notin X} \gamma^{\mathcal{A}_{-i}}_i(\omega)$. Player i with signal s_i believes that payoff feedback associated with outcome X_{a_i} occurs with probability

$$\begin{aligned}\left(\textstyle\sum_{a_{-i}\in\mathcal{A}^*_{-i}} G^{A_{-i}}_{s_i}(a_{-i})\right) \times \left(\textstyle\sum_{v\in\mathcal{V}} G^{V}_{s_i}(v)\right) &= \textstyle\sum_{a_{-i}\in\mathcal{A}^*_{-i}} G^{A_{-i}}_{s_i}(a_{-i}) \\ &= \textstyle\sum_{\{A_{-i}:\omega\in X,\ \gamma^{\mathcal{A}_{-i}}_i(\omega)=A_{-i}\}} G^{0,2^{A_{-i}}}_{s_i,\alpha_{-i}}(A_{-i}) \\ &= \Pr(\{\omega : \omega \in X_{a_i}\}),\end{aligned}$$

where the second inequality follows from adding (1) over all $a_{-i} \in \mathcal{A}^*_{-i}$ and the last line represents the true probability of payoff feedback associated with outcome X_{a_i}. Second, consider some other element $Y \in \mathcal{P}^{\mathcal{V}}_{i,a_i}$ and let $\mathcal{V}^* = \{v \in \gamma^{\mathcal{V}}_i(\omega) : \omega \in Y\}$. There are two subcases to consider. First,

[4] $\mathcal{P}$ is coarser than $\mathcal{P}'$ if every element of $\mathcal{P}'$ is a subset of some element of $\mathcal{P}$.

[5] Formally, the right hand sides of equations (1) and (2) can each be used to define two continuous functions from the unit simplex to itself, so that Brouwer's fixed point theorem implies that there exist solutions $G^{A_{-i}}_{s_i}$ and $G^{V}_{s_i}$.

suppose that for all $\omega' \notin Y$, $\gamma_i^{\mathcal{V}}(\omega) \cap \mathcal{V}^* = \varnothing$. Then Y is the event where some feedback $v \in \mathcal{V}^*$ is obtained, and beliefs over this event must be correct since, by adding (2) over all $v \in \mathcal{V}^*$,

$$\begin{aligned} \textstyle\sum_{v \in \mathcal{V}^*} G_{s_i}^V(v) &= \sum_{\{V: V \subset \mathcal{V}^*\}} G_{s_i,\alpha_{-i}}^{0,2^V}(V) \\ &= \Pr(\{\omega : \omega \in Y\}). \end{aligned}$$

The second subcase occurs if the first does not occur – then, by definition of $\mathcal{P}_{i,a_i}^{\mathcal{V}}$ it must be the case that Y is the event where feedback includes some $v \in \mathcal{V}^*$ but is never given by the (uninformative) entire set $\mathcal{V}$. Player i with signal s_i believes the probability of this event to be

$$\begin{aligned} \left(\textstyle\sum_{v \in \mathcal{V}^*} G_{s_i}^V(v)\right) \times \left(1 - G_{s_i,\alpha_{-i}}^{0,2^V}(\mathcal{V})\right) &= \sum_{\{V: V \subset \mathcal{V}^*, V \neq \mathcal{V}\}} G_{s_i,\alpha_{-i}}^{0,2^V}(V) \\ &= \Pr(\{\omega : \omega \in Y\}), \end{aligned}$$

where the first equality follows by summing over (2). ■

The condition in Theorem B is critical in the sense that if it is not satisfied, then there exists an environment where naive-consistent beliefs do not exist for some (a_i, α_{-i}).

To illustrate, in the example above, where player 1 only has one action,

$$\mathcal{P}_{1,a}^{\mathcal{V}} = \{\{\omega_1, \omega_4\}, \{\omega_2, \omega_3, \omega_5, \omega_6\}, \{\omega_7, \omega_8, \omega_9\}\},$$

$$\mathcal{P}_{1,a}^{\mathcal{A}_{-i}} = \{\{\omega_1, \omega_2, \omega_3\}, \{\omega_4, \omega_5, \omega_6\}, \{\omega_7, \omega_8, \omega_9\}\},$$

and $\mathbb{P} = \{\mathcal{P}_{1,a}^{\mathcal{V}}, \mathcal{P}_{1,a}^{\mathcal{A}_{-i}}\}$. Finally, it is straightforward to establish that $\mathcal{P}_1^{\mathcal{U}}$ is coarser than $\mathcal{P}_{1,a}^{\mathcal{V}}$ if and only if $u_1(a_1, v_1) = u_1(a_2, v_1)$ and $u_1(a_1, v_2) = u_1(a_1, v_3) = u_1(a_2, v_2) = u_1(a_2, v_3)$ – which is true for the example above.

C. Multidimensional action space

I extend the result in Theorem 2 to an *action* space that is partially ordered, as long as the selection effect is still driven by a completely ordered component of the action space. To define the latter, decompose the action a_i into two components, $a_i = (a_i^{NS}, a_i^S) \in \mathcal{A}_i = \mathcal{A}_i^{NS} \times \mathcal{A}_i^S \subset \mathbb{R}^{K+1}$, where $\mathcal{A}_i^{NS}$ and $\mathcal{A}_i^S$ are finite lattices and $\mathcal{A}_i^S \subset \mathbb{R}$. Write the naive profit function from action $a_i = (a_i^{NS}, a_i^S)$ when beliefs are γ_i-naive-consistent for $((\widehat{a}_i^{NS}, \widehat{a}_i^S), \alpha_{-i})$ as $\pi_i^N((a_i^{NS}, a_i^S), (\widehat{a}_i^{NS}, \widehat{a}_i^S), \alpha_{-i}, s_i)$. The *selection effect is unidimensional* if for every player i and signal $s_i \in S_i$, π_i^N is constant in $\widehat{a}_i^{NS}$ for every α_{-i}. In this case, I omit the dependence of the naive profit function π_i^N on $\widehat{a}_i^{NS}$.

Define a game with *strategic complementarities and unidimensional selection effect* to be a game with strategic complementarities (as defined in Section III), but where the assumption on the action space is replaced with the assumption of a unidimensional selection effect and, in addition, i) $\pi_i^{NE}(a_i, \alpha_{-i}, s_i)$ is supermodular in a_i and has increasing differences in (a_i, α_{-i}); ii) $\pi_i^{NE}(a_i, \widehat{a}_i^S, \alpha_{-i}, s_i)$ is supermodular in a_i and has increasing differences in $(a_i, \widehat{a}_i^S)$ and increasing differences in (a_i, α_{-i}).[6]

Theorem C. *The statement of Theorem 2 holds for games with strategic complementarities and unidimensional selection effect.*

Proof. The idea is to convert a "multidimensional" problem to a "unidimensional" one, prove that Theorem 1 holds for this unidimensional problem, and finally use complementarity in own action to show that the comparison for the unidimensional component of the strategy space extends to the multidimensional component. I omit i, α_{-i} from the notation for simplicity. The proof shows that the result in Theorem 1 holds in the new setup, so that the proof in Theorem 2 can be applied (where the only change in the latter proof is to use the additional condition of supermodularity – which is stronger than quasi-supermodularity – for the monotone comparative statics results to be applicable). Define $\rho_{NS}^N(a^S, \widehat{a}^S, s) \equiv \arg\max_{a^{NS}} \pi^N(a^{NS}, a^S, \widehat{a}^S, s)$ and $\rho_{NS}^{NE}(a^S, s) \equiv \arg\max_{a^{NS}} \pi^{NE}(a^{NS}, a^S, s)$. By supermodularity and increasing differences, there exist extreme (i.e. lowest and highest) elements of ρ^N which are nondecreasing in $(a^S, \widehat{a}^S)$ and extreme elements of ρ^{NE} which are nondecreasing in a^S. I denote these elements by $\underline{a}_{NS}^N$, $\overline{a}_{NS}^N$ and $\underline{a}_{NS}^{NE}$, $\overline{a}_{NS}^{NE}$. Define $\pi_*^N(a^S, \widehat{a}^S, s) \equiv \pi^N(\underline{a}_{NS}^N(a^S, \widehat{a}^S), a^S, s)$ and $h_*^N(\widehat{a}^S, s) = \left\{ a_*^S \in A^S : a_*^S \in \arg\max_{a^S \in A^S} \widetilde{\pi}^N(a^S, \widehat{a}^S, s) \right\}$.

The following relationships establish that $\pi_*^N(a^S, \widehat{a}^S, s)$ has increasing differences in $(a^S, \widehat{a}^S)$. Let $a_1^S, a_0^S, \widehat{a}_1^S, \widehat{a}_0^S \in A^S$ such that $a_1^S \geq a_0^S$ and $\widehat{a}_1^S \geq \widehat{a}_0^S$. Then

$$\begin{aligned}
& \pi^N(\underline{a}_{NS}^N(a_1^S, \widehat{a}_1^S), a_1^S, \widehat{a}_1^S) - \pi^N(\underline{a}_{NS}^N(a_0^S, \widehat{a}_1^S), a_0^S, \widehat{a}_1^S) \\
\geq\ & \pi^N(\sup(\underline{a}_{NS}^N(a_1^S, \widehat{a}_0^S), \underline{a}_{NS}^N(a_0^S, \widehat{a}_1^S), a_1^S), \widehat{a}_1^S) - \pi^N(\underline{a}_{NS}^N(a_0^S, \widehat{a}_1^S), a_0^S, \widehat{a}_1^S) \\
\geq\ & \pi^N(\sup(\underline{a}_{NS}^N(a_1^S, \widehat{a}_0^S), \underline{a}_{NS}^N(a_0^S, \widehat{a}_1^S)), a_1^S, \widehat{a}_0^S) - \pi^N(\underline{a}_{NS}^N(a_0^S, \widehat{a}_1^S), a_0^S, \widehat{a}_0^S) \\
\geq\ & \pi^N(\underline{a}_{NS}^N(a_1^S, \widehat{a}_0^S), a_1^S, \widehat{a}_0^S) - \pi^N(\inf(\underline{a}^{NS}(a_1^S, \widehat{a}_0^S), \underline{a}_{NS}^N(a_0^S, \widehat{a}_1^S)), a_0^S, \widehat{a}_0^S) \\
\geq\ & \pi^N(\underline{a}_{NS}^N(a_1^S, \widehat{a}_0^S), a_1^S, \widehat{a}_0^S) - \pi^N(\underline{a}_{NS}^N(a_0^S, \widehat{a}_0^S), a_0^S, \widehat{a}_0^S),
\end{aligned}$$

where the first and last inequalities follow from the optimality of $\underline{a}_{NS}^N$ (and the fact that both the infimum and the supremum are in the choice sets due to the lattice structure), the second inequality follows from the assumption that π^N has increasing differences in the arguments $a = (a^{NS}, a^S)$ and $\widehat{a}^S$, and the third inequality follows from supermodularity of π^N in $a = (a^{NS}, a^S)$. Since increasing differences implies the single-crossing property, the comparison of strategies in Theorem 1 then holds when restricted to the selection component α^S. Finally, by PR1 $\underline{a}_{NS}^N(a^S, a^S, s) = \underline{a}_{NS}^{NE}(a^S, s)$ and $\overline{a}_{NS}^N(a^S, a^S, s) = \overline{a}_{NS}^{NE}(a^S, s)$ and since they are nondecreasing in a^S, then the comparison of

[6]The stronger conditions of supermodularity and increasing differences are used since they are preserved under optimization, while quasi-supermodularity and single-crossing properties may not be preserved.

the selection component extends to the entire strategy $\alpha = (\alpha^{NS}, \alpha^{S})$. The proof in Theorem 2 can then be applied. ■

D. Symmetric games with completely ordered strategy space.

The results in Theorem 2 can also be extended to settings where both action-belief complementarity and strategic complementarities do not hold. Roughly, the first type of complementarity is replaced by continuity and quasiconcavity assumptions, while the second type is replaced by assuming symmetry and a completely ordered strategy space.

Consider a game where: i) for every $i \in \mathcal{N} : \pi_i^N = \pi^N$, $\pi_i^{NE} = \pi^{NE}$, $\gamma_i = \gamma$, $\mathcal{S}_i = \{s\}$ is a singleton, and $\mathcal{A}_i = \mathcal{A} \subset \mathbb{R}$, where $\mathcal{A}$ is nonempty, compact, and convex; ii) $\pi^{NE}(a_i; a_{-i})$ is continuous in (a_i, a_{-i}) and strictly quasiconcave in a_i; iii) $\pi^N(a_i; a_i^*, a_{-i})$ is continuous in (a_i, a_i^*, a_{-i}) and strictly quasiconcave in a_i.

The following fixed-point result is used in the proof.

FP2. (Milgrom and Roberts, 1994) Let $\mathcal{Y}$ be a compact set in $\mathbb{R}$ and $f : \mathcal{Y} \longrightarrow \mathcal{Y}$ a continuous function. Then the set of fixed points of f is nonempty and has a lowest element $\underline{y} = \inf\{y \in \mathcal{Y} : f(y) \leq y\}$ and a highest element $\overline{y} = \sup\{y \in \mathcal{Y} : f(y) \geq y\}$.

Theorem D. In the symmetric environment above, the results in Theorem 2 hold when restricted to the set of symmetric equilibria.

Proof.

Part 1. Let $\mathbf{a}_{-i}$ denote the strategy profile where every player other than i plays a, and let $\mathbf{a}$ denote the profile where every player plays a. Note that a symmetric profile $\mathbf{a}$ is a naive equilibrium if and only if $a \in H^N(a) = \{\widehat{a} : \widehat{a} \in \arg\max_{a'} \pi^N(a', a, \mathbf{a}_{-i})\}$. Similarly, $\mathbf{a}$ is a Nash equilibrium if and only if $a \in H^{NE}(a) = \{\widehat{a} : \widehat{a} \in \arg\max_{a'} \pi^{NE}(a', \mathbf{a}_{-i})\}$. By continuity, convexity, compactness, and strict quasi-concavity, $H_i^N(a_i)$ and $H_i^{NE}(a_i)$ are each single-valued and continuous in a_i. Then by FP2, lowest and highest naive and Nash equilibria exist and are given by $\underline{a}^m = \inf\{a : H^m(a) \leq a\}$ and $\overline{a}^m = \sup\{a : H^m(a) \geq a\}$ for $m \in \{N, NE\}$. A sophisticated equilibrium exists since a Nash equilibrium is always a sophisticated equilibrium.

Part 2. First, I claim that if $H^N(a) > H^{NE}(a)$, then (i) $H^N(a) \neq a$ and (ii) $H^{NE}(a) \neq a$. Consider (i) and suppose, toward a contradiction, that it does not hold: i.e. $a = H^N(a) > H^{NE}(a)$. Then

$$\begin{aligned}
\pi^{NE}(a, \mathbf{a}_{-i}) &= \pi^N(a, a, \mathbf{a}_{-i}) \\
&> \pi^N(H^{NE}(a), a, \mathbf{a}_{-i}) \\
&\geq \pi^N(H^{NE}(a), H^{NE}(a), \mathbf{a}_{-i}) \\
&= \pi^{NE}(H^{NE}(a), \mathbf{a}_{-i}),
\end{aligned}$$

where the first and last equality follows since feedback about own payoffs is observed (PR1), the strict inequality follows since $a = H^N(a)$ is the unique element of $H^N(a)$ (i.e. it is a unique maximizer), and the weak inequality follows from MSP and $a > H^{NE}(a)$. Note that these relationships contradict that $H^{NE}(a)$ is indeed a maximizer of π^{NE} given a. The proof of (ii) is similar: suppose $H^N(a) > H^{NE}(a) = a$. Then

$$\begin{aligned} \pi^{NE}(H^N(a), \mathbf{a}_{-i}) &= \pi^N(H^N(a), H^N(a), \mathbf{a}_{-i}) \\ &\geq \pi^N(H^N(a), a, \mathbf{a}_{-i}) \\ &> \pi^N(a, a, \mathbf{a}_{-i}) \\ &= \pi^{NE}(H^{NE}(a), \mathbf{a}_{-i}), \end{aligned}$$

which once again contradicts that $H^{NE}(a)$ is indeed a maximizer of π^{NE} given a.

Next, I claim that $\overline{a}^{NE} \geq \overline{a}^N$ (these are the extreme points defined in part a.). Suppose not, so that $\overline{a}^{NE} < \overline{a}^N$. But then from the fact that $\overline{a}^{NE}$ is defined as the supremum of a certain set, $H^{NE}(\overline{a}^N) < \overline{a}^N = H^N(\overline{a}^N)$,which by part i) of the previous claim is not possible. Finally, I claim that $\underline{a}^{NE} \geq \underline{a}^N$. Suppose not, so that $\underline{a}^{NE} < \underline{a}^N$. But then $H^N(\overline{a}^{NE}) > \overline{a}^{NE} = H^{NE}(\overline{a}^{NE})$, which now contradicts part ii) of the first claim.

Part 3. Suppose $a < \underline{a}^N$ is part of a symmetric sophisticated equilibrium. Then $H^N(a) > a$, meaning that a naive player would prefer to deviate to a higher action. But then so would a sophisticated player; to see this let $\pi^S \in \Pi^S(a, \mathbf{a}_{-i})$ and note that

$$\begin{aligned} \pi^S(H^N(a), a, \mathbf{a}_{-i}) &\geq \pi^N(H^N(a), a, \mathbf{a}_{-i}) \\ &> \pi^N(a, a, \mathbf{a}_{-i}) \\ &= \pi^S(a, a, \mathbf{a}_{-i}), \end{aligned}$$

where the first inequality follows since sophisticated players know MSP and since $H^N(a) > a$ (PR2), the strict inequality follows from the definition of $H^N(a)$, and the equality follows since players receive feedback about their own payoffs (PR1). Since a is not part of a symmetric sophisticated equilibrium then any sophisticated symmetric equilibrium a^S satisfies $a^S \geq \underline{a}^N$. ■

p	v_1	v_2	v_3
a_1	3/18	2/18	1/18
a_2	2/18	2/18	2/18
a_3	1/18	2/18	3/18

Figure Aa

	v_1	v_2	v_3
a_1	ω_1	ω_2	ω_3
a_2	ω_4	ω_5	ω_6
a_3	ω_7	ω_8	ω_9

Figure Ab

u	v_1	v_2	v_3
a_1	1	2	2
a_2	1	2	2
a_3	0	0	0

Figure Ac