The Mathematics of Credit Derivatives: Firm’s Value Models

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Basic Idea

Black and Scholes (1973) and Merton (1974):
Shares and bonds are derivatives on the firm’s assets.
Limited liability gives shareholders the option to abandon the firm, to put it to the bondholders.
Bondholders have a short position in this put option.
Accounting identity:

\[ \text{Assets} = \text{Equity} + \text{Liabilities} \]

\[ V = E + B \]
# Balance Sheet

<table>
<thead>
<tr>
<th>Assets</th>
<th>Liabilities</th>
</tr>
</thead>
<tbody>
<tr>
<td>Assets (Value of Firm)</td>
<td>$V$</td>
</tr>
<tr>
<td>Equity (Shares)</td>
<td>$E$</td>
</tr>
<tr>
<td>Debt (Bonds)</td>
<td>$\overline{B}$</td>
</tr>
</tbody>
</table>

$$V = E + \overline{B}$$

Note: A real-world balance sheet may not give all the correct numbers because of the accounting rules.
Specification of the Bonds

- Zero Bonds with maturity $T$ and face value (total) $K$

- Price $\overline{B}(t, T)$ at time $t$ per bond

- Firm is solvent at time $T$, if
  \[ V \geq K \]

- Payoff of the bond:
  \[
  \overline{B}(T, T) = \begin{cases} 
  K & \text{if solvent: } V \geq K, \\
  V & \text{if default: } V < K 
  \end{cases}
  \]
Specification of the Share

• Price $E$, no dividends

• Payoff as 'Residual Claim':
  Gets the remainder of the firm's value after paying off the debt (like Call Option):

$$E(T) = \begin{cases} 
V - K & \text{if solvent: } V \geq K, \\
0 & \text{if default: } V < K 
\end{cases}$$

limited liability for $V < K$. 
Valuation with Option Pricing Theory

The value of the whole firm is *tradeable* and available as hedge instrument:

\[ V = E + B \]

Assume lognormal dynamics for the firm's value:

\[ dV = \mu V \, dt + \sigma V \, dW \]

Then \( B \) and \( E \) must satisfy the Black-Scholes p.d.e.:

\[ 0 = \frac{\partial}{\partial t} \overline{B} + \frac{1}{2} \sigma^2 V^2 \frac{\partial^2}{\partial V^2} \overline{B} + rV \frac{\partial}{\partial V} \overline{B} - r \overline{B}. \]

\((r = \text{risk-free interest rate})\) -
Solution for the share price (Black-Scholes formula)

\[
E(t, V) = VN(d_1) - Ke^{-r(T-t)}N(d_2)
\]

\[
d_{1;2} = \frac{\ln(V/K) + (r \pm \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}
\]

Value of the bond:

\[
\bar{B}(t, T) = V - E(t, V) = Ke^{-r(T-t)}N(d_2) + VN(-d_1)
\]

Survival probability: \(N(d_2)\)
Expected recovery payoff: \(VN(-d_1)\)
**Hedging**

Set up portfolio

\[ \Pi = \overline{B}(V, t) + \Delta E(V, t) \]

by Itô’s lemma:

\[
\begin{align*}
d\Pi &= d\overline{B} + \Delta dE \\
&= \left( \frac{\partial \overline{B}}{\partial t} + \frac{1}{2} \frac{\partial^2 \overline{B}}{\partial V^2} + \Delta \frac{\partial E}{\partial t} + \frac{1}{2} \Delta \frac{\partial^2 E}{\partial V^2} \right) dt \\
&\quad + \left( \frac{\partial \overline{B}}{\partial V} + \Delta \frac{\partial E}{\partial V} \right) dV.
\end{align*}
\]

To eliminate the stochastic \(dV\)-term choose

\[ \Delta = -\frac{\partial \overline{B}}{\partial V} \frac{\partial E}{\partial V}. \]
Key Assumptions and Limitations

• Can observe (or imply) the firm’s value $V$
  \textit{Critical! For possible solutions see KPN-Case}

• The firm’s value follows a lognormal random walk
  \textit{Can be relaxed at the cost of tractability (e.g. Zhou).}

• Only zero-coupon debt
  \textit{Can be relaxed at the cost of tractability. (e.g. Geske)}

• Default only at $T$: \textit{Easy to relax. See Black-Cox and others.}

• Constant interest-rates $r$
  \textit{Very simple if independence between interest-rates and $V$. Otherwise see Briys-de Varenne and Longstaff-Schwartz.}
Pricing Accuracy

Eom, Helwege, Huang (2000)

• One-shot pricing of a cross-section of corporate bonds with asset-based models.

• Substantial pricing errors in all models.

• Merton Model:
  • generally underestimates spreads
  • by a significant amount (80% of the spread)
  • parameter variations do not help much

• Geske Model: similar to Merton model:
  severe underestimation of spreads

• Longstaff-Schwartz:
★ overestimates spreads severely for risky bonds
★ but could not raise spreads enough for good quality credits
★ still slightly better than Merton

- Leland-Toft:
  coupon size drives variation in predicted spreads

- all models have problems for short maturities or high quality

- Very poor predictive power in all cases: mean absolute errors in spreads are more than 70% of the true spread
The KMV-Approach

uses a modification of the Black-Scholes / Merton model:
the VK (Vasicek/Kealhofer) model

• equity as perpetuity
• more classes of liabilities:
  short-term, long-term, convertible, preferred equity, equity
• asset dynamics:
  drift is explicitly incorporated, adjusted for cash outflows (coupons, dividends)
• empirical distribution:
  (log)normal is inaccurate: using empirical instead
  much fatter tails than normal: DtD=4 maps into 100bp default risk (0 under Merton).
• details are not published
The Default Point

• 'Asset value, at which the firm will default'

• between total liabilities and short-term liabilities

• rule-of-thumb:

\[
[\text{Short Term Liabilities}] + \frac{1}{2}[\text{Long Term Liabilities}].
\]

• KMV do not say how other liabilities (convertibles, preferred equity) enter this relationship

• Adjustments for debt amortization by adjusting the default point:
★ debt is usually refinanced with other debt (not equity)
★ (conservatively) assume that long-term debt is refinanced short-term (increasing default point)
★ could also assume: payoff through asset value reduction
KMV’s Distance to Default

- **Distance to Default**: Summary Statistic for credit quality

\[
\text{[Distance to Default]} = \frac{\text{[Market Value of Assets]} - \text{[Default Point]}}{\text{[Market Value of Assets]} \times \text{[Asset Volatility]}}
\]

- In BSM- Setup:

\[
\text{DtD} = \frac{\ln(V/K) + (\mu - \frac{1}{2}\sigma_V^2)(T - t)}{\sigma_V \sqrt{T - t}}
\]
Expected Default Frequencies

- **Expected Default Frequency** = Frequency, with which firms of the same distance to default have defaulted in history.

- Calibration to historical data, historical asset value distribution: leaving the modelling framework.

- EDFs depend on the time-horizon. Connection via default probability:
  \[
  EDF = \text{corresponding one-year default probability} = (1 - EDF)^n = 1 - [n\text{-year default probability}]
  \]
### Linking the Firm’s Value Model to Market Variables

<table>
<thead>
<tr>
<th>Model</th>
<th>Market</th>
</tr>
</thead>
<tbody>
<tr>
<td>$V$</td>
<td>unobservable</td>
</tr>
<tr>
<td>$K$</td>
<td>total debt (or default point)</td>
</tr>
<tr>
<td>$\sigma_V$</td>
<td>unobservable</td>
</tr>
<tr>
<td>$r$</td>
<td>observable</td>
</tr>
<tr>
<td>$T$</td>
<td>user choice</td>
</tr>
</tbody>
</table>

Some Outputs

$$E = \frac{\partial E}{\partial V} V$$

$$\sigma_E = \sigma_V \frac{\partial E}{\partial V}$$

Can use market capitalisation and equity volatility to calibrate.
Asset Volatility Estimation

• Inputs: historical equity, historical debt (default points)
  From these: historical asset value time series.

• Problems:
  ★ The relationship equity value — asset value depends on asset volatility itself.
  ★ We must estimate over a period of time (e.g. 2 years): leverage may change.
    Cannot use equity volatility directly.

• Iterative Estimation Steps:
  1. Initial guess for asset volatility $\sigma_V^0$.
  2. . . . combined with time series of (equity, default points):
     $\Rightarrow$ first time series of asset values $\{V_t^0\}$
  3. Estimate next guess $\sigma_V^1$ from $\{V_t^0\}$.
  4. repeat, until convergence.

• KMV also combine this estimate with country, industry, size averages.
Advantages of Firm’s Value Models

✔ Relationships between different securities of same issuer

✔ Convertible bonds

✔ Collateralized Loans

✔ default correlation between different issuers can be modelled realistically.

✔ Fundamental orientation

✔ well-suited for theoretical questions (corporate finance)
Disadvantages of Firm’s Value Models

✘ observability of firm’s value: calibration, fitting

✘ bonds are not inputs but outputs
  defaultable bonds are far from being fundamentals

✘ all data is rarely available

✘ souvereign issuers cannot be priced

✘ often complex and unflexible

✘ unrealistic short-term spreads
Case Study: KPN

KPN is the former Dutch national telecommunications provider. The core business areas are fixed network telephony (in the Netherlands), mobile communication and data/IP services.

In the first half of 2000 KPN embarked upon an ambitious expansion course, mainly through the takeover of E-Plus (a German mobile phone provider) for which KPN paid EUR 9.1 bn in cash and EUR 9.9 bn in share conversion rights. The second large investment was the acquisition of a German UMTS license for which KPN paid EUR 6.5 bn (and its business partner 1.9 bn).

In this case study we try to analyse the effect of this on KPN’s credit risk using a Merton-type firm’s value model.
## KPN’s Balance Sheet: Assets

<table>
<thead>
<tr>
<th>Assets</th>
<th>30 June 00</th>
<th>31 Dec 99</th>
</tr>
</thead>
<tbody>
<tr>
<td>Intangible fixed assets</td>
<td>21,355</td>
<td>1,032</td>
</tr>
<tr>
<td>Property, plant, equip.</td>
<td>10,797</td>
<td>8,896</td>
</tr>
<tr>
<td>Financial fixed assets</td>
<td>1,495</td>
<td>1,376</td>
</tr>
<tr>
<td>Current assets</td>
<td>6,733</td>
<td>6,687</td>
</tr>
<tr>
<td><strong>Total Assets</strong></td>
<td><strong>40,380</strong></td>
<td><strong>17,991</strong></td>
</tr>
</tbody>
</table>

(EUR mn)
KPN’s Balance Sheet: Liabilities

<table>
<thead>
<tr>
<th>Liabilities</th>
<th>30 June 00</th>
<th>31 Dec 99</th>
</tr>
</thead>
<tbody>
<tr>
<td>Equity</td>
<td>8,913</td>
<td>6,364</td>
</tr>
<tr>
<td>Minority interests</td>
<td>152</td>
<td>25</td>
</tr>
<tr>
<td>Conversion rights</td>
<td>7,560</td>
<td></td>
</tr>
<tr>
<td>Provisions</td>
<td>652</td>
<td>795</td>
</tr>
<tr>
<td>Long-term liabilities</td>
<td>15,953</td>
<td>5,412</td>
</tr>
<tr>
<td>Current liabilities</td>
<td>7,150</td>
<td>5,395</td>
</tr>
<tr>
<td><strong>Total Liabilities</strong></td>
<td><strong>40,380</strong></td>
<td><strong>17,991</strong></td>
</tr>
</tbody>
</table>

(EUR mn)
KPN’s Debt Profile in the first Half of 2000 (EUR bn)

<table>
<thead>
<tr>
<th>December 31, 1999</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Long-term interest-bearing debt</td>
<td>5.4</td>
</tr>
<tr>
<td>Short-term interest-bearing debt</td>
<td>1.8</td>
</tr>
<tr>
<td><strong>Total interest-bearing debt</strong></td>
<td><strong>7.2</strong></td>
</tr>
<tr>
<td>+/- Floating Rate Notes</td>
<td>6.0</td>
</tr>
<tr>
<td>+/- Private placement of debt in Japan</td>
<td>1.0</td>
</tr>
<tr>
<td>+/- E-Plus credit facility (remainder)</td>
<td>4.9</td>
</tr>
<tr>
<td>+/- Debt at E-Plus level</td>
<td>1.6</td>
</tr>
<tr>
<td>+/- Redemptions</td>
<td>0.7</td>
</tr>
<tr>
<td><strong>Increase of interest-bearing debt</strong></td>
<td><strong>12.8</strong></td>
</tr>
<tr>
<td><strong>Total interest-bearing debt at June 30, 2000</strong></td>
<td><strong>20.0</strong></td>
</tr>
</tbody>
</table>
KPN’s Share Price

<table>
<thead>
<tr>
<th></th>
<th>30 June 00</th>
<th>31 Dec 99</th>
</tr>
</thead>
<tbody>
<tr>
<td>Shares Outstanding (mn)</td>
<td>958.7</td>
<td>958.7</td>
</tr>
<tr>
<td>Share Price (EUR)</td>
<td>46.87</td>
<td>47.49</td>
</tr>
<tr>
<td>Market Cap. (EUR mn)</td>
<td>44,937</td>
<td>45,534</td>
</tr>
</tbody>
</table>

In the E-Plus takeover (Feb 00), KPN has issued to BellSouth:

- an *Exchange Right*
  to exchange their remaining E-Plus shares in 200m KPN shares
- a *Warrant*
  to buy 92.6m KPN shares at EUR 39.6875
KPN’s Share Price 1999 and 2000
Case Study II: Enron

- 1986: Enron is formed.
- 1989: international expansion begins
- 1994: Enron starts electricity trading
- 1999: Enron forms its broadband services unit, Enron online is formed
- 2000: share price all-time high
- Feb. 2001: Jeff Skilling CEO. EDF: 0.35%
- Aug. 2001: Skiling resigns. EDF: 1.91%
EDF History of Enron close to Default

Source: KMV
The Nine Years Before: Share Price vs. Credit Spread

Statistical Investigation

the connection should be:

[Share] ↑ [Spread] ↓

This is true until end of 1997.

Regression results:

- For no bond did stock prices explain more than $R^2 < 59\%$ of the spreads.
- Slightly better results for changes in share prices and spreads: 4 out of 8 bonds had consistent directions with share price movements. (2 of them with significant coefficients)
- Out of 16,200 data points (spread observations), 46\% were consistent with the Merton model.
- Generally very poor explanatory power
Discussion: What went Wrong?

- Accountancy fraud does not even factor here (yet).
- Problem: DotCom-Bubble:
  - irrationally inflated share prices
  - no more indicators of value of the firm’s assets
  - only indicate expectation to “find a bigger fool”
- Better results outside of bubble. But how do we recognize a bubble in advance?!
- Hedging performance (1st differences) not good.
The Black and Cox Model

Default when firm’s value falls below the value of its liabilities.

\[ \tau = t \iff V(t) \leq K(t) \quad \text{and} \quad V(s) > K(s) \quad \text{before} \ (s < t). \]

Default as soon as insufficient collateral.

• Constant default barrier: \( K(t) = \bar{K} \)  
  (Black/Cox (1976), Longstaff/Schwartz (1997))
• Discounted default barrier: \( K(t) = B(t, T)\bar{K} \)  
  (Briys/de Varenne (1997))
• General stochastic default barrier: \( dK(t) = \ldots dt + \ldots dW \)  
  (Hull/White)
• No maturity, only default barrier: \( T \to \infty \)  
  (Leland and follow-ups)

All these approaches have the same qualitative behaviour.
Default Costs

If $V$ (and $K$) has continuous paths, we are able to predict the default-value of $V$ one moment before default.

Need default costs to have loss in default or stochastic (or lower) recoveries.

Payoff of bonds:

$K$ at maturity $T$, if there was no default previously: $\tau > T$

$cB(\tau, T)$ at $\tau$, if default before maturity: $\tau \leq T$.

$c$ as in recovery models for intensity.
The Survival Probability

The probability, that a BM with drift

$$X(t) = X(0) + \mu t + \sigma W(t)$$

does not hit the barrier $K$ before time $T$, is

$$N \left( \frac{\mu T - (K - X(0))}{\sigma \sqrt{T}} \right) - e^{2\frac{(K-X(0))\mu}{\sigma^2}} N \left( \frac{\mu T + (K - X(0))}{\sigma \sqrt{T}} \right).$$
How to avoid zero short-term spreads?

Cause: If there is a *finite distance* to the barrier, a *continuous process* cannot reach it in the next instance.

- Introduce jumps in the firm’s value $V$
  (Zhou 2001)

- Maybe the default barrier is indeed closer than we thought.
  Duffie/Lando (1997), and partially: Giesecke (2002), Finkelstein, Lardy et.al. (2002)
The Idea of Duffie and Lando

Defaults happen, when the firm’s value $V(t)$ hits a lower barrier $K(t)$ but we do not know the true value of the firm.

We know:

- $V(t) > K(t)$: there has been no default so far
- $f(t, v)$: some prior probability density function for our guess (at time $t$), where $V(t)$ actually is:
  * $f(t, K) = 0$: no default so far
  * $P[V(t) \in [v, v + dv]] = f(t, v)dv$
- the dynamics of $V$ ($\mu_V$ and $\sigma_V$ can be stochastic)

\[
dV = \mu_V dt + \sigma_V dW
\]
Density of $V$

Possible Range of $V$
How can a default happen?

The **law of the iterated logarithm** gives the size of the local fluctuations of a Brownian motion:

*Over a small time interval $[t, t + \Delta t]$ the Brownian motion will fluctuate up and down by $\pm \sqrt{\Delta t}$ with probability 1. Not more, not less.*

This holds in the limit as $\Delta t \to 0$.

It will even hit

$$\pm \sqrt{\Delta t} \ln(\ln(\frac{1}{\Delta t})), \quad 1$$

but it will not exceed these values.

Over $[t, t + \Delta t]$ the worst movement for $V$ is therefore

$$\Delta V^{\text{bad}} = \mu_V \Delta t - \sigma_V \sqrt{\Delta t}$$

---

$^1$We will ignore the $\ln \ln(1/\Delta t)$ - term because it grows far too slowly to have an effect.
If we can observe $V$ with certainty, there are two cases:

(i) $V(t) > K + \sigma_V \sqrt{\Delta t}$:
$V$ is too far away from the barrier. No default will happen, even for the worst-case movement.

(ii) $K < V(t) \leq K + \sigma_V \sqrt{\Delta t}$:
$V$ is very close to the barrier. Here a default can happen over the next time step. As $\Delta t \to 0$ we know, that it will indeed happen.
The Probability of a Default

What is the probability of being in case (ii)?

\[
P \left[ K < V(t) \leq K + \sigma_V \sqrt{\Delta t} \right] = \int_{K}^{K+\sigma_V \sqrt{\Delta t}} f(t, v) dv
\]

Note (Taylor): \( f(t, K) = 0 \) and \( f \) is approximately linear over small intervals

\[
f(t, x) \approx f(t, K) + f'(t, K)(x - K) = f'(t, K)(x - K)
\]

for \( x - K \) small.
The Rate of Defaults

The probability of being close to a default is:

\[
P \left[ K < V(t) \leq K + \sigma_V \sqrt{\Delta t} \right] \\
= \int_K^{K+\sigma_V \sqrt{\Delta t}} f(t, v) dv \\
= f'(t, K) \int_K^{K+\sigma_V \sqrt{\Delta t}} (v - K) dv \\
= f'(t, K) \int_0^{\sigma_V \sqrt{\Delta t}} v \, dv = f'(t, K) \frac{1}{2} \sigma^2_V \Delta t
\]
The Default Intensity

Over a small time interval \([t, t + \Delta t]\), the probability of default is proportional to the length of that time interval.

\[
\lim_{\Delta t \to 0} \frac{1}{\Delta t} P \left[ \text{default in} \ [t, t + \Delta t] \right] = \frac{1}{2} \sigma_V^2 f'(t, KK).
\]

This defines the defaults as an intensity process with intensity

\[
\lambda = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \frac{1}{2} \sigma_V^2 f'(t, K) \Delta t = \frac{1}{2} \sigma_V^2 f'(t, K)
\]
Summary: Duffie and Lando

- The probability of default over a short time-interval is proportional to the size of that time interval.
- From the point of view of the investor:
  Defaults are triggered by a jump process with intensity

\[ \frac{1}{2} \sigma^2 \frac{f'}{f}(t, K) \]

- The 'steeper' \( f \) is at \( K \), the more probability mass close to \( K \), the higher the likelihood of a default.
- Need to update \( f \) after the next time step (can get very complicated).
A Simple Special Case: Delayed Observation

• At time $t$, we do not observe $V(t)$ but only $V(t - \Delta) = v$
  E.g. we only get the numbers of one quarter/one year ago.

• We also observe if there was a default in $[t - \Delta, t]$.

• Our conditional distribution of the firm’s value given this information is therefore

$$P[V(t) \leq H \mid \mathcal{F}_t] = P[V(t) \leq H \mid \{V(t - \Delta) = v\} \land \{V(s) > K \forall s \in [t - \Delta, t]\}]$$

  This is known in closed-form (next slide).

• This allows us to calculate the default intensities directly.
The Joint Distribution

Let \( \frac{dV}{V} = \mu dt + \sigma dW \), \( V(0) = V_0 \) and \( m_V(T) := \min_{t \leq T} V(t) \). Then

\[
\mathbb{P} \left[ V(T) \geq H \land m_V(T) \geq K \right] = N(d_3) - \left( \frac{K}{V_0} \right)^{(2\mu/\sigma^2)-1} N(d_4)
\]

where

\[
d_3 = \frac{\ln(V_0/H) + (\mu - \frac{1}{2}\sigma^2)T}{\sigma \sqrt{T}}
\]

\[
d_4 = \frac{\ln(K^2/(V_0H)) + (\mu - \frac{1}{2}\sigma^2)T}{\sigma \sqrt{T}}
\]
The Idea of Lardy and Finkelstein (CreditGrades)

Defaults happen, when the firm’s value $V(t)$ hits a lower barrier $K(t)$ but we do not know the true value of the lower barrier.

Almost the same as Duffie/Lando, but unfortunately not exactly. With quite unrealistic consequences.

What do we learn now?

• We know today’s (time $t$) firm’s value.
• We know that no default has occurred yet.
• Hence, the barrier must be less than the running minimum of the firm’s value up to now.
Resulting Dynamics
Results: Dynamics

(see Giesecke (2002))

- The default compensator behaves like the running maximum of a diffusion process.

- The time of default is totally inaccessible, but a default intensity does not exist.

- Unless \( V \) equals its current running minimum, we again have zero short-term credit spreads.

- At \( t = 0 \), there is a discrete, positive probability of default.
• The model should not be used for hedging: The spreads change shape drastically (and unrealistically) at $t = 0$.

• For credit pricing at $t = 0$ the model may just be acceptable (because it is reduced to Duffie/Lando with delayed observation).

• A US patent application has been filed for CreditGrades. Personally, I disapprove of this.

• My advice: Duffie/Lando with delayed observations is better anyway.
References


[3] Young Ho Eom, Jean Helwege, and Jin Zhi Huang. Structural models of corporate bond pricing: An empirical analysis. working paper, Finance Department, Ohio State University, Ohio State University, Columbus, OH 43210, USA, October 2000.


