Appendix A: Proofs

PROOF OF PROPOSITION 1: Consider the seller's optimal regret R^{*C} in (6). A lower bound of this regret is obtained by restricting nature's optimization to the subset $\mathcal{D}_0^C \subseteq \mathcal{D}^C$ such that $\{v,\tau\} \in \mathcal{D}_0^C$ if and only if $v_1 = v_2 = \cdots = v_C$ and $\tau_1 = \tau_2 = \cdots = \tau_C$, that is, all C customers have identical valuations and arrival times. But if all customers are identical then the seller's regret is equal to C times the optimal regret with a single customer, that is, $R^{*C} \geq C\, R^*$.

On the other hand, any fixed pricing strategy p_t leads to an upper bound for R^{*C} in the sense that

$$R^{*C} \leq \sup_{\{v,\tau\}\in\mathcal{D}^C} \Pi_F^C(v,\tau) - \sum_{i=1}^{C} e^{-r d(v_i,\tau_i,p)}\, p_{d(v_i,\tau_i,p)}.$$

Furthermore, it is not hard to see that

$$\Pi_F^C(v,\tau) \leq \sum_{i=1}^{C} e^{-r\tau_i}\, v_i$$

since the right-hand side is the payoff that a clairvoyant can get under perfect price discrimination (i.e., charging a different price to every customer). It follows that

$$R^{*C} \leq \sup_{\{v,\tau\}\in\mathcal{D}^C} \sum_{i=1}^{C} e^{-r\tau_i}\, v_i - e^{-rd(v_i,\tau_i,p)}\, p_{d(v_i,\tau_i,p)} = \sum_{i=1}^{C} \sup_{\{v_i,\tau_i\}\in\mathcal{D}} e^{-r\tau_i}\, v_i - e^{-rd(v_i,\tau_i,p)}\, p_{d(v_i,\tau_i,p)},$$

hence the maximization above decouples for every customer. But if the seller selects a pricing vector p_t^* which is an optimal pricing strategy for the case of single customer, then each term in the summation on the right equals R^*, that is, $R^{*C} \leq C\, R^*$. We conclude that $R^{*C} = C\, R^*$. □

PROOF OF LEMMA 1: Consider a price path p_t such that $p_t \geq \underline{p}_t(R)$ for all $t \in [0,T]$. The worst-case valuation regret from p_t satisfies the following inequality

$$\max_t e^{-rt}(\bar{v} - p_t) \leq \max_t e^{-rt}(\bar{v} - \underline{p}_t(R)) \leq \max_t e^{-rt}(\bar{v} - (\bar{v} - e^{rt}R)) = R. \ \square$$

PROOF OF LEMMA 2: Consider an arbitrary price path $p \in \mathcal{P}$ (*i.e.*, a continuous function from $[0,T]$ to $[\underline{v},\bar{v}]$) and let

$$\mathcal{R}_{\mathrm{M}}(p) = \sup_{(v,\tau)\in\mathcal{D}} \{\mathcal{R}_{\mathrm{M}}(v,\tau,p)\} = \sup_{(v,\tau)\in\mathcal{D}} \left\{ e^{-r\tau}\, v - e^{-r\, d_M(v,\tau,p)}\, p_{d_M(v,\tau,p)} \right\},$$

be the corresponding seller's worst-case regret. In what follows, we show that there always exists a (weakly decreasing) price path $\hat{p}$ such that $\mathcal{R}_{\mathrm{M}}(\hat{p}) \leq \mathcal{R}_{\mathrm{M}}(p)$.

Indeed, for a given $p \in \mathcal{P}$, we let $\hat{p}$ be the running minimum price path induced by p which is given by $\hat{p}_t := \min\{p_\tau : \tau \in [0,t]\}$ for $t \in [0,T]$ (the 'min' is well-defined by the continuity of p

and Weierstrass Theorem). By construction, $\hat{p}$ is (weakly) decreasing and continuous (since p is continuous). In order to show that $\mathcal{R}_{\mathrm{M}}(p) \geq \mathcal{R}_{\mathrm{M}}(\hat{p})$ let us partition the customer type's space $\mathcal{D}$ into the following three subsets $\mathcal{D}_1 := \{(v,\tau) \in \mathcal{D} : p_\tau \leq v\}$, $\mathcal{D}_2 := \{(v,\tau) \in \mathcal{D} : \hat{p}_\tau \leq v < p_\tau\}$ and $\mathcal{D}_3 := \{(v,\tau) \in \mathcal{D} : \hat{p}_\tau > v\}$ and define $R_i(p) := \sup_{(v,\tau)\in\mathcal{D}_i} \{\mathcal{R}_{\mathrm{M}}(v,\tau,p)\}$. In what follows, we show that $R_i(p) = R_i(\hat{p})$ for $i = 1,3$ and that $R_2(\hat{p}) \leq R_1(\hat{p})$. As a result,

$$\mathcal{R}_{\mathrm{M}}(p) = \max_{i=1,2,3}\{R_i(p)\} \geq \max_{i=1,3}\{R_i(p)\} = \max_{i=1,3}\{R_i(\hat{p})\} = \max_{i=1,2,3}\{R_i(\hat{p})\} = \mathcal{R}_{\mathrm{M}}(\hat{p}).$$

-) Let $(v,\tau) \in \mathcal{D}_1$. By the definition of $\hat{p}$, there exists a $t \leq \tau$ such that $\hat{p}(t) = p(t) \leq p(\tau) \leq v$. Since consumers are myopic, it follows that $\mathcal{R}_{\mathrm{M}}(v,t,p) = \mathcal{R}_{\mathrm{M}}(v,t,\hat{p}) = e^{-r\,t}(v - p_t) \geq \mathcal{R}_{\mathrm{M}}(v,\tau,p)$. Since $(v,t) \in \mathcal{D}_1$, we conclude that $R_1(p) = R_1(\hat{p})$.

-) Let $(v,\tau) \in \mathcal{D}_3$. In this case, it is easy to see that $d_M(v,\tau,p) = d_M(v,\tau,\hat{p})$ (possibly ∞) and $p(d_M(v,\tau,p)) = \hat{p}(d_M(v,\tau,\hat{p})$. It follows that $\mathcal{R}_{\mathrm{M}}(v,\tau,p) = \mathcal{R}_{\mathrm{M}}(v,\tau,\hat{p})$. We conclude that $R_3(p) = R_3(\hat{p})$.

-) Let $(v,\tau) \in \mathcal{D}_2$. Since $p_\tau > \hat{p}_\tau$ there exists a $t \in [0,\tau)$ such that $\hat{p}(t) = p(t) \leq v$. It follows that $\mathcal{R}_{\mathrm{M}}(v,\tau,\hat{p}) = e^{-r\,\tau}(v - \hat{p}_\tau) \leq e^{-r\,t}(v - \hat{p}_t) = \mathcal{R}_{\mathrm{M}}(v,t,\hat{p})$. Furthermore, $(v,t) \in \mathcal{D}_1$. Hence, $R_2(\hat{p}) \leq R_1(\hat{p})$.

It follows from the three cases above that $\mathcal{R}_{\mathrm{M}}(p) \geq \mathcal{R}_{\mathrm{M}}(\hat{p})$. Hence, without loss of optimality the seller can restrict attention to (weakly) decreasing price functions. □

Proof of Lemma 3: Let $p \in \mathcal{P}$ be a decreasing function. For any customer type (v,τ) with $v < p_\tau$, the delay regret he generates is equal to $(e^{-r\,\tau} - e^{-r\,d_M(v,\tau,p)})\,v$. We distinguish two cases:

- Case 1: $d_M(v,\tau,p) \leq T$. In this case the customer buys the product at a future time $d_M(v,\tau,p)$ such that $p_{d_M(v,\tau,p)} = v$ and he generates a delay regret $(e^{-r\,\tau} - e^{-r\,d_M(v,\tau,p)})\,p_{d_M(v,\tau,p)}$ which is bounded above by $(1 - e^{-r\,d_M(v,\tau,p)})\,p_{d_M(v,\tau,p)}$. But, by hypothesis, the price path p satisfies $p_t \leq R/(1 - e^{-r\,t}) \vee \underline{v}$ for all $t \in [0,T]$. Furthermore, from the monotonicity of p and the fact that $v < p_\tau$ it is not hard to see that $R/(1 - e^{-r\,d_M(v,\tau,p)}) \geq \underline{v}$. As a result, $(1 - e^{-r\,d_M(v,\tau,p)})\,p_{d_M(v,\tau,p)} \leq R$, that is, the delay regret generated by the customer is less than or equal to R.

- Case 2: $d_M(v,\tau,p) = \infty$. In this case the customer is priced out of the market. Since p is a decreasing price path this means that $v < p_T$. But, by hypothesis, $p_T \leq R \vee \underline{v}$, hence this case can only occur if $R > \underline{v}$. So, assuming $R > \underline{v}$, the delay regret in this case is equal to $e^{-r\,\tau}\,v$, which is bounded above by v, which together with the inequalities $v < p_T < R$ imply that the delay regret generated by the customer is less than R. □

PROOF OF PROPOSITION 2: This result follows immediately from Lemmas 1-3. □

PROOF OF THEOREM 1: Proposition 2 characterizes the set of minimax regret price paths. Region A_1 is the set of problem parameters where the solution is given by the price paths $\underline{p}_t(R)$ and $\overline{p}_t(R)$ intersecting tangentially as described by equations (7) and (8). For this solution to be feasible, the ratio $u = \underline{v}/\overline{v}$ must satisfy $u \le 1/2$ in order to ensure the middle equation of (8) is feasible. The solution must also satisfy $\underline{p}_t(R) = \overline{v} - e^{rT} R \le R \vee \underline{v}$, where $R = \overline{v}/4$ from the right-most equation in (8). Therefore, A_1 is characterized by

$$u \le \frac{1}{2} \text{ and } T \ge \min\left\{\frac{\ln 3}{r}, \frac{\ln\big(4(1-u)\big)}{r}\right\}.$$

In regions A_2, A_3 and A_4, the boundary conditions (u and/or T) become binding and, thus, play a role in determining the minimax regret. If $u \ge \frac{1}{2}$ and the selling horizon is sufficiently long, the binding constraint becomes $\underline{p}_{t^*}(R^*_{\mathrm{M}}) = \bar{p}_{t^*}(R^*_{\mathrm{M}}) = \underline{v}$ for some t^*, as given by equation (9). We can solve for R^*_{M} and t^* to get $R^*_{\mathrm{M}} = u(1-u)\bar{v}$ and $t^* = \frac{1}{r}\ln\frac{1}{u}$. Thus, we need $T \ge \frac{1}{r}\ln\frac{1}{u}$ to guarantee to obtain $t^* \le T$. The pair of conditions $u \ge \frac{1}{2}$ and $T \ge \frac{1}{r}\ln\frac{1}{u}$ lead to the definition of A_2.

In the regime where $(u,T) \notin A_1 \cup A_2$, the binding constraint becomes $\underline{p}_T(R^*_{\mathrm{M}}) = \max\{R^*_{\mathrm{M}}, \underline{v}\}$. One can solve the equation

$$\bar{v} - e^{rt} R^*_{\mathrm{M}} = \max\{R^*_{\mathrm{M}}, \underline{v}\} \tag{17}$$

and get the following results: if $T \le \min\{\ln(3)/r, \ln(1/u-1)/r\}$, then the value that maximizes the right-hand side of equation (17) is R^*_{M}, generating the solution associated with region A_3; otherwise, the right-hand side of equation (17) is maximized by $\underline{v}$, producing the solution for region A_4. □

PROOF OF THEOREM 2: Consider an arbitrary price path p with regret $\mathcal{R}_{\mathrm{M}}(p)$ and let us suppose that $\mathcal{R}_{\mathrm{M}}(p) < R^*_{\mathrm{M}}$. In what follows, we will show that the previous inequality leads to a contradiction. For this, consider the critical time t^* (see the discussion that precedes Theorem 1). It is not hard to see that the lower bound $\underline{p}_t(R^*_{\mathrm{M}})$ is strictly decreasing in $[0,t^*]$. It follows, under the assumption that $\mathcal{R}_{\mathrm{M}}(p) < R^*_{\mathrm{M}}$, that we must have that $p_t > \underline{p}_t(R^*_{\mathrm{M}})$ for all $t \in [0,t^*]$. Otherwise, we would be able to find a customer type that generates a valuation regret strictly greater than R^*_{M} under the price path p, and this would violate our hypothesis $\mathcal{R}_{\mathrm{M}}(p) < R^*_{\mathrm{M}}$.

Now, consider a customer with type $(v,\tau) = (\underline{p}_{t^*}(R^*_{\mathrm{M}}), 0)$. Since $p_t > \underline{p}_t(R^*_{\mathrm{M}})$ for all $t \in [0,t^*]$ and $\underline{p}_t(R^*_{\mathrm{M}})$ is strictly decreasing in this interval, it follows that the purchasing time of this customer under the price path p satisfies $d_M(\underline{p}_{t^*}(R^*_{\mathrm{M}}), 0, p) > t^*$. In addition, we trivially must have that $\mathcal{R}_{\mathrm{M}}(p)$ is greater than or equal to the delay regret generated by this consumer. It follows that

$$\mathcal{R}_{\mathrm{M}}(p) \ge \big(1 - \exp(-r\, d_M(\underline{p}_{t^*}(R^*_{\mathrm{M}}), 0, p)\big)\, \underline{p}_{t^*}(R^*_{\mathrm{M}}) > \big(1 - \exp(-r\, t^*\big)\, \underline{p}_{t^*}(R^*_{\mathrm{M}}). \tag{18}$$

Let us now evaluate this expression for each of the four parameter regimes (regions A_1 through A_4) identified in Theorem 1:

- In region A_1, we have that $R^*_{\rm M} = \bar{v}/4$, $t^* = \ln(2)/r$ and $\underline{p}_{t^*}(R^*_{\rm M}) = \bar{v}/2$. As a result, we have that $\big(1-\exp(-r\,t^*\big)\,\underline{p}_{t^*}(R^*_{\rm M}) = R^*_{\rm M}$, which together with (18), contradicts the hypothesis $\mathcal{R}_{\rm M}(p) < R^*_{\rm M}$.

- In region A_2, we have that $R^*_{\rm M} = \underline{v}\,(1-\underline{v}/\bar{v})$, $t^* = \ln(\bar{v}/\underline{v})/r$ and $\underline{p}_{t^*}(R^*_{\rm M}) = \underline{v}$. In this case, we get that $\big(1-\exp(-r\,t^*\big)\,\underline{p}_{t^*}(R^*_{\rm M}) = R^*_{\rm M}$, which again contradicts $\mathcal{R}_{\rm M}(p) < R^*_{\rm M}$.

- In region A_3, we have that $t^* = T$ and so $d_M(\underline{p}_{t^*}(R^*_{\rm M}), 0, p) = \infty$, *i.e.*, the customer is price out of the market. In addition, $\underline{p}_{t^*}(R^*_{\rm M}) = R^*_{\rm M}$. It follows from (18) that $\mathcal{R}_{\rm M}(p) \geq R^*_{\rm M}$, which contradicts our hypothesis.

- In region A_4, we have again that $t^* = T$ and $d_M(\underline{p}_{t^*}(R^*_{\rm M}), 0, p) = \infty$. In this case, we have that $\underline{p}_{t^*}(R^*_{\rm M}) = \underline{v} \geq R^*_{\rm M}$. Once again, it follows from (18) that $\mathcal{R}_{\rm M}(p) \geq R^*_{\rm M}$, which contradicts our hypothesis and the proof is complete. $\square$

Proof of Lemma 4: See Part V in Rockafellar (1997). $\square$

Proof of Proposition 3: Note first that the proposed solution θ^* is feasible in the sense that $\theta^* \in \Theta(\hat{v})$. Let us define $\tilde{v} := \max\{\hat{v}, \bar{v}\,\exp(\theta_0 - 1)\}$. The proposed solution can be written as:

$$\theta^*(x) = \begin{cases} \theta_0 & \text{if} \quad \hat{v} \leq x < \tilde{v} \\ 1 - \ln(\bar{v}) + \ln(x) & \text{if} \quad \tilde{v} \leq x \leq \bar{v}. \end{cases}$$

It follows that the regret of a customer with valuation $v \in [\hat{v}, \bar{v}]$ under θ^* is equal to

$$\int_{\hat{v}}^{v} \theta^*(x)\,\mathrm{d}x + v\,(1-\theta^*(v)) = \min\{v, \tilde{v}\} + \hat{v}\,(\ln(\bar{v}) - \ln(\tilde{v}) - 1).$$

An important property of this regret is that it is increasing in v and it is constant in the interval $[\tilde{v}, \bar{v}]$. Hence, the seller's worst-case regret under θ^* is achieved in the entire interval $[\tilde{v}, \bar{v}]$. We will use this property to prove the optimality of θ^* using a variational arguments. To this end, suppose that θ^* is not optimal. Hence, there is a $\theta \in \Theta(\hat{v})$ that generates a strictly better regret than the one generated by θ^* and let us define $\epsilon(x) := \theta(x) - \theta^*(x)$ for all $x \in [\hat{v}, \bar{v}]$. By feasibility, we need $\epsilon(x)$ to be right-continuous and such that $\theta(x) \in [\theta_0, 1]$ for all $x \in [\hat{v}, \bar{v}]$. For instance, since $\theta^*(\bar{v}) = 1$, we must have $\epsilon(\bar{v}) \leq 0$. Similarly, since $\theta^*(x) = \theta_0$ for all $x \in [\hat{v}, \tilde{v})$, we must have $\epsilon(x) \geq 0$ in this interval.

Now, the assumption that θ generates a strictly better regret than θ^* and the fact that seller's worst-case regret under θ^* is achieved in the entire interval $[\tilde{v}, \bar{v}]$ imply that $\epsilon(x)$ must satisfy

$$\int_{\hat{v}}^{v} \epsilon(x)\,\mathrm{d}x - v\,\epsilon(v) < 0 \qquad \text{for all } v \in [\tilde{v}, \bar{v}]. \tag{19}$$

Since $\epsilon(x) \geq 0$ in the interval $x \in [\hat{v}, \tilde{v})$, we must have $\epsilon(\tilde{v}) > 0$, since $\epsilon(\tilde{v}) > \frac{1}{\tilde{v}} \int_{\hat{v}}^{\tilde{v}} \epsilon(x)\,\mathrm{d}x \geq 0$. Let us show that $\epsilon(x) \geq 0$ for all $x \in [\tilde{v}, \bar{v})$. Suppose, by contradiction, that this is not the case. Then, there exist a $\delta < 0$ and $y = \inf\{v \in (\tilde{v}, \bar{v}) : \epsilon(v) \leq \delta\}$. Also, since $\epsilon(x)$ is right-continuous and $\epsilon(\tilde{v}) > 0$, we must have $y > \tilde{v}$ and

$$\int_{\hat{v}}^{y} \epsilon(x)\,\mathrm{d}x > \delta\,(y - \hat{v}) \geq \epsilon(y)(y - \hat{v}) \geq y\,\epsilon(y).$$

This contradicts (19) and we must have $\epsilon(x) \geq 0$ for all $x \in [\hat{v}, \bar{v})$. Combining the non-negativity and right-continuity of $\epsilon(x)$ and the facts that $\epsilon(\tilde{v}) > 0$ and $\epsilon(\bar{v}) \leq 0$, we get

$$\int_{\hat{v}}^{\bar{v}} \epsilon(x)\,\mathrm{d}x - \bar{v}\,\epsilon(\bar{v}) > 0.$$

This again contradicts (19), which proves the optimality of θ^*. □

Proof of Theorem 3: From Proposition 3 we have that for an arbitrary $\hat{v} \in [\underline{v}, \bar{v}]$

$$R_{\hat{v}} = \hat{v}\,\mathbb{1}(\hat{v} > \underline{v}) \vee \tilde{v} + \hat{v}\,(\ln(\bar{v}) - \ln(\tilde{v}) - 1), \qquad \text{where} \quad \tilde{v} := \max\{\hat{v}, \bar{v}\,\exp(e^{-r\,T} - 1)\}.$$

Hence, it is a matter of relatively straightforward calculations to show that

$$\hat{v}^* := \underset{\underline{v} \leq \hat{v} \leq \bar{v}}{\arg\max}\,\{R_{\hat{v}}\} = \max\left\{\underline{v}\,,\ \frac{\bar{v}\,\exp(e^{-r\,T} - 1)}{1 + e^{-r\,T}}\right\} \quad \text{and}$$

$$R_0^* = R_{\hat{v}^*} = \tilde{v}^* + \hat{v}^*\,(\ln(\bar{v}) - \ln(\tilde{v}^*) - 1), \qquad \text{where} \quad \tilde{v}^* := \max\{\hat{v}^*, \bar{v}\,\exp(e^{-r\,T} - 1)\}.$$

Recall that $\hat{v}^*$ represents is the threshold that separates consumers' valuations into those that buy and those that do not buy the product. From Proposition 3, we have that the corresponding optimal purchasing strategy of those customers that buy the product is given by $\theta^*(v) = \theta_0 \vee 1 - \ln(\bar{v}) + \ln(v)$ for $v \in [\hat{v}^*, \bar{v}]$. We can reverse the change of variable, $\theta = e^{-r\,t}$, to rewrite this consumers' purchasing strategy in term of purchasing time. Indeed, if we let $d^*(v)$ be the time at which a consumer with valuation $v \in [\hat{v}^*, \bar{v}]$ buys the product, we have that

$$d^*(v) = -\frac{1}{r}\,\ln(\theta^*(v)) = -\frac{1}{r}\ln\Big(1 - \ln(\bar{v}) + \ln(v)\Big) \wedge T \qquad \text{for all } v \in [\hat{v}^*, \bar{v}].$$

The last step in the proof is a *verification* step. Essentially, we need to verify that the proposed pricing strategy implements the consumers' optimal strategy $\theta^*(v)$ (or equivalently $d^*(v)$). The optimality of the pricing strategy will then follow from the optimality of $\theta^*(v)$ in Proposition 3. Let us recall from the statement of the theorem that our proposed pricing strategy is equal to

$$p_{\mathrm{S}}^*(t) = \begin{cases} e^{r\,t}\ (\bar{v}\,\exp(e^{-r\,t} - 1) - R_0^*) & \text{for all} \quad t \in [0, d^*(\hat{v}^*)] \\ \underline{v} & \text{for all} \quad t \in (d^*(\hat{v}^*), T]. \end{cases}$$

To verify that $p_S^*(t)$ implements $d^*(v)$ we need to show that the following conditions hold

a) For all $v \in [\underline{v}, \hat{v}^*) : \max_{0\le t\le T} \left\{e^{-r\,t}\,[v - p_S^*(t)]\right\} < 0$,

b) For all $v \in [\hat{v}^*, \bar{v}] : d^*(v) = \max\left\{\operatorname{argmax}_{0\le t\le T} \left\{e^{-r\,t}\,[v - p_S^*(t)]\right\}\right\}$ and

c) $\hat{v}^* = p_S^*(d^*(\hat{v}^*))$.

Condition (a) guarantees that customers with valuation below $\hat{v}^*$ leave the market without buying. Condition (b) guarantees that the price path p_t implements $d^*(v)$ for $v \ge \hat{v}^*$. Finally, condition (c) ensures that the lowest type that buys the product $v = \hat{v}^*$ gets zero utility (recall the definition of $\hat{v}$ in equation (14)).

In order to verify the three conditions above we identify three cases depending on the values of $\underline{v}$, $\bar{v}$ and $\theta_0 = e^{-r\,T}$: (i) $\underline{v} \le \bar{v}\,\exp(\theta_0 - 1)/(1+\theta_0)$, (ii) $\bar{v}\,\exp(\theta_0 - 1)/(1+\theta_0) \le \underline{v} \le \bar{v}\,\exp(\theta_0 - 1)$, and (iii) $\underline{v} \ge \bar{v}\,\exp(\theta_0 - 1)$. We will verify case (i) and leave the other two to the reader as they follow similar steps. These cases correspond to the regions B_1, B_2 and B_3 in Figure 3 respectively.

For case (i), we have that $\hat{v}^* = \bar{v}\,\exp(\theta_0 - 1)/(1+\theta_0)$, $\tilde{v}^* = \bar{v}\,\exp(\theta_0 - 1)$, $R_0^* = \hat{v}^*$ and

$$d^*(v) = \begin{cases} T & \text{for all } v \in [\hat{v}^*, \tilde{v}^*] \\ -\ln(1 - \ln(\bar{v}) + \ln(v))/r & \text{for all } v \in [\tilde{v}^*, \bar{v}]. \end{cases}$$

As a result, the proposed price path takes the form

$$p_S^*(t) = e^{r\,t}\left(\bar{v}\,\exp(e^{-rt} - 1) - \bar{v}\,\frac{\exp(\theta_0 - 1)}{1+\theta_0}\right), \qquad \text{for all } t \in [0, T].$$

Hence the consumer's utility maximization problem is equal to

$$\max_{0\le t\le T} \left\{e^{-r\,t}[v - p_S^*(t)]\right\} = \max_{0\le t\le T} \left\{e^{-r\,t}v - \bar{v}\,\exp(e^{-rt} - 1) - \bar{v}\,\frac{\exp(\theta_0 - 1)}{1+\theta_0}\right\}.$$

Since the function $e^{-r\,t}v - \bar{v}\,\exp(e^{-rt} - 1)$ is unimodal (inverted U-shaped) for $t \in [0, T]$, we can use first-order condition to solve the consumer's problem. This FOC is given by

$$r\,e^{-r\,t}\,\left(\bar{v}\,\exp(e^{-r\,t}) - v\right) = 0 \qquad \Longrightarrow \qquad t(v) = -\frac{1}{r}\,\ln\left(1 - \ln(\bar{v}) + \ln(v)\right).$$

It is a matter of simple calculations to show that $t(v) \le T$ if and only if $v \ge \tilde{v}^*$ and so for values of v in $[\hat{v}^*, \tilde{v}^*]$ the consumer's utility is maximized at $t = T$. Hence, we conclude that the optimal purchasing time of consumers with valuation in $[\hat{v}^*, \bar{v}]$ is precisely

$$d^*(v) = \begin{cases} T & \text{for all } v \in [\hat{v}^*, \tilde{v}^*] \\ -\ln(1 - \ln(\bar{v}) + \ln(v))/r & \text{for all } v \in [\tilde{v}^*, \bar{v}] \end{cases}$$

as required. Finally, it is easy to verify that $p_S^*(d^*(\hat{v}^*)) = \hat{v}^*$ and so customer with valuation equal to $\hat{v}^*$ makes zero utility and all customers with valuation strictly below $\hat{v}^*$ make a negative utility under $p_S^*(t)$ and hence leave the market without purchasing the product. $\square$

PROOF OF LEMMA 5: To alleviate the notation, let us write $p_t = p^*_{\mathrm{S}}(t)$ and $\tilde{p}_t = \tilde{p}^*_{\mathrm{S}}(t)$.

The first step in the proof is to show that the function p_t is continuously differentiable in $[0,T]$. This holds trivially for the case in which $d^*(\hat{v}^*) = T$. On the other hand, if $d^*(\hat{v}^*) < T$, what we need to show is that the left-derivative of p_t at $t = d^*(\hat{v}^*)$ is equal to 0. To see this, note that for all $t \in [0, d^*(\hat{v}^*)]$,

$$\dot{p}_t = r\,(p_t - R^*_0) - r\,e^{-r\,t}\,p_t.$$

Now, the condition $d^*(\hat{v}^*) < T$ implies that $d^*(\hat{v}^*) = -\ln\big(1 - \ln(\bar{v}) + \ln(\underline{v})\big)/r$, $p_{d^*(\hat{v}^*)} = \underline{v}$ and $R^*_0 = \underline{v}\,(\ln(\bar{v}) - \ln(\underline{v}))$. Plugging these values, we get that $\dot{p}_t = 0$ at $t = d^*(\hat{v}^*)$ and so p_t is continuously differentiable in $[0,T]$.

Now, from the fact that p_t is decreasing and continuously differentiable in $[0,T]$, the definition of $\tilde{p}_t$ in (11) implies that

$$\begin{aligned} \tilde{p}_t &= \sup_{s\in(t,T]} \left\{ \frac{e^{-r\,t}\,p_t - e^{-r\,s}\,p_s}{e^{-r\,t} - e^{-r\,s}} \right\} = \sup_{s\in(t,T]} \left\{ \frac{\int_t^s r\,e^{-r\,u}\,(p_u - \dot{p}_u/r)\,\mathrm{d}u}{\int_t^s r\,e^{-r\,u}\,\mathrm{d}u} \right\} \qquad \text{for all } t \in [0,T) \\ \tilde{p}_T &= p_T. \end{aligned}$$

But $p_t - \dot{p}_t/r = R^*_0 + e^{-r\,t}\,p_t$, which is a decreasing function of t. It follows that the supremum above is attained as $s \downarrow t$. Finally, from L'Hôpital's rule, we get then that $\tilde{p}_t = p_t - \dot{p}_t/r$, which (as we just showed) is decreasing in $[0,T]$. □

PROOF OF THEOREM 4: Since the optimal price path $p^*_{\mathrm{S}}(t)$ was derived assuming that all customer arrive at time $t = 0$ (i.e., customers such as customer 'B' in Figure 9), we only need to show that the worst-case regret of a customer with type (v,t) in the upper boundary $\mathcal{D}_{\bar{v}} = \{(\bar{v},t) : t \in [0,T]\}$ is less than or equal to the regret R^*_0 for all $t \in [0,T]$. However, this condition follows trivially since the regret generated by customer $(\bar{v},t)$ if the seller uses price $p^*_{\mathrm{S}}(t)$ is equal to

$$e^{-r\,t}[\bar{v} - p^*_{\mathrm{S}}(t)] = \begin{cases} R^*_0 - \bar{v}\left(\exp(e^{-r\,t} - 1) - e^{-r\,t}\right) & \text{for all} \quad t \in [0, d^*(\hat{v}^*)] \\ e^{-r\,t}(\bar{v} - \underline{v}) & \text{for all} \quad t \in (d^*(\hat{v}^*), T]. \end{cases}$$

It is not hard to see that $\exp(e^{-r\,t} - 1) - e^{-r\,t} \geq 0$ for all $t \geq 0$ and so $e^{-r\,t}[\bar{v} - p^*_{\mathrm{S}}(t)] \leq R^*_0$ for all $t \in [0, d^*(\hat{v}^*)]$. On the other hand, the price path $p^*_{\mathrm{S}}(t)$ is continuous at $t = d^*(\hat{v}^*)$ and $e^{-r\,t}[\bar{v} - p^*_{\mathrm{S}}(t)] = e^{-r\,t}(\bar{v} - \underline{v}) \leq e^{-r\,d^*(\hat{v}^*)}(\bar{v} - \underline{v}) \leq R^*_0$ for all $t \in (d^*(\hat{v}^*), T]$. □

PROOF OF PROPOSITION 4: For a given price path $p \in \mathcal{P}$, let us partition the space of customer types $\mathcal{D}$ into $\mathcal{D}_b$ and $\mathcal{D}_l$, where $\mathcal{D}_b$ represents the set of types that buy the product and $\mathcal{D}_l$ the set of types that are priced out of the market under p. The sets $\mathcal{D}_b$ and $\mathcal{D}_l$ are independent of whether the customers are myopic or strategic. Hence, the regret generated by a customer with type $(v,\tau)\in\mathcal{D}_l$ under p is equal to $\mathcal{R}_\theta(v,\tau,p)=e^{-r\,\tau}\,v$ for any $\theta\in\{M,S\}$.

For a given consume type $(v,\tau)\in\mathcal{D}_b$ and price path p, it is straightforward to see that a strategic consumer will purchase later than a myopic consumer, i.e., $d_{\mathrm{S}}(v,\tau,p)\ge d_{\mathrm{M}}(v,\tau,p)$. We now show that a strategic consumer also buys at a lower price, i.e., $p_{d_{\mathrm{S}}(v,\tau,p)}\le p_{d_{\mathrm{M}}(v,\tau,p)}$. Suppose $p_{d_{\mathrm{S}}(v,\tau,p)}>p_{d_{\mathrm{M}}(v,\tau,p)}$. Then the utility for the strategic customer at time $d_{\mathrm{S}}(v,\tau,p)$ is $e^{-rd_{\mathrm{S}}(v,\tau,p)}(v-p_{d_{\mathrm{S}}(v,\tau,p)})$, which is strictly less than $e^{-rd_{\mathrm{M}}(v,\tau,p)}(v-p_{d_{\mathrm{M}}(v,\tau,p)})$. This implies that the strategic customer could have gained a higher utility if he purchased the product at time $d_{\mathrm{M}}(v,\tau,p)$, which contradicts to the fact that $d_{\mathrm{S}}(v,\tau,p)$ is the time when the utility of strategic customer is maximized.

Therefore, the seller can always extract a higher surplus from a myopic consumer than from a strategic one, i.e., $e^{-rd_{\mathrm{M}}(v,\tau,p)}p_{d_{\mathrm{M}}(v,\tau,p)}\ge e^{-rd_{\mathrm{S}}(v,\tau,p)}p_{d_{\mathrm{S}}(v,\tau,p)}$. Thus, the regret from selling to strategic customer is higher, or $\mathcal{R}_{\mathrm{S}}(v,\tau,p)\ge\mathcal{R}_{\mathrm{M}}(v,\tau,p)$ for all v,τ and p. Since, $R^*_\theta=\inf_{p\in\mathcal{P}}\sup_{(v,\tau)\in\mathcal{D}}\mathcal{R}_\theta(v,\tau,p)$, we obtain that $\mathcal{R}^*_{\mathrm{S}}\ge\mathcal{R}^*_{\mathrm{M}}$. □

Proof of Proposition 5: From the proof of Proposition 4, we know that for under any given price path, a strategic customer yields higher regret than a myopic customer with the same type. We conclude that for any $p\in\mathcal{P}$ and $(v,\tau)\in\mathcal{D}$, nature will select a strategic buyer $\theta=S$, and the result follows. □

Proof of Proposition 6: The proof follows the same steps as the proof of Proposition 1 and is therefore omitted. □

Proof of Proposition 7: The proof follows the same steps as the proof of Proposition 2 and is therefore omitted. □

Proof of Proposition 8: From the identity $\mathcal{D}_0\subseteq\widehat{\mathcal{D}}\subseteq\mathcal{D}$ we have that

$$R^*_0=\inf_{p\in\mathcal{P}}\sup_{(v,\tau)\in\mathcal{D}_0}\mathcal{R}_{\mathrm{S}}(v,\tau,p)\le\inf_{p\in\mathcal{P}}\sup_{(v,\tau)\in\widehat{\mathcal{D}}}\mathcal{R}_{\mathrm{S}}(v,\tau,p)\le\inf_{p\in\mathcal{P}}\sup_{(v,\tau)\in\mathcal{D}}\mathcal{R}_{\mathrm{S}}(v,\tau,p)=R^*_{\mathrm{S}}.$$

However, from Theorem 4 we have that $R^*_0=R^*_{\mathrm{S}}$. □

Appendix B: Modified Gallego and van Ryzin (1994) Model

In this appendix, we compare the optimal price path resulting from our robust minimax approach to one of the most well-cited models of dynamic pricing within the OR/MS literature. It is probably fair to say that one of the most influential papers on this topic is the *Management Science* paper by Gallego and van Ryzin (1994), who consider a monopolist selling a finite inventory of a product over a finite selling horizon. At the core of Gallego and van Ryzin's pricing model is a dynamic programming formulation that produces the following optimality condition for an optimal price $p^*(t,n)$ as function of time-to-go t and available inventory n:

$$p^*(t,n) = \operatorname{argmax}\left\{\lambda(p)\left[p - (J(t,n) - J(t,n-1)\right]\right\}, \qquad (a)$$

where $\lambda(p)$ is the demand intensity (a non-negative and decreasing function of p) and $J(t,n)$ is the value function, that is, the seller's expected optimal revenues as a function of the state (t,n). (In Gallego and van Ryzin (1994), t denotes time-to-go as opposed to calendar time).

One of the key results in Gallego and van Ryzin (1994) is Theorem 1 that shows that the value function is strictly increasing and strictly concave in both t and n. Using this result and equation (a) above, one can show that for many commonly used demand functions, the optimal price path is decreasing concave in calendar time. For example, in Section 2.3 in Gallego and van Ryzin (1994), the authors consider the special case of an exponential demand function and show that $p^*(t,n) = 1 + J(t,n) - J(t,n-1)$, where

$$J(t,n) = \log\left(\sum_{i=0}^{n} \frac{(\lambda^*(T-t))^i}{i!}\right), \quad \text{where } \lambda^* := \lambda(p^*) \text{ and } \quad p^* := \arg\max\{p\,\lambda(p)\}.$$

Figure 12 depicts $p^*(t,n)$ for $n = 1,2,3$ under an exponential demand model with $T = 1$ and $\lambda^* = 10$. A distinctive feature of these prices paths in the Gallego and van Ryzin model is that they exhibit "*accelerating markdowns*". This is in contrast to some of the optimal price paths in our robust minimax setting that have "*decelerating markdowns*" (e.g., upper bound price path depicted in Figure 5.

The attentive reader may have have noticed that there is one important aspect of our model that is not captured by the Gallego and van Ryzin (1994) framework, namely, the fact that the seller uses a discount factor, r, to penalize future cash-flows. Since the use of a discount factor creates pressure on the seller to try to expedite sales and possibly engage in early markdowns, it is plausible that an optimal price path in a modified Gallego and van Ryzin setting that includes discounting could exhibit decelerating markdowns. In what follows we show that this is not the case.

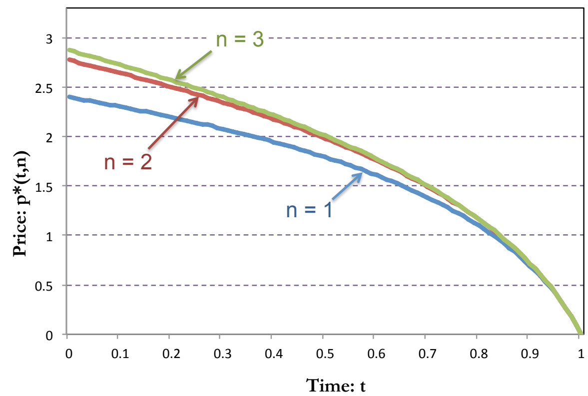

Figure 12 Optimal price paths for the Gallego and van Ryzin model when consumers have exponential willingness to pay.

As we argued in the previous paragraph, for the purpose of having a more direct and fair (apples-to-apples) comparison between the results in Gallego and van Ryzin and our model, we next modify their setting to account for this difference. We would also like to select the values of n and $\lambda(p)$ in a way that it is consistent with our distribution-free analysis. To this end, we first note that regarding our robust prior-free formulation, the "natural" choice for modeling consumers' willingness-to-pay is a Uniform distribution in $[\underline{v},\bar{v}]$, which is equivalent to assuming that the demand function $\lambda(p)$ is linear, or $\lambda(p)=\Lambda\,(\bar{v}-p)/(\bar{v}-\underline{v})$ for $p\in[\underline{v},\bar{v}]$ and for some positive scalar Λ. In addition, by the linearity of the regret on the number of customers (Proposition 1), we can restrict our analysis (w.l.o.g.) to the case of a single consumer. Therefore, we set $n=1$ and select the values of Λ and T (the selling horizon) so that $\Lambda\,T=1$. Under these conditions, one can show that the *modified Gallego and van Ryzin model* (that incorporates discounting) leads to the following optimality (HJB) condition for the value function $J(t)$

$$-\frac{\mathrm{d}J(t)}{\mathrm{d}t}=\max_p\left\{\Lambda\,\frac{(\bar{v}-p)}{(\bar{v}-\underline{v})}\,[p-J(t)]\right\}-r\,J(t),\quad \text{with border condition } J(T)=0.$$

After the solving the maximization on p, this HJB equation reduces to a standard Riccati equation that leads to following solution.

$$J(t)=\bar{v}+4\,\frac{(\bar{v}-\underline{v})}{\Lambda}\left(\frac{A\,\alpha_1\,e^{\alpha_1\,t}+\alpha_2\,e^{\alpha_2\,t}}{A\,e^{\alpha_1\,t}+e^{\alpha_2\,t}}\right),$$

where

$$A=-\left(\frac{4\alpha_2(\bar{v}-\underline{v})+\Lambda\,\bar{v}}{4\alpha_1(\bar{v}-\underline{v})+\Lambda\,\bar{v}}\right)e^{(\alpha_2-\alpha_1)T}\quad\text{and}\qquad \alpha_{1,2}=\frac{r\pm\sqrt{r^2+\Lambda\,r/(\bar{v}-\underline{v})}}{2}.$$

Finally, the optimal price policy is given by

$$p^*(t)=\frac{\bar{v}+J(t)}{2}.$$

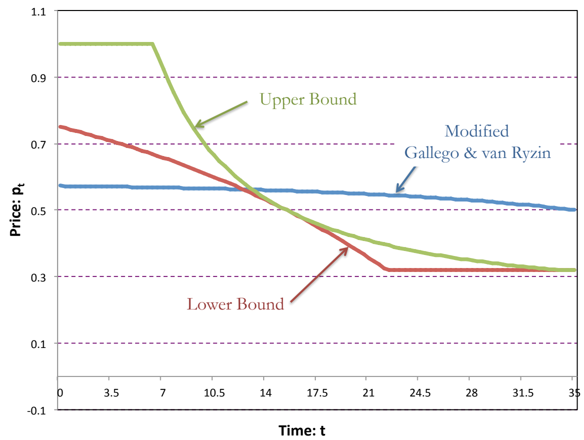

Figure 13 Comparison of robust price paths to modified Gallego and van Ryzin solution.

Figure 13 plots this *modified Gallego and van Ryzin* price path using the data in Figure 5 in the paper, namely $\bar{v} = 1$, $\underline{v} = 0.32$, $T = 35$ and $r = 0.045$. The figure also depicts our distribution-free upper bound $\bar{p}_t^*$ and lower bound $\underline{p}_t^*$ price paths.

Again, we can see that despite the additional pressure to start marking down early created by the discount factor, optimal price paths in the Gallego and van Ryzin framework remain concave, *i.e.*, exhibit *accelerating markdowns*. We also note that the optimal price path in the modified Gallego and van Ryzin model is more stable (flat) than the optimal robust counterpart.